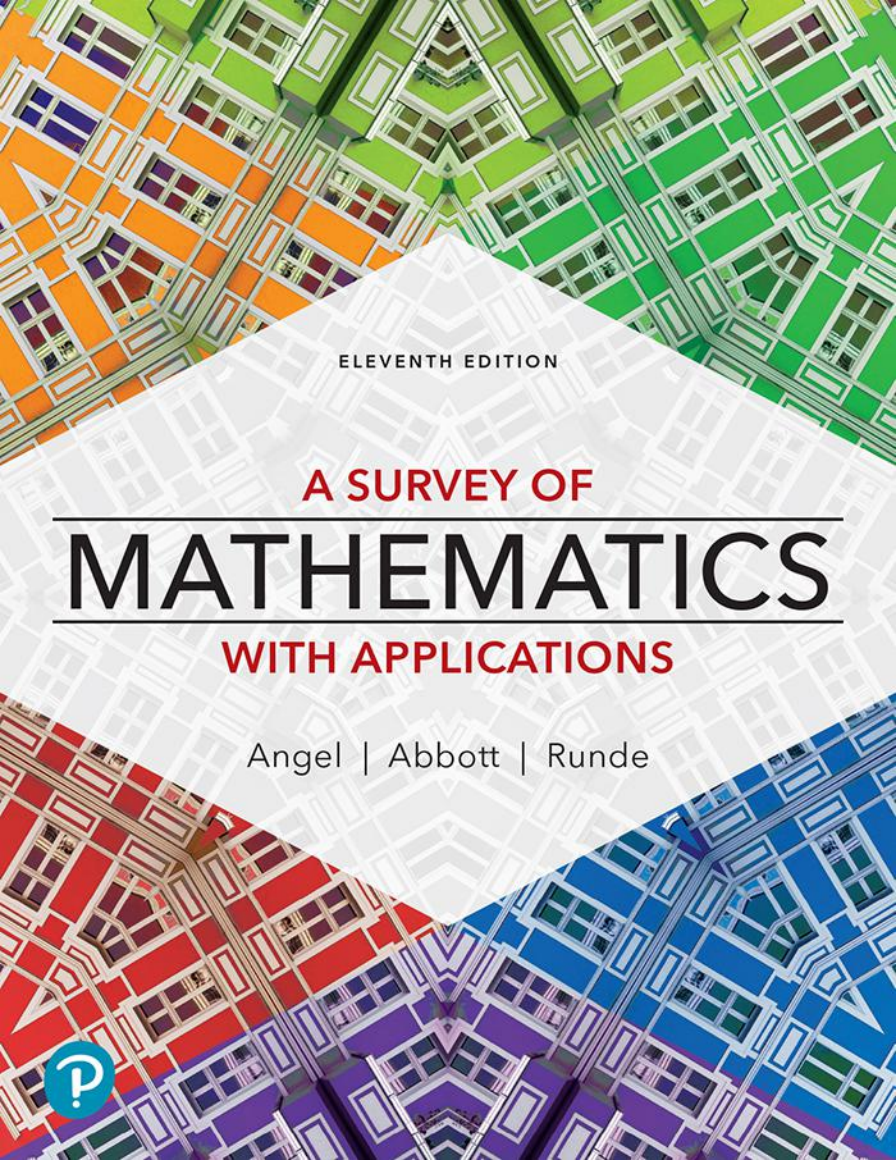
ELEVENTH EDITION
A SURVEY OF
MATHEMATICS
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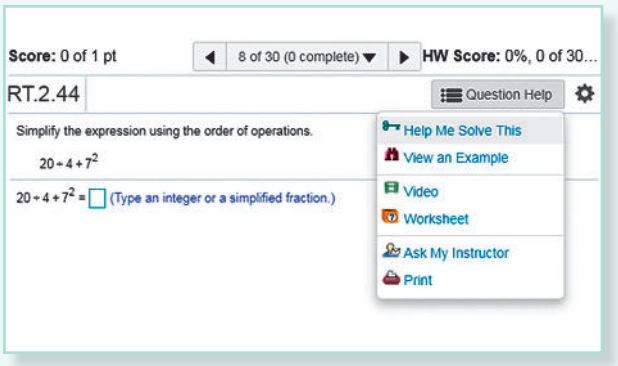

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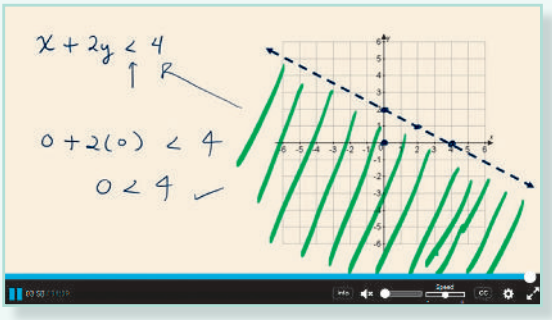
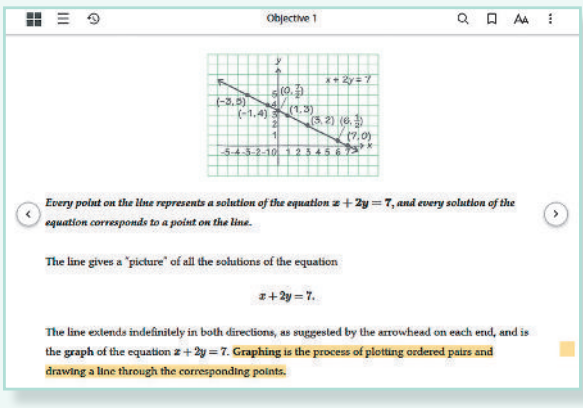
MyLab Math's interactive exercises mirror those in the textbook but are programmed to allow you unlimited practice, leading to mastery. Most exercises include **learning aids**, such as "Help Me Solve This," "View an Example," and "Video," and they offer helpful feedback when you enter incorrect answers.

Instructional Videos

Instructional videos cover key examples from the text and can conveniently be played on any mobile device. These videos are especially helpful if you miss a class or just need further explanation.





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11TH EDITION

A Survey of Mathematics with Applications

ALLEN R. ANGEL
Monroe Community College

CHRISTINE D. ABBOTT
Monroe Community College

DENNIS C. RUNDE
State College of Florida

Pearson

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To my wife, Kathy Angel
A. R. A.

To my sons, Matthew and Jake Abbott
C. D. A.

To my mother, Tina Runde, and the memory of my father, Bud Runde
D. C. R.

Contents

To the Student

Mathematics is an exciting, living study. Its applications shape the world around you and influence your everyday life. We hope that as you read this book you will realize just how important mathematics is and gain an appreciation of both its usefulness and its beauty. We also hope to teach you some practical mathematics that you can use every day and that will prepare you for further mathematics courses.

The primary purpose of this text is to provide material that you can read, understand, and enjoy. To this end, we have used straightforward language and tried to relate the mathematical concepts to everyday experiences. The concepts, definitions, and formulas that deserve special attention are in boxes or are set in boldface, italics, or color type. We have also provided many detailed examples for you to follow.

Be sure to read the chapter summary, work the review exercises, and take the chapter test at the end of each chapter. The answers to the odd-numbered exercises, all review exercises, and all chapter test exercises appear in the answer section in the back of the text. You should, however, use the answers only to check your work. The answers to all Recreational Mathematics exercises are provided either in the Recreational Mathematics boxes themselves or in the back of the book.

It is difficult to learn mathematics without becoming involved. To be successful, we suggest that you read the text carefully and *work each exercise in each assignment in detail*, for it is in doing the math that you really learn and enjoy it. If you are using this text within MyLab Math, you'll find a wealth of other learning aids available there, including tutorial videos and homework help.

We welcome your suggestions and your comments. You may contact us at math@pearson.com. (Please use the subject line "Angel Survey of Math.") Good luck with your adventure in mathematics!

Allen R. Angel

Christine D. Abbott

Dennis C. Runde

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Preface

We present *A Survey of Mathematics with Applications*, Eleventh Edition, with the knowledge that we use mathematics every day. In this edition, we stress how mathematics is used in our daily lives and why it is important. Our primary goal is to give students a text they can read, understand, and enjoy while learning how mathematics affects the world around them. Numerous real-life applications are used to motivate topics. A variety of interesting and useful exercises demonstrate the real-life nature of mathematics and its importance in students' lives.

The text is intended for students who require a broad-based general overview of mathematics, especially those majoring in the liberal arts, elementary education, the social sciences, business, nursing, and allied health fields. It is particularly suitable for those courses that satisfy the minimum competency requirement in mathematics for graduation or transfer.

New to This Edition

New within the Textbook

- **Co-Requisite Course Support** – On the first page of each section in the Annotated Instructor Edition, there is a list of the prerequisite skills needed for that section. Please see the "Integrated Review" note under New to MyLab Math (below) for information on how to address gaps in prerequisite skills and use MyLab Math for co-requisite courses.
- **Data and Context Updates** – We've updated time-sensitive data to the most current available or changed the context of a problem or narrative so that it is more relevant to students.

- **Animations** – Located throughout the narrative you'll find Animation icons like the one pictured at the left. They are designed to facilitate active learning and visualization of key concepts. These Animations are housed within MyLab Math and are ideal for classroom use during lecture or by students independently. They were created in GeoGebra and are usable on any type of device. They are also editable.

- **StatCrunch** – Located throughout the narrative you'll find StatCrunch icons like the one pictured at the left. Like the Animation features, they are designed to facilitate active learning and exploration. They are also housed within MyLab Math. StatCrunch is Pearson's online statistical software.
- **Technology Tips** – This feature of the text has been updated to go beyond graphing calculators and spreadsheets. Content now includes references to apps on smartphones and tablets.
- **Downloadable Data Sets** – For problems and examples in which students are expected to analyze a set of data, you'll see the icon DS, which indicates that the data is available to download in *.txt and *.csv formats. All of the data sets are housed in MyLab Math and also at **bit.ly/2Mxjj23**.
- **Learning Catalytics** – Learning Catalytics is a "bring your own device" student polling and assessment system, available in MyLab Math. Each section of the Annotated Instructor's Edition features keywords that can be entered into Learning Catalytics to bring you directly to questions for use in lecture for that section. Detailed instructions for using these keywords can be found at **bit.ly/31Z0r2p**.

New within MyLab Math

- **Integrated Review** – MyLab Math contains a chapter called "Additional Review" that features all of the necessary prerequisite objectives for this course. To help you use this content, we have also included:
  - **Skills Check Quizzes** by chapter assess the prerequisite skills students need for that chapter.

- **Skills Review Homework**, again by chapter, is personalized (based on the results of the Skills Check Quiz) to provide students with help on the prerequisite skills they are lacking. Students receive just the help they need—no more, no less.
- **Co-Requisite Support** – In addition to the items above, for co-requisite courses (or students who just need more help), we've included videos and worksheets to provide necessary instruction for prerequisite skills. There's no need to go elsewhere for remediation.

- **Tech Help for Stats Exercises** – For statistics-related exercises in which students might use technology to solve, we've included a "Tech Help" button that allows them to get keystroke-by-keystroke instructions for StatCrunch, Excel, and TI Graphing Calculators. Note that this Tech Help button can be suppressed if you don't want students to use these technologies.
- **Animations** – Over 40 animations, referenced in the text, are designed to facilitate active learning and visualization of key concepts. They are ideal for classroom use during lecture or by students independently. They were created in GeoGebra and are usable on any type of device. They are also editable.
- **StatCrunch** – A dozen StatCrunch activities, referenced in the text, are designed to facilitate active learning and exploration. StatCrunch is Pearson's online statistical software.

Continuing and Revised Features

- **Chapter Openers** – Interesting and motivational applications introduce each chapter, which includes the Why This Is Important section, and illustrate the real-world nature of the chapter topics.
- **Problem Solving** – Beginning in Chapter 1, students are introduced to problem solving and critical thinking. We continue the theme of problem solving throughout the text and present special problem-solving exercises in the exercise sets.
- **Critical Thinking Skills** – In addition to a focus on problem solving, this book also features sections on inductive and deductive reasoning, estimation, and dimensional analysis.
- **Profiles in Mathematics** – Brief historical sketches and vignettes present stories of people who have advanced the discipline of mathematics. In this edition, we included more diversity among the mathematicians included.
- **Did You Know?** – These colorful, engaging, and lively features highlight the connections of mathematics to history, the arts and sciences, technology, and a broad variety of disciplines.
- **Mathematics Today** – These features discuss current real-life uses of the mathematical concepts in the chapter. Each box ends with Why This Is Important.
- **Recreational Math** – In these features, students are invited to apply the math in puzzles, games, and brain teasers. Answers are given in upside-down type at the bottom of the feature or (for longer answers) at the back of the book. In addition, Recreational Mathematics problems appear in the exercise sets so that they can be assigned as homework.
- **Technology Tips** – The material in these features explains how students can use technology including calculators, spreadsheets, and smartphone apps to explore various mathematical concepts and solve application problems.
- **Timely Tips** – These easy-to-identify boxes offer helpful information to make the material under discussion more understandable.
- **Key-Idea Boxes** – Important definitions, formulas, and procedures are boxed, making key information easy to identify for students.
- **Chapter Summaries, Review Exercises, and Chapter Tests** – The end-of-chapter summary charts provide an easy study experience by directing students to the location in the text where specific concepts are discussed. Review Exercises and Chapter Tests also help students review material and prepare for exams.

Acknowledgments

We thank the reviewers from all editions of the book and all the students who have offered suggestions for improving it. A list of reviewers for all editions of this book follows, with reviewers of this edition noted with an asterisk (*). Thanks to you all for helping make *A Survey of Mathematics with Applications* one of the most successful liberal arts mathematics textbooks in the country.

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James Wooland, Florida State University
*Kelly Young, Lander University
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We thank James Lapp, Lauri Semarne, and Rick Ponticelli for their conscientious job of checking the text and answers for accuracy.

James Lapp also deserves our thanks for the excellent work he did on the Student's Solutions Manual and Instructor's Solutions Manual. We'd also like to thank Barbara Burke of Hawaii Pacific University and Deborah Doucette of Erie Community College for developing the Integrated Review worksheets to accompany the text.

Finally, we thank our family members Kathy, Robert, and Steve Angel, Mathew and Jake Abbott, and Kris, Alex, Nick, and Max Runde for their support and encouragement throughout this project. We are grateful for their wonderful support and understanding while we worked on the book. They also gave us support and encouragement and were very understanding when we could not spend as much time with them as we wished because of book deadlines. Without the support and understanding of our families, this book would not be a reality.

Allen R. Angel
Christine D. Abbott
Dennis C. Runde

Get the most out of

MyLab Math

Pearson MyLab

MyLab Math for *Survey of Mathematics* 11e by Angel/Abbott/Runde *(access code required)*

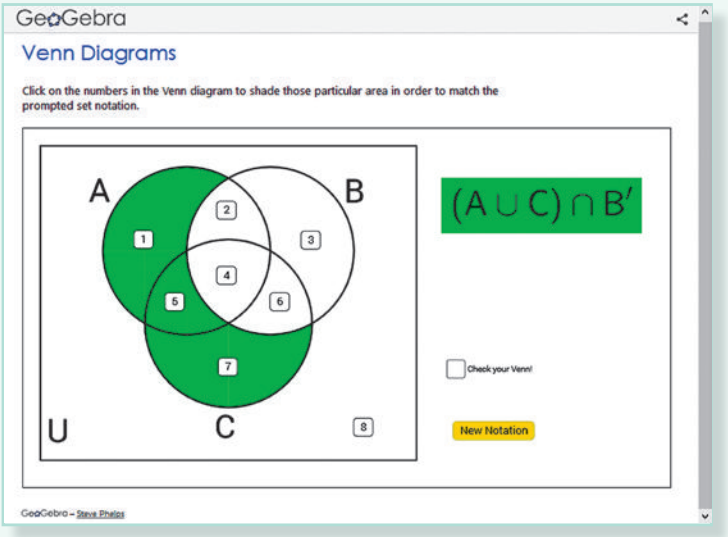
MyLab Math is tightly integrated with each author team's style, offering a learning and practice experience that gives students a consistent experience from text to MyLab.

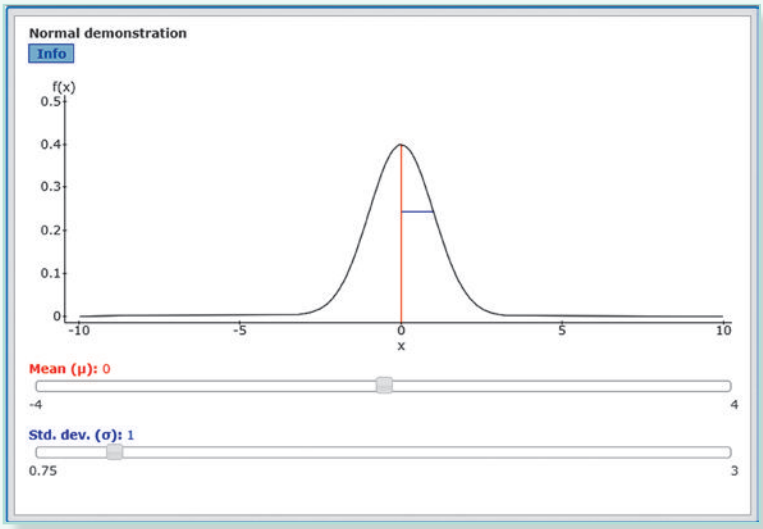
MyLab Math includes **assignable algorithmic exercises** for unlimited practice opportunities. The **complete eText** is available for learning in any environment. And a **comprehensive gradebook** gives instructors and students alike insight into how they are doing at any time.

Additionally, the following resources are available to enrich student learning.

Animations

NEW! Animations let students interact with the math in a visual, tangible way. These animations allow students to explore and manipulate the mathematical concepts, leading to more durable understanding. They can also be used by instructors in the classroom to enhance instruction.

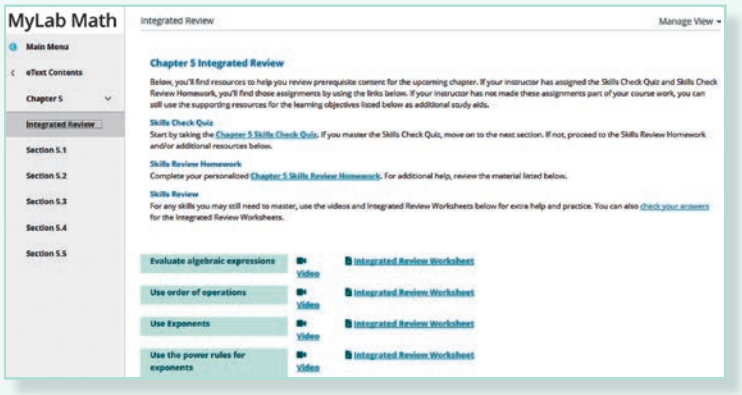




StatCrunch

NEW! StatCrunch® is a powerful web-based statistical software that allows users to collect, crunch, and communicate with data. Now integrated into this MyLab Math course, StatCrunch can be used to analyze and understand statistical concepts. DS icons in the text indicate that a data set is available to download online and analyze in statistical software, such as StatCrunch or Excel.

pearson.com/mylab/math

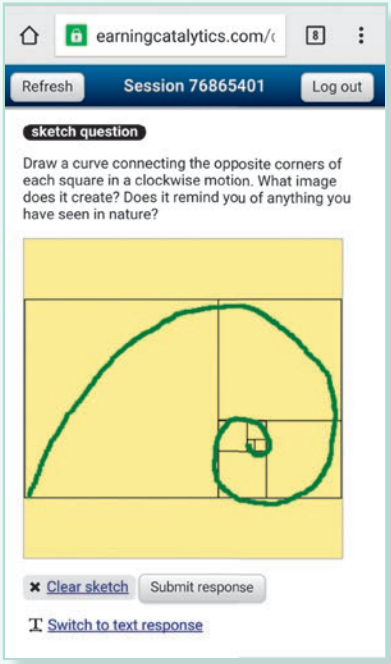


Integrated Review

Ideal for a corequisite course, or simply to get underprepared students up to speed, Integrated Review includes assessments, assignments, and resources on prerequisite topics. For each chapter, pre-made, assignable (and editable) quizzes assess students' understanding of the prerequisite skills needed for that chapter. Personalized follow-up homework assignments and remediation resources, in the form of videos and worksheets, are available for any gaps in skills that are identified.

Skills for Success Modules

These modules offer resources and assignments that aim to bolster students' ability to succeed in college courses and prepare for future professions. These modules now include a Mindset section with growth mindset-focused videos and exercises that encourage students to maintain a positive attitude about learning, value their own ability to grow, and view mistakes as a learning opportunity.



Learning Catalytics

Integrated into the MyLab course, Learning Catalytics uses students' devices in the classroom for an engagement, assessment, and classroom intelligence system that gives instructors real-time feedback on student learning. To make it easy to integrate this learning tool into the classroom, keywords are available in the Annotated Instructor's Edition at point of use. Detailed instructions for using these keywords can be found at **bit.ly/31Z0r2p**.

pearson.com/mylab/math

Resources for

Success

Pearson MyLab

Instructor Resources

Annotated Instructor's Edition

All answers are included. When possible, answers are on the page with the exercises. Longer answers are in the back of the book. The AIE now includes references to Learning Catalytics to help instructors incorporate those resources. Corequisite support in the form of Integrated Review prerequisite skills are also referenced in the AIE for instructors who need to provide corequisite support in their course.

The following instructor resources are available to download from **www.pearson.com** or the MyLab Math course.

PowerPoint® Lecture Slides

Fully editable slides correlated with the textbook are available. Accessible, screen reader-friendly versions of the slides are also available.

Instructor's Solutions Manual

This manual includes fully worked solutions to all text exercises.

Instructor's Testing Manual

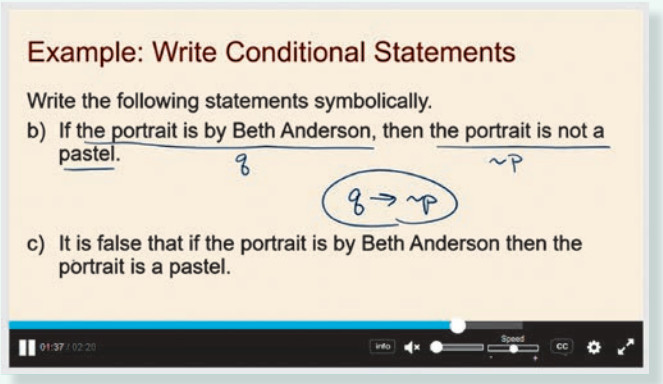
This manual includes tests with answer keys for each chapter of the text.

TestGen®

TestGen® (www.pearson.com/testgen) enables instructors to build, edit, print, and administer tests using a computerized bank of questions developed to cover all the objectives of the text.

Student Resources

Video Program



Available in MyLab Math, a modern, accessible video tutor program covers every section in the text, providing students with a video tutor at home, in a lab, or on the go. All videos have closed captioning available. Video assessment questions in the Assignment Manager check for student understanding of the video they just watched, making the videos assignable.

Student Solutions Manual

Available in MyLab Math and in print, this manual provides detailed worked-out solutions to odd-numbered exercises, and to all Chapter Review and Test exercises. (ISBN 10: 0-13-574045-2; ISBN 13: 978-0-13-574045-3)

Integrated Review Workbook

This workbook provides additional practice and study support for students. Worksheets cover prerequisite developmental-level topics necessary to move on to each chapter. This workbook is available in MyLab Math or as a printed, unbound, three-hole-punched workbook. (ISBN 10: 0-13-574028-2 ISBN 13: 978-0-13-574028-6)

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Graduating from college requires the solution to many problems. Using critical thinking skills is vital when solving these problems.

1

Critical Thinking Skills

What You Will Learn

- Inductive and deductive reasoning
- Estimation techniques
- Problem-solving procedures

Why This Is Important

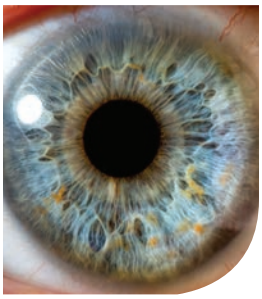
Everyday life presents us with a wide range of problems that we must solve such as deciding which classes to take in the next semester of college or deciding if you can afford to buy a new car. Making good decisions can lead to desirable outcomes. For example, choosing classes that are part of your program of study may help you stay on track to graduate from college. Delaying a new car purchase may help you afford more important things in your life like college tuition.

The information you encounter in this chapter will help sharpen your critical thinking skills. These skills will help you make better decisions as you solve the problems that you encounter in your everyday life.

SECTION 1.1 Inductive and Deductive Reasoning

Upon completion of this section, you will be able to:

- Understand and use inductive reasoning to solve problems.
- Understand and use deductive reasoning to solve problems.



The science of biometrics involves the measurement and analysis of unique physical characteristics. Biometrics are usually used as a means of verifying personal identity. Fingerprints, iris patterns in eyes, facial recognition, DNA, and voice patterns can all be used for personal identification. Some smartphones can be unlocked by pressing a button that recognizes a unique fingerprint or by scanning an eye to recognize a unique iris pattern. Crime scene investigation often involves fingerprint and DNA evidence. Voice recognition software uses voice patterns of callers to help prevent fraud and to improve customer service.

Why This Is Important Using biometrics for personal identification involves reasoning to a general conclusion through observation of specific cases. In this section, we will discuss how inductive and deductive reasoning are essential critical thinking skills used in biometrics and in many other applications.

Inductive Reasoning

Before looking at some examples of inductive reasoning and problem solving, let us first review a few facts about certain numbers. The ***natural numbers*** or ***counting numbers*** are the numbers 1, 2, 3, 4, 5, 6, 7, 8, ... The three dots, called an ***ellipsis***, mean that 8 is not the last number but that the numbers continue in the same manner. A word that we sometimes use when discussing the counting numbers is "divisible." If $a \div b$ has a remainder of zero, then *a is divisible by b*. The counting numbers that are divisible by 2 are 2, 4, 6, 8, ... These numbers are called the *even counting numbers*. The counting numbers that are not divisible by 2 are 1, 3, 5, 7, 9, ... These numbers are the *odd counting numbers*. When we refer to *odd numbers* or *even numbers*, we mean odd or even counting numbers.

Recognizing patterns is sometimes helpful in solving problems, as Examples 1 and 2 illustrate.

Example 1 *The Product of Two Odd Numbers*

If two odd numbers are multiplied together, will the product always be an odd number?

Solution To answer this question, we will examine the products of several pairs of odd numbers to see if there is a pattern.

$$\begin{aligned} 1 \times 5 &= 5 & 3 \times 7 &= 21 & 5 \times 9 &= 45 \\ 1 \times 7 &= 7 & 3 \times 9 &= 27 & 5 \times 11 &= 55 \\ 1 \times 9 &= 9 & 3 \times 11 &= 33 & 5 \times 13 &= 65 \\ 1 \times 11 &= 11 & 3 \times 13 &= 39 & 5 \times 15 &= 75 \end{aligned}$$

All the products are odd numbers. Thus, we might predict from these examples that the product of any two odd numbers is an odd number. ■

Example 2 *The Sum of an Odd Number and an Even Number*

If an odd number and an even number are added, will the sum be an odd number or an even number?

Solution Let's look at a few examples in which one number is odd and the other number is even.

$$3 + 4 = 7 \qquad 9 + 6 = 15 \qquad 23 + 18 = 41$$
$$5 + 12 = 17 \qquad 5 + 14 = 19 \qquad 81 + 32 = 113$$

All these sums are odd numbers. Therefore, we might predict that the sum of an odd number and an even number is an odd number. ■

In Examples 1 and 2, we cannot conclude that the results are true for all counting numbers. From the patterns developed, however, we can make predictions. This type of reasoning process, arriving at a general conclusion from specific observations or examples, is called ***inductive reasoning***, or ***induction***.

Definition: Inductive Reasoning
Inductive reasoning is the process of reasoning to a general conclusion through observations of specific cases.

Induction often involves observing a pattern and from that pattern predicting a conclusion. Imagine an endless row of dominoes. You knock down the first, which knocks down the second, which knocks down the third, and so on. Assuming the pattern will continue uninterrupted, you conclude that any one domino that you select in the row will eventually fall, even though you may not witness the event.

Inductive reasoning is often used by mathematicians and scientists to develop theories and predict answers to complicated problems. For this reason, inductive reasoning is part of the ***scientific method***. When a scientist or mathematician makes a prediction based on specific observations, it is called a ***hypothesis*** or ***conjecture***. After looking at the products in Example 1, we might conjecture that the product of two odd numbers will be an odd number. After looking at the sums in Example 2, we might conjecture that the sum of an odd number and an even number is an odd number.

Examples 3 and 4 illustrate how we arrive at a conclusion using inductive reasoning.

Example 3 *Biometrics*

As described in the opening paragraph of this section, the science of biometrics is used for personal identification. What reasoning process has led to the conclusion that no two people have the same fingerprints, iris patterns, DNA, or voice patterns?

Solution By studying the biometrics of millions of people, scientists have never found two people who have the exact same fingerprints, iris patterns, DNA, or voice patterns. By induction, then, a conclusion can be reached that each of these biometrics provides a unique identification. A general conclusion is reached through the observation of specific cases. Therefore, the science of biometrics makes use of inductive reasoning. ■

Did You Know?

An Experiment Revisited

▲ *David Scott on the moon*

Apollo 15 astronaut David Scott used the moon as his laboratory to show that a heavy object (a hammer) does indeed fall at the same rate as a light object (a feather). Had Galileo dropped a hammer and feather from the Tower of Pisa, the hammer would have fallen more quickly to the ground and he still would have concluded that a heavy object falls faster than a lighter one. If it is not the object's mass that is affecting the outcome, then what is it? The answer is air resistance or friction: Earth has an atmosphere that creates friction on falling objects. The moon does not have an atmosphere; therefore, no friction is created.

Example 4 *Divisibility by 4*

Consider the conjecture "If the last two digits of a number form a number that is divisible by 4, then the number itself is divisible by 4." We will test several numbers to see if the conjecture appears to be true or false.

Solution Let's look at some numbers whose last two digits form a number that is divisible by 4.

Number	Do the Last Two Digits Form a Number That is Divisible by 4?	Is the Number Divisible by 4?
324	Yes; $24 \div 4 = 6$	Yes; $324 \div 4 = 81$
4328	Yes; $28 \div 4 = 7$	Yes; $4328 \div 4 = 1082$
10,612	Yes; $12 \div 4 = 3$	Yes; $10{,}612 \div 4 = 2653$
21,104	Yes; $4 \div 4 = 1$	Yes; $21{,}104 \div 4 = 5276$

In each case, we find that if the last two digits of a number are divisible by 4, then the number itself is divisible by 4. From these examples, we might be tempted to generalize that the conjecture "If the last two digits of a number are divisible by 4, then the number itself is divisible by 4" is true.* ■

Example 5 *Pick a Number, Any Number*

Pick any number, multiply the number by 4, add 2 to the product, divide the sum by 2, and subtract 1 from the quotient. Repeat this procedure for several different numbers and then make a conjecture about the relationship between the original number and the final number.

Solution Let's go through this one together.

Pick a number:	say, 5
Multiply the number by 4:	$4 \times 5 = 20$
Add 2 to the product:	$20 + 2 = 22$
Divide the sum by 2:	$22 \div 2 = 11$
Subtract 1 from the quotient:	$11 - 1 = 10$

Note that we started with the number 5 and finished with the number 10. If you start with the number 2, you will end with the number 4. Starting with 3 would result in a final number of 6, 4 would result in 8, and so on. On the basis of these few examples, we may conjecture that when you follow the given procedure, the number you end with will always be twice the original number. ■

The result reached by inductive reasoning is often correct for the specific cases studied but not correct for all cases. History has shown that not all conclusions arrived at by inductive reasoning are correct. For example, Aristotle (384–322 B.C.) reasoned inductively that heavy objects fall at a faster rate than light objects. About 2000 years later, Galileo (1564–1642) dropped two pieces of metal—one 10 times heavier than the other—from the Leaning Tower of Pisa in Italy. He found that both hit the ground at exactly the same moment, so they must have traveled at the same rate.

* This statement is in fact true, as is discussed in Section 5.1.

When forming a general conclusion using inductive reasoning, you should test it with several special cases to see whether the conclusion appears correct. If a special case is found that satisfies the conditions of the conjecture but produces a different result, such a case is called a ***counterexample***. A counterexample proves that the conjecture is false because only one exception is needed to show that a conjecture is not valid. Galileo's counterexample disproved Aristotle's conjecture. If a counterexample cannot be found, the conjecture is neither proven nor disproven.

Consider the statement "All birds fly." A penguin is a bird that does not fly. Therefore, a penguin is a counterexample to the statement "All birds fly."

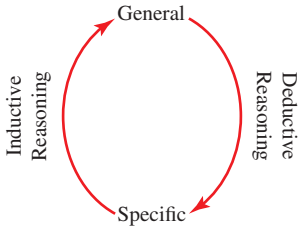
Deductive Reasoning

A second type of reasoning process is called ***deductive reasoning***, or ***deduction***. Mathematicians use deductive reasoning to *prove* conjectures true or false.

Definition: Deductive Reasoning
Deductive reasoning is the process of reasoning to a specific conclusion from a general statement.

Timely Tip

The following diagram helps explain the difference between inductive reasoning and deductive reasoning. Inductive reasoning is the process of reasoning to a general conclusion through observations of specific cases. Deductive reasoning is the process of reasoning to a specific conclusion from a general statement.



Example 6 illustrates deductive reasoning.

Example 6 *Pick a Number, n*

Prove, using deductive reasoning, that the procedure given in Example 5 will always result in twice the original number selected.

Solution To use deductive reasoning, we begin with the *general* case rather than specific examples. In Example 5, specific cases were used. Let's select the letter n to represent *any number*.

Pick any number:	n	
Multiply the number by 4:	$4n$	$4n$ means 4 times n.
Add 2 to the product:	$4n + 2$	
Divide the sum by 2:	$\frac{4n + 2}{2} = \frac{\overset{2}{\cancel{4}}n}{\underset{1}{\cancel{2}}} + \frac{\overset{1}{\cancel{2}}}{\underset{1}{\cancel{2}}} = 2n + 1$	
Subtract 1 from the quotient:	$2n + 1 - 1 = 2n$	

Note that for any number n selected, the result is $2n$, or twice the original number selected. Since n represented any number, we are beginning with the general case. Thus, this is deductive reasoning. ■

Animation

Number Tricks Using Deductive Reasoning

In Example 5, you may have *conjectured*, using specific examples and inductive reasoning, that the result would be twice the original number selected. In Example 6, we *proved*, using deductive reasoning, that the result will always be twice the original number selected.