A Problem Solving Approach to

Mathematics

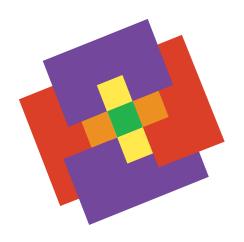for Elementary School Teachers

13th Edition

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A Problem Solving Approach to

Mathematics

for Elementary School Teachers

13th Edition

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Dedication

To Connor and Cole and all other future problem solvers of the world.
—Rick Billstein

To my papa for always believing in me.
To my husband Brian—it's a joy to share my love, life, and career with you.
To my daughters Isabelle, Noelle, and Juliette—you provide me with constant joy, pride, and inspiration.
—Barbara Boschmans
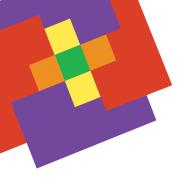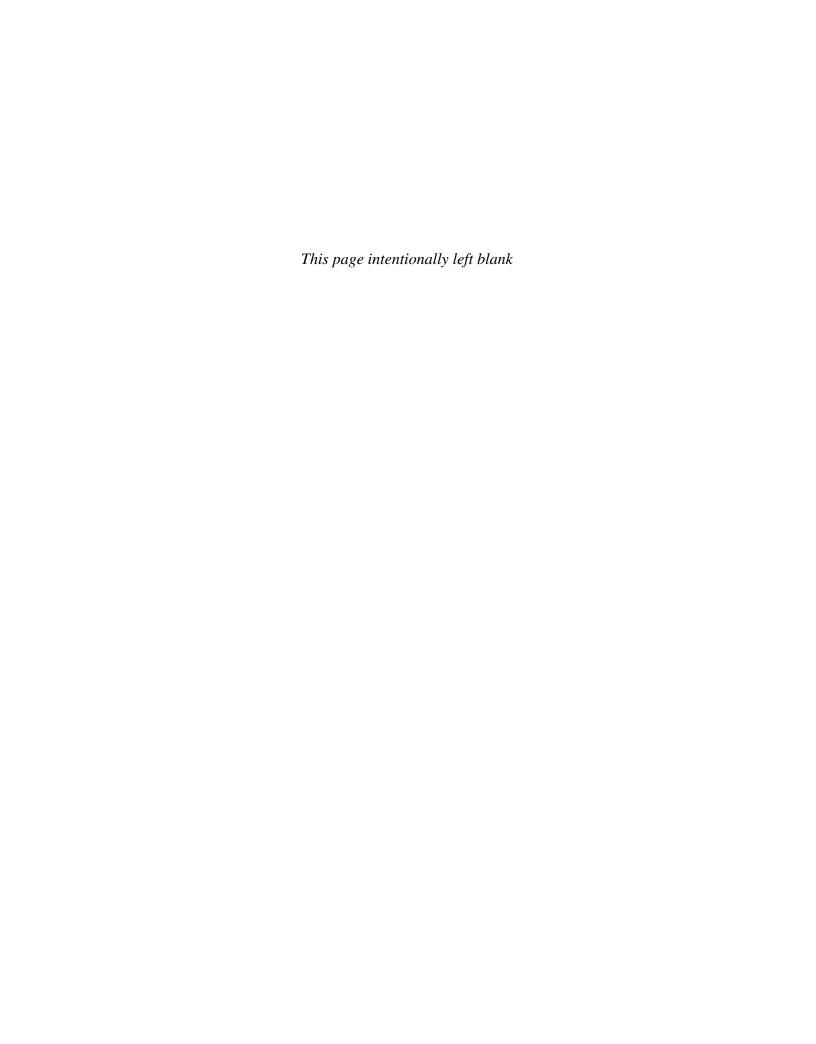
To my dear cousin Paweł Białowąs.
—Shlomo Libeskind

To my family, and to the next generation of prospective mathematics teachers
without whom mathematics would be in serious trouble.
—Johnny W. Lott

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Contents

Chapter 8   Algebraic Thinking   429

Chapter 9   Probability   495

Chapter 10   Data Analysis/Statistics: An Introduction   573

* Online modules are available in MyLab Math or at www.pearsonhighered.com/mathstatsresources

Chapter 11   Introductory Geometry   659

Chapter 12   Congruence and Similarity with Constructions   737

Chapter 13   Area, Pythagorean Theorem, and Volume   802

Chapter 14   Transformations   894

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Preface

The 13th edition of *A Problem Solving Approach to Mathematics for Elementary School Teachers* is designed to prepare outstanding future elementary and middle school teachers. This edition, heavy on concept and skill based, has an emphasis on active and collaborative learning. The content has been revised and updated to better prepare prospective students as future teachers in their own classrooms.

National Standards for Mathematics

- **Common Core State Standards (CCSS) for Mathematics** The National Governors Association spearheaded the effort to develop the *Common Core Standards* (2010); they are used in this text to highlight concepts. The complete text of the *Common Core Standards* is found at www.corestandards.org.
- **Principles and Standards** The National Council of Teachers of Mathematics (NCTM) publications, *Principles and Standards of School Mathematics* (2000) and *Principles to Actions: Ensuring Mathematical Success for All* (2014) are guides for the book. The complete text of the NCTM *Principles and Standards* can be found online at www.nctm.org.

Our Goals

- To present appropriate mathematics in an intellectually honest and mathematically correct manner.
- To use problem solving as an integral part of mathematics.
- To approach mathematics in a sequence that instills confidence and challenges students.
- To provide opportunities for alternate forms of teaching and learning.
- To provide communication problems to develop writing skills that allow students to practice reasoning and explanation through mathematical exposition.
- To provide core mathematics for prospective elementary and middle school teachers in a way that challenges them to determine why mathematics is done as it is.
- To provide core mathematics that allows instructors to use methods integrated with content.
- To assist prospective teachers with connecting mathematics, its ideas, and its applications.
- To assist future teachers in becoming familiar with the content and philosophy of national standards.

The 13th edition provides instructors a variety of approaches to teaching, encourages discussion and collaboration among future teachers and with their instructors, and aids the integration of projects into the curriculum. Most importantly, it promotes discovery and active learning.

New to This Edition

- The Chapter Opener Problems use examples from social media and are more relevant and engaging.
- Chapter 3 has been completely reorganized to more closely model the way operations are taught in elementary school. See Content Highlights for more details.
- The Mathematical Connections portion of the exercise sets has been reorganized and upgraded to reflect current thinking in mathematics education.
- StatCrunch **StatCrunch** has been referenced where relevant to provide the opportunity for students to interact with data.
- All chapters have many new solved examples and new problems in assessments.
- The treatment of many topics has been enhanced to reflect a tighter connection to the CCSS. Examples include
  - ► Chapter 1: Expanded the Four-Step Polya Problem-Solving Process with input from Standards for Mathematical Practice. The process is referred to in examples throughout the chapter and the book. The section on patterns has been shortened.
  - ► Chapter 2: The section on valid reasoning has been revised. Logical reasoning is now an integral part of Chapter 2.
  - ► Chapter 5: Now includes a definition of addition for integers that uses absolute value—included because it is one of the techniques used in operations on integers in CCSS.
  - ► Chapter 6: As called for in CCSS, a section on Using Bar Models to Solve Rational Proportion Problems has been added.

▶ Chapter 8: Algebraic Thinking is streamlined, with greater emphasis on variables and equation solving.
▶ Chapter 10: Reorganization and the inclusion of misuses throughout the chapter more clearly follow recommendations of statistical educators.
▶ Chapter 13: Due to the central role of measurement in geometry, Chapter 13, Area, Pythagorean Theorem, and Volume, has been moved to come before Chapter 14, Transformations.

• The chapter summary charts are more comprehensive as resources for students.
• New media assets are available in MyLab Math. All media assets include assignable questions.
  ▶ Completely new section lecture videos incorporate the various resources found in the MyLab Math course, such as animations and IMAP videos.
  ▶ Common Core in Action videos
  ▶ Animations library
  ▶ Math Education Insights videos
  ▶ Common Core Assessment Analysis questions enable students to analyze real CCSS assessment questions.
  ▶ Mindset materials support students in developing a math mindset and an understanding of how important it is to do so with their future students.

Content Highlights

Chapter 1 An Introduction to Problem Solving
This chapter has been reorganized and shortened to make it friendlier. The section on patterns has been shortened.

Chapter 2 Introduction to Logic and Sets
This section on logic works hand in hand with the ideas of set operations and enhances reasoning. Set theory and set operations with properties are introduced as a basis for learning whole number concepts. The section on Valid Reasoning has been shortened.

Chapter 3 Numeration Systems and Whole Number Operations
This chapter has been reorganized to model more closely the way the operations are taught in elementary school. The chapter opens with a brief overview of different numeration systems and an emphasis on place value through the study of different bases. New addition models were added. Subtraction models were extensively expanded. Now multiplication with algorithms is followed by division. Operations in different bases, mental math, and estimation are included with each operation as well as the whole number properties for each operation.

Chapter 4 Number Theory
In the 13th edition, a separate chapter on number theory does not depend on integers, which are introduced in Chapter 5. Concepts of divisibility with divisibility tests are discovered. Prime numbers, prime factorization, greatest common divisor and least common multiple as well as the Euclidean Algorithm are explored with many new exercises added. A module on Clock Arithmetic is available online.*

Chapter 5 Integers
This chapter concentrates only on integers, their representation, their operations, and properties.

Chapter 6 Rational Numbers and Proportional Reasoning
This chapter has been revised to follow many recommendations in the *Common Core Standards*. Videos showing elementary students learning fraction concepts are included so that future teachers can observe what happens when elementary students absorb what is taught and how they work with those concepts. Proportional reasoning, one of the most important concepts taught in middle school mathematics, is covered in great depth in its natural setting. A new section on using Bar Models to Solve Ratio and Proportion Problems has been added.

Chapter 7 Decimals, Percents, and Real Numbers
This chapter focuses on the decimal representation of rational numbers and their operations. Percent applications are solved with percent bars. Mental math and estimations with percents are also included. The chapter also includes a development of real numbers.

Chapter 8 Algebraic Thinking
With an introduction to real numbers moved to an earlier chapter, the chapter now gives a review of algebra needed to teach in grades K through 8. This includes work in the coordinate plane. A module on Using Real Numbers in Equations is available online.*

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* Online modules are available in MyLab Math or at www.pearsonhighered.com/mathstatsresources

Chapter 9 Probability
This chapter discusses the probabilities of single stage and multistage experiments. Counting techniques have been expanded in this edition. StatCrunch **StatCrunch** references link to applets providing a hands-on way for students to interact with probability concepts.

Chapter 10 Data Analysis/Statistics: An Introduction
Chapter 10 has been reorganized and simplified. Designing Experiments/Collecting Data is based on *Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report: A preK–12 Curriculum Framework* (2005) by the American Statistical Association. In the following sections, data, graphs, examples, and assessment exercises have been updated and new material added. The action formerly called Abuses of Statistics has been integrated into earlier sections in this edition. StatCrunch **StatCrunch** references aid students in data display.

Chapter 11 Introductory Geometry
This chapter allows students to explore some ramifications of different definitions in mathematics used in schools. Linear measure is introduced to emphasize its importance in the curriculum. Also symmetries are introduced as a concept to form geometrical definitions. The Networks module is now offered online.*

Chapter 12 Congruence and Similarity with Constructions
Congruence and constructions sections have been expanded to allow more exploration. Many new exercises have been added. A module on Trigonometric Ratios via Similarity is available online.*

Chapter 13 Area, Pythagorean Theorem, and Volume
Concepts of linear measure are included with the topics of area, the Pythagorean Theorem, and volume. Many topics have been shifted and new material added. Assessment sets and examples have been updated.

Chapter 14 Transformations
The order of the geometry chapters has been changed, with transformations now being the subject of the final chapter. Tessellations have been added to the section on translations and rotations. Reflections, glide reflections, and dilations follow this section.

Features

In creating the 13th edition of this text, we used strengths of previous editions, incorporating feedback from users, and making improvements to better prepare future teachers for teaching mathematics.

Learning the Mathematics in the New Standards

- **New!** In this edition, we have made judicious cuts to even more effectively bring key ideas to the forefront. **A streamlined narrative** keeps students focused on the important ideas.
- **Chapter openers** from social media are thought-provoking and set the tone to prepare students for the material ahead.
- **Learning Objectives** at the beginning of every section focus student attention on key ideas.
- **Problem-Solving Strategies** are highlighted in italics and are used throughout the text to help students put the strategies to work.
- **Chapter Summaries** are organized in a student-friendly chart format for easy reference. These summaries also provide specific pages for additional help on individual concepts.
- **Chapter Review** questions allow students to test themselves.

Focusing on the Standards for Mathematical Practice

- The **Activity Manual** includes classroom-tested activities and a pouch of perforated, printed color manipulatives.
  - ▶ **Activity Manual annotations** in the Annotated Instructor's Edition clarify when specific activities should be used for each lesson, making it easier to teach a more hands-on course.
  - ▶ The manual is available as a value-pack option. Ask your Pearson representative for details.
- **Now Try This** exercises follow key examples and help students in their learning, facilitate the development of critical thinking and problem-solving skills, and stimulate class discussion. Answers are in both the Annotated Instructor's Edition and student text.

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* Online modules are available in MyLab Math or at www.pearsonhighered.com/mathstatsresources

- **13 eManipulatives,** available in MyLab Math, allow students to investigate, explore, practice, build conceptual understanding, and solve specific problems without the mess or cost of physical manipulatives. Annotations in the student edition indicate where these eManipulatives are relevant. Exercises related to the eManipulatives are assignable within MyLab Math.
- **Integrating Mathematics and Pedagogy (IMAP)** videos, available in MyLab Math, feature elementary school children working problems. Margin notes in the student edition indicate where these videos are relevant. Exercises related to the IMAP videos are assignable within MyLab Math.
- **New! StatCrunch** StatCrunch technology has been referenced in various chapters as an aid to learning.

Teaching the Mathematics in the New Standards

- **Common Core State Standards (CCSS)** are cited in sections to focus student attention and provide a springboard for discussion.
- **New! More exercises** have been added, particularly in the reorganized Mathematical Connections exercise sets.
- **Connecting Mathematics to the Classroom** exercises require interpretation and analysis of the thinking of typical K-8 students.
- **Updated! School Book Pages** are included to show how various topics are introduced to the K-8 pupil. Icons within the text link the narrative to the appropriate School Book Page. Students are asked to complete many of the activities on the student pages so they can see what is expected in elementary school.
- **Historical Notes** have been updated to include living mathematicians as well as to add context and humanize the mathematics.

Assessing the Mathematics Content and Procedures

- **Extensive Problem Sets** are organized into three categories for maximum instructor flexibility when assigning homework and that address the standards.
  - ▶ **Assessment A** has problems with answers in the text, so that students can check their work.
  - ▶ **Assessment B** contains parallel problems to those in Assessment A, but answers are not given in the student text.
  - ▶ **Mathematical Connections** problems include the following categories: Connecting Mathematics to the Classroom, Review Problems, School Book Pages, Group Work, and NAEP sample questions. Odd-numbered answers to Connecting Mathematics to the Classroom, Review Problems, and School Book Pages are in the student answer section.
- **Hundreds of assignable algorithmic exercises.** The MyLab Math course for the 13th edition contains even more assignable exercises to meet students' needs. Assignable exercise types include the following:
  - ▶ **Textbook** exercises—over 2000 algorithmically generated exercises parallel those in the text.
  - ▶ **Lecture video** exercises allow instructors to assign the videos with confidence that students will watch them.
  - ▶ **New! Common Core in Action** video exercises help students see the content of the CCSS in context.
  - ▶ **New! Common Core Assessment Analysis** exercises require analysis and interpretation of sample CCSS exercises.
  - ▶ **New! Animations** exercises encourage students to interact with the Animations to support understanding.
  - ▶ **eManipulative** exercises require use of the eManipulatives within MyLab Math so students can be familiar with this important teaching and learning tool.
  - ▶ **Integrating Mathematics and Pedagogy (IMAP)** video exercises require analysis of student work.
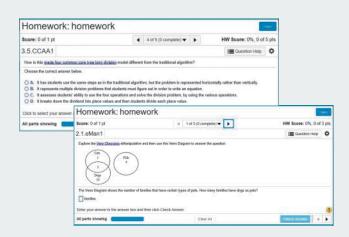
MyLab Math Online Course for *A Problem Solving Approach to Mathematics for Elementary School Teachers*
(access code required)
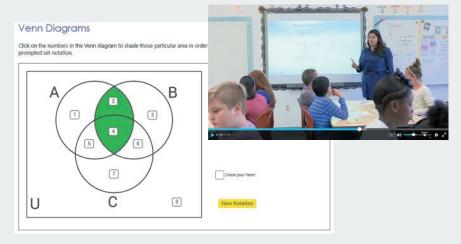
MyLab™ Math is the teaching and learning platform that empowers you to reach *every* student. By combining trusted author content with digital tools and a flexible platform, MyLab Math personalizes the learning experience and improves results for each student.

Unlimited Practice & Unique Assessment

Over 2000 exercises correlated to the textbook provide immediate feedback, to give students unlimited opportunities for practice.

Additional exercises complement the various media elements in the course, such as Animations, Common Core in Action videos, IMAP videos, eManipulatives, and more. **Common Core Assessment Analysis** questions were developed to require analysis and interpretation of sample CCSS test problems.
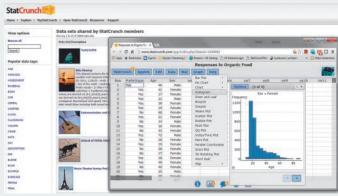
Learning Tools

Resources in the MyLab course support the text, including a variety of videos, eManipulatives, animations, and StatCrunch. New Mindset materials support students in developing a math mindset and how to do so with their future students.

StatCrunch

Integrated into the MyLab Math course and accompanying eText as an aid to learning, StatCrunch® is a powerful web-based statistical software that allows users to interact with data.

Resources for Success

Instructor Resources

Annotated Instructor's Edition

- Includes answers to text exercises on the pages where they appear, when possible. Answers not on the page are found in the back of the book.
- Provides answers to new chapter opening problems, Now Try This problems, and Mathematical Connections questions.
- Annotations throughout help integrate use of the activities in the accompanying Activity Manual.

Online Supplements

The following instructor material is available for download from Pearson's Instructor Resource Center (www.pearsonhighered.com/irc) or within the text's MyLab Math course.

Instructor's Solution Manual

Brian Beaudrie, *Northern Arizona University*

- Provides detailed, worked-out solutions to all of the problems in Assessments A and B, Mathematical Connections Review Problems, and Chapter Review exercises.

Instructor's Testing Manual

- Comprehensive worksheets contain two forms of chapter assessments with answers for each.

Instructor's Guide for *Mathematics Activities for Elementary School Teachers*, 13th edition

Dan Dolan, *Project to Increase Mastery of Mathematics and Science, Wesleyan University*; Jim Williamson, *University of Montana*; and Mari Muri, *Project to Increase Mastery of Mathematics and Science, Wesleyan University*

- Contains answers and teaching suggestions for activities in the student Activity Manual.

PowerPoint Lecture Slides

- Fully editable slides provide section-by-section coverage of key topics and concepts along with examples.
- An Image Resource Library is also available within MyLab Math and contains art files from the text enabling further customization of the PowerPoint lectures.

TestGen®

- TestGen® enables instructors to build, edit, and administer tests using a computerized bank of questions developed to cover all the objectives of the text. Instructors can also modify or add new questions.

Student Resources

Mathematics Activities for Elementary School Teachers: 13th edition
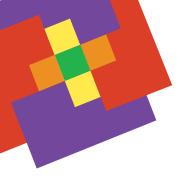
Dan Dolan, *Project to Increase Mastery of Mathematics and Science, Wesleyan University*; Jim Williamson, *University of Montana*; and Mari Muri, *Project to Increase Mastery of Mathematics and Science, Wesleyan University*
ISBN: 0134995619 / 9780134995618

- Provides hands-on, manipulative-based activities keyed to the text that involve future elementary school teachers discovering concepts, solving problems, and exploring mathematical ideas.
- Colorful, perforated paper manipulatives in a convenient storage pouch.
- Activities can also be adapted for use with elementary students at a later time.
- References to these activities are in the margin of the Annotated Instructor's Edition.

Student Solutions Manual

Brian Beaudrie, *Northern Arizona University*
ISBN: 0135184207 / 9780135184202

- Provides detailed, worked-out solutions to all of the problems in Assessment A, odd Mathematical Connections Review Problems, and all Chapter Review exercises.

Acknowledgments

For past editions of this book, many noted and illustrious mathematics educators and mathematicians have served as reviewers. To honor the work of the past as well as to honor the reviewers of this edition, we list all but place asterisks by this edition's reviewers.

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Jane Schielack
Barbara Shabell
M. Geralda Shaefer

An Introduction to Problem Solving

1

Find and explain a pattern that will replace the question mark.

$$1 + 4 \rightarrow 5$$

$$2 + 5 \rightarrow 12$$

$$3 + 6 \rightarrow 21$$

$$8 + 11 \rightarrow \ ?$$

If needed, see Hint on page 34.

P roblem solving has long been central in the learning of mathematics at all levels. George Pólya (1887–1985), a great mathematician of the twentieth century, is the father of mathematical problem solving. He pointed out that "solving a problem means finding a way out of difficulty, a way around an obstacle, attaining an aim which was not immediately attainable." (Pólya 1981, p.ix)

Pólya developed a four-step problem-solving process that has been adopted by many. A modified version is given here.

1. Understanding the problem
2. Devising a plan
3. Carrying out the plan
4. Looking back

Problem solving is one of the five process standards identified in *Principles and Standards of School Mathematics* (2000) by the *National Council of Teachers of Mathematics (NCTM)* and shown in Figure 1.

Figure 1
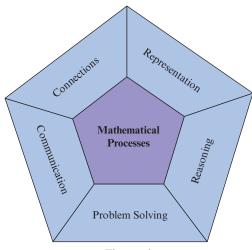
CCSS Problem solving is also one of the eight *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics* (hereafter identified as Common Core Standards and abbreviated as CCSS). The Common Core Standards, developed in 2010 through the National Governors Association and the Council of Chief State School Officers, are built around the Standards for Mathematical Practice seen in Table 1. Table 1 is an abridged version of these *Standards*.

Students learn mathematics by solving problems. *Exercises* are routine practice for skill building and serve a purpose in learning mathematics, but problem solving must be a focus of school mathematics. A reasonable amount of tension and discomfort improves problem-solving performance.

▶ **IMAP Video**

Watch Elise talk about problem-solving with her students.

Table 1

1. **Make sense of problems and persevere in solving them.**
   Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt.

2. **Reason abstractly and quantitatively.**
   Mathematically proficient students make sense of quantities and their relationships in problem situations.

3. **Construct viable arguments and critique the reasoning of others.**
   Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples.

4. **Model with mathematics.**
   Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community.

5. **Use appropriate tools strategically.**
   Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software.

6. **Attend to precision.**
   Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context.

7. **Look for and make use of structure.**
   Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$.

8. **Look for and express regularity in repeated reasoning.**
   Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal.

Mathematical problem solving may occur when:

1. Students are presented with a situation that they understand but do not know how to proceed directly to a solution.
2. Students are interested in finding the solution and attempt to do so.
3. Students are required to use mathematical ideas to solve the problem.

We present many opportunities in this text to solve problems. Each chapter in this text opens with a problem that can be solved using concepts developed in the chapter. Throughout the text, some problems are solved using a four-step process described in Section 1-1.

1-1 Mathematics and Problem Solving

IMAP Video

Watch Tonya's class model strategies for solving a problem.

If problems are approached repeatedly in only one way, a mind-set may be formed. For example, consider the following:

> Spell the word spot *three times out loud.* "S-P-O-T! S-P-O-T! S-P-O-T!" *Now answer the question* "What do we do when we come to a green light?" *Write an answer.*

If we answer "Stop," we may be guilty of having formed a mind-set. We do not stop at a *green* light.

Consider the following: "A shepherd had 36 sheep. All but 10 died. How many lived?" If we answer "10," we are ready to try some problems. If not, we probably did not understand the question by not reading it carefully. *Understanding the problem* is the first step in the four-step problem-solving process developed by George Pólya. Using this process does not guarantee a solution to a problem, but it does provide a systematic approach for finding a solution.

Four-Step Problem-Solving Process

1. **Understanding the problem**
   a. Can the problem be stated differently?
   b. What is to be found or what is needed?
   c. What are the unknowns?
   d. What information is obtained from the problem?
   e. What information, if any, is missing or not needed?

2. **Devising a plan**
   The following list of strategies, although not exhaustive, is very useful:
   a. Look for a pattern.
   b. Examine related problems and determine whether the same techniques applied to them can be applied to the current problem.
   c. Examine a simpler or special case of the problem to gain insight into the solution of the original problem.
   d. Make a table or list.
   e. Identify a subgoal.
   f. Make a diagram.
   g. Use guess and check.
   h. Work backward.
   i. Write an equation.

3. **Carrying out the plan**
   a. Implement the strategy or strategies in step 2 and perform any necessary actions or computations.
   b. Attend to precision in language and mathematics used.
   c. Check each step of the plan along the way. This may be intuitive checking or a formal proof of each step.
   d. Keep an accurate record of all work.

4. **Looking back**
   a. Check the results in the original problem. (In some cases, this will require a proof.)
   b. Interpret the solution in terms of the original problem. Does the answer make sense? Is it reasonable? Does it answer the question that was asked?
   c. Determine whether there is another method of finding the solution.
   d. If possible, determine other related or more general problems for which the techniques will work.

For the four-step problem-solving process to be successful in schools, it must receive instructional attention in the mathematics curriculum. The process wording may change from book to book, but the ideas are basically the same. Look at the sixth grade School Book Page on page 5, which refers to *Mathematical Practice 1*, and notice how Pólya's four-step process is used. See exercise 11 in Mathematical Connection 1-1.

School Book Page — Problem Solving Process

Math Practices and Problem Solving Handbook

1 ▶ Make sense of problems and persevere in solving them.

Jon earns $15.50 per week for helping his dad deliver newspapers. He has helped his dad for 3 weeks. Jon uses part of his earnings to buy a new video game that costs $42.39, including tax. How much of his earnings does he have left?

NEWS

What am I asked to find? How much money Jon has left.

What are the quantities and variables? How do they relate? The cost of the video game is an expense.

The amount Jon earned by helping his father is income.

What can I do if I get stuck? Start by finding out how much money Jon earned in two weeks, then in three weeks.

What is a good plan for solving the problem? Find the total Jon earned in three weeks. Then subtract the cost of the video game.

Other questions to consider:
- Have I solved a similar problem before?
- What information is necessary and what is unnecessary?
- How can I check that my answer makes sense?
- How is my solution pathway the same as or different from my classmate's?

F16

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IMAP Video

Watch Tonya's class model strategies for solving a problem.

Strategies for Problem Solving

A variety of problems in different contexts provide experience in problem solving. Strategies are used to discover or construct the means to achieve a solution. For each strategy described, we give an example that can be solved with that strategy. Often, problems can be solved in more than one way. There is no one best strategy to use.

In many of the examples, we use the **natural numbers**, 1, 2, 3, ... The first three dots, an *ellipsis,* are used to represent missing terms. The expanded problem-solving steps highlighting some strategies are shown next.

Strategy: Look for a Pattern

IMAP Video

Watch a fourth–grade class model Gauss's strategy.

Problem Solving **Gauss's Problem**

As young students, Carl Gauss and his fellow classmates were asked to find the sum of the first 100 natural numbers. The teacher expected to keep the class occupied for some time, but Gauss gave the answer almost immediately. How might he have done it?

Understanding the Problem The natural numbers are 1, 2, 3, 4, ... Thus, the problem is to find the sum $1 + 2 + 3 + 4 + \ldots + 100$.

Devising a Plan The strategy *look for a pattern* is useful here. One story about young Gauss reports that he listed the sum, and wrote the same sum backward, as in Figure 2. If $S = 1 + 2 + 3 + 4 + 5 + \ldots + 98 + 99 + 100$, then Gauss could have seen the following pattern.

$$
\begin{array}{rrrrrrrrr}
S = & 1 + & 2 + & 3 + & 4 + & 5 + \ldots + & 98 + & 99 + & 100 \\
+\ S = & 100 + & 99 + & 98 + & 97 + & 96 + \ldots + & 3 + & 2 + & 1 \\
\hline
2S = & 101 + & 101 + & 101 + & 101 + & 101 + \ldots + & 101 + & 101 + & 101
\end{array}
$$

Figure 2

To discover the original sum from the last equation, Gauss could have divided the sum, $2S$, in Figure 2 by 2.

Carrying Out the Plan Note that the sum in each pair, $(1, 100), (2, 99), (3, 98), \ldots , (100, 1)$, is always 101, and there are 100 pairs with this sum. Thus, $2S = 100 \cdot 101$ and $S = \dfrac{100 \cdot 101}{2} = 5050$.

Looking Back This technique can be used to solve a more general problem of finding the sum of the first n natural numbers $1 + 2 + 3 + 4 + 5 + 6 + \ldots + n$. We use the same plan as before and notice the relationship in Figure 3. Because there are n sums of $(n + 1)$, we have $2S = n(n + 1)$ and $S = \dfrac{n(n + 1)}{2}$.

$$
\begin{array}{rrrrrr}
S = & 1 + & 2 + & 3 + & 4 + \ldots + & n \\
+\ S = & n + & (n - 1) + & (n - 2) + & (n - 3) + \ldots + & 1 \\
\hline
2S = & (n + 1) + & (n + 1) + & (n + 1) + & (n + 1) + \ldots + & (n + 1)
\end{array}
$$

Figure 3

A different strategy for finding a sum of consecutive natural numbers involves the strategy of *making a diagram* and thinking of the sum geometrically as a stack of blocks. This alternative method is explored in exercise 2 of Assessment 1-1A.

▶ **NOW TRY THIS 1**

Explain whether the approach in Gauss's Problem of writing the sum backward and applying the strategy *look for a pattern* will or will not work in finding the sum: $1^2 + 2^2 + \ldots + 100^2$.

Historical Note

Carl Gauss (1777–1855), one of the greatest mathematicians of all time, was born to humble parents in Brunswick, Germany. He was an infant prodigy who later made contributions in many areas of science as well as mathematics. After Gauss's death, the King of Hanover honored him with a commemorative medal with the inscription "Prince of Mathematics." ●

Strategy: Examine a Related Problem

Problem Solving   **Sums of Even Natural Numbers**

Find the sum of the even natural numbers less than or equal to 100.

Understanding the Problem Even natural numbers are $2, 4, 6, 8, 10, \ldots$. The problem is to find the sum of these numbers: $2 + 4 + 6 + 8 + \ldots + 100$.

Devising a Plan Recognizing that the sum can be *related to Gauss's original problem* helps us devise a plan. Consider the following:

$$2 + 4 + 6 + 8 + \ldots + 100 = 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + 2 \cdot 4 + \ldots + 2 \cdot 50$$
$$= 2(1 + 2 + 3 + 4 + \ldots + 50)$$

Thus, we can use Gauss's method to find the sum of the first 50 natural numbers and then double that result.

Carrying Out the Plan We carry out the plan as follows:

$$2 + 4 + 6 + 8 + \ldots + 100 = 2(1 + 2 + 3 + 4 + \ldots + 50)$$
$$= 2\left[\frac{50(50 + 1)}{2}\right]$$
$$= 2550$$

Thus, the sum of the even natural numbers less than or equal to 100 is 2550.

Looking Back A different way to approach this problem is to realize that there are 25 sums of 102, as shown in Figure 4. (Why are there 25 sums to consider, and why is the sum in each pair always 102?)

$$
\begin{array}{c}
102 \\
102 \\
102 \\
102 \\
2 \; + \; 4 \; + \; 6 \; + \; 8 \; + \ldots + \; 94 \; + \; 96 \; + \; 98 \; + 100
\end{array}
$$
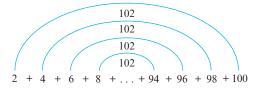
Figure 4

Thus, the sum is $25 \cdot 102 = 2550$.

▶ **NOW TRY THIS 2**
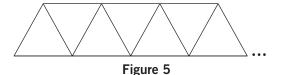
Find the sum of consecutive natural numbers shown: $25 + 26 + 27 + \ldots + 120$. Solve this problem in two different ways.

Strategies: Examine a Simpler Case; Make a Table

Often-used strategies in problem solving are *examine a simpler case* and *make a table*. A table can be used to look for patterns that emerge in the problem, which in turn can lead to a solution. An example of these strategies is shown next.

Problem Solving   **Perimeter of a Figure Made Up of Triangles**

The length of each side of the triangles shown in Figure 5 is 1 in. If 100 triangles are placed as shown, what is the perimeter of (distance around) the figure that is formed?

Figure 5

Understanding the Problem The perimeter of 1 triangle is 3 in. If the triangles are placed as shown in Figure 5, what is the perimeter of the figure that is formed?

Devising a Plan Simpler cases show that with 1 triangle the perimeter is 3 in., with 2 triangles the perimeter is 4 in., and with 3 triangles the perimeter is 5 in. If these data are placed in Table 2, then a pattern can be found and used to find the perimeter of 100 triangles placed as shown in Figure 5. Note that some sides of the triangles are not part of the perimeter.

Table 2

Number of triangles	1 △	2 △▽	3 △▽△
Perimeter (inches)	3	4	5

Carrying Out the Plan In Table 2, the perimeter of the figure formed is always 2 more than the number of triangles. This is because when one more triangle is added 1 inch is lost but 2 more inches are added. When 10 triangles are used, the perimeter is $10 + 2 = 12$ in. When 100 triangles are used, the perimeter is $100 + 2 = 102$ in.

Looking Back Part of *Looking Back* is to determine if the problem can be generalized. In this case, if there are n triangles placed as shown in Figure 5, then the perimeter is $(n + 2)$ in.

Strategy: Identify a Subgoal

In attempting to devise a plan for solving a problem, a solution to a somewhat easier or more familiar related problem could make it easier. In such a case, finding the solution to the easier problem may become a *subgoal*. The magic square problem on page 9 is an example.

Problem Solving | **A Magic Square**

Arrange the numbers 1 through 9 into a square subdivided into nine cells, as in Figure 6, so that the sum of every row, column, and major diagonal is the same. The result is a *magic square*.

Figure 6

Understanding the Problem Each of the nine numbers 1, 2, 3, . . . , 9 must be placed in the cells, a different number in each cell, so that the sums of the numbers in each row, in each column, and in each of the two major diagonals are the same.

Devising a Plan If we knew the fixed sum of the numbers in each row, column, and diagonal, we would have a better idea of which numbers can appear together in a single row, column, or diagonal. Thus the *subgoal* is to find that fixed sum. The sum of the nine numbers, $1 + 2 + 3 + \ldots + 9$, equals 3 times the sum in one row. (Why?) Consequently, the fixed sum can be found using the process developed by Gauss. We have $\dfrac{1 + 2 + 3 + \ldots + 9}{3} = \dfrac{(9 \cdot 10) \div 2}{3} = 15$. Next, we need to decide what numbers could occupy the various cells. The number in the center cell will appear in four sums, each adding to 15 (two diagonals, the second row, and the second column). Each number in the corner cells appears in three sums of 15. (Why?) If we write 15 as a sum of three different numbers 1 through 9 in all possible ways, we could count how many sums contain each of the numbers 1 through 9. The numbers that appear in at least four sums are candidates for placement in the center cell, whereas the numbers that appear in at least three sums are candidates for the corner cells. Thus the new *subgoal* is to write 15 in as many ways as possible as a sum of three different numbers from 1, 2, 3, . . . , 9.

Carrying Out the Plan The sums of 15 can be written systematically as follows:

$$9 + 5 + 1$$
$$9 + 4 + 2$$
$$8 + 6 + 1$$
$$8 + 5 + 2$$
$$8 + 4 + 3$$
$$7 + 6 + 2$$
$$7 + 5 + 3$$
$$6 + 5 + 4$$

Note that the order of the numbers in sums like $9 + 5 + 1$ is irrelevant because the order in which additions are done does not matter. In the list, 1 appears in only two sums, 2 in three sums, 3 in two sums, and so on. Table 3 summarizes this information.

Table 3

Number	1	2	3	4	5	6	7	8	9
Number of sums containing the number	2	3	2	3	4	3	2	3	2

The only number that appears in four sums is 5; hence, 5 must be in the center cell. (Why?) Because 2, 4, 6, and 8 appear 3 times each, they must go in the corner cells. Suppose we choose 2 for the upper left corner. Then 8 must be in the lower right corner. This is shown in Figure 7(a). Now we could place 6 in the lower left corner or upper right corner. If we choose the upper right corner, we obtain the result in Figure 7(b). The magic square can now be completed, as shown in Figure 7(c).

(a)   (b)   (c)
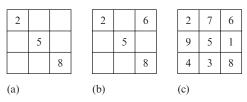
Figure 7

Looking Back We have seen that 5 was the only number among the given numbers that could appear in the center. However, we had various choices for a corner, and so it seems that the magic square we found is not the only one possible. Can you find all the others?

Strategy: Make a Diagram

In the following problem, *making a diagram* helps us to understand the problem and work toward a solution.
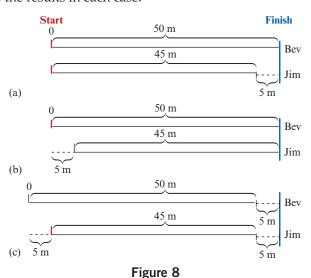
Problem Solving   **50-m Race Problem**

Bev and Jim ran a 50-m race three times. The speed of the runners did not vary. In the first race, Jim was at the 45-m mark when Bev crossed the finish line.

  a. In the second race, Jim started 5 m ahead of Bev, who lined up at the starting line. Who won?
  b. In the third race, Jim started at the starting line and Bev started 5 m behind. Who won?

Understanding the Problem When Bev and Jim ran a 50-m race, Bev won by 5 m; that is, whenever Bev covered 50 m, at the same time Jim covered only 45 m. If Bev started at the starting line and Jim started at the 5-m line or if Jim started at the starting line and Bev started 5 m behind, we are to determine who would win in each case.

Devising a Plan A strategy to determine the winner under each condition is to *make a diagram*. A diagram for the first 50-m race is given in Figure 8(a). In this case, Bev won by 5 m. In the second race, Jim had a 5-m head start and hence when Bev ran 50 m to the finish line, Jim ran only 45 m. Because Jim is 45 m from the finish line, he reached the finish line at the same time as Bev did. This is shown in Figure 8(b). In the third race, because Bev started 5 m behind, we use Figure 8(a) but move Bev back 5 m, as shown in Figure 8(c). From the diagram we determine the results in each case.

Figure 8

Carrying Out the Plan From Figure 8(b) we see that if Jim had a 5-m head start, then the race ends in a tie. If Bev started 5 m behind Jim, then at 45 m they would be tied. Because Bev is faster than Jim, Bev would cover the last 5 m faster than Jim and win the race.

Looking Back The diagrams show that the solution makes sense and is appropriate. Other problems can be investigated involving racing and handicaps. For example, if Bev and Jim run on a 100-m oval track, how many laps will it take for Bev to lead Jim by one full lap? (Assume the same speeds as earlier.)

> ▶ **NOW TRY THIS 3**

An elevator stopped at the middle floor of a building. It then moved up 4 floors, stopped, moved down 6 floors, stopped, and then moved up 10 floors and stopped. The elevator was now 3 floors from the top floor. How many floors does the building have?

Strategy: Work Backward

In some problems, it is easier to start with the result and to *work backward*. This is demonstrated in the following problem.

Problem Solving   **Bank Withdrawals**

On Monday, Sally withdrew half her money from the bank. On Tuesday, she withdrew $3000. On Wednesday, she withdrew half of her remaining money. She had $7000 left. How much money did she have in her bank account on Monday?

Understanding the Problem Sally made 3 withdrawals from the bank and on the last day she had $7000 remaining. The question is, how much money did she start with?

Devising a Plan Since the problem involves finding the initial amount given the final amount, the strategy of *working backward* seems appropriate.

Carrying Out the Plan The final amount, $7000, is $\frac{1}{2}$ the amount Sally had on Tuesday, so she had 2($7000) = $14,000 on Tuesday. On Tuesday, she withdrew $3000, so she had $14,000 + $3000 = $17,000 on Monday. On Monday, she withdrew half her money, so she had 2($17,000) = $34,000 to start.

Looking Back Part of *looking back* involves checking the answer. If Sally had $34,000 on Monday and withdrew $\frac{1}{2}$ of it, she would have $17,000 left. On Tuesday, she withdrew $3000, leaving $14,000. On Wednesday, she withdrew $\frac{1}{2}$ of her remaining money, or $7000, leaving $7000, which is the amount left.

Strategy: Use Guess and Check

In the strategy of *guess and check,* we first guess at a reasonable answer. Then we check to see whether the guess is correct. If not, the next step is to learn as much as possible about the answer based on this guess before making a next guess. This strategy can be regarded as a form of trial and error, in which the information about the error helps us choose what to try next. The guess-and-check strategy is often used when a student does not know how to solve the problem more efficiently or if the student does not yet have the tools to solve the problem in a faster way.

   Consider the School Book Page on page 12. One strategy students could use to solve this problem is *guess and check.* Because 234 is too small and 240 is too great, students know the value of x is between 234 and 240. They could try the values 235, 236, 237, 238, and 239 or they could use *guess and check* again. Because 5(238) = 1190, $x = 238$. Notice how the strategy of *guess and check* is used with the next strategy, *write an equation*. See exercise 12 in Mathematical Connection 1-1.
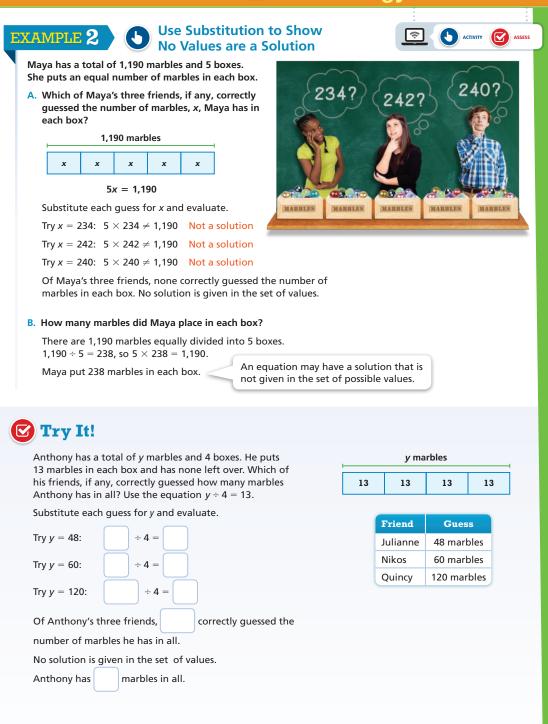
Strategy: Write an Equation

Even though algebraic thinking is involved in the strategy *write an equation* and may evoke thoughts of traditional algebra, a closer look reveals that algebraic thinking starts very early in students' school lives. For example, on page 12, Example 2 starts with the *guess and check* strategy and evolves into solving the equation $5x = 1190$. We use algebraic thinking long before formal algebra is taught. A more formal approach to algebra is seen in Chapter 8. An example of the strategy *write an equation* is seen on the School Book Page on page 13.

School Book Page — Guess and Check Strategy

EXAMPLE 2 — Use Substitution to Show No Values are a Solution

ACTIVITY   ASSESS

Maya has a total of 1,190 marbles and 5 boxes. She puts an equal number of marbles in each box.

A. Which of Maya's three friends, if any, correctly guessed the number of marbles, x, Maya has in each box?

1,190 marbles

x	x	x	x	x

$$5x = 1{,}190$$

Substitute each guess for x and evaluate.

Try $x = 234$: $5 \times 234 \neq 1{,}190$ Not a solution

Try $x = 242$: $5 \times 242 \neq 1{,}190$ Not a solution

Try $x = 240$: $5 \times 240 \neq 1{,}190$ Not a solution

Of Maya's three friends, none correctly guessed the number of marbles in each box. No solution is given in the set of values.

234?   242?   240?

B. How many marbles did Maya place in each box?

There are 1,190 marbles equally divided into 5 boxes.
$1{,}190 \div 5 = 238$, so $5 \times 238 = 1{,}190$.

Maya put 238 marbles in each box.

An equation may have a solution that is not given in the set of possible values.

✓ Try It!

Anthony has a total of y marbles and 4 boxes. He puts 13 marbles in each box and has none left over. Which of his friends, if any, correctly guessed how many marbles Anthony has in all? Use the equation $y \div 4 = 13$.

Substitute each guess for y and evaluate.

y marbles

13	13	13	13

Try $y = 48$: ☐ $\div 4 =$ ☐

Try $y = 60$: ☐ $\div 4 =$ ☐

Try $y = 120$: ☐ $\div 4 =$ ☐

Friend	Guess
Julianne	48 marbles
Nikos	60 marbles
Quincy	120 marbles

Of Anthony's three friends, ☐ correctly guessed the number of marbles he has in all.

No solution is given in the set of values.

Anthony has ☐ marbles in all.

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4-1 Understand Equations and Solutions   **179**

School Book Page — Write an Equation

1 Make sense of problems and persevere in solving them.

2 Reason abstractly and quantitatively.

Math Practices and Problem Solving Handbook

Nori, her friend, and her mother bought a baseball game ticket package. The package includes good seats, lunch, and a chance to get autographs from players.

The total cost for the three of them is $375.

They each paid a $50 deposit. Write an equation that shows how much each of them still owes.

What am I asked to find? An equation that shows how much each of them still owes.

What can I do if I get stuck? I can start subtracting each $50 deposit to find out how much all 3 still owe.

What are the quantities and variables? How do they relate? The cost of the ticket package and the deposit amount are their expenses.

The amount of the deposits is subtracted from the price of the ticket package.

What is a good plan for solving the problem? Define a variable for the unknown and use the quantities I know to write an equation that relates these quantities.

Other questions to consider:
- Have I solved a similar problem before?
- What information is necessary and unnecessary?
- How can I check that my answer makes sense?
- How is my solution pathway the same as or different from my classmate's?

Nori finds out that if 6 people buy the ticket package it only costs $684. Each person still needs to pay a $50 deposit. How does that change the equation? Does each person pay less if 6 people go?

How can I represent this problem situation using numbers, variables, and symbols? I can use the equation I wrote before and change the number of people and the total cost of the ticket package.

What do the numbers, variables, and symbols in the expression or equation mean/represent in the problem situation? $6(p + 50)$ represents the total cost of the package, where p is each person's share.

3-Person Ticket Package
$$3(p + 50) = 375$$
$$3p + 150 = 375$$
$$3p + 150 - 150 = 375 - 150$$
$$3p = 225$$
$$p = 75$$

6-Person Ticket Package
$$6(p + 50) = 684$$
$$6p + 300 = 684$$
$$6p + 300 - 300 = 684 - 300$$
$$6p = 384$$
$$p = 64$$

The 6-person plan is less expensive per person.

Assessment 1-1A

1. Use the approach in Gauss's Problem to find the following sums of arithmetic sequences.
   a. $1 + 2 + 3 + 4 + \ldots + 99$
   b. $1 + 3 + 5 + 7 + \ldots + 1001$
   c. $3 + 6 + 9 + 12 + \ldots + 300$
   d. $4 + 8 + 12 + 16 + \ldots + 400$
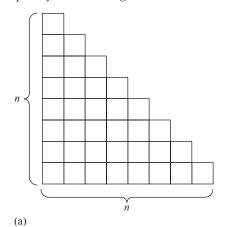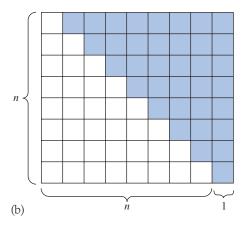
2. Use the ideas in drawings (a) and (b) to find the solution to Gauss's Problem for the sum $1 + 2 + 3 + \ldots + n$. Explain your reasoning.

(a)

(b)
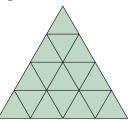
3. Find the sum
   $36 + 37 + 38 + 39 + \ldots + 146 + 147$.

4. Cookies are sold singly or in packages of 2 or 6. With this packaging, how many ways can you buy
   a. 10 cookies?
   b. a dozen cookies?

5. In a big red box, there are 7 smaller blue boxes. In each of the blue boxes, there are 7 black boxes. In each of the black boxes, there are 7 yellow boxes. In each of those yellow boxes, there are 7 tiny gold boxes. How many boxes are there altogether? Explain your answer.

6. How many triangles are in the following figure?

7. Without computing each sum, find which is greater, O or E, and by how much.
$$O = 1 + 3 + 5 + 7 + \ldots + 97$$
$$E = 2 + 4 + 6 + 8 + \ldots + 98$$
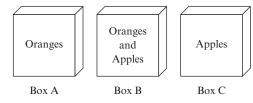
8. Alababa, Bubba, Cory, and Dandy are in a horse race. Bubba is the slowest; Cory is faster than Alababa but slower than Dandy. Name the finishing order of the horses.

9. How many ways can you make change for a $50 bill using $5, $10, and $20 bills?

10. The following is a magic square (all rows, columns, and diagonals sum to the same number). Find the value of each letter.

17	a	7
12	22	b
c	d	27

11. Debbie and Amy began reading a novel on the same day. Debbie reads 9 pages a day and Amy reads 6 pages a day. If Debbie is on page 72, on what page is Amy?

12. The 14 digits of a credit card are written in the boxes shown. If the sum of any three consecutive digits is 20, what is the value of A?

A		7									7		4

13. Three closed boxes (A, B, and C) of fruit arrive as a gift from a friend. Each box is mislabeled. How could you choose only one fruit from one box to decide how the boxes should be labeled?

Box A — Oranges
Box B — Oranges and Apples
Box C — Apples
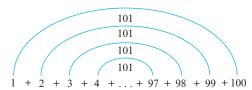
14. An electrician charges $50 per hour and spends $15 a day on gasoline. If she netted $1315 in 4 days, how many hours did she work?

15. Kathy stood on the middle rung of a ladder. She climbed up three rungs, moved down five rungs, and then climbed up seven rungs. Then she climbed up the remaining six rungs to the top of the ladder. How many rungs are there in the whole ladder?
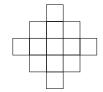
16. Alex made 4 pies, some apple and some cherry. There were 9 slices in each apple pie and 7 slices in each cherry pie. If he had 34 slices of pie, how many of each type of pie were there?

17. Al bought a CD player for $100, then sold it for $125. He then bought it back for $150. Later he sold it for $175. Did he make money, lose money, or break even? Explain.

18. A baseball bat and ball cost $50. If the bat costs $49 more than the ball, what is the cost of each item?

Assessment 1-1B

1. Use the approach in Gauss's Problem to find the following sums of arithmetic sequences.
   a. $1 + 2 + 3 + 4 + \ldots + 49$
   b. $1 + 3 + 5 + 7 + \ldots + 2009$
   c. $6 + 12 + 18 + \ldots + 600$
   d. $1000 + 995 + 990 + \ldots + 5$
2. Use the diagram below to explain how to find the sum of
   a. the first 100 natural numbers.


```
              101
            101
          101
        101
   1 + 2 + 3 + 4 + ... + 97 + 98 + 99 + 100
```

   b. Use this technique to find $1 + 2 + 3 + 4 + \ldots + 201$
3. Find the sum of $58 + 59 + 60 + 61 + \ldots + 203$.
4. Eve Merriam[*] titled her children's book *12 Ways to Get to 11* (1993). Using only addition and natural numbers, describe 12 ways that one can arrive at the sum of 11.
5. Explain why in a drawer containing only two different colors of socks one must draw only three socks to find a matching pair.
6. How many squares are in the following figure?

7. If $P = 1 + 3 + 5 + 7 + \ldots + 99$ and $Q = 5 + 7 + 9 + \ldots + 101$ are sums, determine which is greater, P or Q, and by how much.
8. The sign says that you are leaving Missoula, Butte is 120 mi away, and Bozeman is 200 mi away. There is a rest stop halfway between Butte and Bozeman. How far is the rest stop from Missoula if both Butte and Bozeman are in the same direction?
9. Marc goes to the store with exactly $1.00 in change. He has at least one of each coin less than a half-dollar coin, but he does not have a half-dollar coin.
   a. What is the least number of coins he could have?
   b. What is the greatest number of coins he could have?
10. Find a 3-by-3 magic square using the numbers 3, 5, 7, 9, 11, 13, 15, 17, and 19.
11. Eight marbles look alike, but one is slightly heavier than the others. Using a balance scale, explain how you can determine the heavier one in exactly three weighings.
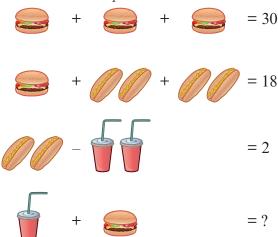
12. Recall the song "The Twelve Days of Christmas":
   On the first day of Christmas my true love gave to me a partridge in a pear tree.
   On the second day of Christmas my true love gave to me two turtle doves and a partridge in a pear tree.
   On the third day of Christmas my true love gave to me three French hens, two turtle doves, and a partridge in a pear tree.
   This pattern continues for 9 more days. After 12 days,
   a. which gifts did my true love give the most? (Yes, you have to remember the song.)
   b. how many total gifts did my true love give to me?

13. a. Suppose you have quarters, dimes, and pennies with a total value of $1.19. How many of each coin can you have without being able to make change for a dollar?
    b. Tell why one of the combinations of coin you have in part (a) is the least number of coins that you can have without being able to make change for a dollar.
14. Suppose you buy lunch for the math club. You have enough money to buy 20 salads or 15 sandwiches. The group wants 12 sandwiches. How many salads can you buy?
15. One winter night the temperature fell 15 degrees between midnight and 5 A.M. By 9 A.M., the temperature had doubled from what it was at 5 A.M. By noon, it had risen another 10 degrees to 32 degrees. What was the temperature at midnight?
16. Seth bought gifts at a toy store and spent $33. He bought puzzles and trucks. The puzzles cost $9 each and the trucks cost $5 each. If he bought 5 gifts, how many of each did he buy?
17. Find the value of the question mark.
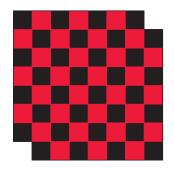
18. You are given a cube that is made of $10 \times 10 \times 10$ smaller cubes for a total of 1000 smaller cubes. If you take off one layer of small cubes all around the larger cube, how many smaller cubes remain?

[*]Merriam, E. *12 ways to Get to 11.* New York: Aladdin Paperbacks, 1993.

Mathematical Connections 1-1

1. Create a 3-by-3 magic square using nine of the ten numbers 20, 21, 22, 23, 24, 25, 26, 27, 28, and 29. Explain your solution and reasoning. List the strategies you have used.

2. In the checkerboard, two squares on opposite corners have been removed. A domino can cover two adjacent squares on the board. Can dominoes be arranged in such a way that all the remaining squares on the board can be covered with no dominoes overlapping or hanging off the board? If not, why not? (*Hint:* Each domino must cover one black and one red square. Compare this with the number of each color of squares on the board.)
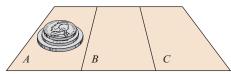
3. **a.** If 10 people shake hands with one another exactly once, how many handshakes take place?
   b. Find as many ways as possible to do the problem.
   c. Generalize the solution for *n* people.

4. An *unmagic square* is one in which the rows, columns, and diagonals must have different sums. Each of the digits 1 through 9 must be used. Complete the unmagic square shown below.

9		7
	1	
3		5

5. Place a half-dollar, a quarter, and a nickel in position *A* as shown in the figure below. Try to move these coins, one at a time, to position *C*. At no time may a larger coin be placed on a smaller coin. Coins may be placed in position *B*.
   a. How many moves does it take to get them to position *C*?
   b. Now add a penny to the pile and see how many moves are required. This is a simple case of the famous Tower of Hanoi problem, in which ancient Brahman priests were required to move a pile of 64 disks of decreasing size, after which the world would end. How long would it take at a rate of one move per second?

6. Choose a problem-solving strategy and make up a problem that would use this strategy. Write the solution using Pólya's four-step approach.

7. The distance around the world is approximately 40,000 km. Approximately how many people of average size would it take to stretch around the world if they were holding hands?

Connecting Mathematics to the Classroom

8. John asks why the last step of Pólya's four-step problem-solving process, *looking back*, is necessary, since he has already given the answer. What could you tell him?

9. A student asks why he just can't make "random guesses" rather than "intelligent guesses" when using the guess-and-check problem-solving strategy. How do you respond?

10. Rob says that it is possible to create a magic square with the numbers 1, 3, 4, 5, 6, 7, 8, 9, and 10. How do you respond?

School Book Pages

11. Refer to the School Book Page on page 5.
    a. Go through each of the questions asked on the School Book Page and solve the original problem.
    b. How do the steps laid out on the School Book Page on page 5 compare to Pólya's four-step problem-solving process?

12. Refer to the School Book Page on page 12.
    a. Solve the *Try It!* problem using only *guess and check*.
    b. Solve the problem using the given equation.

Group Work

1. Work in pairs on the following versions of a game called NIM. A calculator is needed for each pair.
   a. Player 1 presses 1 and $+$ or 2 and $+$. Player 2 does the same. The players take turns until the target number of 21 is reached. The first player to make the display read 21 is the winner. Determine a strategy for deciding who always wins.
   b. Try a game of NIM using the numbers 1, 2, 3, and 4, with a target number of 104. The first player to reach 104 wins. What is the winning strategy?
   c. Try a game of NIM using the numbers 3, 5, and 7, with a target number of 73. The first player to exceed 73 loses. What is the winning strategy?
   d. Now play Reverse NIM with the keys 1 and 2. Instead of $+$, use $-$. Put 21 on the display. Let the target number be 0. Determine a strategy for winning Reverse NIM.
   e. Try Reverse NIM using the numbers 1, 2, and 3 and starting with 24 on the display. The target number is 0. What is the winning strategy?
   f. Try Reverse NIM using the numbers 3, 5, and 7 and starting with 73 on the display. The first player to display a negative number loses. What is the winning strategy?

2. Work as a group. You need 15 index cards numbered 1 to 15. Start with a stack of cards numbered in order from least to greatest. Then put the top card (1) face up on the table. Put the next card on the bottom of the stack. Continue to alternate in this way until all cards are face up.
   a. Suppose you start with 5 cards. Which card do you think will be last? Check your guess.
   b. Suppose you start with 16 cards. What card will be last?
   c. How can you predict the last card if you know how many cards you start with?

National Assessment of Educational Progress (NAEP)

1. Susie said, "I have 83¢ but fewer than 10 coins." Show in the chart how many of each coin she could have to total 83¢.

Total number of coins	25¢	25¢	25¢	25¢

NAEP, Grade 4, 2009

2. Ms. Kim has 45 stickers that she wants to give out to 6 students. The students are sitting in a circle. Ms. Kim gives out one sticker at a time and keeps going around the circle until all the stickers are gone. How many of the students will get more than 7 stickers?
   A. 2
   B. 3
   C. 5
   D. 6

NAEP, Grade 4, 2011

3. Tim, Bob, and Molly joined a club between March and July.

Tim joined the club after Bob.
Molly joined the club before Bob.
Which time line shows the order in which they joined the club?

A. March _____ July
     Molly   Bob   Tim

B. March _____ July
     Molly   Tim   Bob

C. March _____ July
      Tim   Molly   Bob

D. March _____ July
      Bob   Tim   Molly

NAEP, Grade 4, 2013

4. A yellow box holds 72 pencils.
   $\boxed{Y}$ is the same as $\boxed{72}$.
   Two red boxes holds as many pencils as 1 yellow box.
   $\boxed{R}$ $\boxed{R}$ is the same as $\boxed{Y}$.
   Three blue boxes hold as many pencils as 1 red box.
   $\boxed{B}$ $\boxed{B}$ $\boxed{B}$ is the same as $\boxed{R}$.
   How many pencils does 1 blue box hold?
   Answer: _____
   Show or explain how you got your answer.

NAEP, Grade 4, 2013

1-2 Explorations with Patterns

Mathematics has been described as the study of patterns. Patterns are everywhere—in wallpaper, tiles, traffic, and even television schedules. Scientists look for patterns to isolate variables so that they can reach valid conclusions in their research.

Non-numerical patterns abound. For young children, a pattern could appear in non-numerical form, as shown in Now Try This 4.

▶ **NOW TRY THIS 4**

a. Find three more terms to continue a pattern:

$$\bigcirc, \triangle, \triangle, \bigcirc, \triangle, \triangle, \bigcirc, \_\_, \_\_, \_\_$$

b. Describe in words a possible pattern found in part (a).

Patterns can be surprising, as seen in the following example.

Example 1

a. Describe any patterns seen in the following:

$$1 + 0 \cdot 9 = 1$$
$$2 + 1 \cdot 9 = 11$$
$$3 + 12 \cdot 9 = 111$$
$$4 + 123 \cdot 9 = 1111$$
$$5 + 1234 \cdot 9 = 11111$$

b. Do the patterns continue? Why or why not?

Solution

a. There are several possible patterns. For example, the numbers on the far left are natural numbers. The pattern starts with 1 and continues to the next greater natural number in each successive line. The numbers "in the middle" are products of two numbers, the second of which is 9; the left-most number in the first product is 0; after that the left-most number in each product is formed using successive natural numbers as digits, including an additional digit in each successive line. The five computations given above result in the numbers that are formed using 1s and include an additional 1 in each successive line.

b. The pattern in the complete equation appears to continue for a number of cases, but it does not continue in general; for example,

$$13 + 123456789101112 \cdot 9 = 1{,}111{,}111{,}101{,}910{,}021.$$

This pattern breaks down when the pattern of digits in the number being multiplied by 9 contains previously used digits.

E-Manipulative Activity

Additional practice with patterns can be found in the *Patterns* activity on the E-Manipulatives disk (or MML). The activity involves completing simple and complex patterns of symbols and colored blocks.

As seen in Example 1, determining a pattern on the basis of a few cases is not reliable. For all patterns found, we should either show that the pattern does not hold in general or justify that the pattern always works. Reasoning is used in both cases.

Reasoning

CCSS Some books list various types of reasoning as problem-solving strategies. However, we think that reasoning underlies problem solving. The *Common Core Standards for Mathematical Practice* lists the following:

- **Reason abstractly and quantitatively.**
- **Construct viable arguments and critique the reasoning of others.**
- **Look for and express regularity in repeated reasoning. (p. 1)**

For students to recognize reasoning and proof as fundamental aspects of mathematics, it is necessary that they use both reasoning and proof in their studies. However, it must be recognized that the level of use depends on the grade level of the students and their understanding of mathematics. For example, from very early ages, students use *inductive reasoning* to look for regularities in patterns based on a very few cases and to develop **conjectures**—statements or conclusions that have not been proven. **Inductive reasoning** is the method of making generalizations based on observations and patterns. Such reasoning may or may not be valid, and conjectures based on inductive reasoning may or may not be true. The validity, or truth, of conjectures in mathematics relies on **deductive reasoning**—the use of definitions, undefined terms, mathematical axioms that are assumed to be true, and previously proved theorems, together with logic to prove these conjectures.

Throughout mathematics, there is a fine interweaving of inductive reasoning and conjecturing to develop conclusions thought to be true. Deductive reasoning is required to prove those statements. In this section, we show how inductive reasoning may lead to false conclusions or false conjectures. We show how deductive reasoning is used to prove true conjectures.

Inductive and Deductive Reasoning

Scientists make observations and propose general laws based on patterns. Statisticians use patterns when they draw conclusions based on collected data. This process of *inductive reasoning* may lead to new discoveries; its weakness is that conclusions are drawn only from the collected evidence. If not all cases have been checked, another case may prove the conclusion false. For example, considering only that $0^2 = 0$ and that $1^2 = 1$, we might conjecture that *every number squared is equal to itself.* However, $2^2 \neq 2$. Thus we found an example that contradicts the conjecture. Such an example is a **counterexample**; it shows that the conjecture is false. Sometimes finding a counterexample is difficult, but not finding one immediately does not prove that a conjecture is true. A dramatic example of a conjecture that holds true for a very large number of cases but still fails to be true for all cases involves the concept of *perfect squares.* A natural number that is a square of some natural number is a perfect square. For example, 9 is a perfect square because $9 = 3^2$. The conjecture that $1 + 1141n^2$ is never a perfect square is true for every natural number n until $n = 30,693,385,322,765,657,197,397,208$ when it fails.

Next, consider a pattern that does work and helps solve a problem. How can you find the sum of three consecutive natural numbers without performing the addition? Three examples are given below.

$$14 + 15 + 16; \qquad \text{Sum} = 45$$
$$19 + 20 + 21; \qquad \text{Sum} = 60$$
$$99 + 100 + 101; \qquad \text{Sum} = 300$$

After studying the sums, a pattern of multiplying the middle number by 3 emerges. The pattern suggests other mathematical questions to consider. For example,

1. Does this work for any 3 consecutive natural numbers?
2. How can we find the sum of any odd number of consecutive natural numbers?
3. What happens if there is an even number of consecutive natural numbers?

We answer the first question, and give a deductive proof showing that the sum of three consecutive natural numbers is equal to three times the middle number.

Proof

Let n be the first of three consecutive natural numbers. Then the three numbers are $n, n + 1$, and $n + 2$. The sum of these three numbers is $n + (n + 1) + (n + 2) = 3n + 3 = 3(n + 1)$. Therefore, the sum of the three consecutive natural numbers is three times the middle number.

A somewhat different way is to let the middle number be m. Then the three consecutive numbers are $m - 1, m$, and $m + 1$. Their sum is $(m - 1) + m + (m + 1) = 3m$, that is, three times the middle number.

The Danger of Making Conjectures Based on a Few Cases

The following discussion illustrates the danger of making a conjecture based on a few cases. In Figure 9, we choose points on a circle and connect them to form distinct, nonoverlapping regions. In this figure, 2 points determine 2 regions, 3 points determine 4 regions, and 4 points determine 8 regions. What is the maximum number of regions that would be determined by 10 points?

The data from Figure 9 are recorded in Table 4. It appears that each time the number of points increases by 1, the number of regions doubles. If this were true, then for 5 points

(a) 2 points, 2 regions   (b) 3 points, 4 regions   (c) 4 points, 8 regions
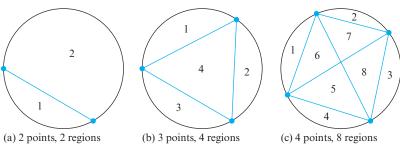
Figure 9

we would have determined the number of regions by doubling the number of regions with 4 points, or $2 \cdot 8 = 16 = 2^4$, and so on. If we base our conjecture on this pattern, we might believe that for 10 points, we would have 2^9, or 512 regions. (Why?)

An initial check for this conjecture is to see whether we obtain 16 regions for 5 points. The diagram for 5 points in Figure 10 confirms our guess of 16 regions. For 6 points, the pattern

Table 4							
Number of points	2	3	4	5	6	. . .	10
Maximum number of regions	2	4	8				?
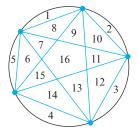
Figure 10

predicts that the number of regions is 32. Choosing the points so that they are neither symmetrically arranged nor equally spaced and counting the regions carefully, we get 31 regions, not 32 as predicted. No matter how the points are located on the circle, the guess of 32 regions is not correct. This counterexample tells us that the doubling pattern is not correct. Note that it does not tell us whether there are 512 regions with 10 points; it tells us only that this conjecture is not true.

▶ **NOW TRY THIS 5**

A *prime number* is a natural number with exactly two distinct positive numbers, 1 and the number itself, that divide it with 0 remainder; for example, 2, 3, 5, 7, 11, and 13 are primes. One day, Amy makes a *conjecture* that the formula $y = x^2 + x + 11$ will produce only prime numbers if she substitutes the natural numbers, 1, 2, 3, 4, 5, . . . for x. She shows her work in Table 5 for $x = 1, 2, 3, 4$.

Table 5				
x	1	2	3	4
y	13	17	23	31

a. What type of reasoning is Amy using?
b. Try the next several natural numbers and see whether they seem to work.
c. Show that Amy's conjecture is false for $x = 11$.

Arithmetic Sequences

A **sequence** is an ordered arrangement of numbers, figures, or objects. A sequence has items or *terms* identified as *1st, 2nd, 3rd*, and so on. Often, sequences can be classified by their properties. For example, what property do the following first three sequences have that the fourth does not?

a. 1, 2, 3, 4, 5, 6, . . .
b. 0, 5, 10, 15, 20, 25, . . .
c. 2, 6, 10, 14, 18, 22, ...
d. 1, 11, 111, 1111, 11111, 111111, . . .

In each of the first three sequences, each term—starting from the second term—is obtained from the preceding term by adding a fixed number, the **common difference** or **difference**. In part (a) the difference is 1, in part (b) the difference is 5, and in part (c) the difference is 4. Sequences such as the first three are arithmetic sequences. An **arithmetic sequence** is a sequence in which each successive term from the second term on is obtained from the previous term by the addition or subtraction of a fixed number. The sequence in part (d) is not arithmetic because there is no single fixed number that can be added to or subtracted from the previous term to obtain the next term.

It is convenient to denote the terms of a sequence by a single letter with a subscript. For example, in the sequence (b) above $a_1 = 0$, $a_2 = 5$, $a_3 = 10$, and so on. Arithmetic sequences can be generated from objects, as shown in Example 2.

Example 2

Find a numerical pattern in the number of matchsticks required to continue the sequence shown in Figure 11 if each subsequent figure has one more square.
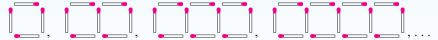
Figure 11

Solution
Assume the matchsticks are arranged so each figure has one more square on the right than the preceding figure. The addition of a square to an arrangement requires the addition of three matchsticks each time. With this assumption, the numerical pattern obtained is 4, 7, 10, 13, 16, 19, . . . , an arithmetic sequence starting at 4 and having a difference of 3.

An informal description of an arithmetic sequence is one that can be described as an "add *d*" pattern, where *d* is the common difference. In Example 2, $d = 3$. In the language of children, the pattern in Example 2 is "add 3." This is an example of a **recursive pattern**. In a recursive pattern, after one or more consecutive terms are given, each successive term of the sequence is obtained from the previous term(s). For example, 11, 14, 17, . . . is another "add 3" sequence starting with 11.

A recursive pattern is typically used in a spreadsheet, as seen in Table 6, where the index column tracks the order of the terms. The headers for the columns are A, B, and so on. The first entry in the A column (in the A1 cell) is 4; to find the term in the A2 cell, we use the number in the A1 cell and add 3. The pattern is continued using the *Fill Down* command. In spreadsheet language, the formula $= A1 + 3$ finds any term after the first by adding 3 to the previous term and using *Fill Down*. A formula based on a recursive pattern is a **recursive formula**.

Table 6

Index → Column		A	B
	1	4	
	2	7	
	3	10	
	4	13	
	5	16	
	6	19	
	7	22	
	8	25	
	9	28	
	10		
	11		
	12		
	13		

If we want to find the number of matchsticks in the 100th figure in Example 2, we use the spreadsheet or we find an explicit formula or a general rule for finding the number of match-sticks when given the position of the term, the term number. The problem-solving strategy *making a table* is again helpful here.

The spreadsheet in Table 6 provides an easy way to *make a table*. The index column gives the term numbers, and column A gives the terms of the sequence. If we are building such a table without a spreadsheet, it might look like Table 7. Each term is a sum of 4 and a certain number of 3s. We see that the number of 3s is 1 less than the term number. This pattern should continue, since the first term is $4 + 0 \cdot 3$ and each time we increase the number of the term by 1, we add one *more* 3. Thus, it seems that the 100th term is $4 + (100 - 1)3$; and, in general, the nth term is $4 + (n - 1)3$. Note that $4 + (n - 1)3$ could be written as $3n + 1$.

Table 7

Term Number	Term
1	$4 = 4 + 0 \cdot 3$
2	$7 = 4 + 3 = 4 + 1 \cdot 3$
3	$10 = (4 + 1 \cdot 3) + 3 = 4 + 2 \cdot 3$
4	$13 = (4 + 2 \cdot 3) + 3 = 4 + 3 \cdot 3$
.	.
.	.
.	.
n	$4 + (n - 1)3 = 3n + 1$

Still a different approach to finding the number of matchsticks in the 100th term of Figure 11 might be as follows: If the matchstick figure has 100 squares, we could find the total num-ber of matchsticks by adding the number of horizontal and vertical sticks. There are $2 \cdot 100$ placed horizontally. (Why?) In the first figure, there are 2 matchsticks placed vertically; in the second, 3; and in the third, 4. In the 100th figure, there should be $(100 + 1)$ vertical matchsticks. Altogether, there will be $2 \cdot 100 + (100 + 1)$, or 301, matchsticks in the 100th figure. Similarly, in the nth figure, there would be $2n$ horizontal and $(n + 1)$ vertical matchsticks, for a total of $3n + 1$. This discussion is summarized in Table 8.

Table 8

Term Number	Number of Matchsticks Horizontally	Number of Matchsticks Vertically	Total
1	2	2	4
2	4	3	7
3	6	4	10
4	8	5	13
.	.	.	.
.	.	.	.
.	.	.	.
100	200	101	301
.	.	.	.
.	.	.	.
.	.	.	.
n	$2n$	$n + 1$	$2n + (n + 1) = 3n + 1$

If we are given the value of the term, we can use the formula $3n + 1$ for the nth term in Table 8 to *work backward* and find the term number. For example, given the term 1798, we can write an equation: $3n + 1 = 1798$. Therefore, $3n = 1797$ and $n = 599$. Consequently, 1798 is the 599th term. We could obtain the same answer by solving $4 + (n - 1)3 = 1798$ for n.

In the matchstick problem, we found the nth term of a sequence. If the nth term of a sequence is given, we can find any term of the sequence, as shown in Example 3.

Example 3

Find the first four terms of a sequence, the nth term of which is given by the following, and determine whether the sequence seems to be arithmetic:

a. $4n + 3$ **b.** $n^2 - 1$

Solution

Table 9

Term Number	Term
1	$4 \cdot 1 + 3 = 7$
2	$4 \cdot 2 + 3 = 11$
3	$4 \cdot 3 + 3 = 15$
4	$4 \cdot 4 + 3 = 19$

a. Table 9 shows the first four terms of the sequence are 7, 11, 15, 19. This sequence seems arithmetic, with difference 4.

Table 10

Term Number	Term
1	$1^2 - 1 = 0$
2	$2^2 - 1 = 3$
3	$3^2 - 1 = 8$
4	$4^2 - 1 = 15$

b. Table 10 shows the first four terms of the sequence are 0, 3, 8, 15. This sequence is not arithmetic, because it has no common difference.

We generalize our work with arithmetic sequences in Chapter 8.

Example 4

The diagrams in Figure 12 show the molecular structure of alkanes, a class of hydrocarbons. C represents a carbon atom and H a hydrogen atom. A connecting segment shows a chemical bond.

```
        H                  H   H                H   H   H
        |                  |   |                |   |   |
   H — C — H  ,       H — C — C — H  ,     H — C — C — C — H   ,  ...
        |                  |   |                |   |   |
        H                  H   H                H   H   H
  methane (C₁H₄)       ethane (C₂H₆)         propane (C₃H₈)
```

methane (C_1H_4)   ethane (C_2H_6)   propane (C_3H_8)
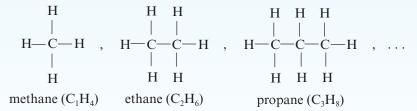
Figure 12

a. Hectane is an alkane with 100 carbon atoms. How many hydrogen atoms does it have?
b. Write a general rule for alkanes C_nH_m showing the relationship between m and n.

Solution

a. To determine the relationship between the number of carbon and hydrogen atoms, we study the drawing of the alkanes and disregard the extreme left and right hydrogen atoms in each. With this restriction, we see that for every carbon atom, there are two hydrogen atoms. Therefore, there are twice as many hydrogen atoms as carbon atoms plus the two hydrogen atoms at the extremes. For example, when there are 3 carbon atoms, there are $(2 \cdot 3) + 2 = 8$ hydrogen atoms. This notion is summarized in Table 11. If we extend the table for 4 carbon atoms, we get $(2 \cdot 4) + 2$, or 10, hydrogen atoms. For 100 carbon atoms, there are $(2 \cdot 100) + 2$, or 202, hydrogen atoms.

Table 11

No. of Carbon Atoms	No. of Hydrogen Atoms
1	$2 \cdot 1 + 2 = 4$
2	$2 \cdot 2 + 2 = 6$
3	$2 \cdot 3 + 2 = 8$
.	.
.	.
.	.
100	$2 \cdot 100 + 2 = 202$
.	.
.	.
.	.
n	$2n + 2 = m$

b. In general, for n carbon atoms there would be n hydrogen atoms attached above, n attached below, and 2 attached on the sides. Hence, the total number of hydrogen atoms m would be $2n + 2$. It follows that the number of hydrogen atoms is $m = 2n + 2$.

Example 5

A theater is arranged so that there are 20 seats in the first row and 4 additional seats in each consecutive row to the back of the theater, where there are 144 seats. How many rows are there in the theater?

Solution   Two strategies lend themselves to this problem. One is to *build a table* and to consider the entries as seen in Table 12.

Table 12

Row Number	Number of Seats
1	20
2	$20 + 4$
3	$20 + 2 \cdot 4$
4	$20 + 3 \cdot 4$
5	$20 + 4 \cdot 4$
.	.
.	.
.	.
n	$20 + (n - 1)4$

Observe that in Table 12, when we write the number of seats as 20 plus the number of additional 4 seats in consecutive rows, the number of 4s added is one less than the number of the row. We know that in the last row there are 144 seats. Thus, we have the following:

$$144 = 20 + (n - 1)4$$

Subtracting 20 from each side of the equation, we get

$$124 = (n - 1)4 \text{ or } n - 1 = 31$$

Therefore, $n = 32$, and there are 32 rows in the theater.

Fibonacci Sequence

Dan Brown's popular book *The Da Vinci Code*[*] brought renewed interest to one of the most famous sequences of all time, the **Fibonacci sequence**.

The Fibonacci sequence is

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots .$$

This sequence is not *arithmetic*, as there is no fixed difference, d. The first two terms of the Fibonacci sequence are 1, 1 and each subsequent term is the sum of the previous two. If we denote the terms of the Fibonacci sequence by $F_1, F_2, F_3, \ldots$, we have

$$F_1 = F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, \text{ and so on.}$$

Also $F_3 = F_2 + F_1, F_4 = F_3 + F_2, \ldots$, and in general $F_n = F_{n-1} + F_{n-2}$ for $n = 3, 4, 5, \ldots$. The numbers in the sequence are known as *Fibonacci numbers*.

Historical Note

Leonardo de Pisa was born around 1170. His real family name was Bonaccio but he preferred the nickname Fibonacci, derived from *filius Bonacci,* meaning "son of Bonacci." In his book *Liber Abaci* (1202), he described a now-famous rabbit problem, whose solution, the sequence 1, 1, 2, 3, 5, 8, 13, 21, . . . , became known as the *Fibonacci sequence*.   ●

▶ **NOW TRY THIS 6**

Here is the Fibonacci sequence:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	. . .
F_n	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	. . .

There are some interesting patterns. Look at the number $F_3 = 2$. Every 3rd number is a multiple of 2: $(2, 8, 34, 144, 610, \ldots)$. Find other patterns for every fourth number, every fifth number, and conjecture a pattern for every nth number.

Geometric Sequences

A child has 2 biological parents, 4 grandparents, 8 great grandparents, 16 great-great grandparents, and so on. The number of generational ancestors form the **geometric sequence** 2, 4, 8, 16, 32, ... Each successive term of a geometric sequence is obtained from its predecessor by multiplying by a fixed nonzero number, the **ratio**. In this example, both the first term and the ratio are 2. (The ratio is 2 because each person has two parents.) To find the nth term, examine the pattern in Table 13.

Table 13

Term Number	Term
1	$2 = 2^1$
2	$4 = 2 \cdot 2 = 2^2$
3	$8 = (2 \cdot 2) \cdot 2 = 2^3$
4	$16 = (2 \cdot 2 \cdot 2) \cdot 2 = 2^4$
5	$32 = (2 \cdot 2 \cdot 2 \cdot 2) \cdot 2 = 2^5$
.	.
.	.
.	.

---

[*]Brown, Dan. *The Da Vinci Code*. Doubleday, 2003.

In Table 13, when the given term is written as a power of 2, the term number is the **exponent**. Following this pattern, the 10th term is 2^{10}, or 1024, the 100th term is 2^{100}, and the nth term is 2^n. Thus, the number of generated ancestors in the nth generation is 2^n. The notation used in Table 13 can be generalized as follows.

Definition of a^n

$$\overbrace{}^{n \text{ factors}}$$

If n is a natural number, then $a^n = a \cdot a \cdot a \cdot \ldots \cdot a.$

If $n = 0$ and $a \neq 0$, then $a^0 = 1.$

Geometric sequences play an important role in everyday life. For example, suppose we have $1000 in a bank that pays 5% interest annually. (Note that $5\% = 0.05$.) If no money is added or withdrawn, then at the end of the first year we have all of the money we deposited plus 5% more.

$$\text{Year 1: } \$1000 + 0.05(\$1000) = \$1000(1 + 0.05) = \$1000(1.05) = \$1050$$

If no money is added or taken out, then at the end of the second year we would have 5% more money than the previous year.

$$\text{Year 2: } \$1050 + 0.05(\$1050) = \$1050(1 + 0.05) = \$1050(1.05) = \$1102.50$$

The amount of money in the account after any number of years can be found by noting that every dollar invested for one year becomes $1 + 0.05 \cdot 1$, or 1.05 dollars. Therefore, the amount in each year is obtained by multiplying the amount from the previous year by 1.05. The amounts in the bank after each year form a geometric sequence because the amount in each year (starting from year 2) is obtained by multiplying the amount in the previous year by the same number, 1.05. This is summarized in Table 14.
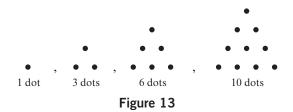
Table 14

Term Number (Year)	Term (Amount at the End of Each Year)
1	$\$1000(1.05)^1$
2	$\$1000(1.05)^2$
3	$\$1000(1.05)^3$
4	$\$1000(1.05)^4$
.	.
.	.
.	.
n	$\$1000(1.05)^n$

a. Two bacteria are in a dish. The number of bacteria triples every hour. Following this pattern, find the number of bacteria in the dish after 10 hours and after n hours.

b. Suppose that instead of increasing geometrically as in part (a), the number of bacteria increases arithmetically by 3 each hour. Compare the growth after 10 hours and after n hours. Comment on the difference in growth of a geometric sequence versus an arithmetic sequence.

Other Sequences

Figurate numbers, based on geometrical patterns, also provide examples of sequences that are neither arithmetic nor geometric. Such numbers can be represented by dots arranged in the shape of certain geometric figures. The number 1 is the beginning of most patterns involving figurate numbers. The arrays in Figure 13 represent the first four terms of the sequence of **triangular numbers**.

| | 1 dot | 3 dots | 6 dots | 10 dots |

Figure 13

The triangular numbers can be written numerically as 1, 3, 6, 10, 15, ... The sequence 1, 3, 6, 10, 15, . . . is not an arithmetic sequence because there is no common difference. It is not a geometric sequence because there is no common ratio. It is not the Fibonacci sequence.

Table 15 suggests a pattern for finding the next terms and the nth term for the triangular numbers. The second term is obtained from the first term by adding 2; the third term is obtained from the second term by adding 3; and so on.

Table 15

Term Number	Term
1	1
2	$3 = 1 + 2$
3	$6 = 1 + 2 + 3$
4	$10 = 1 + 2 + 3 + 4$
5	$15 = 1 + 2 + 3 + 4 + 5$
.	.
.	.
.	.
10	$55 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$

We could approach the problem differently. Because the nth triangular number has n dots in the nth row, it is equal to the sum of the dots in the previous triangular

number (the $(n - 1)$st one) plus the n dots in the nth row. Following this pattern, the 10th term is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$, or 55, and the nth term is $1 + 2 + 3 + 4 + 5 + \ldots + (n - 1) + n$. This problem is similar to Gauss's Problem in Section 1-1. Because of the work done in Section 1-1, we know that this sum can be expressed as

$$\frac{n(n + 1)}{2}.$$

Next consider the first four *square numbers* in Figure 14. These square numbers, 1, 4, 9, 16 can be written as $1^2, 2^2, 3^2, 4^2$. Continuing, the number of dots in the 10th array would be 10^2, the number of dots in the 100th array would be 100^2, and the number of dots in the nth array would be n^2. The sequence of square numbers is neither arithmetic nor geometric.
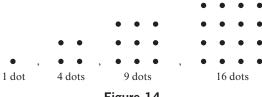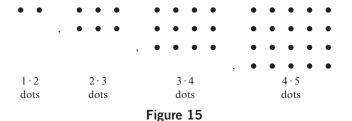
Figure 14

▶ **NOW TRY THIS 8**

Consider the rectangular numbers in Figure 15, in which the number of columns and the number of rows increase by 1 with each successive "rectangle." What is the 10th rectangular number, and what is the nth rectangular number?

| $1 \cdot 2$ | $2 \cdot 3$ | $3 \cdot 4$ | $4 \cdot 5$ |
| dots | dots | dots | dots |

Figure 15

Example 6

Figure 16 shows the first three figures of arrays of matchsticks with the number of matchsticks written below the figures. If the next figure consists of a 4-by-4 square arrangement, and each subsequent figure has one more row and one more column of matchsticks squares than the preceding figure, without actually counting, find the number of matchsticks in

  a. the 7th figure.
  b. the nth figure.
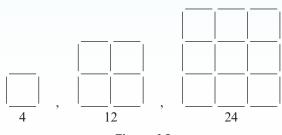
Figure 16

Solution

One way to solve the problem is to count the matchsticks in rows and columns. The number of columns is the same as the number of rows. Thus, we find the number of matchsticks in the rows and multiply the result by 2. In the first figure, we have 2 rows and 1 matchstick in each. Because we are adding one row and one column to get the subsequent figure, we can write the number of matchsticks in each figure, as shown in Table 16.

Table 16

Figure Number	Number of Rows	Number in Each Row	Total in the Rows
1	2	1	$1 \cdot 2$
2	3	2	$2 \cdot 3$
3	4	3	$3 \cdot 4$
4	5	4	$4 \cdot 5$
5	6	5	$5 \cdot 6$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$n + 1$	n	$n(n + 1)$

For part (b), because the number of matchsticks in the columns is the same as in the rows, the total number is $2n(n + 1)$. If $n = 7$, then we have the answer for part (a) as $2 \cdot 7(7 + 1)$ or 112.

Assessment 1-2A

1. For each of the following sequences of figures, determine a possible pattern and draw the next figure according to that pattern:

a. , . . .

b. , . . .

c. , . . .

2. Each of the following sequences is either arithmetic or geometric. Identify the sequences and list the next three terms for each.
a. 1, 5, 9, 13, . . .
b. 70, 120, 170, . . .
c. 1, 3, 9, . . .
**d.** $10, 10^3, 10^5, 10^7, \ldots$
e. $193 + 7 \cdot 2^{30}, 193 + 8 \cdot 2^{30}, 193 + 9 \cdot 2^{30}, \ldots$

3. Find the 100th term and the nth term for each of the sequences in exercise 2.

4. Use a traditional clock face to determine the next three terms in the following sequence.

$$1, 6, 11, 4, 9, \ldots$$
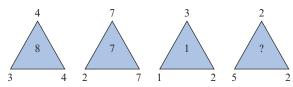
5. The pattern 1, 8, 27, 64, 125, . . . is a cubic pattern named because $1 = 1 \cdot 1 \cdot 1$ or 1^3, $8 = 2 \cdot 2 \cdot 2$ or 2^3, and so on.
a. What is the least 4-digit number greater than 1000 in this pattern?
b. What is the greatest 3-digit number in this pattern?
c. What is the greatest number in this pattern that is less than 10^4?
d. If this pattern was produced in a normal spreadsheet, what is the number in cell A14?

6. The first windmill has 5 matchstick squares, the second has 9, and the third has 13, as shown. How many matchstick squares are in **(a)** the 10th windmill? **(b)** the nth windmill? **(c)** How many matchsticks will it take to build the nth windmill?

7. In the following sequence, the figures are made of cubes that are glued together. If the exposed surface needs to be painted, how many squares will be painted in **(a)** the 15th figure? **(b)** the nth figure?

8. The school population for a certain school is predicted to increase by 60 students per year for the next 12 years. If the current enrollment is 700 students, what will the enrollment be after 12 years?

9. Juan's annual income has been increasing each year by the same dollar amount. The first year his income was $24,000, and the ninth year his income was $31,680. In which year was his income $45,120?

10. Find a number to continue the pattern and replace the question mark. Explain your thinking.

11. One block is needed to make an up-down-up staircase with one step up and one step down.
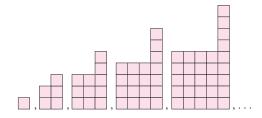
Four blocks are needed to build an up-down-up staircase with 2 steps up and two steps down.

   a. How many blocks are needed to build an up-down-up staircase with 5 steps up and 5 steps down?
   b. How many blocks are needed to build an up-down-up staircase with n steps up and n steps down?

12. How many terms are there in each of the following sequences?
   a. 51, 52, 53, 54, . . . , 251
   b. 1, 2, 2^2, 2^3, . . . , 2^{60}
   c. 10, 20, 30, 40, . . . , 2000
   d. 1, 2, 4, 8, 16, 32, . . . 1024

13. Find the first five terms in sequences with the following nth terms.
   a. $n^2 + 2$
   b. $5n + 1$
   c. $10^n - 1$
   d. $3n - 2$

14. Find a counterexample for each of the following:
   a. If n is a natural number, then $(n + 5)/5 = n + 1$.
   b. If n is a natural number, then $(n + 4)^2 = n^2 + 4^2$.

15. Assume that the following patterns are built of square tiles and the pattern continues. Answer the questions that follow.

   a. How many square tiles are there in the sixth figure?
   b. How many square tiles are in the nth figure?
   c. Is there a figure that has exactly 1259 square tiles? If so, which one?

16. Consider the sequences given in the table below. Find the least number, n, such that the nth term of the geometric sequence is greater than the corresponding term in the arithmetic sequence.
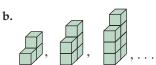
Term Number	1	2	3	4	5	6	...	n
Arithmetic	400	600	800	1000	1200	1400	...	
Geometric	2	4	8	16	32	64	...	

17. A sheet of paper is cut into 5 same-size parts. Each of the parts is then cut into 5 same-size parts and so on. Answer the following.
   a. After the 5th cut, how many of the smallest pieces of paper are there?
   b. After the nth cut, how many of the smallest pieces are there?

18. Each of the following sequences is labeled either arithmetic or geometric. In each part, find the missing terms.
   a. __, 39, __, __, 69 (arithmetic)
   b. __, 200, __, __, 800 (arithmetic)
   c. __, 5^4, __, __, 5^{10} (geometric)

19. A *Fibonacci-type sequence* is a sequence in which the first two terms are arbitrary and in which every term starting from the third is the sum of the two previous terms. Each of the following is a Fibonacci-type sequence. In each part, find the missing terms.
   a. __, __, 1, 1, __, __, __, __
   b. __, __, __, 10, 13, __, 36, 59
   c. 0, 2, __, __, __, __, __, __

20. A new pair of tennis shoes cost $80. If the price increases each year by 5% of the previous year's price, find the following:
   a. The price after 5 years
   b. The price after n years

Assessment 1-2B

1. In each of the following, determine a possible pattern and draw the next figure according to that pattern if the sequence continues.
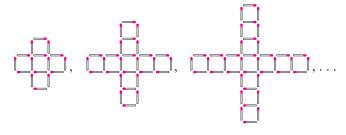
   a. , . . .

   b. , . . .

   c. , . . .

2. Each of the following sequences is either arithmetic or geometric. Identify the sequences and list the next three terms for each.

   a. 2, 6, 10, 14, . . .   **b.** 0, 13, 26, . . .

   c. 4, 16, 64, . . .   **d.** $2^2, 2^6, 2^{10}, \ldots$

   e. $100 + 4 \cdot 2^{50}, 100 + 6 \cdot 2^{50}, 100 + 8 \cdot 2^{50}, \ldots$

3. Find the 100th term and the nth term for each of the sequences in exercise 2.

4. Use a traditional clock face to determine the next three terms in the following sequence:

   $$1, 9, 5, 1, \ldots$$
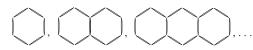
5. Observe the following pattern:

   $$1 + 3 = 2^2,$$
   $$1 + 3 + 5 = 3^2,$$
   $$1 + 3 + 5 + 7 = 4^2$$

   a. Conjecture a generalization based on this pattern.
   b. Based on the generalization in (a), find

   $$1 + 3 + 5 + 7 + \ldots + 35.$$

6. In the following pattern, one hexagon takes 6 toothpicks to build, two hexagons take 11 toothpicks to build, and so on. How many toothpicks would it take to build

   a. 10 hexagons?
   b. n hexagons?

7. Each successive figure below is made of small triangles like the first one in the sequence. Conjecture the number of small triangles needed to make

   a. the 100th figure?
   b. the nth figure?

8. A tank contains 15,360 L of water. At the end of each subsequent day, half of the water is removed and not replaced. How much water is left in the tank after 10 days?
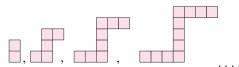
9. The Washington Middle School time schedule is an arithmetic sequence. Each period is the same length and includes a 4th period lunch. The first three periods begin at 8:10 A.M., 9:00 A.M., and 9:50 A.M., respectively. At what time does the eighth period begin?

10. There are nine points drawn as shown below. Can you connect all nine points with four straight line segments without lifting a pen from the paper.

11. The triangular figures shown below are constructed with toothpicks. The pattern shows what happens with 1 triangle, with 2 triangles, and with 3 triangles. Describe a possible pattern. Assuming the pattern continues, answer the following questions.

    a. How many toothpicks would be needed for the 10th figure?
    b. How many toothpicks are needed for the nth figure?
    c. In which figure are exactly 102 toothpicks used?

12. How many terms are there in each of the following sequences?

    a. $1, 3, 3^2, 3^3, \ldots, 3^{99}$
    b. 9, 13, 17, 21, 25, . . . , 353
    c. 38, 39, 40, 41, . . . , 238

13. Find the first five terms in sequences with the following nth terms.

    a. $5n - 1$
    b. $6n - 2$
    c. $5n + 1$
    d. $n^2 - 1$

14. Find a counterexample for each of the following:

    a. If n is a natural number, then $(3 + n)/3 = n$.
    b. If n is a natural number, then $(n - 2)^2 = n^2 - 2^2$.

15. Assume the following pattern with terms built of square tiles continues and answer the questions that follow.

    a. How many square tiles are there in the seventh figure?
    b. How many square tiles are in the nth figure?
    c. Is there a figure that has exactly 449 square tiles? If so, which one?

16. Consider the sequences given in the table below. Find the least number, n, such that the nth term of the geometric

sequence is greater than the corresponding term in the arithmetic sequence.

Term Number	1	2	3	4	5	6	. . .	n
Arithmetic	200	500	800	1100	1400	1700	. . .	
Geometric	1	3	9	27	81	243	. . .	

17. Female bees are born from fertilized eggs, and male bees are born from unfertilized eggs. This means that a male bee has only a mother, whereas a female bee has a mother and a father. If the ancestry of a male bee is traced 10 generations including the generation of the male bee, how many bees are there in all 10 generations? (*Hint:* The Fibonacci sequence might be helpful.)
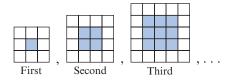
18. Each of the following sequences is labeled either arithmetic or geometric. In each part, find the missing terms.
    a. __, 49, __, __, 64 (arithmetic)
    b. 1, __, __, __, 625 (geometric)
    c. __, 3^{10}, __, __, 3^{19} (geometric)
    d. a, __, __, __, $5a$ (arithmetic)
19. Each of the following sequences is a Fibonacci-type sequence. Find the missing terms.
    a. 1, __, __, 7, 11
    b. __, 2, __, 4, __
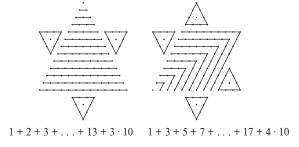   c. __, __, 3, 4, __

Mathematical Connections 1-2

1. a. If a fixed number is added to each term of an arithmetic sequence, is the resulting sequence an arithmetic sequence? Justify the answer.
   b. If each term of an arithmetic sequence is multiplied by a fixed number, will the resulting sequence always be an arithmetic sequence? Justify the answer.
   c. If the corresponding terms of two arithmetic sequences are added, is the resulting sequence arithmetic?
2. A student says she read that Thomas Robert Malthus (1766–1834), a renowned British economist and demographer, claimed that the increase of population will take place, if unchecked, in a geometric sequence, whereas the supply of food will increase in only an arithmetic sequence. This theory implies that population increases faster than food production. The student is wondering why. How do you respond?
3. Abby and Dan are preparing for a GRE (Graduate Record Exam) to take place in 5 months. Abby starts by studying 10 hours the first week and increases her study by 30 minutes per week. Dan starts at 6 hours per week, but increases his time every week by 45 minutes per week. Answer the following.
   a. How many hours did each student study in week 8?
   b. In which week will Dan first catch up with Abby in the number of hours spent studying per week?
4. The *arithmetic average* of two numbers x and y is $\frac{x + y}{2}$.
   Use *deductive reasoning* to explain why if three numbers a, b, and c form an arithmetic sequence, then b is the arithmetic average of a and c.

5. A mathematician named Christian Goldbach (1690–1764) made a conjecture that has not been proven. Millions of examples have been found that support his conjecture, but no counterexample has ever been found. Until a counterexample is found or someone presents a logical proof, this will remain a conjecture.

   Goldbach's Conjecture: *Every even number greater than 2 can be written as a sum of two prime numbers.*

Test Goldbach's Conjecture for various even numbers, for example, $4 = 2 + 2$ and $10 = 7 + 3$.
6. The figure below shows the first three terms of a sequence of figures containing small square tiles. Some of the tiles are shaded. Notice that the first figure has one shaded tile. The second figure has $2 \cdot 2$, or 2^2, shaded tiles. The third figure has $3 \cdot 3$, or 3^2, shaded tiles. If this pattern of having shaded squares surrounded by white borders continues, answer the following:
   a. How many shaded tiles are there in the nth figure?
   b. How many white tiles are there in the nth figure?

First , Second , Third , . . .

7. Patterns can be used to count the number of dots on the Chinese checkerboard; two patterns are shown here. Determine several other patterns to count the dots.

$1 + 2 + 3 + \ldots + 13 + 3 \cdot 10$ $1 + 3 + 5 + 7 + \ldots + 17 + 4 \cdot 10$

8. Make up a pattern involving figurate numbers and find the number of dots in the 100th figure. Describe the pattern and how to find the 100th term.

Connecting Mathematics to the Classroom

9. Joey said that 4, 24, 44, and 64 all have remainder 0 when divided by 4, so all numbers that end in 4 must have 0 remainder when divided by 4. How do you respond?

10. Al and Betty were asked to extend the sequence 2, 4, 8, ... Al said his answer of 2, 4, 8, 16, 32, 64, . . . was the correct one. Betty said Al was wrong and it should be 2, 4, 8, 14, 22, 32, 44, ... What do you tell these students?

11. A student claims the sequence 1, 1, 1, 1, . . . is both arithmetic and geometric and would like to know if there are other such sequences. How do you respond?

12. A student claims that she has found an easy way to find the number of terms in an arithmetic sequence: "You take the difference between the last and first terms and divide by the common difference." How do you respond?

13. Suppose flu breaks out in a school system and the number of students becoming ill is increasing. A student asks which is worse: that the number of students with the flu is increasing arithmetically or geometrically? Explain your answer.

14. a. Students noticed that the digits of numbers in the 9 times table always add to 9, for example, $9 \times 4 = 36$ and $3 + 6 = 9$. They conjecture that all multiples of 9 have digits that add to 9. What would you do with students investigating this conjecture?

b. After exploring the conjecture in part (a), students changed their conjecture to the sum of the digits in all multiples of 9 add to 9 or 18. How do you respond?

c. What conjecture might work for the sum of the digits for the multiples of 9?

Review Problems

15. In a baseball league consisting of 10 teams, each team plays each of the other teams twice. How many games will be played?

16. How many ways can you make change for 40¢ using only nickels, dimes, and quarters?

17. Tents hold 2, 3, 5, 6, or 12 people. What combinations of tents are possible to sleep 26 people if all tents are fully occupied and only one 12-person tent is used?

Group Work

1. The following pattern is called *Pascal's triangle.* It was named for the mathematician Blaise Pascal (1623–1662).

$$
\begin{array}{ccccccccccccccc}
&&&&&&& 1 \\
&&&&&& 1 && 1 \\
&&&&& 1 && 2 && 1 \\
&&&& 1 && 3 && 3 && 1 \\
&&& 1 && 4 && 6 && 4 && 1 \\
&& 1 && 5 && 10 && 10 && 5 && 1 \\
& 1 && 6 && 15 && 20 && 15 && 6 && 1 \\
1 && 7 && 21 && 35 && 35 && 21 && 7 && 1
\end{array}
$$

a. Have each person in the group find four different patterns in the triangle and then share them with the rest of the group.

b. Add the numbers in each horizontal row. Discuss the pattern that occurs.

c. Use part (b) to find the sum in the 16th row.

d. What is the sum of the numbers in the nth row?

2. If the following pattern continued indefinitely, the resulting figure would be called the *Sierpinski triangle,* or *Sierpinski gasket.*

In a group, determine each of the following. Discuss different counting strategies.

a. How many black triangles would be in the fifth figure?

b. How many white triangles would be in the fifth figure?

c. How many black triangles are in the nth figure?

National Assessment of Educational Progress (NAEP)

1. The growing number pattern 3, 4, 6, 9, 13, . . . follows a rule. Explain the rule.

Write a new growing pattern beginning with 21 that follows the same rule. 21, _____, _____,_____, _____ , _____

NAEP, Grade 4, 2013

2. Write the next two numbers in the number pattern.

1 6 4 9 7 12 10 _____ _____

Write the rule that you used to find the two numbers you wrote.
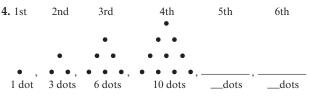
NAEP, Grade 4, 2009

3. $\dfrac{1}{20}, \dfrac{4}{20}, \dfrac{7}{20}, \dfrac{10}{20}, \dfrac{13}{20}, \ldots$

If the pattern shown continues, what is the first fraction in the pattern that will be *greater* than 1?

a. $\dfrac{20}{20}$

b. $\dfrac{21}{20}$

c. $\dfrac{22}{20}$

d. $\dfrac{25}{20}$

NAEP, Grade 4, 2013

4.

1st	2nd	3rd	4th	5th	6th

1 dot , 3 dots , 6 dots , 10 dots , ___dots , ___dots

A pattern of dots is shown above. How many dots would be in the 6th picture?

Answer: _____

Explain how you found your answer.

NAEP, Grade 4, 2009

Hint for Solving the Chapter Opener Problem

Answer vary, for example, consider $a + b \rightarrow ab + a$.

Chapter 1 Summary

key concepts	definitions, descriptions, and theorems
Section 1-1	
CCSS *Standards for Mathematical Practice* **(p. 3)**	• See Table 1
Four-step problem-solving process (p. 4)	• Understanding the problem • Carrying out the plan • Devising a plan • Looking back
Natural numbers (p. 6)	The numbers 1, 2, 3, 4, 5, . . . The three dots above, called an *ellipsis*, indicate a continuation.
Problem-solving strategies (p. 4)	• Look for a pattern • Make a diagram • Examine a related problem • Use guess and check • Examine a simpler case • Work backward • Make a table • Write an equation • Identify a subgoal
Section 1-2	
Conjecture (p. 18)	A statement thought to be true but not yet proven.
Inductive reasoning (p. 18)	Method of making generalizations based on observations and patterns, which may or may not be true.
Deductive reasoning (p. 18)	Logical method of proving the truth of conjectures through the use of axioms, theorems, undefined terms.
Counterexample (p. 19)	An example that shows a conjecture is false.
Sequence (p. 21)	An ordered arrangement of numbers, figures, or objects. The individual items of a sequence are the *terms* of the sequence. The *term number* indicates the position of the term in the sequence.
Arithmetic sequence (p. 21)	A sequence in which each successive term from the second term on is obtained from the previous term by adding a fixed number. The fixed number is the *common difference* or *difference*.
Recursive pattern (p. 21)	*A pattern* in which each successive term of the sequence is obtained from the previous term(s).
Fibonacci sequence (p. 26)	The sequence in which the first two terms are both 1 and each subsequent term is the sum of the previous two terms: 1, 1, 2, 3, 5, 8, 13, 21, . . . In general, a sequence in which the first two terms are arbitrary and each subsequent term is the sum of the previous two terms is a *Fibonacci-type sequence.*
Geometric sequence (p. 26)	A sequence in which each successive term from the second term on is obtained from the previous term by multiplying by a fixed nonzero number. The fixed number is the *ratio*.
Figurate numbers (p. 28)	Numbers that can be represented by dots arranged in the shape of geometric figures. Figurate numbers provide examples of sequences that are neither arithmetic nor geometric.

Chapter 1 Review

1. If today is Sunday, July 4, and next year is not a leap year, what day of the week will July 4 be on next year?

2. Jackie spent $40 on two items. If she spent $5.90 more on the more expensive item, how much did this item cost? What strategy have you used to solve this problem?

3. List three more terms that complete a pattern in each of the following; explain your reasoning, and tell whether each sequence is arithmetic or geometric, or neither.
   a. 0, 1, 3, 6, 10, ___, ___, ___,
   b. 52, 47, 42, 37, ___, ___, ___,
   c. 6400, 3200, 1600, 800, ___, ___, ___,
   d. 1, 2, 3, 5, 8, 13, ___, ___, ___,
   e. 2, 5, 8, 11, 14, ___, ___, ___,
   f. 1, 4, 16, 64, ___, ___, ___,
   g. 0, 4, 8, 12, ___, ___, ___,
   h. 1, 8, 27, 64, ___, ___, ___,

4. Find a possible nth term in each of the following.
   a. 5, 8, 11, 14, . . .
   b. 3, 9, 27, 81, 243, . . .
   c. $2^2 - 1, 2^3 - 1, 2^4 - 1, 2^5 - 1, \ldots$

5. Find the first five terms of the sequences whose nth term is given as follows:
   a. $3n - 2$
   b. $n^2 + n$
   c. $4n - 1$

6. Find the following sums:
   a. $2 + 4 + 6 + 8 + 10 + \ldots + 200$
   b. $51 + 52 + 53 + 54 + \ldots + 151$

7. Produce a counterexample, if possible, to disprove each of the following. If a statement is true, justify it.
   a. If two odd numbers are added, then the sum is odd.
   b. If a number is odd, then the last digit is a 1 or a 3.
   c. If two even numbers are added, then the sum is even.

8. Complete the following magic square; that is, complete the square so that the sum in each row, column, and diagonal is the same.
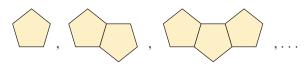
16	3	2	13
	10		
9		7	12
4		14	

9. How many people can be seated at 12 square tables lined up end to end if each table individually holds four persons?

10. Solve the following equations:
    a. $\boxed{} + 2^{60} = 2^{61}$
    b. $\boxed{}^2 = 625$

11. If fence posts are to be placed in a row 5 m apart, how many posts are needed for 100 m of fence?

12. If a complete rotation of a car tire moves a car forward 6 ft, how many rotations of the tire occur before the tire goes off its 50,000 mi warranty?

13. The members of Mrs. Grant's class are standing in a circle; they are evenly spaced and are numbered in order. The student with number 7 is standing directly across from the student with number 17. How many students are in the class?

14. A carpenter has three large boxes. Inside each large box are two medium-sized boxes. Inside each medium-sized box are five small boxes. How many boxes are there altogether?

15. An ant farm can hold 100,000 ants. If the farm held 1500 ants on the first day, 3000 ants on the second day, 6000 ants on the third day, and so on forming a geometric sequence, in how many days will the farm be full?

16. Toma's team entered a mathematics contest in which teams of students compete by answering questions that are worth either 3 points or 5 points. No partial credit was given. Toma's team scored 44 points on 12 questions. How many 5-point questions did the team answer correctly?

17. Three pieces of wood are needed for a project. They are to be cut from a 90-cm-long piece of wood. The longest piece is to be 3 times as long as the middle-sized piece and the shortest piece is to be 10 cm shorter than the middle-sized piece. How long are the pieces?

18. How many four-digit numbers have the same digits as 1993?

19. We have two containers, one of which holds 7 cups and the other holds 4 cups. How can we measure exactly 5 cups of water, if we have an unlimited amount of water with which to start?

20. The following geometric arrays suggest a sequence of numbers: 2, 6, 12, 20, . . .

    a. Find the next three terms.
    b. Find the 100th term.
    c. Find the nth term.

21. Each side of each pentagon below is 1 unit long.

    a. Draw a possible next figure in the sequence.
    b. What is the perimeter (distance around) of each of the first four figures?
    c. What is the perimeter of the 100th figure?
    d. What is the perimeter of the nth figure?

22. a. If every second term in an arithmetic sequence is circled, do the circled terms always constitute an arithmetic sequence? Justify your answer.
    b. Answer the question in part (a) again, but replace "arithmetic" with "geometric."