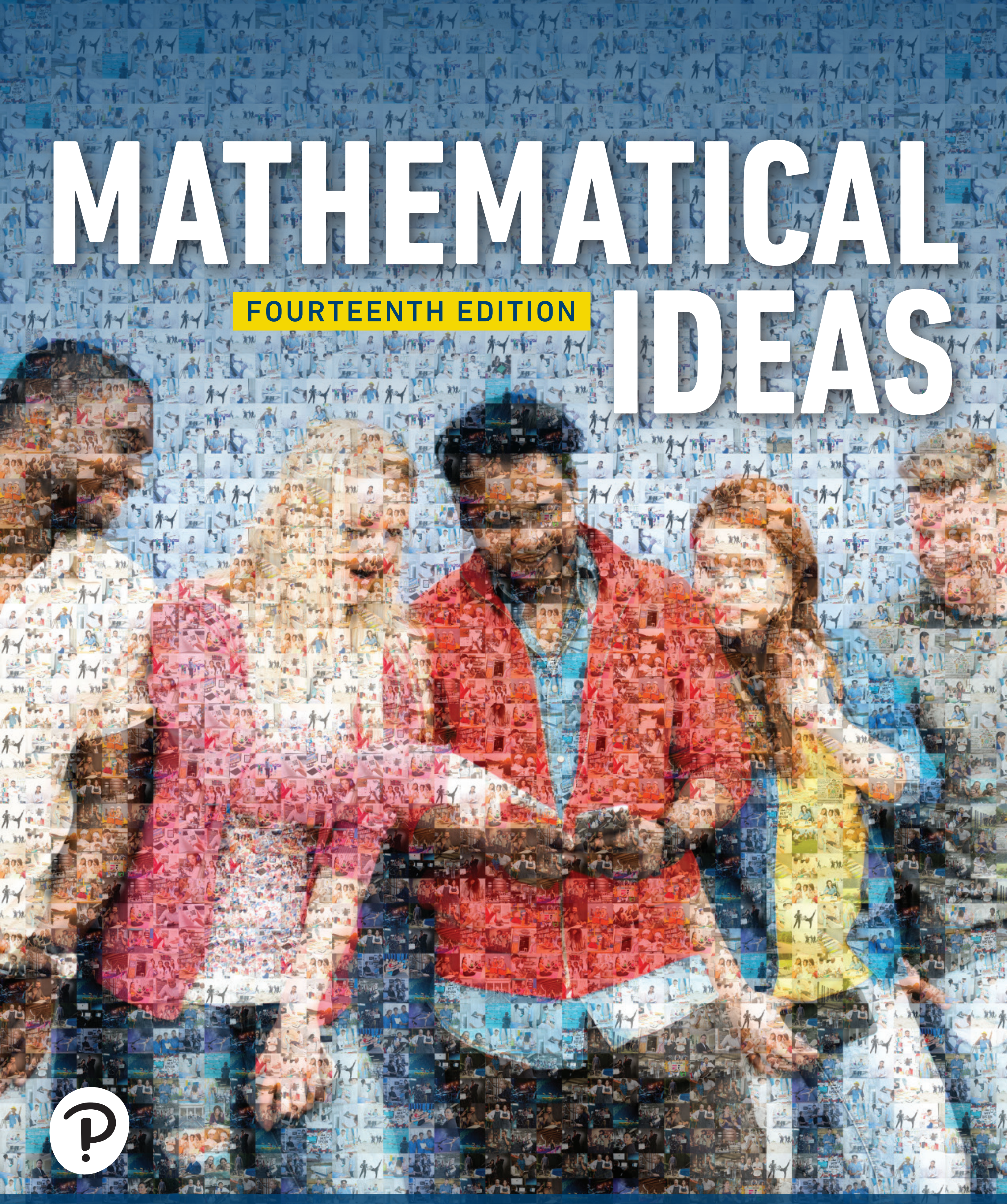
MATHEMATICAL IDEAS

FOURTEENTH EDITION



Charles D. Miller | Vern E. Heeren | John Hornsby | Christopher Heeren

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14th EDITION

MATHEMATICAL IDEAS

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Pearson

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To Sweet Baby Jess—we love you—PPJ

To my beloved wife, Carole, and to our three sons, their wives, and our ten grandchildren, each one a continuing inspiration in my life—VERN

To my children—you continually inspire me with your courage and creativity—CHRIS

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*Online section can be found in the MyLab Math course or at www.pearsonhighered.com/mathstatsresources

NOTE: Trigonometry module and Metrics module available in MyLab Math or online at www.pearsonhighered.com/mathstatsresources.

PREFACE

After thirteen editions and over five decades, *Mathematical Ideas* continues to be one of the most popular texts in liberal arts mathematics education. We are proud to present the fourteenth edition of a text that offers non-STEM students a practical coverage that connects mathematics to the world around them. It is a flexible text that has evolved alongside changing trends but remains steadfast to its original objectives.

Mathematical Ideas is written with a variety of students in mind. It is well suited for several courses, including those geared toward the aforementioned liberal arts audience and survey courses in mathematics or finite mathematics. Students taking these courses will pursue careers in nursing and healthcare, the construction trades, communications, hospitality, information technology, criminal justice, retail management and sales, computer programming, political science, school administration, and myriad other fields. Accordingly, we have chosen to continue showcasing how the math in this course will be relevant in this wide array of career options. Chapter openers continue to focus on particular careers commonly anticipated by students.

- Chapter openers address how the chapter topics can be applied within the context of work and future careers.
- We made sure to retain the hundreds of examples and exercises from the previous edition that pertain to these interests.
- Every chapter also contains from one to three of the popular ***When Will I Ever Use This?*** features that help students connect mathematics to the workplace.

Interesting and mathematically pertinent movie and television applications and references are still interspersed throughout the chapters.

Ample topics are included for a two-term course, yet the variety of topics and flexibility of sequence make the text suitable for shorter courses as well. Our main objectives continue to be comprehensive coverage, appropriate organization, clear exposition, an abundance of examples, and well-planned exercise sets with numerous applications.

New and Expanded in This Edition

- Career applications have taken on greater prominence, especially through the inclusion of additional *When Will I Ever Use This?* feature boxes.
- Many of the learning objectives that begin each section have been reworded for clarity and specificity.
- All exercise sets now begin with **Concept Check** questions to confirm that students truly understand concepts before diving right into homework. Exercise sets have once again been updated, and there are over 1000 new or modified exercises, many with an emphasis on career applications.
- Most **chapter summaries** have been reorganized to improve readability. They still include the following components:
  - A list of **Key Terms** for each section of the chapter
  - **New Symbols** with definitions, to clarify newly introduced symbols
  - **Test Your Word Power** questions that enable students to test their knowledge of new vocabulary
  - A **Quick Review** that gives a brief summary of concepts covered in the chapter, along with examples illustrating those concepts
- Actual screen captures from the **TI-84 Plus C** model calculator are now included.
- The presentation has been made more uniform whenever clarity for the reader could be served.

- The general style has been freshened with more pedagogical use of color, new photos and art, and opening of the exposition.
- A new **Workbook with Investigations and Integrated Review Worksheets** accompanies the text. The workbook contains worksheets for every objective in the text, Investigation activities students can do in or out of class on their own or in a group, and worksheets to complement the Integrated Review content in the MyLab Math course.
- **NEW online resources**
  - **StatCrunch** applets and activities are available as indicated by the StatCrunch icon in the margin. Data sets will also be available for ease of use in the StatCrunch program.
  - **Group review modules in Learning Catalytics Modules** have been developed to complement each chapter's review test for use as classroom review. Using this student engagement, assessment, and classroom intelligence system gives instructors real-time feedback on student learning before the test. Instructions on how to use the Learning Catalytics Chapter Test modules are provided in the MyLab Math course.
  - The **Corequisite MyLab Math Support Course** takes the traditional integrated review content a step further to provide one complete solution for instructors considering a corequiste model. By mapping relevant algebra objectives from the *Mathematical Ideas* text to particular chapters and sections, students have access to the foundational support they need to successfully complete the course. The Corequisite Support Course includes over 30 review topics and enables students to access material for both their credit and support course with a single access code.
  - **Section lecture videos** will cover the section objectives with a modern, clear, and engaging approach. The videos will incorporate the use of animations, applets, and StatCrunch to demonstrate concepts, making use of resources available to students in the MyLab Math course.
  - The **Animations Library** will help students explore concepts more deeply, encouraging them to visualize and interact with concepts such as Venn diagrams, probability, logic, debt, graph theory, geometry, statistics, and more. An Animation icon in the text indicates where the animations can aid in understanding.
  - **Mindset** materials are included in the **Video & Resource Library: Skills for Success** section of the MyLab Math course, along with Math-Reading Connections, College Success Modules, and Professionalism Tools.

Overview of Chapters

- **Chapter 1 (The Art of Problem Solving)** introduces the student to inductive reasoning, pattern recognition, and problem-solving techniques. We continue to provide exercises based on the monthly Calendar from *Mathematics Teacher* and have added new ones throughout this edition. The chapter opener recounts the solving of the Rubik's cube by a college professor. The *When Will I Ever Use This?* feature shows how estimation techniques may be used by a group home employee charged with holiday grocery shopping.
- **Chapter 2 (The Basic Concepts of Set Theory)** includes updated examples and exercises on surveys. The chapter opener suggests ways in which set theory can apply to exploring career opportunities in nursing and other health-related careers. One *When Will I Ever Use This?* feature applies Venn diagrams to a genetic brain disease study. Another applies survey techniques to the allocation of work crews in the building trade.

- **Chapter 3 (Introduction to Logic)** introduces the fundamental concepts of inductive and deductive logic. The chapter opener connects logic with fantasy literature, and new exercises further illustrate this relationship. *For Further Thought* addresses common fallacies in everyday life. One *When Will I Ever Use This?* feature connects circuit logic to the design and installation of home monitoring systems. Another shows a pediatric nurse applying a logical flowchart and truth tables to a child's vaccination protocol.
- **Chapter 4 (Numeration Systems)** covers historical numeration systems, including Egyptian, Roman, Chinese, Babylonian, Mayan, Greek, and Hindu-Arabic systems. A connection between base conversions in positional numeration systems and Web design is suggested in the new chapter opener and illustrated in the *When Will I Ever Use This?* feature and a new example.
- **Chapter 5 (Number Theory)** presents an introduction to the prime and composite numbers, the Fibonacci sequence, and a cross section of related historical developments. The largest currently known prime numbers of various categories are identified, and recent progress on Goldbach's conjecture and the twin prime conjecture are noted. The chapter opener and one *When Will I Ever Use This?* feature apply cryptography and modular arithmetic to criminal justice in the context of cyber security. Another such feature shows how a health practicioner might use the concept of least common denominator in determining proper drug dosage.
- **Chapter 6 (The Real Numbers and Their Representations)** introduces some of the basic concepts of real numbers, their various forms of representation, and operations of arithmetic with them. The chapter opener and *When Will I Ever Use This?* feature discuss applying percents to tipping and retail pricing markup.
- **Chapter 7 (The Basic Concepts of Algebra)** can be used to present the basics of algebra (linear and quadratic equations, applications, exponents, polynomials, and factoring) to students for the first time, or as a review of previous courses. One *When Will I Ever Use This?* feature applies proportions to an automobile owner's determination of fuel mileage, and another relates inequalities to computation of the score needed to maintain a certain grade point average.
- **Chapter 8 (Graphs, Functions, and Systems of Equations and Inequalities)** is the second of our two algebra chapters. It continues with graphs, equations, and applications of linear, quadratic, exponential, and logarithmic functions and models, along with systems of equations. The chapter opener shows how an automobile owner can use a linear graph to relate price per gallon, amount purchased, and total cost. One *When Will I Ever Use This?* feature discusses a "Eureka moment" pertaining to pumping gasoline, and another connects logarithms with the interpretation of earthquake reporting in the news.
- **Chapter 9 (Geometry)** covers elementary plane geometry, transformational geometry, basic geometric constructions, non-Euclidean geometry (including projective geometry), and chaos and fractals. The chapter opener and one *When Will I Ever Use This?* feature connect geometric volume formulas to a video game developer's job of designing the visual field of a game screen. Another *When Will I Ever Use This?* feature relates right triangle geometry to a forester's determination of safe tree-felling parameters, and a third connects perimeter and circumference formulas to a pest control specialist's task of determining the quantity of pesticide needed to prevent termite infestation.
- **Chapter 10 (Counting Methods)** focuses on elementary counting techniques, in preparation for the probability chapter. The chapter opener relates how a restaurateur used counting methods to help design the sales counter signage in a new restaurant. One *When Will I Ever Use This?* feature describes an entrepreneur's use of probability and sports statistics in designing a game and in building a successful company based on it. Another explains the categories of hands in 5-card poker. The binomial theorem is now discussed in more detail.

- **Chapter 11 (Probability)** covers the basics of probability, odds, and expected value. The chapter opener relates to the professions of weather forecaster, actuary, baseball manager, and corporate manager, applying probability, statistics, and expected value to interpreting forecasts, determining insurance rates, selecting optimum strategies, and making business decisions. One *When Will I Ever Use This?* feature shows how a tree diagram helps a decision maker provide equal chances of winning to three players in a game of chance. A second such feature shows how knowledge of probability can help some TV game show contestants to choose a strategy that will boost their odds of winning! Many additional exercises have been added to Section 11.3.

- **Chapter 12 (Statistics)** is an introduction to statistics that focuses on the measures of central tendency, dispersion, and position and discusses the normal distribution and its applications. The chapter opener and two *When Will I Ever Use This?* features connect probability and graph construction and interpretation to how a psychological therapist may motivate and carry out treatment for alcohol and tobacco addiction. A third such feature applies normal curves to analyze worker skills in an electronics assembly plant.

- **Chapter 13 (Personal Financial Management)** provides the student with the basics of the mathematics of finance as applied to inflation, consumer debt, and home mortgages. We also include a section on investing, with emphasis on stocks, bonds, and mutual funds. Tables, examples, and exercises have been updated to reflect current interest rates and investment returns. Margin notes feature smart apps for financial calculations. In response to reviewer requests, added emphasis has been placed on the use of financial calculators in examples and exercises. This chapter includes two *When Will I Ever Use This?* features. The first explores the cost-effectiveness of solar energy, applying chapter topics essential for a sales representative in the solar energy industry. The second shows how a financial planner might apply several topics addressed in the chapter to compare the cost of renting a house to the cost of buying one. The chapter opener connects the time value of money to how a financial planner can help clients make wise financial decisions.

- **Chapter 14 (Graph Theory)** covers the basic concepts of graph theory and its applications. The chapter opener shows how a writer can apply graph theory to the analysis of poetic rhyme. One *When Will I Ever Use This?* feature connects graph theory to how a postal or delivery service manager could determine the most efficient delivery routes. Another tells of a unique use by an entrepreneur who developed a business based on finding time-efficient ways to navigate theme parks.

- **Chapter 15 (Voting and Apportionment)** deals with issues in voting methods and apportionment of representation, topics that have become increasingly popular in liberal arts mathematics courses. To illustrate the important work of a political consultant, the chapter opener connects different methods of analyzing votes. One *When Will I Ever Use This?* feature relates voting methods to the functioning of governing boards. Another gives an example of how understanding apportionment methods can help in the work of a school administrator. A new margin note and a new *For Further Thought* feature address the increasingly contentious issues related to the electoral college and gerrymandering, respectively.

Course Outline Considerations

Chapters in the text are, in most cases, independent and may be covered in the order chosen by the instructor. The few exceptions are as follows:

- Chapter 6 contains some material dependent on the ideas found in Chapter 5.
- Chapter 6 should be covered before Chapter 7 if student background so dictates.
- Chapters 7 and 8 form an algebraic "package" and should be covered in sequential order.
- A thorough coverage of Chapter 11 depends on knowledge of Chapter 10 material, although probability can be covered without teaching extensive counting methods by avoiding the more difficult exercises.

Features of the Fourteenth Edition

Chapter Openers In keeping with the career theme, chapter openers address a situation related to a particular career. Some are new to this edition, and some include a problem that the reader is asked to solve. We hope that you find these chapter openers useful and practical.

ENHANCED! Varied Exercise Sets We continue to present a variety of exercises that integrate drill, conceptual, and applied problems, and there are over 1000 new or modified exercises in this edition. The text contains a wealth of exercises to provide students with opportunities to practice, apply, connect, and extend the mathematical skills they are learning. We have updated the exercises that focus on real-life data and have retained their titles for easy identification. Several chapters are enriched with new applications, particularly Chapters 6, 7, 8, 11, 12, and 13. We continue to use graphs, tables, and charts where appropriate. Many of the graphs are presented in a style similar to that seen by students in today's print and electronic media.

UPDATED! Emphasis on Real Data in the Form of Graphs, Charts, and Tables We continue to use up-to-date information from magazines, newspapers, and the Internet to create real applications that are relevant and meaningful.

Problem-Solving Strategies Special paragraphs labeled "Problem-Solving Strategy" relate the discussion of problem-solving strategies to techniques that have been presented earlier.

For Further Thought These entries, following the exercise sets of many sections, encourage students to share their reasoning processes among themselves to gain a deeper understanding of key mathematical concepts.

ENHANCED! When Will I Ever Use This? These features in each chapter connect chapter topics to career or workplace situations (and answer that age-old question)!

ENHANCED! Margin Notes This popular feature is a hallmark of this text and has been retained and updated where appropriate. These notes are interspersed throughout the text and are drawn from various sources, such as the lives of mathematicians, historical vignettes, anecdotes on mathematics textbooks of the past, newspaper and magazine articles, and current research in mathematics.

Optional Graphing Technology We continue to provide graphing calculator screens to show how technology can be used to support results found analytically. It is not essential, however, that a student have a graphing calculator to study from this text. The technology component is optional.

ENHANCED! Chapter Summaries Extensive summaries at the end of each chapter include Key Terms, New Symbols with definitions, Test Your Word Power vocabulary checks, and a Quick Review that provides a brief summary of concepts (with examples) covered in the chapter.

Chapter Tests Each chapter concludes with a chapter test so that students can check their mastery of the material.

Online Resources

- **NEW! StatCrunch** applets and activities are available as indicated by the StatCrunch icon in the margin. Data sets will also be available for ease of use in the StatCrunch program.
- ***Trigonometry, Metrics, and Magic Squares** content that was previously in the text is now found in the MyLab Math course, including the assignable MyLab Math questions.
- ***Extensions** cover topics such as Infinite Sets and their Cardinalities, Logic Problems, Sudoku, Z, and Modern Cryptography. Extensions include instruction, exercises, and homework problems in the MyLab Math course.
- **NEW! Group chapter review modules are available in Learning Catalytics** The team-based assessment format facilitates both individual review outside of class and group interaction during class. As students enter responses using smartphones or other connected devices, instructors receive immediate feedback, allowing them to focus on concepts that need attention.
- **ENHANCED! Integrated Review** content in the MyLab Math course uses newly developed worksheets including key terms, summaries, notes, guided problems, and practice problems. This MyLab™ includes a full suite of supporting Integrated Review resources for the Mathematical Ideas course, including pre-made, assignable (and editable) quizzes to assess the prerequisite skills needed for each chapter, and personalized remediation for any gaps in skills that are identified. Each student, therefore, receives just the help that he or she needs–no more, no less.
- **NEW! Corequisite MyLab Math** takes the traditional integrated review content a step further to provide one complete solution for instructors considering a corequiste model. By mapping relevant algebra objectives from the *Mathematical Ideas* text to particular chapters and sections, students have access to the foundational support they need to successfully complete the course. The Corequisite Support Course includes over 30 review topics and enables students to access material for both their credit and support course with a single access code.
- **NEW!** The **Animations Library** will help students explore concepts more deeply, encouraging them to visualize and interact with concepts such as Venn diagrams, probability, logic, debt, graph theory, geometry, statistics, and more. An Animation icon in the text indicates where the animations can aid in understanding.

*Online material can be found in the MyLabMath course or at www.pearsonhighered.com/mathstatsresources

- Culinary math review, basic math in baking, methods to calculate drug doses, pharmacology mathematics review, carpentry, and nursing modules are all available in the MyLab Math course to further support the connection to careers and the workplace.
- **NEW! Mindset** materials are included in the **Video & Resource Library: Skills for Success** section of the MyLab Math course, along with Math-Reading Connections, College Success Modules, and Professionalism Tools.
- **Video Program**
    - **NEW! Section lecture videos** will cover the section objectives with a modern, clear, and engaging approach. The videos will incorporate the use of animations, applets, and StatCrunch to demonstrate concepts, making use of resources available to students in the MyLab Math course.
    - **Interactive Concept Check videos with assignable MyLab Math questions** These videos walk students through a concept and then ask them to answer a question within the video. If students answer correctly, the concept is summarized. If a student selects one of the two incorrect answers, the video continues focusing on why students probably selected that answer and works to correct that line of thinking and explain the concept. Then students get another chance to answer a question to prove mastery.
    - **When Will I Ever Use This? videos** bring the ideas in the *When Will I Ever Use This?* feature to life in a fun, memorable way.

Resources for Success

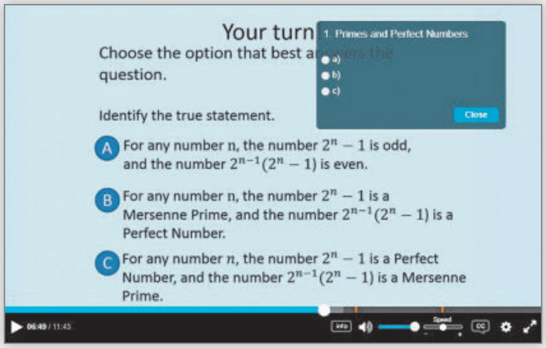
Pearson MyLab

MyLab Math Online Course for Math Ideas
by Miller, Heeren, Hornsby & Heeren (access code required)

MyLab™ Math is the teaching and learning platform that empowers you to reach *every* student. By combining trusted author content with digital tools and a flexible platform, MyLab Math personalizes the learning experience and improves results for each student.

Interactive Concept Videos

These videos walk students through key concepts and pause to ask them questions, requiring student interaction throughout. For each incorrect answer, the video follows a different path focusing on the reason or misconception for selecting that particular incorrect answer. Students are then presented with another interactive question to gauge understanding. Assignable exercises are available for instructors to assess student understanding.



Skills for Success and Mindset Modules

These modules are designed to help students succeed in college courses and prepare for life after college. In addition to **assignable Study Skills and Math/Reading Connections** material, new **Mindset material** has been added to the course. This content encourages students to maintain a positive attitude about learning, value their own ability to grow, and view mistakes as a learning opportunity.

StatCrunch

StatCrunch® is a powerful web-based statistical software program with pre-built applets, experiments, and simulations that bring concepts to life. StatCrunch applets and activities are integrated into the text, giving students an opportunity to visualize and interact with concepts at the point of use. StatCrunch resources are also provided to instructors to demonstrate additional ways to use StatCrunch in the classroom.

Group Review Modules in Learning Catalytics

Learning Catalytics™ is an interactive student response tool that uses students' smartphones, tablets, or laptops to engage students and promote peer-to-peer learning. The author has developed modules to complement each chapter's review test for use as a group review activity in class.

Pearson MyLab

Resources for Success

Instructor Resources

Annotated Instructor's Edition

When possible, answers are on the page with the exercises. Longer answers are in the back of the book.

Online Supplements

The following instructor material is available for download from Pearson's Instructor Resource Center (www.pearsonhighered.com/irc) or within the text's MyLab Math course.

Instructor's Solutions Manual

This Manual includes fully worked solutions to all text exercises.

PowerPoint® Lecture Slides

These fully editable lecture slides present key concepts and definitions from the text. Fully accessible PowerPoint lecture slides are also available. The **Image Resource Library**, found in the Instructor Resources section of the MyLab Math course, includes art from the text that instructors can use when editing the existing lecture slides or for developing their own slides or worksheets.

Instructor's Testing Manual

This manual includes tests with answer keys for each chapter of the text.

TestGen®

Enables instructors to build, edit, print, and administer tests using a computerized bank of questions developed to cover all the objectives of the text.

Student Resources

New! Workbook with Investigations and Integrated Review Worksheets

ISBN: 0-13-499726-3 / 978-0-13-499726-1

Worksheets for every objective in the text include key terms, summaries, a notes section, guided problems and practice problems. Also included are Investigations students can do in or out of class, on their own or in a group. Worksheets to support the integrated review content in the MyLab Math course are also included.

Student Solutions Manual

ISBN: 0-13-499736-0 / 978-0-13-499736-0

This manual provides detailed worked-out solutions to odd-numbered exercises.

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Henry Wyzinski, *Indiana University Northwest*

A project of this magnitude cannot be accomplished without the help of many other dedicated individuals. Marnie Greenhut served as portfolio manager for this edition. Carol Merrigan provided excellent production supervision. Anne Kelly and Patty Bergin of Pearson gave us their unwavering support.

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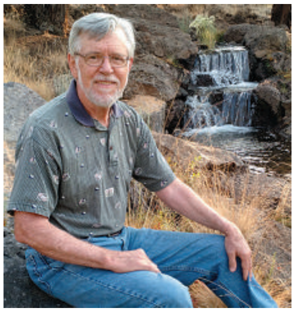
Vern E. Heeren
John Hornsby
Christopher Heeren

ABOUT THE AUTHORS



Vern Heeren grew up in the Sacramento Valley of California. After earning a Bachelor of Arts degree in mathematics, with a minor in physics, at Occidental College, and completing his Master of Arts degree in mathematics at the University of California, Davis, he began a 38-year teaching career at American River College, teaching math and a little physics. He coauthored *Mathematical Ideas* in 1968 with office mate Charles Miller, and he has enjoyed researching and revising it over the years. It has been a joy for him to complete the fourteenth edition, along with long-time coauthor John Hornsby, and with son Christopher.

These days, besides pursuing his mathematical interests, Vern enjoys spending time with his wife Carole and their family, exploring the wonders of nature at and near their home in central Oregon.

John Hornsby joined the author team of Margaret Lial, Charles Miller, and Vern Heeren in 1988. In 1990, the sixth edition of *Mathematical Ideas* became the first of nearly 150 titles he has coauthored for Scott Foresman, HarperCollins, Addison-Wesley, and Pearson in the years that have followed. His books cover the areas of developmental and college algebra, precalculus, trigonometry, and mathematics for the liberal arts. He is a native and resident of New Roads, Louisiana.

Christopher Heeren is a native of Sacramento, California. While studying engineering in college, he had an opportunity to teach a math class at a local high school, and this sparked both a passion for teaching and a change of major. He received a Bachelor of Arts degree and a Master of Arts degree, both in mathematics, from California State University, Sacramento. Chris has taught mathematics at the middle school, high school, and college levels, and he currently teaches at American River College in Sacramento. He has a continuing interest in using technology to bring mathematics to life. When not writing, teaching, or preparing to teach, Chris enjoys spending time with his lovely wife Heather and their three children.

1 The Art of Problem Solving

Professor Terry Krieger, of Rochester (Minnesota) Community College, shares his thoughts about why he decided to become a mathematics teacher. He is an expert at the Rubik's Cube. Here, he explains how he mastered this classic puzzle.

From a very young age I always enjoyed solving problems, especially problems involving numbers and patterns. There is something inherently beautiful in the process of discovering mathematical truth. Mathematics may be the only discipline in which different people, using wildly varied but logically sound methods, will arrive at the same correct result—not just once, but every time! It is this aspect of mathematics that led me to my career as an educator. As a mathematics instructor, I get to be part of, and sometimes guide, the discovery process.

I received a Rubik's Cube as a gift my junior year of high school. I was fascinated by it. I devoted the better part of three months to solving it for the first time, sometimes working 3 or 4 hours per day on it.

There was a lot of trial and error involved. I devised a process that allowed me to move only a small number of pieces at a time while keeping other pieces in their places. Most of my moves affect only three or four of the 26 unique pieces of the puzzle. What sets my solution apart from those found in many books is that I hold the cube in a consistent position and work from the top to the bottom. Most book solutions work upward from the bottom.

I worked on the solution so much that I started seeing cube moves in my sleep. In fact, I figured out the moves for one of my most frustrating sticking points while sleeping. I just woke up knowing how to do it.

The eight corners of the cube represented a particularly difficult challenge for me. Finding a consistent method for placing the corners appropriately took many, many hours. To this day, the amount of time that it takes for me to solve a scrambled cube depends largely on the amount of time that it takes for me to place the corners.

When I first honed my technique, I was able to consistently solve the cube in 2 to 3 minutes. My average time is now about 65 seconds. My fastest time is 42 seconds.

1.1 SOLVING PROBLEMS BY INDUCTIVE REASONING

OBJECTIVES

1 Distinguish between inductive and deductive reasoning.

2 Recognize that inductive reasoning may not always lead to valid conclusions.

Characteristics of Inductive and Deductive Reasoning

The development of mathematics can be traced to the Egyptian and Babylonian cultures (3000 B.C.–A.D. 260) as a necessity for counting and problem solving. To solve a problem, a cookbook-like recipe was given, and it was followed repeatedly to solve similar problems. By observing that a specific method worked for a certain type of problem, the Babylonians and the Egyptians concluded that the same method would work for any similar type of problem. Such a conclusion is called a *conjecture*. A **conjecture** is an educated guess based on repeated observations of a particular process or pattern.

The method of reasoning just described is called *inductive reasoning*.

INDUCTIVE REASONING

Inductive reasoning is characterized by drawing a general conclusion (making a conjecture) from repeated observations of specific examples. The conjecture may or may not be true.

In testing a conjecture obtained by inductive reasoning, it takes only one example that does not work to prove the conjecture false. Such an example is called a **counterexample.**

Inductive reasoning provides a powerful method of drawing conclusions, but there is no assurance that the observed conjecture will always be true. For this reason, mathematicians do not accept a conjecture as an absolute truth until it is formally proved using methods of *deductive reasoning*. Deductive reasoning characterized the development and approach of Greek mathematics, as seen in the works of Euclid, Pythagoras, Archimedes, and others. During the classical Greek period (600 B.C.–A.D. 450), general concepts were applied to specific problems, resulting in a structured, logical development of mathematics.

Animation: Explaining Number Tricks Using Deductive Reasoning

DEDUCTIVE REASONING

Deductive reasoning is characterized by applying general principles to specific examples.

We now look at examples of these two types of reasoning. In this chapter, we often refer to the **natural, or counting, numbers.**

$$1, 2, 3, \ldots \quad \text{Natural (counting) numbers}$$

↑ Ellipsis points

The three dots (*ellipsis points*) indicate that the numbers continue indefinitely in the pattern that has been established. The most probable rule for continuing this pattern is "Add 1 to the previous number," and this is indeed the rule that we follow.

Now consider the following list of natural numbers:

$$2, 9, 16, 23, 30.$$

What is the next number of this list? What is the pattern? After studying the numbers, we might see that

$$2 + 7 = 9, \quad \text{and} \quad 9 + 7 = 16.$$

Do we add 16 and 7 to get 23? Do we add 23 and 7 to get 30? Yes. It seems that any number in the given list can be found by adding 7 to the preceding number, so the next number in the list would be

$$30 + 7 = 37.$$

We set out to find the "next number" by reasoning from observation of the numbers in the list. We may have jumped from these observations to the general statement that any number in the list is 7 more than the preceding number. This is an example of inductive reasoning.

By using inductive reasoning, we concluded that 37 was the next number. Suppose the person making up the list has another answer in mind. The list of numbers

$$2, 9, 16, 23, 30$$

actually gives the dates of Mondays in June if June 1 falls on a Sunday. The next Monday after June 30 is July 7. With this pattern, the list continues as

$$2, 9, 16, 23, 30, 7, 14, 21, 28, \ldots.$$

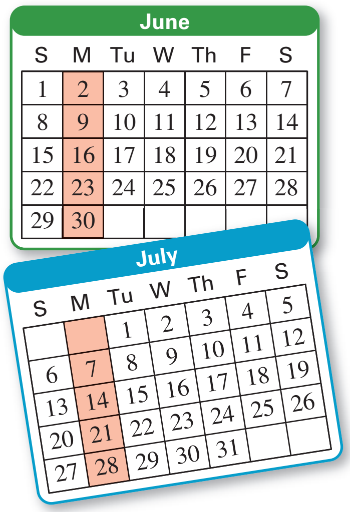


Figure 1

See the calendar in **Figure 1.** The correct answer would then be 7. The process used to obtain the rule "add 7" in the preceding list reveals a main flaw of inductive reasoning.

We can never be sure that what is true in a specific case will be true in general. Inductive reasoning does not guarantee a true result, but it does provide a means of making a conjecture.

We now review some basic notation. Throughout this book, we use *exponents* to represent repeated multiplication.

$$\text{Base} \rightarrow 4^3 = 4 \cdot 4 \cdot 4 = 64 \quad \text{4 is used as a factor 3 times.}$$

↑ Exponent

EXPONENTIAL EXPRESSION

If a is a number and n is a counting number (1, 2, 3, . . .), then the exponential expression a^n is defined as follows.

$$a^n = \underbrace{a \cdot a \cdot a \cdot \ \ldots \ \cdot a}_{n \text{ factors of } a}$$

The number a is the **base** and n is the **exponent.**

With deductive reasoning, we use general statements and apply them to specific situations. For example, a basic rule for converting feet to inches is to multiply the number of feet by 12 in order to obtain the equivalent number of inches. This can be expressed as a formula.

$$\text{Number of inches} = 12 \times \text{number of feet}$$

This general rule can be applied to any specific case. For example, the number of inches in 3 feet is $12 \times 3 = 36$ inches.

Reasoning through a problem usually requires certain *premises*. A **premise** can be an assumption, law, rule, widely held idea, or observation. Then reason inductively or deductively from the premises to obtain a **conclusion.** The premises and conclusion make up a **logical argument.**

EXAMPLE 1 Identifying Premises and Conclusions

Identify each premise and the conclusion in each of the following arguments. Then tell whether each argument is an example of inductive or deductive reasoning.

(a) Our house is made of brick. Both of my next-door neighbors have brick houses. Therefore, all houses in our neighborhood are made of brick.

(b) All keyboards have the symbol @. I have a keyboard. My keyboard has the symbol @.

(c) Today is Tuesday. Tomorrow will be Wednesday.

Solution

(a) The premises are "Our house is made of brick," and "Both of my next-door neighbors have brick houses." The conclusion is "Therefore, all houses in our neighborhood are made of brick." Because the reasoning goes from specific examples to a general statement, the argument is an example of inductive reasoning (although it may very well give a faulty conclusion).

(b) Here the premises are "All keyboards have the symbol @" and "I have a keyboard." The conclusion is "My keyboard has the symbol @." This reasoning goes from general to specific, so deductive reasoning was used.

(c) There is only one premise here, "Today is Tuesday." The conclusion is "Tomorrow will be Wednesday." The fact that Wednesday immediately follows Tuesday is being used, even though this fact is not explicitly stated. Because the conclusion comes from general facts that apply to this special case, deductive reasoning was used.

While inductive reasoning may, at times, lead to false conclusions, in many cases it does provide correct results if we look for the most *probable* answer.

Fibonacci (1170–1250) discovered the sequence named after him in solving a problem on rabbits. Fibonacci ("son of Bonaccio") is one of several names for **Leonardo of Pisa**. His father managed a warehouse in present-day Bougie (or Bejaia), in Algeria. Thus it was that Leonardo Pisano studied with a Moorish teacher and learned the "Indian" numbers that the Moors and other followers of Mohammed brought with them in their westward drive.

Fibonacci wrote books on algebra, geometry, and trigonometry.

EXAMPLE 2 Predicting the Next Number in a List

Use inductive reasoning to determine the *probable* next number in each list below.

(a) 5, 9, 13, 17, 21, 25, 29 **(b)** 1, 1, 2, 3, 5, 8, 13, 21 **(c)** 2, 4, 8, 16, 32

Solution

(a) Each number in the list is obtained by adding 4 to the previous number. The probable next number is $29 + 4 = 33$. (This is an example of an *arithmetic sequence*.)

(b) Beginning with the third number in the list, 2, each number is obtained by adding the two previous numbers in the list. That is,

$$1 + 1 = 2, \quad 1 + 2 = 3, \quad 2 + 3 = 5,$$

and so on. The probable next number is $13 + 21 = 34$. (These are the first few terms of the *Fibonacci sequence*.)

(c) It appears here that to obtain each number after the first, we must double the previous number. Therefore, the probable next number is $32 \times 2 = 64$. (This is an example of a *geometric sequence*.)

EXAMPLE 3 Predicting the Product of Two Numbers

Consider the list of equations. Predict the next multiplication fact in the list.

$$37 \times 3 = 111$$
$$37 \times 6 = 222$$
$$37 \times 9 = 333$$
$$37 \times 12 = 444$$

Solution

The left side of each equation has two factors, the first 37 and the second a multiple of 3, beginning with 3. Each product (answer) consists of three digits, all the same, beginning with 111 for 37×3. Thus, the next multiplication fact would be

$$37 \times 15 = 555, \quad \text{which is indeed true.}$$

Pitfalls of Inductive Reasoning

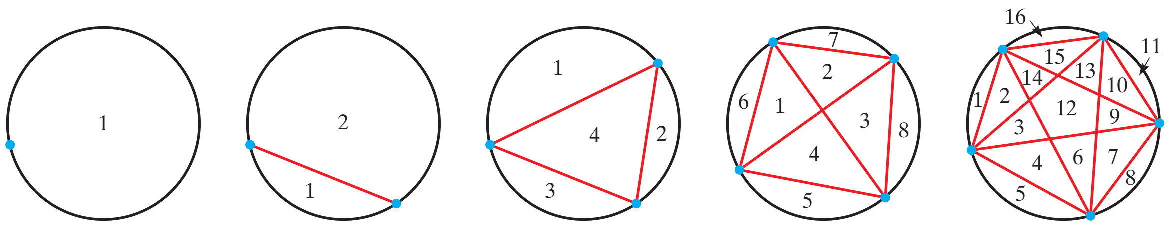
There are pitfalls associated with inductive reasoning. A classic example involves the maximum number of regions formed when chords are constructed in a circle. When two points on a circle are joined with a line segment, a *chord* is formed.

Locate a single point on a circle. Because no chords are formed, a single interior region is formed. See **Figure 2(a)** on the next page. Locate two points and draw a chord. Two interior regions are formed, as shown in **Figure 2(b).** Continue this pattern. Locate three points, and draw all possible chords. Four interior regions are formed, as shown in **Figure 2(c).** Four points yield 8 regions and five points yield 16 regions. See **Figures 2(d) and 2(e).**

The results of the preceding observations are summarized in **Table 1.** The pattern formed in the column headed "Number of Regions" is the same one we saw in **Example 2(c),** where we predicted that the next number would be 64. It seems here that for each additional point on the circle, the number of regions doubles.

Table 1

Number of Points	Number of Regions
1	1
2	2
3	4
4	8
5	16



(a) (b) (c) (d) (e)

Figure 2

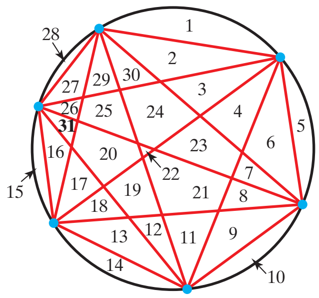


Figure 3

A reasonable inductive conjecture would be that for six points, 32 regions would be formed. But as **Figure 3** indicates, there are *only 31 regions*. The pattern of doubling ends when the sixth point is considered. Adding a seventh point would yield 57 regions. The numbers obtained here are

$$1, 2, 4, 8, 16, 31, 57.$$

For n points on the circle, the number of regions is given by the expression

$$\frac{n^4 - 6n^3 + 23n^2 - 18n + 24}{24}.$$

1.1 EXERCISES

CONCEPT CHECK *Fill in the blank with the correct response.*

1. The first five natural numbers are ________.

2. "If today is Friday, then tomorrow is ________," is an example of valid deductive reasoning.

3. The most *probable* next number in the list 1, 4, 7, 10, 13 is ____.

4. "Donna's first three children were named Anna, Bernard, and Cary. Therefore, her next child's name will start with the letter ____," is an example of inductive reasoning.

5. If the natural numbers between 1 and 8 inclusive are multiplied by 9, the first digit of the product increases by 1 each time, and the second digit decreases by 1 each time, as shown in the following list. Inductive reasoning suggests that the product 9×9 is equal to ____.

$$1 \times 9 = 09$$
$$2 \times 9 = 18$$
$$3 \times 9 = 27$$
$$4 \times 9 = 36$$
$$5 \times 9 = 45$$
$$6 \times 9 = 54$$
$$7 \times 9 = 63$$
$$8 \times 9 = 72$$

6. See **Example 2(b).** The number following 34 in the Fibonacci sequence is ____.

Determine whether the reasoning is an example of deductive or inductive reasoning.

7. The next number in the pattern 2, 4, 6, 8, 10 is 12.

8. My dog barked and woke me up at 1:02 a.m., 2:03 a.m., and 3:04 a.m. So he will bark again and wake me up at 4:05 a.m.

9. To find the perimeter P of a square with side of length s, I can use the formula $P = 4s$. So the perimeter of a square with side of length 7 inches is $4 \times 7 = 28$ inches.

10. A company charges a 10% re-stocking fee for returning an item. So when I return a radio that cost \$150, I will get only \$135 back.

11. If a mechanic says that it will take seven days to repair your SUV, then it will actually take ten days. The mechanic says, "I figure it'll take exactly one week to fix it, ma'am." Then you can expect it to be ready ten days from now.

12. If you take your medicine, you'll feel a lot better. You take your medicine. Therefore, you'll feel a lot better.

13. It has rained every day for the past seven days, and it is raining today as well. So it will also rain tomorrow.

14. Rhonda's first five children were boys. If she has another baby, it will be a boy.

15. The 2000 movie *Cast Away* stars Tom Hanks as the only human survivor of a plane crash, stranded on a tropical island. He estimates that his distance from where the plane lost radio contact is about 400 miles (a radius), and then he uses the formula for the area of a circle,

$$\text{Area} = \pi\,(\text{radius})^2,$$

to determine that a search party would have to cover an area of over 500,000 square miles to look for him and his "pal" Wilson.

16. If the same number is subtracted from both sides of a true equation, the new equation is also true. I know that $11 + 18 = 29$. Therefore, $(11 + 18) - 13 = 29 - 13$.

17. If you build it, they will come. You build it. Therefore, they will come.

18. All men are mortal. Socrates is a man. Therefore, Socrates is mortal.

19. It is a fact that every student who ever attended Southern University was accepted into graduate school. Because I am attending Southern, I can expect to be accepted into graduate school, too.

20. For the past 57 years, a rare plant has bloomed in Colombia each summer, alternating between yellow and green flowers. Last summer, it bloomed with green flowers, so this summer it will bloom with yellow flowers.

21. In the sequence 5, 10, 15, 20, 25, . . . , the most probable next number is 30.

22. (This anecdote is adapted from a story by Howard Eves in *In Mathematical Circles*.) A scientist had a group of 100 fleas, and one by one he would tell each flea "Jump," and the flea would jump. Then with the same fleas, he yanked off their hind legs and repeated "Jump," but the fleas would not jump. He concluded that when a flea has its hind legs yanked off, it cannot hear.

23. Discuss the differences between inductive and deductive reasoning. Give an example of each.

24. Give an example of inductive reasoning with a faulty conclusion.

Determine the most probable next term in each of the following lists of numbers.

25. 6, 9, 12, 15, 18

26. 13, 18, 23, 28, 33

27. 3, 12, 48, 192, 768

28. 32, 16, 8, 4, 2

29. 3, 6, 9, 15, 24, 39

30. $\frac{1}{3}, \frac{3}{5}, \frac{5}{7}, \frac{7}{9}, \frac{9}{11}$

31. $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \frac{9}{10}$

32. 1, 4, 9, 16, 25

33. 1, 8, 27, 64, 125

34. 2, 6, 12, 20, 30, 42

35. 4, 7, 12, 19, 28, 39

36. 27, 21, 16, 12, 9

37. Construct a list of numbers similar to that in **Exercise 25** such that the most probable next number in the list is 60.

38. Construct a list of numbers similar to that in **Exercise 36** such that the most probable next number in the list is 8.

Use the list of equations and inductive reasoning to predict the next equation, and then verify your conjecture.

39.
$$(9 \times 9) + 7 = 88$$
$$(98 \times 9) + 6 = 888$$
$$(987 \times 9) + 5 = 8888$$
$$(9876 \times 9) + 4 = 88{,}888$$

40.
$$(1 \times 9) + 2 = 11$$
$$(12 \times 9) + 3 = 111$$
$$(123 \times 9) + 4 = 1111$$
$$(1234 \times 9) + 5 = 11{,}111$$

41.
$$3367 \times 3 = 10{,}101$$
$$3367 \times 6 = 20{,}202$$
$$3367 \times 9 = 30{,}303$$
$$3367 \times 12 = 40{,}404$$

42.
$$15{,}873 \times 7 = 111{,}111$$
$$15{,}873 \times 14 = 222{,}222$$
$$15{,}873 \times 21 = 333{,}333$$
$$15{,}873 \times 28 = 444{,}444$$

43.
$$11 \times 11 = 121$$
$$111 \times 111 = 12{,}321$$
$$1111 \times 1111 = 1{,}234{,}321$$

44.
$$34 \times 34 = 1156$$
$$334 \times 334 = 111{,}556$$
$$3334 \times 3334 = 11{,}115{,}556$$

45.
$$3 = \frac{3(2)}{2}$$
$$3 + 6 = \frac{6(3)}{2}$$
$$3 + 6 + 9 = \frac{9(4)}{2}$$
$$3 + 6 + 9 + 12 = \frac{12(5)}{2}$$

46.
$$2 = 4 - 2$$
$$2 + 4 = 8 - 2$$
$$2 + 4 + 8 = 16 - 2$$
$$2 + 4 + 8 + 16 = 32 - 2$$

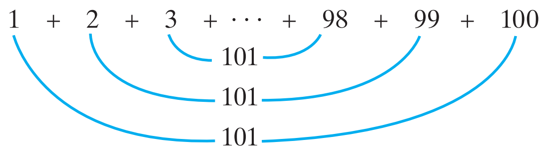
47.
$$5(6) = 6(6 - 1)$$
$$5(6) + 5(36) = 6(36 - 1)$$
$$5(6) + 5(36) + 5(216) = 6(216 - 1)$$
$$5(6) + 5(36) + 5(216) + 5(1296) = 6(1296 - 1)$$

48.
$$3 = \frac{3(3 - 1)}{2}$$
$$3 + 9 = \frac{3(9 - 1)}{2}$$
$$3 + 9 + 27 = \frac{3(27 - 1)}{2}$$
$$3 + 9 + 27 + 81 = \frac{3(81 - 1)}{2}$$

49.
$$\frac{1}{2} = 1 - \frac{1}{2}$$
$$\frac{1}{2} + \frac{1}{4} = 1 - \frac{1}{4}$$
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1 - \frac{1}{8}$$
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 1 - \frac{1}{16}$$

50.
$$\frac{1}{1 \cdot 2} = \frac{1}{2}$$
$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{2}{3}$$
$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \frac{3}{4}$$
$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} = \frac{4}{5}$$

Legend has it that the great mathematician Carl Friedrich Gauss (1777–1855) at a very young age was told by his teacher to find the sum of the first 100 *counting numbers. While his classmates toiled at the problem, Carl simply wrote down a single number and handed the correct answer in to his teacher. The young Carl explained that he observed that there were* 50 *pairs of numbers that each added up to* 101. *(See below.) So the sum of all the numbers must be* $50 \times 101 = 5050$.



50 sums of 101 = 50 × 101 = 5050

Use the method of Gauss to find each sum.

51. $1 + 2 + 3 + \cdots + 200$

52. $1 + 2 + 3 + \cdots + 400$

53. $1 + 2 + 3 + \cdots + 800$

54. $1 + 2 + 3 + \cdots + 2000$

55. Modify the procedure of Gauss to find the sum $1 + 2 + 3 + \cdots + 175$.

56. Explain in your own words how the procedure of Gauss can be modified to find the sum $1 + 2 + 3 + \cdots + n$, where n is an odd natural number. (When an odd natural number is divided by 2, it leaves a remainder of 1.)

57. Modify the procedure of Gauss to find the sum $2 + 4 + 6 + \cdots + 100$.

58. Use the result of **Exercise 57** to find the sum $4 + 8 + 12 + \cdots + 200$.

Solve each problem.

59. What is the most probable next number in this list?

12, 1, 1, 1, 2, 1, 3

(*Hint:* Think about a clock with chimes.)

60. What is the next term in this list?

O, T, T, F, F, S, S, E, N, T

(*Hint:* Think about words and their relationship to numbers.)

61. Choose any three-digit number with all different digits, and follow these steps.

(a) Reverse the digits, and subtract the smaller from the larger. Record your result.

(b) Choose another three-digit number and repeat this process. Do this as many times as it takes for you to see a pattern in the different results you obtain. (*Hint:* What is the middle digit? What is the sum of the first and third digits?)

(c) Write an explanation of this pattern.

62. Choose any number, and follow these steps.

(a) Multiply by 2.
(b) Add 6.
(c) Divide by 2.
(d) Subtract the number you started with.
(e) Record your result.

Repeat the process, except in Step (b), add 8. Record your final result. Repeat the process once more, except in Step (b), add 10. Record your final result.

(f) Observe what you have done. Then use inductive reasoning to explain how to predict the final result.

63. Complete the following.

$142{,}857 \times 1 =$ _______ $142{,}857 \times 2 =$ _______

$142{,}857 \times 3 =$ _______ $142{,}857 \times 4 =$ _______

$142{,}857 \times 5 =$ _______ $142{,}857 \times 6 =$ _______

What pattern exists in the successive answers? Now multiply 142,857 by 7 to obtain an interesting result.

64. Refer to **Figures 2(b)–(e)** and **Figure 3.** Instead of counting interior regions of the circle, count the chords formed. Use inductive reasoning to predict the number of chords that would be formed if seven points were used.

1.2 AN APPLICATION OF INDUCTIVE REASONING: NUMBER PATTERNS

OBJECTIVES

1 Recognize arithmetic and geometric sequences.
2 Apply the method of successive differences to predict the next term in a sequence.
3 Recognize number patterns.
4 Use sum formulas.
5 Recognize triangular, square, and pentagonal numbers.

Number Sequences

An ordered list of numbers such as

$$3, 9, 15, 21, 27, \ldots$$

is called a *sequence*. A **number sequence** is a list of numbers having a first number, a second number, a third number, and so on. These are the **terms of the sequence.**

The sequence that begins

$$5, 9, 13, 17, 21, \ldots$$

is an *arithmetic sequence,* or *arithmetic progression*. In an **arithmetic sequence,** each term after the first is obtained by adding the same number, the **common difference,** to the preceding term. To find the common difference, choose any term after the first and subtract from it the preceding term. If we choose

$$9 - 5 \quad \text{(the second term minus the first term)},$$

for example, we see that the common difference is 4. To find the term following 21, we add 4 to get

$$21 + 4 = 25.$$

The sequence that begins

$$2, 4, 8, 16, 32, \ldots$$

is a *geometric sequence,* or *geometric progression*. In a **geometric sequence,** each term after the first is obtained by multiplying the preceding term by the same number, the **common ratio.** To find the common ratio, choose any term after the first and divide it by the preceding term. If we choose

$$\frac{4}{2} \quad \text{(the second term divided by the first term)},$$

for example, we see that the common ratio is 2. To find the term following 32, we multiply by 2 to get

$$32 \cdot 2 = 64.$$

StatCrunch Identifying Arithmetic and Geometric Sequences

EXAMPLE 1 Identifying Arithmetic and Geometric Sequences

For each sequence, determine whether it is an *arithmetic sequence,* a *geometric sequence,* or *neither.* If it is either arithmetic or geometric, give the next term.

(a) 5, 10, 15, 20, 25, . . . **(b)** 3, 12, 48, 192, 768, . . . **(c)** 1, 4, 9, 16, 25, . . .

Solution

(a) If we choose *any* term after the first term, and subtract the preceding term, we find that the common difference is 5.

$$10 - 5 = 5 \qquad 15 - 10 = 5 \qquad 20 - 15 = 5 \qquad 25 - 20 = 5$$

Therefore, this is an arithmetic sequence. The next term is $25 + 5 = 30$.

(b) If any term after the first is divided by the previous term, we find the common ratio 4.

$$\frac{12}{3} = 4 \qquad \frac{48}{12} = 4 \qquad \frac{192}{48} = 4 \qquad \frac{768}{192} = 4$$

This is a geometric sequence. The next term is $768 \cdot 4 = 3072$.

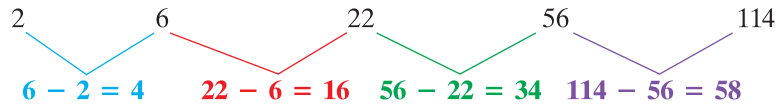
(c) Although there is a pattern here (the terms are the squares of the first five counting numbers), there is neither a common difference nor a common ratio. This is neither an arithmetic nor a geometric sequence.

Successive Differences

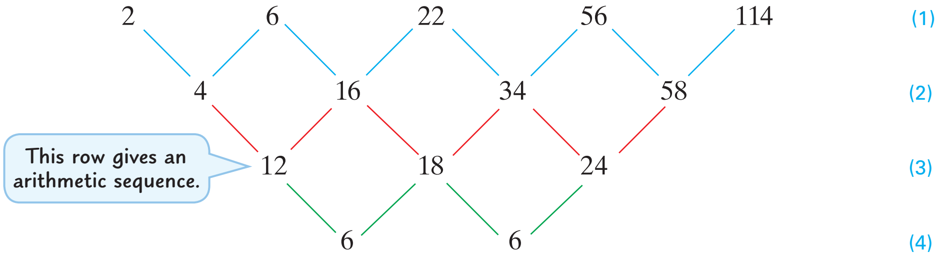
Some sequences present more difficulty than our earlier examples when we are making a conjecture about the next term. Often the **method of successive differences** may be applied in such cases. Consider the sequence

$$2, 6, 22, 56, 114, \ldots .$$

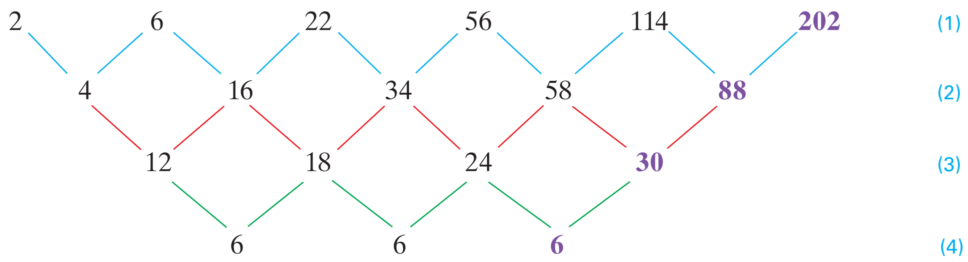
Because the next term is not obvious, subtract the first term from the second term, the second from the third, the third from the fourth, and so on.



Now repeat the process with the sequence 4, 16, 34, 58, and continue repeating until the difference is a constant value, as shown in line (4).



Once a line of constant values is obtained, simply work "backward" by adding until the desired term of the given sequence is obtained. Thus, for this pattern to continue, another 6 should appear in line (4), meaning that the next term in line (3) would have to be $24 + 6 = 30$. The next term in line (2) would be $58 + 30 = 88$. Finally, the next term in the given sequence would be $114 + 88 = \mathbf{202}$.



StatCrunch Successive Differences

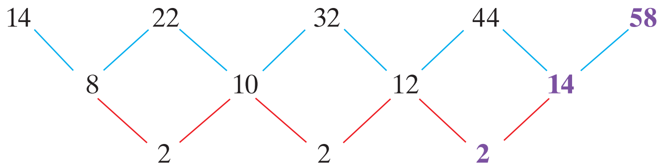
EXAMPLE 2 Using Successive Differences

Use successive differences to determine the next number in each sequence.

(a) 14, 22, 32, 44, . . . **(b)** 5, 15, 37, 77, 141, . . .

Solution

(a) Subtract a term from the one that follows it, and continue until a pattern is observed.

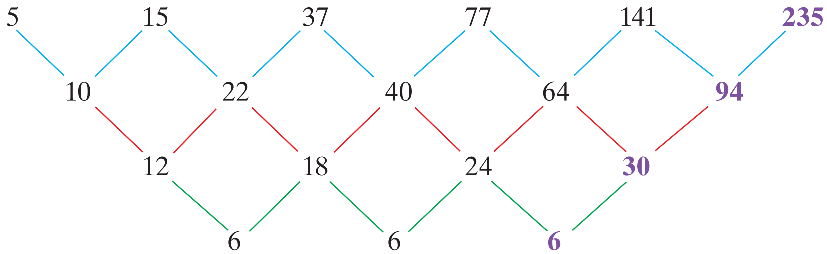


Once the row of 2s was obtained and extended, we were able to obtain

$$12 + 2 = 14, \quad \text{and} \quad 44 + 14 = 58 \quad \text{as shown above.}$$

The next number in the sequence is **58**.

(b) Proceed as before to obtain the following diagram.



The numbers in the "diagonal" at the far right were obtained by adding:

$$24 + 6 = 30,$$

$$64 + 30 = 94,$$

and $$141 + 94 = 235.$$

The next number in the sequence is **235**.

The method of successive differences does not always work. For example, try it on the Fibonacci sequence in **Example 2(b)** of **Section 1.1** and notice what happens. The resulting sequence is also a Fibonacci-type sequence, beginning with 0, 1.

$$0, 1, 1, 2, 3, 5, 8, \ldots$$

Continuing this way never does lead to a sequence of constant values.

A Prime Pattern A natural number greater than 1 that has only 1 and itself as factors is called a **prime number.** It was proved thousands of years ago that there are infinitely many prime numbers. The first few are 2, 3, 5, 7, 11, 13, and 17.

The number 31 is prime and the number 331 is prime. They can be factored only as 1×31 and 1×331, respectively. In fact, the following numbers are all prime.

3331

33,331

333,331

3,333,331

33,333,331

There is an obvious pattern here. But don't be fooled. You might be tempted to believe that

333,333,331

is also prime. But it can be written as the product

$17 \times 19{,}607{,}843$

and is an example of a composite number—a natural number greater than 1 that is not prime.

There are many Web sites that will enable you to check whether a number is prime or composite, and you may want to verify the statements above. One such site is http://www.onlineconversion.com/prime.htm

Number Patterns and Sum Formulas

Observe the number pattern at the right. In each case, the left side of the equation is the indicated sum of consecutive odd counting numbers beginning with 1, and the right side is the square of the number of terms on the left side. Inductive reasoning would suggest that the next line in this pattern is as follows.

$$1 = 1^2$$
$$1 + 3 = 2^2$$
$$1 + 3 + 5 = 3^2$$
$$1 + 3 + 5 + 7 = 4^2$$
$$1 + 3 + 5 + 7 + 9 = 5^2$$

$$1 + 3 + 5 + 7 + 9 + 11 = 6^2$$ Each side simplifies to 36.

We cannot conclude that this pattern will continue indefinitely, because observation of a finite number of examples does *not* guarantee that the pattern will continue. However, mathematicians have proved that this pattern does indeed continue indefinitely, using a method of proof called **mathematical induction.** (See any comprehensive college algebra text.)

Any even counting number may be written in the form $2k$, where k is a counting number. It follows that the kth odd counting number is written $2k - 1$. For example, the **third** odd counting number, 5, can be written $2(\mathbf{3}) - 1$.

SUM OF THE FIRST *n* ODD COUNTING NUMBERS

If n is any counting number, then the following is true.

$$\mathbf{1 + 3 + 5 + \cdots + (2n - 1) = n^2}$$

EXAMPLE 3 Predicting the Next Equation in a List

Several equations are given illustrating a suspected number pattern. Determine what the next equation would be, and verify that it is indeed a true statement.

(a)
$$1^2 = 1^3$$
$$(1 + 2)^2 = 1^3 + 2^3$$
$$(1 + 2 + 3)^2 = 1^3 + 2^3 + 3^3$$
$$(1 + 2 + 3 + 4)^2 = 1^3 + 2^3 + 3^3 + 4^3$$

(b)
$$1 = 1^3$$
$$3 + 5 = 2^3$$
$$7 + 9 + 11 = 3^3$$
$$13 + 15 + 17 + 19 = 4^3$$

(c)
$$1 = \frac{1 \cdot 2}{2}$$
$$1 + 2 = \frac{2 \cdot 3}{2}$$
$$1 + 2 + 3 = \frac{3 \cdot 4}{2}$$
$$1 + 2 + 3 + 4 = \frac{4 \cdot 5}{2}$$

(d)
$$12{,}345{,}679 \times 9 = 111{,}111{,}111$$
$$12{,}345{,}679 \times 18 = 222{,}222{,}222$$
$$12{,}345{,}679 \times 27 = 333{,}333{,}333$$
$$12{,}345{,}679 \times 36 = 444{,}444{,}444$$
$$12{,}345{,}679 \times 45 = 555{,}555{,}555$$

Notice that there is no 8 here.

Solution

(a) The left side of each equation is the square of the sum of the first n counting numbers. The right side is the sum of their cubes. The next equation would be

$$(1 + 2 + 3 + 4 + 5)^2 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3.$$

Each side simplifies to 225, so the pattern is true for this equation.

(b) The left sides of the equations contain the sum of odd counting numbers, starting with the first (1) in the first equation, the second and third (3 and 5) in the second equation, the fourth, fifth, and sixth (7, 9, and 11) in the third equation, and so on. Each right side contains the cube (third power) of the number of terms on the left side. Following this pattern, the next equation would be

$$21 + 23 + 25 + 27 + 29 = 5^3,$$

which can be verified by computation.

(c) The left side of each equation gives the indicated sum of the first n counting numbers, and the right side is always of the form

$$\frac{n(n+1)}{2}.$$

For the pattern to continue, the next equation would be

$$1 + 2 + 3 + 4 + 5 = \frac{5 \cdot 6}{2}.$$

Because each side simplifies to 15, the pattern is true for this equation.

(d) In each case, the first factor on the left is 12,345,679 and the second factor is a multiple of 9 (that is, 9, 18, 27, 36, 45). The right side consists of a nine-digit number, all digits of which are the same (that is, 1, 2, 3, 4, 5). For the pattern to continue, the next equation would be as follows.

$$12{,}345{,}679 \times 54 = 666{,}666{,}666$$

Verify that this is a true statement.

The patterns established in **Examples 3(a) and 3(c)** can be written as follows.

SPECIAL SUM FORMULAS

For any counting number n, the following are true.

$$(1 + 2 + 3 + \cdots + n)^2 = 1^3 + 2^3 + 3^3 + \cdots + n^3$$

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

We can provide a general deductive argument showing how the second equation is obtained. Let S represent the sum $1 + 2 + 3 + \cdots + n$. This sum can also be written as

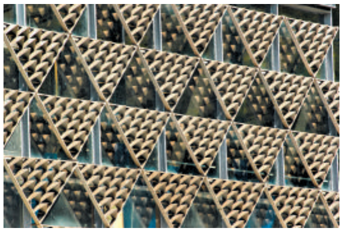
$$S = n + (n-1) + (n-2) + \cdots + 1.$$

Write these two equations as follows.

$$\begin{aligned} S &= 1 + 2 + 3 + \cdots + n \\ S &= n + (n-1) + (n-2) + \cdots + 1 \\ \hline 2S &= (n+1) + (n+1) + (n+1) + \cdots + (n+1) \end{aligned}$$ Add the corresponding sides.

$$2S = n(n+1)$$ There are n terms of $n + 1$.

$$S = \frac{n(n+1)}{2}$$ Divide both sides by 2.



Figurate Numbers

Pythagoras and his Pythagorean brotherhood studied numbers of geometric arrangements of points, such as **triangular numbers, square numbers,** and **pentagonal numbers. Figure 4** illustrates the first few of each of these types of numbers.

The **figurate numbers** possess numerous interesting patterns. For example, every square number greater than 1 is the sum of two consecutive triangular numbers. ($9 = 3 + 6$, $25 = 10 + 15$, and so on.)

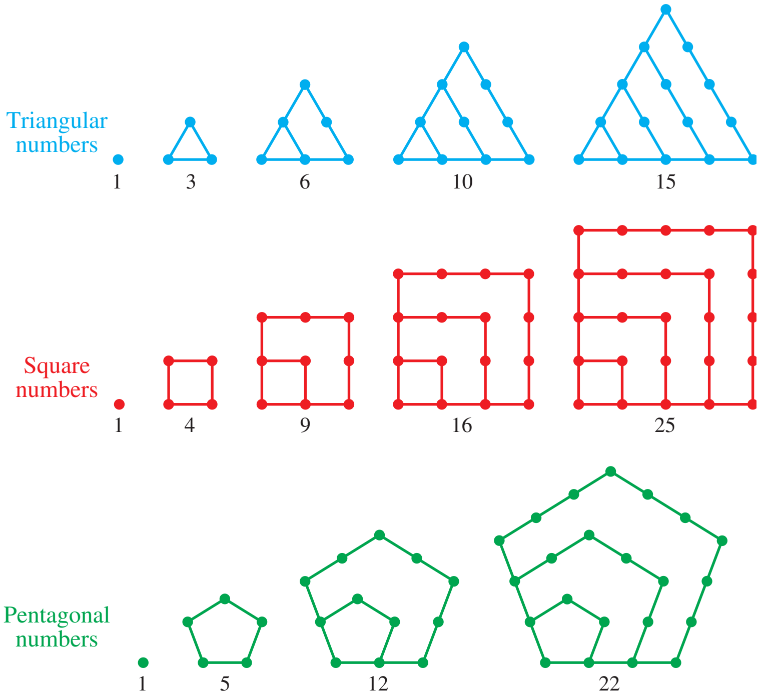


Figure 4

Every pentagonal number can be represented as the sum of a square number and a triangular number. (For example, $5 = 4 + 1$ and $12 = 9 + 3$.) Many other such relationships exist.

In the expression T_n, n is called a **subscript.** T_n is read **"T sub *n*,"** and it represents the triangular number in the nth position in the sequence. For example,

$$T_1 = 1, \quad T_2 = 3, \quad T_3 = 6, \quad \text{and} \quad T_4 = 10.$$

S_n and P_n represent the nth square and pentagonal numbers, respectively.

FORMULAS FOR TRIANGULAR, SQUARE, AND PENTAGONAL NUMBERS

For any natural number n, the following are true.

The nth triangular number is given by $\mathbf{T_n = \frac{n(n+1)}{2}}$.

The nth square number is given by $\mathbf{S_n = n^2}$.

The nth pentagonal number is given by $\mathbf{P_n = \frac{n(3n-1)}{2}}$.

EXAMPLE 4 Using the Formulas for Figurate Numbers

Use the formulas to find each of the following.

(a) the seventh triangular number **(b)** the twelfth square number

(c) the sixth pentagonal number

Solution

(a) $T_7 = \frac{n(n+1)}{2} = \frac{7(7+1)}{2} = \frac{7(8)}{2} = \frac{56}{2} = 28$ Formula for a triangular number, with $n = 7$

(b) $S_{12} = n^2 = 12^2 = 144$ Formula for a square number, with $n = 12$

$12^2 = 12 \cdot 12$

Inside the brackets, multiply first and then subtract.

(c) $P_6 = \frac{n(3n-1)}{2} = \frac{6[3(6)-1]}{2} = \frac{6(17)}{2} = 51$ Formula for a pentagonal number, with $n = 6$

EXAMPLE 5 Illustrating a Figurate Number Relationship

Show that the sixth pentagonal number is equal to the sum of 6 and 3 times the fifth triangular number.

Solution

From **Example 4(c),** $P_6 = 51$. The fifth triangular number is 15. Thus,

$$51 = 6 + 3(15) = 6 + 45 = 51.$$

The general relationship examined in **Example 5** can be written as follows.

$$P_n = n + 3 \cdot T_{n-1} \quad (n \geq 2)$$

(The equation above is read "P sub-*n* is equal to *n* plus 3 times T sub-*n* minus 1, *n* is greater than or equal to 2.")

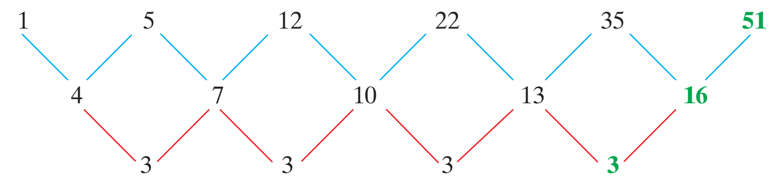
EXAMPLE 6 Predicting the Value of a Pentagonal Number

The first five pentagonal numbers are

1, 5, 12, 22, 35.

Use the method of successive differences to predict the sixth pentagonal number.

Solution



After the second line of successive differences, we work backward to find that the sixth pentagonal number is 51, which was also found in **Example 4(c).**

1.2 EXERCISES

CONCEPT CHECK *Fill in the blank with the correct response.*

1. If 5, 8 are two adjacent terms in an arithmetic sequence (in that order), then the next term is ____.

2. If 5, 25 are two adjacent terms in a geometric sequence (in that order), then the next term is ____.

3. The expression for finding the sum $1 + 2 + 3 + 4 + 5 + 6$ without actually performing the addition is

$$\frac{6(\_\_\_\_\_)}{2}.$$

4. The expression for finding the sum $1 + 3 + 5 + 7 + 9 + 11$ without actually performing the addition is

____.

(*Hint:* There are six terms to be added here.)

For each sequence, determine whether it is an arithmetic *sequence, a* geometric *sequence, or* neither. *If it is either arithmetic or geometric, give the next term in the sequence.*

5. 6, 16, 26, 36, 46, . . .

6. 8, 16, 24, 32, 40, . . .

7. 5, 15, 45, 135, 405, . . .

8. 2, 12, 72, 432, 2592, . . .

9. 1, 8, 27, 81, 243, . . .

10. 2, 8, 18, 32, 50, . . .

11. 256, 128, 64, 32, 16, . . .

12. 4096, 1024, 256, 64, 16, . . .

13. 1, 3, 4, 7, 11, . . .

14. 0, 1, 1, 2, 3, . . .

15. 12, 14, 16, 18, 20, . . .

16. 10, 50, 90, 130, 170, . . .

Use the method of successive differences to determine the next number in each sequence.

17. 1, 4, 11, 22, 37, 56, . . .

18. 3, 14, 31, 54, 83, 118, . . .

19. 6, 20, 50, 102, 182, 296, . . .

20. 1, 11, 35, 79, 149, 251, . . .

21. 0, 12, 72, 240, 600, 1260, 2352, . . .

22. 2, 57, 220, 575, 1230, 2317, . . .

23. 5, 34, 243, 1022, 3121, 7770, 16,799, . . .

24. 3, 19, 165, 771, 2503, 6483, 14,409, . . .

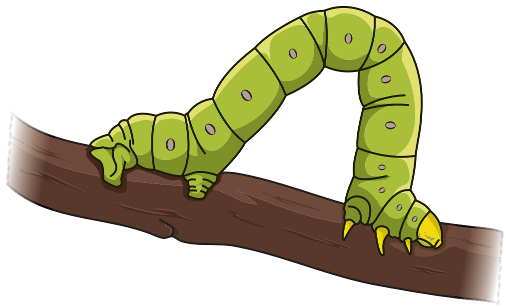
Solve each problem.

25. Refer to **Figures 2 and 3** in **Section 1.1.** The method of successive differences can be applied to the sequence of interior regions,

1, 2, 4, 8, 16, 31,

to find the number of regions determined by seven points on the circle. What is the next term in this sequence? How many regions would be determined by eight points? Verify this using the formula given at the end of that section.

26. The 1952 film *Hans Christian Andersen* stars Danny Kaye as the Danish writer of fairy tales. In a scene outside a schoolhouse window, Kaye sings a song to an inchworm. "Inchworm" was written for the film by the composer Frank Loesser and has been recorded by many artists, including Paul McCartney and Kenny Loggins. It was once featured on an episode of *The Muppets* and sung by Charles Aznavour.



As Kaye sings the song, the children in the school room are heard chanting addition facts:

$2 + 2 = 4, \quad 4 + 4 = 8, \quad 8 + 8 = 16,$ and so on.

(a) Use patterns to state the next addition fact (as heard in the movie).

(b) If the children were to extend their facts to the next four in the pattern, what would those facts be?

Several equations are given illustrating a suspected number pattern. Determine what the next equation would be, and verify that it is indeed a true statement.

27. $(1 \times 9) - 1 = 8$
$(21 \times 9) - 1 = 188$
$(321 \times 9) - 1 = 2888$

28. $(1 \times 8) + 1 = 9$
$(12 \times 8) + 2 = 98$
$(123 \times 8) + 3 = 987$

29. $999{,}999 \times 2 = 1{,}999{,}998$
$999{,}999 \times 3 = 2{,}999{,}997$

30. $101 \times 101 = 10{,}201$
$10{,}101 \times 10{,}101 = 102{,}030{,}201$

31. $3^2 - 1^2 = 2^3$
$6^2 - 3^2 = 3^3$
$10^2 - 6^2 = 4^3$
$15^2 - 10^2 = 5^3$

32. $1 = 1^2$
$1 + 2 + 1 = 2^2$
$1 + 2 + 3 + 2 + 1 = 3^2$
$1 + 2 + 3 + 4 + 3 + 2 + 1 = 4^2$

33. $2^2 - 1^2 = 2 + 1$
$3^2 - 2^2 = 3 + 2$
$4^2 - 3^2 = 4 + 3$

34. $1^2 + 1 = 2^2 - 2$
$2^2 + 2 = 3^2 - 3$
$3^2 + 3 = 4^2 - 4$

35. $1 = 1 \times 1$
$1 + 5 = 2 \times 3$
$1 + 5 + 9 = 3 \times 5$

36. $1 + 2 = 3$
$4 + 5 + 6 = 7 + 8$
$9 + 10 + 11 + 12 = 13 + 14 + 15$

Use the formula $S = \frac{n(n+1)}{2}$ *to find each sum.*

37. $1 + 2 + 3 + \cdots + 300$

38. $1 + 2 + 3 + \cdots + 500$

39. $1 + 2 + 3 + \cdots + 675$

40. $1 + 2 + 3 + \cdots + 825$

Use the formula $S = n^2$ *to find each sum. (Hint: To find n, add 1 to the last term and divide by 2.)*

41. $1 + 3 + 5 + \cdots + 101$

42. $1 + 3 + 5 + \cdots + 49$

43. $1 + 3 + 5 + \cdots + 999$

44. $1 + 3 + 5 + \cdots + 301$

Solve each problem.

45. Use the formula for finding the sum

$$1 + 2 + 3 + \cdots + n$$

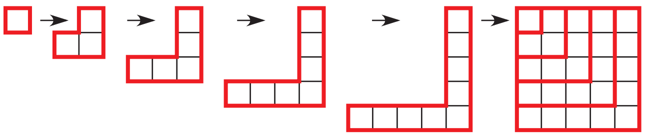
to discover a formula for finding the sum

$$2 + 4 + 6 + \cdots + 2n.$$

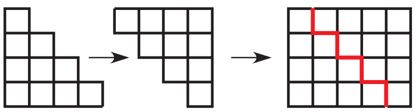
46. State in your own words the following formula discussed in this section.

$$(1 + 2 + 3 + \cdots + n)^2 = 1^3 + 2^3 + 3^3 + \cdots + n^3$$

47. Explain how the following diagram geometrically illustrates the formula $1 + 3 + 5 + 7 + 9 = 5^2$.



48. Explain how the following diagram geometrically illustrates the formula $1 + 2 + 3 + 4 = \frac{4 \times 5}{2}$.



49. Use patterns to complete the table below.

Figurate Number	1st	2nd	3rd	4th	5th	6th	7th	8th
Triangular	1	3	6	10	15	21		
Square	1	4	9	16	25			
Pentagonal	1	5	12	22				
Hexagonal	1	6	15					
Heptagonal	1	7						
Octagonal	1							

50. The first five triangular, square, and pentagonal numbers can be obtained using sums of terms of sequences as shown below.

Triangular	Square	Pentagonal
$1 = 1$	$1 = 1$	$1 = 1$
$3 = 1 + 2$	$4 = 1 + 3$	$5 = 1 + 4$
$6 = 1 + 2 + 3$	$9 = 1 + 3 + 5$	$12 = 1 + 4 + 7$
$10 = 1 + 2 + 3 + 4$	$16 = 1 + 3 + 5 + 7$	$22 = 1 + 4 + 7 + 10$
$15 = 1 + 2 + 3 + 4 + 5$	$25 = 1 + 3 + 5 + 7 + 9$	$35 = 1 + 4 + 7 + 10 + 13$

Notice the successive differences of the added terms on the right sides of the equations. The next type of figurate number is the **hexagonal** number. (A hexagon has six sides.) Use the patterns above to predict the first five hexagonal numbers.

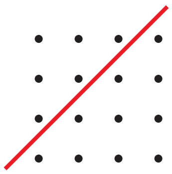
51. Eight times any triangular number, plus 1, is a square number. Show that this is true for the first four triangular numbers.

52. Divide the first triangular number by 3 and record the remainder. Divide the second triangular number by 3 and record the remainder. Repeat this procedure several more times. Do you notice a pattern?

53. Repeat **Exercise 52,** but instead use square numbers and divide by 4. What pattern is determined?

54. **Exercises 52 and 53** are specific cases of the following: When the numbers in the sequence of n-agonal numbers are divided by n, the sequence of remainders obtained is a repeating sequence. Verify this for $n = 5$ and $n = 6$.

55. Every square number can be written as the sum of two triangular numbers. For example, $16 = 6 + 10$. This can be represented geometrically by dividing a square array of dots with a line as shown.



The triangular arrangement above the line represents 6, the one below the line represents 10, and the whole arrangement represents 16. Show how the square numbers 25 and 36 may likewise be geometrically represented as the sum of two triangular numbers.

56. A fraction is in *lowest terms* if the greatest common factor of its numerator and its denominator is 1. For example, $\frac{3}{8}$ is in lowest terms, but $\frac{4}{12}$ is not.

(a) For $n = 2$ to $n = 8$, form the fractions

$$\frac{n\text{th square number}}{(n+1)\text{st square number}}.$$

(b) Repeat part (a) with triangular numbers.

(c) Use inductive reasoning to make a conjecture based on your results from parts (a) and (b), observing whether the fractions are in lowest terms.

*In addition to the formulas for T_n, S_n, and P_n, the following formulas are true for **hexagonal** numbers (H), **heptagonal** numbers (Hp), and **octagonal** numbers (O).*

$$\mathbf{H}_n = \frac{n(4n-2)}{2}, \quad \mathbf{Hp}_n = \frac{n(5n-3)}{2}, \quad \mathbf{O}_n = \frac{n(6n-4)}{2}$$

Use these formulas to find each of the following.

57. the sixteenth square number

58. the eleventh triangular number

59. the ninth pentagonal number

60. the seventh hexagonal number

61. the tenth heptagonal number

62. the twelfth octagonal number

63. Observe the formulas given for H_n, Hp_n, and O_n, and use patterns and inductive reasoning to predict the formula for N_n, the nth **nonagonal** number. (A nonagon has nine sides.) Then use the fact that the sixth nonagonal number is 111 to further confirm your conjecture.

64. Use the result of **Exercise 63** to find the tenth nonagonal number.

Use inductive reasoning to answer each question.

65. If you add two consecutive triangular numbers, what kind of figurate number do you get?

66. If you add the squares of two consecutive triangular numbers, what kind of figurate number do you get?

67. Square a triangular number. Square the next triangular number. Subtract the smaller result from the larger. What kind of number do you get?

68. Choose a value of n greater than or equal to 2. Find T_{n-1}, multiply it by 3, and add n. What kind of figurate number do you get?

In an arithmetic sequence, the nth term a_n is given by the formula

$$a_n = a_1 + (n - 1)d,$$

where a_1 is the first term and d is the common difference. Similarly, in a geometric sequence, the nth term is given by

$$a_n = a_1 \cdot r^{n-1}.$$

Here r is the common ratio. Use these formulas to determine the indicated term in the given sequence.

69. the eleventh term of 2, 6, 10, 14, . . .

70. the sixteenth term of 5, 15, 25, 35, . . .

71. the 21st term of 19, 39, 59, 79, . . .

72. the 36th term of 8, 38, 68, 98, . . .

73. the 101st term of $\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

74. the 151st term of 0.75, 1.50, 2.25, 3.00, . . .

75. the eleventh term of 2, 4, 8, 16, . . .

76. the ninth term of 1, 4, 16, 64, . . .

77. the 12th term of $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

78. the 10th term of $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$

79. the 8th term of $40, 10, \frac{5}{2}, \frac{5}{8}, \ldots$

80. the 9th term of $10, 2, \frac{2}{5}, \frac{2}{25}, \ldots$

*The mathematical array of numbers known as **Pascal's triangle** consists of rows of numbers, each of which contains one more entry than the previous row. The first six rows are shown here.*

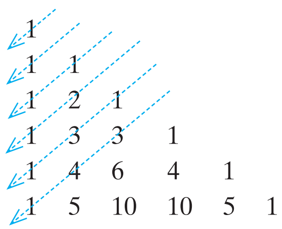
1
1 1
1 2 1
1 **3** **3** 1
1 4 **6** 4 1
1 5 10 10 5 1

Refer to this array to answer the following.

81. Each row begins and ends with a 1. Discover a method whereby the other entries in a row can be determined from the entries in the row immediately above it. (*Hint:* See the entries in color above.) Find the next three rows of the triangle, and prepare a copy of the first nine rows for later reference.

82. Find the sum of the entries in each of the first eight rows. What is the pattern that emerges? Predict the sum of the entries in the ninth row, and confirm your prediction.

83. The first six rows of the triangle are arranged "flush left" here.



Add along the blue diagonal lines. Write these sums in order from left to right. What sequence is this?

84. Find the values of the first four powers of the number 11, starting with 11^0, which by definition is equal to 1. Predict what the next power of 11 will equal by observing the rows of Pascal's triangle. Confirm your prediction by actual computation.

FOR FURTHER THOUGHT

Dattathreya Kaprekar (1905–1986)

Kaprekar Constants

Take any four-digit number whose digits are all different. Arrange the digits in decreasing order, and then arrange them in increasing order. Now subtract. Repeat the process, called the **Kaprekar routine,** until the same result appears.

For example, suppose that we choose a number whose digits are

1, 5, 7, and 9, such as 1579.

9751	8721	7443
−1579	−1278	−3447
8172	7443	3996

9963	6642	7641
−3699	−2466	−1467
6264	4176	6174

Note that we have obtained the number 6174, and the process will lead to 6174 again. The number 6174 is called a **Kaprekar constant.** This number 6174 will always be generated eventually if this process is applied to such a four-digit number.

For Group or Individual Investigation

1. Apply the Kaprekar routine to a four-digit number of your choice, in which the digits are all different. How many steps did it take for you to arrive at 6174?
2. Apply the Kaprekar routine, starting with a *three*-digit number of your choice whose digits are all different. You should arrive at a particular three-digit number that has the same property described for 6174. What is this three-digit number?
3. Applying the Kaprekar routine to a five-digit number does not reach a single repeating result. Instead, it reaches one of the following ten numbers and then cycles repeatedly through a subset of these ten numbers.

53,955	59,994
61,974	62,964
63,954	71,973
74,943	75,933
82,962	83,952

(a) Start the routine with the five-digit number 45,986, and determine which one of the ten numbers above is reached first.

(b) Start with a five-digit number of your own, and determine which one of the ten numbers is eventually reached first.

1.3 STRATEGIES FOR PROBLEM SOLVING

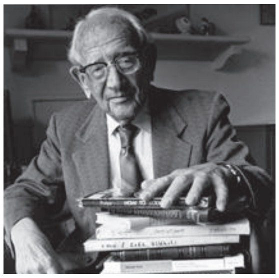
OBJECTIVES

1 Use George Polya's four-step method of problem solving.

2 Apply various strategies for solving problems.

A General Problem-Solving Method

In the first two sections of this chapter we stressed the importance of pattern recognition and the use of inductive reasoning in solving problems. Probably the most famous study of problem-solving techniques was developed by George Polya (1888–1985), among whose many publications was the modern classic *How to Solve It*. In this book, Polya proposed a four-step method for problem solving.



George Polya, author of the classic *How to Solve It*, died at the age of 97 on September 7, 1985. A native of Budapest, Hungary, he was once asked why there were so many good mathematicians to come out of Hungary at the turn of the century. He theorized that it was because mathematics is the cheapest science. It does not require any expensive equipment, only pencil and paper.

POLYA'S FOUR-STEP METHOD FOR PROBLEM SOLVING

Step 1 **Understand the problem.** You cannot solve a problem if you do not understand what you are asked to find. The problem must be read and analyzed carefully. You may need to read it several times. After you have done so, ask yourself, "*What must I find?*"

Step 2 **Devise a plan.** There are many ways to attack a problem. Decide what plan is appropriate for the particular problem you are solving.

Step 3 **Carry out the plan.** Once you know how to approach the problem, carry out your plan. You may run into "dead ends" and unforeseen roadblocks, but be persistent.

Step 4 **Look back and check.** Check your answer to see that it is reasonable. *Does it satisfy the conditions of the problem? Have you answered all the questions the problem asks? Can you solve the problem a different way and come up with the same answer?*

In Step 2 of Polya's problem-solving method, we are told to devise a plan. Here are some strategies that may prove useful.

Problem-Solving Strategies

- Make a table or a chart.
- Look for a pattern.
- Solve a similar, simpler problem.
- Draw a sketch.
- Use inductive reasoning.
- Write an equation and solve it.
- If a formula applies, use it.
- Work backward.
- Guess and check.
- Use trial and error.
- Use common sense.
- Look for a "catch" if an answer seems too obvious or impossible.

Using a Table or Chart

EXAMPLE 1 Solving Fibonacci's Rabbit Problem

A man put a pair of rabbits in a cage. During the first month the rabbits produced no offspring but each month thereafter produced one new pair of rabbits. If each new pair thus produced *reproduces* in the same manner, how many pairs of rabbits will there be at the end of 1 year? (This problem is a famous one in the history of mathematics and first appeared in *Liber Abaci,* a book written by the Italian mathematician Leonardo Pisano (also known as Fibonacci) in the year 1202.)

1, 1, 2, 3, 5, 8, 13, 21, ...

In the 2003 movie *A Wrinkle in Time*, young Charles Wallace, played by David Dorfman, is challenged to identify a particular sequence of numbers. He correctly identifies it as the **Fibonacci sequence.**

Solution

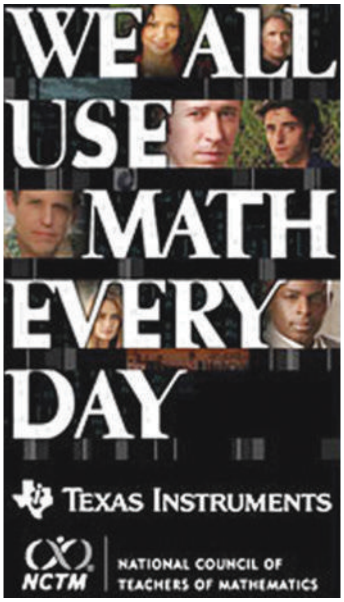
Step 1 **Understand the problem.** We can reword the problem as follows:

How many pairs of rabbits will the man have at the end of one year if he starts with one pair, and they reproduce in the following way? During the first month of life, each pair produces no new rabbits, but each month thereafter each pair produces one new pair.

Step 2 **Devise a plan.** Because there is a definite pattern to how the rabbits will reproduce, we can construct **Table 2** on the next page.

StatCrunch Problem Solving—Using a Table

Animation Scheduling Classes Using an Organized List



NUMB3RS On January 23, 2005, the CBS television network presented the first episode of *NUMB3RS*, a show focusing on how mathematics is used in solving crimes. David Krumholtz plays Charlie Eppes, a brilliant mathematician who assists his FBI agent brother (Rob Morrow).

In the first-season episode "Sabotage" (2/25/2005), one of the agents admits that she was not a good math student, and Charlie uses the **Fibonacci sequence** and its relationship to nature to enlighten her.

The sequence shown in color in **Table 3** is the Fibonacci sequence, mentioned in **Example 2(b)** of **Section 1.1.**

Table 2

Month	Number of Pairs at Start	Number of New Pairs Produced	Number of Pairs at End of Month
1st			
2nd			
3rd			
4th			
5th			
6th			
7th			
8th			
9th			
10th			
11th			
12th			

The answer will go here.

Step 3 **Carry out the plan.** At the start of the first month, there is only one pair of rabbits. No new pairs are produced during the first month, so there is $1 + 0 = 1$ pair present at the end of the first month. This pattern continues. In **Table 3,** we add the number in the first column of numbers to the number in the second column to get the number in the third.

Table 3

Month	Number of Pairs at Start	+	Number of New Pairs Produced	=	Number of Pairs at End of Month	
1st	**1**		0		1	$1 + 0 = 1$
2nd	**1**		1		2	$1 + 1 = 2$
3rd	**2**		1		3	$2 + 1 = 3$
4th	**3**		2		5	·
5th	**5**		3		8	·
6th	**8**		5		13	·
7th	**13**		8		21	·
8th	**21**		13		34	·
9th	**34**		21		55	·
10th	**55**		34		89	·
11th	**89**		55		144	·
12th	**144**		89		**233**	$144 + 89 = 233$

The answer is the final entry.

There will be **233** pairs of rabbits at the end of one year.

Step 4 **Look back and check.** Go back and make sure that we have interpreted the problem correctly. Double-check the arithmetic. We have answered the question posed by the problem, so the problem is solved.

Working Backward

EXAMPLE 2 Determining Money for a Craft Fair

For three consecutive days, Ronnie set up a booth at a crafts fair. On the first day, he tripled the amount of money he brought with him, but spent \$12 at another booth. The second day he brought the money with him, doubled it, and spent \$40 at another booth. On the third day, he again brought the money with him, quadrupled it, and spent nothing. He ended up with \$224. How much did he bring with him the first day?

Solution

This problem asks us to find Ronnie's starting amount. Since we know his final amount, the method of working backward can be applied.

Because his final amount was \$224 and this represents four times the amount he started with on the third day, we *divide* \$224 by 4 to find that he started the third day with \$56. Before he spent \$40 the second day, he had this \$56 plus the \$40 he spent, giving him \$96.

The \$96 represented double what he started with, so he started with \$96 *divided by* 2, or \$48, the second day. Repeating this process once more for the first day, before his \$12 expenditure he had \$48 + \$12 = \$60, which represents triple what he started with. Now divide by 3 to find that he started with

$$\$60 \div 3 = \$20. \quad \text{Answer}$$

CHECK Observe the following equations.

First week: $(3 \times \$20) - \$12 = \$60 - \$12 = \$48$

Second week: $(2 \times \$48) - \$40 = \$96 - \$40 = \$56$

Third week: $(4 \times \$56) = \$224$ His final amount

Using Trial and Error

Recall that $5^2 = 5 \cdot 5 = 25$. That is, 5 squared is 25. Thus, 25 is called a **perfect square.** Other perfect squares are 1, 4, 9, 16, 25, 36, and so on.

Augustus De Morgan was an English mathematician and philosopher who served as professor at the University of London. He wrote numerous books, one of which was *A Budget of Paradoxes*. His work in **set theory** and **logic** led to laws that bear his name and are covered in other chapters.

EXAMPLE 3 Finding Augustus De Morgan's Birth Year

The mathematician Augustus De Morgan lived in the nineteenth century. He made the following statement: *"I was x years old in the year x^2."* In what year was he born?

Solution

We must find the year of De Morgan's birth. The problem tells us that he lived in the nineteenth century, which is another way of saying that he lived during the 1800s. One year of his life was a *perfect square,* so we must find a number between 1800 and 1900 that is a perfect square. Use trial and error.

$$42^2 = 42 \cdot 42 = 1764$$
$$43^2 = 43 \cdot 43 = 1849$$
$$44^2 = 44 \cdot 44 = 1936$$

1849 is between 1800 and 1900.

The only natural number whose square is between 1800 and 1900 is 43, because $43^2 = 1849$. Therefore, De Morgan was 43 years old in 1849. The final step in solving the problem is to subtract 43 from 1849 to find the year of his birth.

$$1849 - 43 = 1806$$

He was born in 1806.

CHECK Look up De Morgan's birth date in a mathematics history book, such as *An Introduction to the History of Mathematics,* Sixth Edition, by Howard W. Eves.

Guessing and Checking

As mentioned above, $5^2 = 25$. The inverse procedure for squaring a number is called taking the **square root.** We indicate the positive square root using a **radical symbol** $\sqrt{\ }$. Thus, $\sqrt{25} = 5$. Also,

$$\sqrt{4} = 2, \quad \sqrt{9} = 3, \quad \sqrt{16} = 4, \quad \text{and so on.} \qquad \text{Square roots}$$

The next problem deals with a square root and dates back to Hindu mathematics, circa 850.

EXAMPLE 4 Finding the Number of Camels

One-fourth of a herd of camels was seen in the forest. Twice the square root of that herd had gone to the mountain slopes, and 3 times 5 camels remained on the riverbank. What is the numerical measure of that herd of camels?

Solution

The numerical measure of a herd of camels must be a counting number. Because the problem mentions

"one-fourth of a herd" and "the square root of that herd,"

the number of camels must be both a multiple of 4 and a perfect square, so only whole numbers are used. The least counting number that satisfies both conditions is 4.

We write an equation where x represents the numerical measure of the herd, and then substitute 4 for x to see whether it is a solution.

$$\underbrace{\frac{1}{4}x}_{\text{One-fourth of the herd}} + \underbrace{2\sqrt{x}}_{\text{Twice the square root of that herd}} + \underbrace{3 \cdot 5}_{\text{3 times 5 camels}} \; = \; \underbrace{x}_{\text{The numerical measure of the herd.}}$$

$$\frac{1}{4}(4) + 2\sqrt{4} + 3 \cdot 5 \stackrel{?}{=} 4 \qquad \text{Let } x = 4.$$

$$1 + 4 + 15 \stackrel{?}{=} 4 \qquad \sqrt{4} = 2$$

$$20 \neq 4$$

Because 4 is not the solution, try **16**, the next perfect square that is a multiple of 4.

$$\frac{1}{4}(16) + 2\sqrt{16} + 3 \cdot 5 \stackrel{?}{=} 16 \qquad \text{Let } x = 16.$$

$$4 + 8 + 15 \stackrel{?}{=} 16 \qquad \sqrt{16} = 4$$

$$27 \neq 16$$

Because 16 is not a solution, try **36**.

$$\frac{1}{4}(36) + 2\sqrt{36} + 3 \cdot 5 \stackrel{?}{=} 36 \qquad \text{Let } x = 36.$$

$$9 + 12 + 15 \stackrel{?}{=} 36 \qquad \sqrt{36} = 6$$

$$36 = 36$$

Thus, 36 is the numerical measure of the herd.

CHECK "One-fourth of 36, plus twice the square root of 36, plus 3 times 5" gives

$$9 + 12 + 15, \quad \text{which equals} \quad 36.$$

Mathematics to Die For In the 1995 movie *Die Hard: With a Vengeance*, detective John McClane (Bruce Willis) is tormented by the villain Simon Gruber (Jeremy Irons), who has planted a bomb in a park. Gruber presents McClane and his partner Zeus Carver (Samuel L. Jackson) with a riddle for disarming it, but there is a time limit. The riddle requires that they get exactly 4 gallons of water using 3-gallon and 5-gallon jugs having no markers. They are able to solve it and defuse the bomb. *Can you do it?* (See the next page for the answer.)

Considering a Similar, Simpler Problem

EXAMPLE 5 Finding the Units Digit of a Power

The digit farthest to the right in a counting number is called the *ones* or *units* digit, because it tells how many ones are contained in the number when grouping by tens is considered. What is the ones (or units) digit in 2^{4000}?

Solution

Recall that 2^{4000} means that 2 is used as a factor 4000 times.

$$2^{4000} = \underbrace{2 \times 2 \times 2 \times \cdots \times 2}_{\text{4000 factors}}$$

To answer the question, we examine some smaller powers of 2 and then look for a pattern. We start with the exponent 1 and look at the first twelve powers of 2.

$$2^1 = 2 \qquad 2^5 = 32 \qquad 2^9 = 512$$
$$2^2 = 4 \qquad 2^6 = 64 \qquad 2^{10} = 1024$$
$$2^3 = 8 \qquad 2^7 = 128 \qquad 2^{11} = 2048$$
$$2^4 = 16 \qquad 2^8 = 256 \qquad 2^{12} = 4096$$

Notice that in any one of the four rows above, the ones digit is the same all the way across the row. The final row, which contains the exponents 4, 8, and 12, has the ones digit 6. Each of these exponents is divisible by 4, and because 4000 is divisible by 4, we can use inductive reasoning to predict that the ones digit in 2^{4000} is **6**.

(*Note:* The ones digit for any other power can be found if we divide the exponent by 4 and consider the remainder. Then compare the result to the list of powers above. For example, to find the ones digit of 2^{543}, divide 543 by 4 to get a quotient of 135 and a remainder of **3**. The ones digit is the same as that of 2^3, which is **8**.)

Drawing a Sketch

EXAMPLE 6 Connecting the Dots

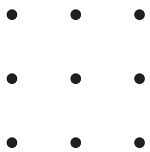


Figure 5

An array of nine dots is arranged in a 3×3 square, as shown in **Figure 5**. Is it possible to join the dots with exactly four straight line segments if you are not allowed to pick up your pencil from the paper and may not trace over a segment that has already been drawn? If so, show how.

Solution

Figure 6 shows three attempts. In each case, something is wrong. In the first sketch, one dot is not joined. In the second, the figure cannot be drawn without picking up your pencil from the paper or tracing over a line that has already been drawn. In the third figure, all dots have been joined, but you have used five line segments as well as retraced over the figure.

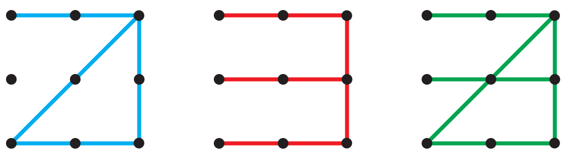


Figure 6

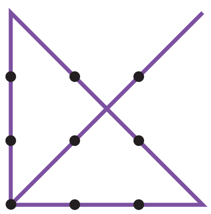


Figure 7

The conditions of the problem can be satisfied, as shown in **Figure 7**. We "went outside of the box," which was not prohibited by the conditions of the problem. This is an example of creative thinking—we used a strategy that often is not considered at first.

Solution to the Jugs-of-Water Riddle
This is one way to do it: With both jugs empty, fill the 3-gallon jug and pour its contents into the 5-gallon jug. Then fill the 3-gallon jug again, and pour it into the 5-gallon jug until the latter is filled. There is now $(3 + 3) - 5 = 1$ gallon in the 3-gallon jug. Empty the 5-gallon jug, and pour the 1 gallon of water from the 3-gallon jug into the 5-gallon jug. Finally, fill the 3-gallon jug and pour all of it into the 5-gallon jug, resulting in $1 + 3 = 4$ gallons in the 5-gallon jug.

(*Note:* There is another way to solve this problem. See if you can discover the alternative solution.)

Using Common Sense

Problem-Solving Strategies

Some problems involve a "catch." They seem too easy or perhaps impossible at first because we tend to overlook an obvious situation. Look carefully at the use of language in such problems. And, of course, never forget to use common sense.

EXAMPLE 7 Determining Coin Denominations

Two currently minted United States coins together have a total value of \$1.05. One is not a dollar. What are the two coins?

Solution

Our initial reaction might be, "The only way to have two such coins with a total of \$1.05 is to have a nickel and a dollar, but the problem says that one of them is not a dollar." This statement is indeed true. The one that is not a dollar is the nickel, and the *other* coin is a dollar! So the two coins are a dollar and a nickel.

1.3 EXERCISES

One of the most popular features of the journal Mathematics Teacher, *published by the National Council of Teachers of Mathematics, is the monthly calendar. It provides an interesting, unusual, or challenging problem for each day of the month. Some of these exercises, and others to follow in this text, are chosen from these calendars and designated "*MT *calendar problem." The authors thank the many contributors for permission to use these problems.*

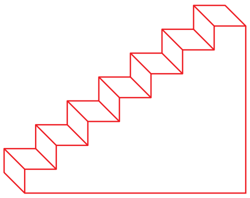
Use the various problem-solving strategies to solve each problem. In many cases there is more than one possible approach, so be creative.

1. ***Digits of a Year*** The year 2013 contains four digits whose values are consecutive integers (0, 1, 2, 3). How many years after 2013 will this event occur next? (*MT* calendar problem)

2. ***Final Digits of a Power of 5*** What are the last two digits of 5^{2007}? (*MT* calendar problem)

3. ***Making Change*** Given nine nickels and five pennies, how many different sums of money less than 50 cents can be formed using one or more coins? (*MT* calendar problem)

4. ***Taking Steps*** In how many ways can you walk up a stairway that has 7 steps if you take only 1 or 2 steps at a time? (*MT* calendar problem)



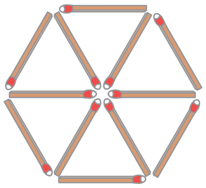
5. ***Number Trick*** Take a two-digit number, represented by ab, such that neither a nor b is zero and a is not equal to b. Reverse the digits and add the result to the original number. Divide this sum by $a + b$. What is the quotient? (*MT* calendar problem)

6. ***Favorite Foods*** Violet, Kathy, Lucy, Charles, and John are at a barbecue with their favorite foods: hot dogs, hamburgers, meatless pasta salad, veggie burgers, and ribs. Charles and Kathy are carnivores, but neither likes hamburgers. Violet is a vegetarian. The women dislike ribs. Lucy loves pasta salad. If no two people have the same favorite food, match each of the people to her or his food. (*MT* calendar problem)

7. ***Broken Elevator*** A man enters a building on the first floor and runs up to the third floor in 20 seconds. At this rate, how many seconds would it take for the man to run from the first floor up to the sixth floor? (*MT* calendar problem)

8. ***Saving Her Dollars*** Every day Sally saved a penny, a dime, and a quarter. What is the least number of days required for her to save an amount equal to an integral (counting) number of dollars? (*MT* calendar problem)

9. ***Do You Have a Match?*** Move 4 of the matches in the figure to create exactly 3 equilateral triangles. (An *equilateral triangle* has all three sides the same length.) (*MT* calendar problem)



10. ***Sudoku*** Sudoku is an $n \times n$ puzzle that requires the solver to fill in all the squares using the integers 1 through n. Each row, column, and subrectangle contains exactly one of each number. Complete the $n \times n$ puzzle. (*MT* calendar problem)

			4
2			
	1		
		1	

11. ***Break This Code*** Each letter of the alphabet is assigned an integer, starting with A = 0, B = 1, and so on. The numbers repeat after every seven letters, so that G = 6, H = 0, and I = 1, continuing on to Z. What two-letter word is represented by the digits 16? (*MT* calendar problem)

12. ***A Real Problem*** We are given the following sequence:

PROBLEMSOLVINGPROBLEMSOLVINGPROB . . .

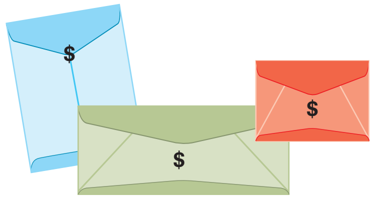
If the pattern continues, what letter will be in the 2012th position? (*MT* calendar problem)

13. ***How Old Is Mommy?*** A mother has two children whose ages differ by 5 years. The sum of the squares of their ages is 97. The square of the mother's age can be found by writing the squares of the children's ages one after the other as a four-digit number. How old is the mother? (*MT* calendar problem)

14. ***An Alarming Situation*** You have three alarms in your room. Your cell phone alarm is set to ring every 30 minutes, your computer alarm is set to ring every 20 minutes, and your clock alarm is set to ring every 45 minutes. If all three alarms go off simultaneously at 12:34 p.m., when is the next time that they will go off simultaneously? (*MT* calendar problem)

15. ***Laundry Day*** Every Monday evening, a mathematics teacher stops by the dry cleaners, drops off the shirts that he wore for the week, and picks up his previous week's load. If he wears a clean shirt every day, including Saturday and Sunday, what is the minimum number of shirts that he needs to own? (*MT* calendar problem)

16. ***Pick an Envelope*** Three envelopes contain a total of six bills. One envelope contains two \$10 bills, one contains two \$20 bills, and the third contains one \$10 and one \$20 bill. A label on each envelope indicates the sum of money in one of the other envelopes. It is possible to select one envelope, see one bill in that envelope, and then state the contents of all of the envelopes. Which envelope should you choose? (*MT* calendar problem)



17. ***Class Members*** A classroom contains an equal number of boys and girls. If 8 girls leave, twice as many boys as girls remain. What was the original number of students present? (*MT* calendar problem)

18. ***Give Me a Digit*** Given a two-digit number, make a three-digit number by putting a 6 as the rightmost digit. Then add 6 to the resulting three-digit number, and remove the rightmost digit to obtain another two-digit number. If the result is 76, what is the original two-digit number? (*MT* calendar problem)

19. ***Missing Digit*** Look for a pattern and find the missing digit x.

3	2	4	8
7	2	1	3
8	4	x	5
4	3	6	9

(*MT* calendar problem)

20. ***Abundancy*** An integer $n > 1$ is **abundant** if the sum of its proper divisors (positive integer divisors smaller than n) is greater than n. Find the smallest abundant integer. (*MT* calendar problem)

21. ***Cross-Country Competition*** The schools in an athletic conference compete in a cross-country meet to which each school sends three participants. Erin, Katelyn, and Iliana are the three representatives from one school. Erin finished the race in the middle position; Katelyn finished after Erin, in the 19th position; and Iliana finished 28th. How many schools took part in the race? (*MT* calendar problem)

22. ***Gone Fishing*** Four friends go fishing one day and bring home a total of 11 fish. If each person caught at least 1 fish, then which of the following *must* be true?

A. One person caught exactly 2 fish.

B. One person caught exactly 3 fish.

C. One person caught fewer than 3 fish.

D. One person caught more than 3 fish.

E. Two people each caught more than 1 fish.

(*MT* calendar problem)

23. ***Bookworm Snack*** A 26-volume encyclopedia (one for each letter) is placed on a bookshelf in alphabetical order from left to right. Each volume is 2 inches thick, including the front and back covers. Each cover is $\frac{1}{4}$ inch thick. A bookworm eats straight through the encyclopedia, beginning inside the front cover of volume A and ending after eating through the back cover of volume Z. How many inches of book did the bookworm eat? (*MT* calendar problem)

24. ***You Lie!*** Max, Sam, and Brett were playing basketball. One of them broke a window, and the other two saw him break it. Max said, "I am innocent." Sam said, "Max and I are both innocent." Brett said, "Max and Sam are both innocent." If only one of them is telling the truth, who broke the window? (*MT* calendar problem)

25. ***Catwoman's Cats*** If you ask Batman's nemesis, Catwoman, how many cats she has, she answers with a riddle: "Five-sixths of my cats plus seven." How many cats does Catwoman have? (*MT* calendar problem)

26. ***Pencil Collection*** Bob gave four-fifths of his pencils to Barbara; then he gave two-thirds of the remaining pencils to Bonnie. If he ended up with ten pencils for himself, with how many did he start? (*MT* calendar problem)

27. ***Adding Gasoline*** The gasoline gauge on a van initially read $\frac{1}{8}$ full. When 15 gallons were added to the tank, the gauge read $\frac{3}{4}$ full. How many more gallons are needed to fill the tank? (*MT* calendar problem)

28. ***Gasoline Tank Capacity*** When 6 gallons of gasoline are put into a car's tank, the indicator goes from $\frac{1}{4}$ of a tank to $\frac{5}{8}$. What is the total capacity of the gasoline tank? (*MT* calendar problem)

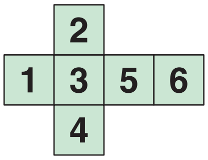
29. ***Number Pattern*** What is the relationship between the rows of numbers?

18,	38,	24,	46,	42
8,	24,	8,	24,	8

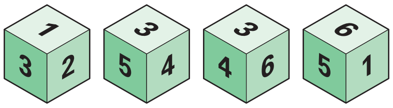
(*MT* calendar problem)

30. ***Locking Boxes*** You and I each have one lock and a corresponding key. I want to mail you a box with a ring in it, but any box that is not locked will be emptied before it reaches its recipient. How can I safely send you the ring? (Note that you and I each have keys to our own lock but not to the other lock.) (*MT* calendar problem)

31. ***Unfolding and Folding a Box*** An unfolded box is shown below.



Which figure shows the box folded up? (*MT* calendar problem)



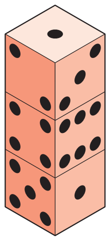
32. ***Unknown Number*** Cindy was asked by her teacher to subtract 3 from a certain number and then divide the result by 9. Instead, she subtracted 9 and then divided the result by 3, giving an answer of 43. What would her answer have been if she had worked the problem correctly? (*MT* calendar problem)

33. ***Labeling Boxes*** You are working in a store that has been very careless with the stock. Three boxes of socks are each incorrectly labeled. The labels say *red socks, green socks,* and *red and green socks*. How can you relabel the boxes correctly by taking only one sock out of one box, without looking inside the boxes? (*MT* calendar problem)

34. ***Vertical Symmetry in States' Names*** (If a vertical line is drawn through the center of a figure, and the left and right sides are reflections of each other across this line, the figure is said to have vertical symmetry.) When HAWAII is spelled with all capital letters, each letter has vertical symmetry. Find the name of a state whose capital letters all have both vertical and horizontal symmetry. (*MT* calendar problem)

35. ***Sum of Hidden Dots on Dice*** Three dice with faces numbered 1 through 6 are stacked as shown. Seven of the eighteen faces are visible, leaving eleven faces hidden on the back, on the bottom, and between dice. The total number of dots not visible in this view is _____.

A. 21
B. 22
C. 31
D. 41
E. 53



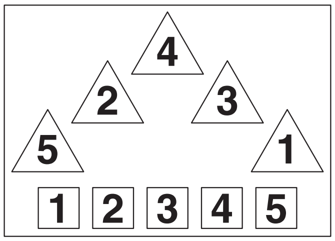
(*MT* calendar problem)

36. ***Mr. Green's Age*** At his birthday party, Mr. Green would not directly tell how old he was. He said,

> "If you add the year of my birth to this year, subtract the year of my tenth birthday and the year of my fiftieth birthday, and then add my present age, the result is eighty."

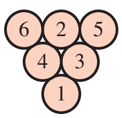
How old was Mr. Green? (*MT* calendar problem)

37. ***Matching Triangles and Squares*** How can you connect each square with the triangle that has the same number? Lines cannot cross, enter a square or triangle, or go outside the diagram. (*MT* calendar problem)

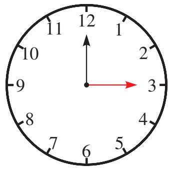


38. ***Age of the Bus Driver*** Today is your first day driving a city bus. When you leave downtown, you have twenty-three passengers. At the first stop, three people exit and five people get on the bus. At the second stop, eleven people exit and eight people get on the bus. At the third stop, five people exit and ten people get on. How old is the bus driver? (*MT* calendar problem)

39. ***Difference Triangle*** Balls numbered 1 through 6 are arranged in a **difference triangle.** Note that in any row, the difference between the larger and the smaller of two successive balls is the number of the ball that appears below them. Arrange balls numbered 1 through 10 in a difference triangle. (*MT* calendar problem)



40. ***Clock Face*** By drawing two straight lines, divide the face of a clock into three regions such that the numbers in the regions have the same total. (*MT* calendar problem)



41. ***Alphametric*** If a, b, and c are digits for which

$$\begin{array}{r} 7\ \ a\ \ 2 \\ -4\ \ 8\ \ b \\ \hline c\ \ 7\ \ 3, \end{array}$$

then $a + b + c =$ _____.

A. 14 **B.** 15 **C.** 16 **D.** 17 **E.** 18

(*MT* calendar problem)

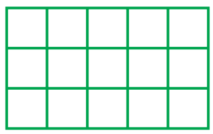
42. ***Perfect Square*** Only one of these numbers is a perfect square. Which one is it? (*MT* calendar problem)

329,476 389,372 964,328
326,047 724,203

43. ***Sleeping on the Way to Grandma's House*** While traveling to his grandmother's for Christmas, George fell asleep halfway through the journey. When he awoke, he still had to travel half the distance that he had traveled while sleeping. For what part of the entire journey had he been asleep? (*MT* calendar problem)

44. ***Buckets of Water*** You have brought two unmarked buckets to a stream. The buckets hold 7 gallons and 3 gallons of water, respectively. How can you obtain exactly 5 gallons of water to take home? (*MT* calendar problem)

45. ***Counting Puzzle (Rectangles)*** How many rectangles are in the figure? (*MT* calendar problem)



46. ***Digit Puzzle*** Place each of the digits

1, 2, 3, 4, 5, 6, 7, and 8

in separate boxes so that boxes that share common corners do not contain successive digits. (*MT* calendar problem)