Calculus with Applications

twelfth edition

Lial
Greenwell
Ritchey

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PREFACE

Calculus with Applications is a thorough, applications-oriented text for students majoring in business, management, economics, or the life or social sciences. In addition to its clear exposition, this text consistently connects mathematics to career and everyday-life situations. A prerequisite of two years of high school algebra is assumed. For this twelfth edition, several new authors, introduced below, have been included to improve and update the MyLab Math course as an even more integrated and rich learning resource for students.

Our Approach

Our main goal is to present applied calculus in a concise and meaningful way so that students can understand the full picture of the concepts they are learning and apply these concepts to real-life situations. This is done through a variety of means.

Focus on Applications Making this course meaningful is critical to students' success. Applications of the mathematics are integrated throughout the text in the exposition, examples, exercise sets, and supplementary resources. We are constantly on the lookout for novel applications, and this is reflected in our efforts to infuse the text with relevance. Our research is showcased in the Index of Applications in the back of the book and the extended list of sources of real-world data on **bit.ly/2HBIxO3**. *Calculus with Applications* presents students with myriad opportunities to relate what they're learning to career situations through the *Apply It* question at the beginning of sections, the applied examples and exercises, and the *Extended Application* at the end of each chapter.

Pedagogy to Support Students Students need careful explanations of the mathematics along with examples presented in a clear and consistent manner. Additionally, students and instructors should have a means to assess the basic prerequisite skills needed for the course content. This can be done with the *Prerequisite Skills Diagnostic Test*, located just prior to Chapter R. If the diagnostic test reveals gaps in basic skills, students can find help within the text. Within MyLab Math there are additional diagnostic quizzes (one per chapter), and remediation is automatically personalized to meet student needs. Students will appreciate the many annotated examples within the text, the *Your Turn* exercises that follow examples, the *For Review* references, and the wealth of learning resources within MyLab Math.

Beyond the Textbook Students want resources at their fingertips and, for them, that means digital. So Pearson has developed a robust MyLab Math course for *Calculus with Applications*. MyLab Math has a well-established and well-documented track record of helping students succeed in mathematics. The MyLab Math online course for this text contains more than 3000 exercises to challenge students and provides help when they need it. Students who learn best through video can view (and review) section- and example-level videos within MyLab Math. These and other resources are available to students as a unified and reliable tool for their success.

New to the Twelfth Edition

We welcome to this edition co-author Katherine Ritchey of University of Mount Union (OH). Katherine's primary focus was updating the contents of the MyLab Math course for the text. Detailed analysis of exercises and their solutions has led to improvements across the course in MyLab Math.

We also welcome Blain Patterson and Sarah Ritchey Patterson of Virgina Military Institute. Blain and Sarah identified interactive figures to be included in MyLab, created additional exercises for the interactive figures and Setup & Solve exercises, and suggested improvements to the PowerPoint slides.

New to the Text and MyLab Math Course

- All MyLab Math **exercises have been reviewed and edited** where necessary by author Katherine Ritchey for improved quality and fidelity to the text. Solutions are more consistent with the pedagogy of the text and the language in exercises and solutions has been updated to reflect more modern conventions.
- The **suite of interactive figures has been expanded** to support teaching and learning. These figures (created in GeoGebra) illustrate key concepts and can be manipulated by users. They have been designed to be used in lectures as well as by students independently.
- **Enhanced Assignments** are section-level assignments that (1) address gaps in prerequisite skills with personalized prerequisite review, (2) help keep skills fresh with spaced practice of key Calculus concepts, and (3) provide opportunities to work exercises without learning aids so that students can check their understanding. They are assignable and editable.
- We added more "help text" annotations to examples. These notes, set in small blue type, appear next to the steps within worked-out examples and provide an additional aid for students with weaker algebra skills.
- The text has always stood out for using **real data** in examples and exercises. The twelfth edition will not disappoint in this area. We have added or updated 12 percent of the exercises, improving data and applications. This includes hundreds of new exercises written for the twelfth edition.
- **Concept Check** exercises have been added to the beginning of the exercises in every section to ensure that students understand the basic concepts before proceeding.
- We heard from users that the Annotated Instructor's Edition for the previous edition required too much flipping of pages to find answers, so now we have included all of the answers in one location—at the back of the Instructor's Edition. For your convenience, all of the answers are also available for you to download (as PDF) in MyLab Math.
- MyLab Math contains a wealth of new resources to help students learn and help you as you teach. Some resources were added or revised based on student usage of the *previous* edition of the MyLab Math course. For example, more exercises were added to chapters and sections that are more widely assigned.
    - Hundreds of new exercises were added to the course to provide you with more options for assignments, including:
        - More application exercises throughout the text.
        - *Setup & Solve* exercises that require students to specify how to set up a problem as well as how to solve it.
    - An Integrated Review option is built into the MyLab Math course. Integrated Review contains pre-loaded diagnostic and remediation resources for key prerequisite skills. Skills Check Quizzes help diagnose gaps in skills prior to each chapter. MyLab Math then provides personalized help for only those skills that a student has not mastered.

New and Revised Content

The chapters and sections in the text are in the same order as the previous edition, making it easier for users to transition to the new edition. In the twelfth edition we've made numerous changes throughout to clarify and simplify the exposition. In addition to revising exercises and examples and updating and adding real-world data, we made the following changes:

Chapter R

- Rearranged material on multiplying polynomials for greater clarity.
- Expanded coverage of factoring.
- Revised or expanded several examples.

Chapter 1

- Expanded coverage of different forms of the equations of lines.
- Updated three examples and the Extended Application with new data.

Chapter 2

- Examples have been updated with new data.
- Additional parts have been added to some examples.

Chapter 3

- Revised introductory limit examples, illustrating how to find limits with graphs, tables, and algebra. Added more details to examples of finding limits at infinity.
- Added labels to the rules for limits for easier reference for the students.
- Revised Tech Note for finding limits, emphasizing selection of viewing window to accurately determine the behavior of a function.
- Added discussion and corresponding exercises about removable and nonremovable discontinuities and the process of determining continuity on an interval. Revised solutions for continuity examples to include more detail, especially with piecewise functions.
- Revised section on graphical differentiation. Added an initial example with a simple function and provided a detailed solution for sketching the derivative from the graph of a function. Rearranged existing examples in increasing difficulty. Revised application example to be more student friendly. Added exercises that guide students step by step through the process of graphical differentiation. Added several interpretation questions throughout the exercises.

Chapter 4

- Revised solutions to many examples of finding the derivative to include more details of the process. Added discussion about the simplification of derivatives and presented alternate forms for writing the solution. Added guided questions to many exercises to give students the chance to interpret their results.
- Added examples and corresponding exercises throughout the chapter for determining the slope and equation of a tangent line.
- Revised chain rule examples and solutions to help students determine the composition of functions and to apply the chain rule. Added definition and examples of the general power rule, a specific case of the chain rule, which can be applied to many exercises.
- Added examples and corresponding exercises for finding the derivative of logarithms by first applying logarithm rules.

Chapter 5

- Added examples with figures to illustrate the following concepts:
  - A function can be increasing (or decreasing) on an interval even though the derivative is not positive (or negative) at every point in the interval;
  - A relative extrema does not necessarily occur at every critical number; and
  - A function does not necessarily have an inflection point at all values where $f''(x) = 0$.
- Revised example to introduce the concept of relative extrema, including a graph. Added more detail in solutions throughout the chapter.

- Revised the example on velocity and acceleration to illustrate how to determine when a vehicle is moving forward and backward, speeding up and slowing down, with the use of derivatives.
- Revised steps for curve sketching to guide students and included new guidelines in the corresponding examples. Added steps in the exercises to guide students in curve sketching.

Chapter 6

- Included an example and discussion to illustrate that an absolute extremum can occur at more than 1 point.
- Added an example where an absolute extremum is determined by applying the critical point theorem.
- Revised Tech Note on how to find extrema using a graphing calculator, providing details on the calculator commands.
- Revised discussion and derivation of formulas for the economic lot size. Added example that investigates how the elasticity changes as the price changes. Added an expanded interpretation of elasticity to examples and exercises.
- Added an example and corresponding exercises to illustrate how to find and simplify second derivatives implicitly.
- Revised error estimation example to include relative error and tolerance

Chapter 7

- Added details and guidance to the example solutions for indefinite and definite integrals, especially those containing exponential functions and x^{-1}.
- Revised section on integration by substitution. Added explicit steps to guide students in the process. Changed ordering of discussions to emphasize the proper selection of u, with corresponding examples.
- Added a discussion and example of a Lorenz curve and the Gini index of income inequality; added corresponding exercises. Revised the solutions to examples of determining the area between two curves, emphasizing the set-up of the definite integral.

Chapter 8

- Added more application exercises to Sections 8.1, 8.2, and 8.4.
- Expanded the exploration of convergent and divergent improper integrals.

Chapter 9

- Simplified the solution in an example of Lagrange multipliers.

Chapter 10

- Coverage of Euler's method now includes the percentage error.

Chapter 11

- Revised discussion of probabilities. Added explanations and steps in solutions throughout the chapter for clarity. Added interpretation of z-score.

Chapter 12

- Added application to introduce the concept of a sequence.
- Revised solutions to annuity examples to include both actuarial notations along with mathematical formulas. Revised review so that present formulas connect to formulas introduced earlier. Updated Tech Notes for annuity calculations.

- Added a Caution box for l'Hospital's rule, to guide students on the correct use of the rule, and added a relatable justification for the rule. Added two business applications for l'Hospital's rule.

Chapter 13

- Unified the triangular and circular points of view of the trigonometric functions.

Features of *Calculus with Applications*

Chapter Opener

Each chapter opens with a quick introduction that relates to an application presented in the chapter.

Apply It

An Apply It question, typically at the start of a section, motivates the math content of the section by posing a real-world question that is then answered within the examples or exercises.

For Review

For Review boxes are provided in the margin as appropriate, giving students just-in-time help with skills they should already know but may have forgotten. For Review comments sometimes include an explanation while others refer students to earlier parts of the textbook for a more thorough review.

FOR REVIEW

Recall from Section 1.1 the formula for the slope of a line through two points (x_1, y_1) and (x_2, y_2):

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

Find the slopes of the lines through the following points.

$(0.5, 30)$	and	$(1, 55)$
$(0.5, 30)$	and	$(1.5, 80)$
$(1, 55)$	and	$(2, 104)$

Compare your answers to the average speeds shown in the table. ■

Caution

Caution notes provide students with a quick "heads up" to common difficulties and errors.

CAUTION Simply because the expression in a limit is approaching 0/0, as in Examples 8 and 9, does *not* mean that the limit is 0 or that the limit does not exist. For such a limit, try to simplify the expression using the following principle: **To calculate the limit of $f(x)/g(x)$ as x approaches a, where $f(a) = g(a) = 0$, you should attempt to factor $x - a$ from both the numerator and the denominator.** ■

Your Turn Exercises

These exercises follow selected examples and provide students with an easy way to quickly stop and check their understanding. Answers are provided at the end of the section's exercises.

Tech Notes

Material on graphing calculators and Microsoft Excel is clearly labeled to make it easier for instructors to use this material (or not). The figures depicting calculator screens are taken from the TI-84 calculator, which features color and higher pixel counts.

TECH NOTE: **CALCULATORS**

Some graphing calculators have the ability to draw piecewise functions. On the TI-84 calculator, letting

$$Y_1 = (X + 1)(X < 1) + (X^2 - 3X + 4)(1 \le X)(X \le 3) + (5 - X)(X > 3)$$

produces the graph shown in Figure 8.

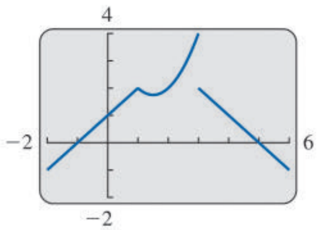


Figure 8

Exercise Sets

Concept Check and Practice and Explorations exercises are followed by Applications exercises, which are grouped by application such as "Business and Economics." Other types of exercises include:

- **Connections** exercises integrate topics presented in different sections or chapters and are indicated with .
- **Technology** exercises are labeled for graphing calculator and for spreadsheets.
- **Skills You'll Need** exercises are included before many exercise sheets to provide an opportunity for students to refresh key prerequisite skills needed.
- **Writing** exercises, labeled with , provide students with an opportunity to explain important mathematical ideas.

Chapter Summary and Review

- The end-of-chapter **Summary** provides students with a quick recap of the important ideas of the chapter followed by a list of key definitions, terms, and examples.
- Chapter **Review Exercises** include Concept Check exercises and an ample set of Practice and Exploration exercises. This provides students with a comprehensive set of exercises to prepare for chapter exams.

Extended Applications

- Extended Applications are provided at the end of each chapter as in-depth applied exercises to help stimulate student interest. These activities can be completed individually or as a group project. Additional Extended Applications for the text can be found in MyLab Math.

MyLab Math Resources for Success

MyLab Math is available to accompany Pearson's market-leading text options, including ***Calculus with Applications*, 12e** (access code required; pearson.com/mylab/math).

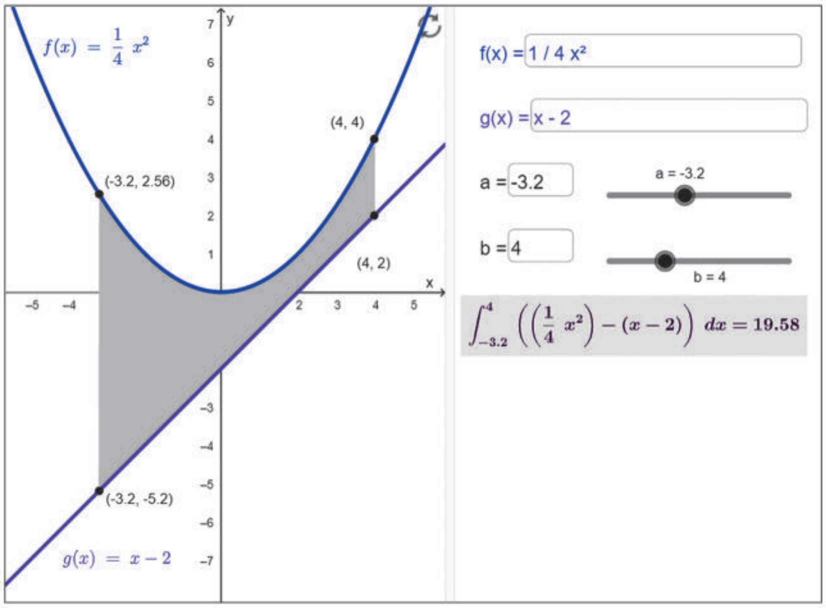
MyLab™ is the teaching and learning platform that empowers you to reach every student. MyLab Math combines trusted author content—including full eText and assessment with immediate feedback—with digital tools and a flexible platform to personalize the learning experience and improve results for each student.

Student Resources

Each student learns at a different pace. Personalized learning pinpoints the precise areas where each student needs practice, giving all students the support they need—when and where they need it—to be successful.

Integrated Review is now included within MyLab Math and can be used in corequisite courses or simply to help students who enter without a full understanding of prerequisite skills. Each student receives the help that they need—no more; no less. Integrated review provides videos on review topics, along with premade, assignable skills-check quizzes and personalized review homework assignments.

New! The **suite of interactive figures has been expanded** to support teaching and learning. These figures (created in GeoGebra) illustrate key concepts and can be manipulated by users. They have been designed to be used in lectures as well as by students independently.



Setup & Solve exercises require students to first describe how they will set up and approach the problem. This reinforces conceptual understanding of the process applied in approaching the problem, promotes long-term retention of the skill, and mirrors what students will be expected to do on a test.

New! Mindset videos and assignable, open-ended **exercises** foster a growth mindset in students. This material encourages them to maintain a positive attitude about learning, value their own ability to grow, and view mistakes as learning opportunities—so often a hurdle for math students.

New! Personal Inventory Assessments are a collection of online exercises designed to promote self-reflection and engagement in students. These 33 assessments include topics such as a Stress Management Assessment, Diagnosing Poor Performance and Enhancing Motivation, and Time Management Assessment.

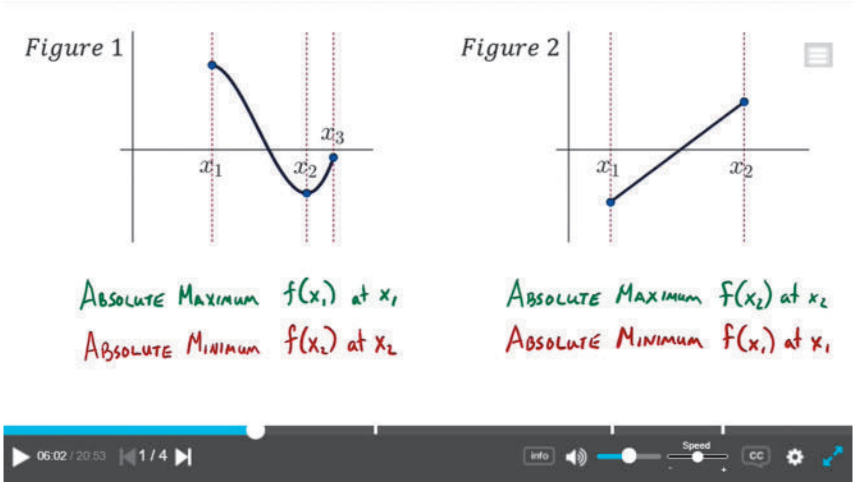
Student's Solutions Manual provides detailed solutions to all odd-numbered text exercises and sample chapter tests with answers. Available within MyLab Math.

Graphing Calculator and Excel Manuals are available within MyLab Math under Video & Resource Library > Learning Tools as well as at bit.ly/2HE5ws2.

Instructor Resources

Your course is unique. So whether you'd like to build your own assignments, teach multiple sections, or set prerequisites, MyLab gives you the flexibility to easily create your course to fit your needs.

New! This edition continues to expand the comprehensive auto-graded exercise options. Exercises were carefully reviewed, vetted, and improved using aggregated student usage and performance data over time.



Videos are available for instructor and student use. Videos are separated into Introduction, Example, and full Section Lectures.

Performance Analytics enable instructors to see and analyze student performance across multiple courses. Based on their current course progress, a student's performance is identified above, at, or below expectations through a variety of graphs and visualizations.

New! Now included with Performance Analytics, **Early Alerts** use predictive analytics to identify struggling students—even if their assignment scores are not a cause for concern. In both Performance Analytics and Early Alerts, instructors can email students individually or by group to provide feedback.

Enhanced Assignments are section-level assignments geared to maximize students' performance with just-in-time prerequisite review. They help keep skills fresh with spaced practice of key concepts and provide opportunities to work exercises without learning aids, so students can check their understanding.

Application labels within exercise sets (e.g. "Business/Econ") enable instructors to easily find types of applications appropriate to their students. Applications enable students to see the relevance of math, helping them become more effective problem-solvers.

Additional Conceptual Questions provide support for assessing concepts and vocabulary. Many of these questions are application oriented.

With **Learning Catalytics™**, you'll hear from every student when it matters most. You pose a variety of questions that help students recall ideas, apply concepts, and develop critical-thinking skills. Students respond using their own smartphones, tablets, or laptops.

New! Complete Instructor Answers includes answers to every textbook exercise. The Complete Instructor Answers is available in MyLab under Instructor Resources and downloadable from the Instructor Resource Center at Pearson.com. The Complete Instructor Answers is also included at the back of the print Instructor Edition.

Instructor's Solutions and Resource Manual provides complete solutions to all exercises, two versions of a pre-test and final exam, and teaching tips.

A **Guide to Video-Based Assignments** shows which exercises correspond to each video clip, making it easy to assess students after they watch an instructional video. This is perfect for flipped-classroom situations.

PowerPoint Presentations include lecture content and key graphics from the textbook. Accessible PowerPoint slides are also available and are built to align with WCAG 2.0 AA standards and Section 508 guidelines.

TestGen® (www.pearsoned.com/testgen) enables instructors to build, edit, print, and administer tests using a computerized bank of questions developed to cover the objectives of the text.

Pearson works continuously to ensure our products are as accessible as possible to all students. Currently we work toward achieving WCAG 2.0 AA standards and meeting Section 508 guidelines for our existing products. To learn more about Pearson's commitment to accessibility, go to pearson.com/us/accessibility.

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Raymond N. Greenwell
Nathan P. Ritchey

Prerequisite Skills Diagnostic Test

Below is a very brief test to help you recognize which, if any, prerequisite skills you may need to remediate in order to be successful in this course. After completing the test, check your answers in the back of the book. In addition to the answers, we have also provided the solutions to these problems in Appendix A. These solutions should help remind you how to solve the problems. For problems 5-26, the answers are followed by references to sections within Chapter R where you can find guidance on how to solve the problem and/or additional instruction. Addressing any weak prerequisite skills now will make a positive impact on your success as you progress through this course.

1. What percent of 50 is 10?

2. Simplify $\frac{13}{7} - \frac{2}{5}$.

3. Let x be the number of apples and y be the number of oranges. Write the following statement as an algebraic equation: "The total number of apples and oranges is 75."

4. Let s be the number of students and p be the number of professors. Write the following statement as an algebraic equation: "There are at least four times as many students as professors."

5. Solve for k: $7k + 8 = -4(3 - k)$.

6. Solve for x: $\frac{5}{8}x + \frac{1}{16}x = \frac{11}{16} + x$.

7. Write in interval notation: $-2 < x \leq 5$.

8. Using the variable x, write the following interval as an inequality: $(-\infty, -3]$.

9. Solve for y: $5(y - 2) + 1 \leq 7y + 8$.

10. Solve for p: $\frac{2}{3}(5p - 3) > \frac{3}{4}(2p + 1)$.

11. Carry out the operations and simplify: $(5y^2 - 6y - 4) - 2(3y^2 - 5y + 1)$.

12. Multiply out and simplify $(x^2 - 2x + 3)(x + 1)$.

13. Multiply out and simplify $(a - 2b)^2$.

14. Factor $3pq + 6p^2q + 9pq^2$.

15. Factor $3x^2 - x - 10$.

16. Perform the operation and simplify: $\frac{a^2 - 6a}{a^2 - 4} \cdot \frac{a - 2}{a}$.

17. Perform the operation and simplify: $\frac{x + 3}{x^2 - 1} + \frac{2}{x^2 + x}$.

18. Solve for x: $3x^2 + 4x = 1$.

19. Solve for z: $\dfrac{8z}{z + 3} \leq 2$.

20. Simplify $\dfrac{4^{-1}(x^2y^3)^2}{x^{-2}y^5}$.

21. Simplify $\dfrac{4^{1/4}(p^{2/3}q^{-1/3})^{-1}}{4^{-1/4}\, p^{4/3}q^{4/3}}$.

22. Simplify as a single term without negative exponents: $k^{-1} - m^{-1}$.

23. Factor $(x^2 + 1)^{-1/2}(x + 2) + 3(x^2 + 1)^{1/2}$.

24. Simplify $\sqrt[3]{64b^6}$.

25. Rationalize the denominator: $\dfrac{2}{4 - \sqrt{10}}$.

26. Simplify $\sqrt{y^2 - 10y + 25}$.

R

Algebra Reference

In this chapter, we will review the most important topics in algebra. Knowing algebra is a fundamental prerequisite to success in higher mathematics. This algebra reference is designed for self-study; study it all at once or refer to it when needed throughout the course. Since this is a review, answers to all exercises are given in the answer section at the back of the book.

R.1 Polynomials

An expression such as $9p^4$ is a **term**; the number 9 is the **coefficient**, p is the **variable**, and 4 is the **exponent**. The expression p^4 means $p \cdot p \cdot p \cdot p$, while p^2 means $p \cdot p$, and so on. Terms having the same variable and the same exponent, such as $9x^4$ and $-3x^4$, are **like terms**. Terms that do not have both the same variable and the same exponent, such as m^2 and m^4, are **unlike terms**.

A **polynomial** is a term or a finite sum of terms in which all variables have whole number exponents, and no variables appear in denominators. Examples of polynomials include

$$5x^4 + 2x^3 + 6x, \qquad 8m^3 + 9m^2n - 6mn^2 + 3n^3, \qquad 10p, \qquad \text{and} \qquad -9.$$

A **binomial** is a polynomial with exactly two terms, such as $2x + 1$ or $m + n$.

Order of Operations

Algebra is a language, and you must be familiar with its rules to correctly interpret algebraic statements. The following order of operations has been agreed upon through centuries of usage.

- Expressions in **parentheses** (or other grouping symbols) are calculated first, working from the inside out. The numerator and denominator of a fraction are treated as expressions in parentheses.
- **Exponents** (or **powers**) are performed next, going from left to right.
- **Multiplication** and **division** are performed next, going from left to right.
- **Addition** and **subtraction** are performed last, going from left to right.

For example, in the expression $[6(x + 1)^2 + 3x - 22]^2$, suppose x has the value of 2. We would evaluate this as follows:

$$\begin{aligned} [6(2 + 1)^2 + 3(2) - 22]^2 &= [6(3)^2 + 3(2) - 22]^2 && \text{Evaluate the expression in the innermost parentheses.} \\ &= [6(9) + 3(2) - 22]^2 && \text{Evaluate 3 raised to a power.} \\ &= (54 + 6 - 22)^2 && \text{Perform the multiplications.} \\ &= (38)^2 && \text{Perform the addition and subtraction from left to right.} \\ &= 1444 && \text{Evaluate the power.} \end{aligned}$$

In the expression $\dfrac{x^2 + 3x + 6}{x + 6}$, suppose x has the value of 2. We would evaluate this as follows:

$$\begin{aligned} \frac{2^2 + 3(2) + 6}{2 + 6} &= \frac{16}{8} && \text{Evaluate the numerator and the denominator.} \\ &= 2 && \text{Simplify the fraction.} \end{aligned}$$

Adding and Subtracting Polynomials

The following properties of real numbers are useful for performing operations on polynomials.

Properties of Real Numbers

For all real numbers a, b, and c:

1. $a + b = b + a$; $ab = ba$;	**Commutative properties**
2. $(a + b) + c = a + (b + c)$; $(ab)c = a(bc)$;	**Associative properties**
3. $a(b + c) = ab + ac$.	**Distributive property**

EXAMPLE 1 Properties of Real Numbers

(a) $2 + x = x + 2$ — **Commutative property of addition**

(b) $x \cdot 3 = 3x$ — **Commutative property of multiplication**

(c) $(7x)x = 7(x \cdot x) = 7x^2$ — **Associative property of multiplication**

(d) $3(x + 4) = 3x + 12$ — **Distributive property** ■

One use of the distributive property is to add or subtract polynomials. Only like terms may be added or subtracted. For example,

$$12y^4 + 6y^4 = (12 + 6)y^4 = 18y^4,$$

and

$$-2m^2 + 8m^2 = (-2 + 8)m^2 = 6m^2,$$

but the polynomial $8y^4 + 2y^5$ cannot be further simplified. To subtract polynomials, we use the facts that $-(a + b) = -a - b$ and $-(a - b) = -a + b$. In the next example, we show how to add and subtract polynomials.

EXAMPLE 2 Adding and Subtracting Polynomials

Add or subtract as indicated.

(a) $(8x^3 - 4x^2 + 6x) + (3x^3 + 5x^2 - 9x + 8)$

SOLUTION Combine like terms.

$$\begin{aligned}&(8x^3 - 4x^2 + 6x) + (3x^3 + 5x^2 - 9x + 8)\\ &= (8x^3 + 3x^3) + (-4x^2 + 5x^2) + (6x - 9x) + 8\\ &= 11x^3 + x^2 - 3x + 8\end{aligned}$$

(b) $2(-4x^4 + 6x^3 - 9x^2 - 12) + 3(-3x^3 + 8x^2 - 11x + 7)$

SOLUTION Multiply each polynomial by the factor in front of the polynomial, and then combine terms as before.

$$\begin{aligned}&2(-4x^4 + 6x^3 - 9x^2 - 12) + 3(-3x^3 + 8x^2 - 11x + 7)\\ &= -8x^4 + 12x^3 - 18x^2 - 24 - 9x^3 + 24x^2 - 33x + 21\\ &= -8x^4 + 3x^3 + 6x^2 - 33x - 3\end{aligned}$$

(c) $(2x^2 - 11x + 8) - (7x^2 - 6x + 2)$

SOLUTION Distributing the minus sign and combining like terms yields

$$\begin{aligned}&(2x^2 - 11x + 8) + (-7x^2 + 6x - 2)\\ &= -5x^2 - 5x + 6.\end{aligned}$$

Try YOUR TURN 1 ■

YOUR TURN 1 Perform the operation $3(x^2 - 4x - 5) - 4(3x^2 - 5x - 7)$.

Multiplying Polynomials

The distributive property is also used to multiply polynomials, along with the fact that $a^m \cdot a^n = a^{m+n}$. For example,

$$x \cdot x = x^1 \cdot x^1 = x^{1+1} = x^2 \qquad \text{and} \qquad x^2 \cdot x^5 = x^{2+5} = x^7.$$

EXAMPLE 3 Multiplying Polynomials

Multiply.

(a) $8x(6x - 4)$

SOLUTION Using the distributive property yields

$$\begin{aligned}8x(6x - 4) &= 8x(6x) - 8x(4)\\ &= 48x^2 - 32x.\end{aligned}$$

(b) $(2m - 5)(m + 4)$

SOLUTION Using the distributive property yields

$$\begin{aligned}(2m - 5)(m + 4) &= 2m(m + 4) - 5(m + 4) \\ &= 2m(m) + 2m(4) - 5(m) - 5(4) && \text{Use the distributive property again.} \\ &= 2m^2 + 8m - 5m - 20 \\ &= 2m^2 + 3m - 20. && \text{Combine like terms.}\end{aligned}$$

■

Notice in Example 3(b) that when we multiplied two binomials, the use of the distributive property twice led to four terms. From each binomial factor, we find the product of the first terms $(2m$ and $m)$, the product of the two outer terms $(2m$ and $4)$, the product of the two inner terms $(-5$ and $m)$, and the product of the last terms $(-5$ and $4)$. Multiplying any two binomials can similarly be done by the FOIL method (First, Outer, Inner, Last), which is just a memory aid for using the distributive property twice.

EXAMPLE 4 Multiplying Binomials

Find $(2m - 5)(m + 4)$ using the FOIL method.

SOLUTION

$$\begin{aligned}(2m - 5)(m + 4) &= \overset{\text{F}}{(2m)(m)} + \overset{\text{O}}{(2m)(4)} + \overset{\text{I}}{(-5)(m)} + \overset{\text{L}}{(-5)(4)} \\ &= 2m^2 + 8m - 5m - 20 \\ &= 2m^2 + 3m - 20\end{aligned}$$

Try YOUR TURN 2 ■

YOUR TURN 2 Find $(2x + 7)(3x - 1)$ using the FOIL method.

When multiplying two polynomials that are *not* binomials, the FOIL method doesn't work, so we use the distributive property.

EXAMPLE 5 Multiplying Polynomials

(a) $(3p - 2)(p^2 + 5p - 1)$

SOLUTION Using the distributive property yields

$$\begin{aligned}&(3p - 2)(p^2 + 5p - 1) \\ &= 3p(p^2 + 5p - 1) - 2(p^2 + 5p - 1) \\ &= 3p(p^2) + 3p(5p) + 3p(-1) - 2(p^2) - 2(5p) - 2(-1) \\ &= 3p^3 + 15p^2 - 3p - 2p^2 - 10p + 2 \\ &= 3p^3 + 13p^2 - 13p + 2.\end{aligned}$$

(b) $(x + 2)(x + 3)(x - 4)$

SOLUTION Multiplying the first two polynomials and then multiplying their product by the third polynomial yields

$$\begin{aligned}&(x + 2)(x + 3)(x - 4) \\ &= [(x + 2)(x + 3)](x - 4) \\ &= (x^2 + 2x + 3x + 6)(x - 4) \\ &= (x^2 + 5x + 6)(x - 4) \\ &= x^3 - 4x^2 + 5x^2 - 20x + 6x - 24 \\ &= x^3 + x^2 - 14x - 24.\end{aligned}$$

Try YOUR TURN 3 ■

YOUR TURN 3 Perform the operation $(3y + 2)(4y^2 - 2y - 5)$.

EXAMPLE 6 Multiplying Polynomials

Find $(2k - 5m)^3$.

SOLUTION Write $(2k - 5m)^3$ as $(2k - 5m)(2k - 5m)(2k - 5m)$. Then multiply the first two factors using FOIL.

$$(2k - 5m)(2k - 5m) = 4k^2 - 10km - 10km + 25m^2$$
$$= 4k^2 - 20km + 25m^2$$

Now multiply this last result by $(2k - 5m)$ using the distributive property, as in Example 5(a).

$$(4k^2 - 20km + 25m^2)(2k - 5m)$$
$$= 4k^2(2k - 5m) - 20km(2k - 5m) + 25m^2(2k - 5m)$$
$$= 8k^3 - 20k^2m - 40k^2m + 100km^2 + 50km^2 - 125m^3$$
$$= 8k^3 - 60k^2m + 150km^2 - 125m^3 \quad \text{Combine like terms.}$$

Try **YOUR TURN 4** ■

YOUR TURN 4 Find $(3x + 2y)^3$.

Notice in the first part of Example 6, when we multiplied $(2k - 5m)$ by itself, that the product of the square of a binomial is the square of the first term, $(2k)^2$, plus twice the product of the two terms, $(2)(2k)(-5m)$, plus the square of the last term, $(-5m)^2$.

CAUTION Avoid the common error of writing $(x + y)^2 = x^2 + y^2$. As the first step of Example 6 shows, the square of a binomial has three terms, so

$$(x + y)^2 = x^2 + 2xy + y^2.$$

Furthermore, higher powers of a binomial also result in more than two terms. For example, verify by multiplication that

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

Remember, for any value of $n \neq 1$,

$$(x + y)^n \neq x^n + y^n. \ ■$$

R.1 Exercises

Perform the indicated operations.

1. $(2x^2 - 6x + 11) + (-3x^2 + 7x - 2)$
2. $(-4y^2 - 3y + 8) - (2y^2 - 6y - 2)$
3. $\left(\frac{1}{2}z^2 + \frac{1}{3}z + \frac{2}{3}\right) + \left(z^2 + \frac{1}{2}z + \frac{4}{3}\right)$
4. $\left(\frac{2}{3}t^2 + \frac{3}{2}t + \frac{5}{6}\right) - \left(\frac{1}{6}t^2 + t + \frac{1}{6}\right)$
5. $-6(2q^2 + 4q - 3) + 4(-q^2 + 7q - 3)$
6. $2(3r^2 + 4r + 2) - 3(-r^2 + 4r - 5)$
7. $(0.613x^2 - 4.215x + 0.892) - 0.47(2x^2 - 3x + 5)$
8. $0.5(5r^2 + 3.2r - 6) - (1.7r^2 - 2r - 1.5)$
9. $-9m(2m^2 + 3m - 1)$
10. $6x(-2x^3 + 5x + 6)$
11. $(3t - 2y)(3t + 5y)$
12. $(9k + q)(2k - q)$
13. $(2 - 3x)(2 + 3x)$
14. $(6m + 5)(6m - 5)$
15. $\left(\frac{2}{5}y + \frac{1}{8}z\right)\left(\frac{3}{5}y + \frac{1}{2}z\right)$
16. $\left(\frac{3}{4}r - \frac{2}{3}s\right)\left(\frac{5}{4}r + \frac{1}{3}s\right)$
17. $(3p - 1)(9p^2 + 3p + 1)$
18. $(3p + 2)(5p^2 + p - 4)$
19. $(2m + 1)(4m^2 - 2m + 1)$

20. $(k + 2)(12k^3 - 3k^2 + k + 1)$

21. $(x + y + z)(3x - 2y - z)$

22. $(r + 2s - 3t)(2r - 2s + t)$

23. $(x + 1)(x + 2)(x + 3)$

24. $(x - 1)(x + 2)(x - 3)$

25. $(x + 2)^2$

26. $(2a - 4b)^2$

27. $(x - 2y)^3$

28. $(3x + y)^3$

YOUR TURN Answers

1. $-9x^2 + 8x + 13$ **2.** $6x^2 + 19x - 7$

3. $12y^3 + 2y^2 - 19y - 10$ **4.** $27x^3 + 54x^2y + 36xy^2 + 8y^3$

R.2 Factoring

Multiplication of polynomials relies on the distributive property. The reverse process, where a polynomial is written as a product of other polynomials, is called **factoring**. For example, one way to factor the number 18 is to write it as the product $9 \cdot 2$; both 9 and 2 are **factors** of 18. Usually, only integers are used as factors of integers. The number 18 can also be written with three integer factors as $2 \cdot 3 \cdot 3$.

The Greatest Common Factor

To factor the algebraic expression $15m + 45$, first note that both $15m$ and 45 are divisible by 15; $15m = 15 \cdot m$ and $45 = 15 \cdot 3$. By the distributive property,

$$15m + 45 = \mathbf{15} \cdot m + \mathbf{15} \cdot 3 = \mathbf{15}(m + 3).$$

Both 15 and $m + 3$ are factors of $15m + 45$. Since 15 divides into both terms of $15m + 45$ (and is the largest number that will do so), 15 is the **greatest common factor** for the polynomial $15m + 45$. The process of writing $15m + 45$ as $15(m + 3)$ is often called **factoring out** the greatest common factor. When finding the greatest common divisor, look for the largest number and the largest power of any variable that divides into each term.

EXAMPLE 1 Factoring

Factor out the greatest common factor.

(a) $12p - 18q$

SOLUTION Both $12p$ and $18q$ are divisible by 6. Therefore,

$$12p - 18q = \mathbf{6} \cdot 2p - \mathbf{6} \cdot 3q = \mathbf{6}(2p - 3q).$$

(b) $6x^3 - 9x^2 + 15x$

SOLUTION Each of these terms is divisible by x as well as by 3.

$$6x^3 - 9x^2 + 15x = \mathbf{3x} \cdot (2x^2) - \mathbf{3x} \cdot (3x) + \mathbf{3x} \cdot 5$$
$$= \mathbf{3x}(2x^2 - 3x + 5)$$

Try YOUR TURN 1

YOUR TURN 1 Factor $4z^4 + 4z^3 + 18z^2$.

One can always check factorization by finding the product of the factors and comparing it to the original expression.

CAUTION When factoring out the greatest common factor in an expression like $2x^2 + x$, be careful to remember the 1 in the second term.

$$2x^2 + x = 2x^2 + 1x = x(2x + 1), \quad \text{not } x(2x).$$

Factoring Trinomials

A polynomial that has no greatest common factor (other than 1) may still be factorable. For example, the polynomial $x^2 + 5x + 6$ can be factored as $(x + 2)(x + 3)$. To see that this is correct, find the product $(x + 2)(x + 3)$; you should get $x^2 + 5x + 6$. A polynomial such as this with three terms is called a **trinomial**. To factor a trinomial of the form $x^2 + bx + c$ (if possible), where the coefficient of x^2 is 1, use FOIL backwards. We look for two factors of c whose sum is b. When c is positive, its factors must have the same sign. Since b is the sum of these two factors, the factors will have the same sign as b. When c is negative, its factors have opposite signs. Again, since b is the sum of these two factors, the factor with the greater absolute value will have the same sign as b.

EXAMPLE 2 Factoring a Trinomial

Factor $y^2 + 8y + 15$.

SOLUTION Since the coefficient of y^2 is 1, factor by finding two numbers whose *product* is 15 and whose *sum* is 8. Because the constant and the middle term are positive, the numbers must both be positive. Begin by listing all pairs of positive integers having a product of 15. As you do this, also form the sum of each pair of numbers.

Products	Sums
$15 \cdot 1 = 15$	$15 + 1 = 16$
$\mathbf{5 \cdot 3 = 15}$	$\mathbf{5 + 3 = 8}$

The numbers 5 and 3 have a product of 15 and a sum of 8. Thus, $y^2 + 8y + 15$ factors as

$$y^2 + 8y + 15 = (y + 5)(y + 3).$$

The answer can also be written as $(y + 3)(y + 5)$. Try **YOUR TURN 2**

YOUR TURN 2 Factor $x^2 - 3x - 10$.

If the coefficient of the squared term is *not* 1, work as shown in Example 3.

EXAMPLE 3 Factoring a Trinomial

Factor $4x^2 + 8xy - 5y^2$.

SOLUTION The possible factors of $4x^2$ are $4x$ and x or $2x$ and $2x$; the possible factors of $-5y^2$ are $-5y$ and y or $5y$ and $-y$. Try various combinations of these factors until one works (if, indeed, any work). For example, try the product $(x + 5y)(4x - y)$.

$$\begin{aligned}(x + 5y)(4x - y) &= 4x^2 - xy + 20xy - 5y^2 \\ &= 4x^2 + 19xy - 5y^2\end{aligned}$$

This product is not correct, so try another combination.

$$\begin{aligned}(2x - y)(2x + 5y) &= 4x^2 + 10xy - 2xy - 5y^2 \\ &= 4x^2 + 8xy - 5y^2\end{aligned}$$

Since this combination gives the correct polynomial,

$$4x^2 + 8xy - 5y^2 = (2x - y)(2x + 5y).$$

Try **YOUR TURN 3**

YOUR TURN 3 Factor $6a^2 + 5ab - 4b^2$.

Factoring by Grouping For a more organized approach to factoring $ax^2 + bx + c$ or $ax^2 + bxy + cy^2$, where a, b, and c are integers, first list the factors of the product ac, similar to the procedure in Example 2. Next, list the sums of those factors, stopping when you find a sum equal to b. In Example 3, $ac = 4(-5) = -20$, so we proceed as follows.

Products	Sums	
$1(-20) = -20$	$1 + (-20) = -19$	
$2(-10) = -20$	$2 + (-10) = -8$	**Almost; just change the signs.**
$\mathbf{(-2)10 = -20}$	$\mathbf{(-2) + 10 = 8}$	**Stop, because $b = 8$.**

Using the fact that $8 = (-2) + 10$, split the $8xy$ terms into $-2xy + 10xy$ and group the first two terms, as well as the last two. Factor each pair of terms, and then factor the entire trinomial as follows.

$$\begin{aligned} 4x^2 + 8xy - 5y^2 &= 4x^2 - 2xy + 10xy - 5y^2 \\ &= (4x^2 - 2xy) + (10xy - 5y^2) \\ &= 2x(2x - y) + 5y(2x - y) \\ &= (2x + 5y)(2x - y) \end{aligned}$$

Special Factorizations

Four special factorizations occur so often that they are listed here for future reference.

Special Factorizations

$x^2 - y^2 = (x + y)(x - y)$	**Difference of two squares**
$x^2 + 2xy + y^2 = (x + y)^2$	**Perfect square**
$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$	**Difference of two cubes**
$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$	**Sum of two cubes**

A polynomial that cannot be factored is called a **prime polynomial**.

EXAMPLE 4 Factoring Polynomials

Factor each polynomial, if possible.

(a) $64p^2 - 49q^2 = (8p)^2 - (7q)^2 = (8p + 7q)(8p - 7q)$ — Difference of two squares

(b) $x^2 + 36$ is a prime polynomial.

(c) $x^2 + 12x + 36 = x^2 + 2(x)(6) + 6^2 = (x + 6)^2$ — Perfect square

(d) $9y^2 - 24yz + 16z^2 = (3y)^2 + 2(3y)(-4z) + (-4z)^2$ — Perfect square

$= [3y + (-4z)]^2 = (3y - 4z)^2$

(e) $y^3 - 8 = y^3 - 2^3 = (y - 2)(y^2 + 2y + 4)$ — Difference of two cubes

(f) $m^3 + 125 = m^3 + 5^3 = (m + 5)(m^2 - 5m + 25)$ — Sum of two cubes

(g) $8k^3 - 27z^3 = (2k)^3 - (3z)^3 = (2k - 3z)(4k^2 + 6kz + 9z^2)$ — Difference of two cubes

(h) $p^4 - 1 = (p^2 + 1)(p^2 - 1) = (p^2 + 1)(p + 1)(p - 1)$ — Difference of two squares ■

Note that $x^2 + y^2$ can not be factored, since there are no two binomials that can be multiplied together to get $x^2 + y^2$.

CAUTION In factoring, always look for a common factor first. Since $36x^2 - 4y^2$ has a common factor of 4,

$$36x^2 - 4y^2 = 4(9x^2 - y^2) = 4(3x + y)(3x - y).$$

It would be incomplete to factor it as

$$36x^2 - 4y^2 = (6x + 2y)(6x - 2y),$$

since each factor can be factored still further. To *factor* means to factor completely, so that each polynomial factor is prime. ■

R.2 Exercises

Factor each polynomial. If a polynomial cannot be factored, write *prime*. Factor out the greatest common factor as necessary.

1. $7a^3 + 14a^2$

2. $3y^3 + 24y^2 + 9y$

3. $13p^4q^2 - 39p^3q + 26p^2q^2$

4. $60m^4 - 120m^3n + 50m^2n^2$

5. $m^2 - 5m - 14$

6. $x^2 + 4x - 5$

7. $z^2 + 9z + 20$

8. $b^2 - 8b + 7$

9. $a^2 - 6ab + 5b^2$

10. $s^2 + 2st - 35t^2$

11. $y^2 - 4yz - 21z^2$

12. $3x^2 + 4x - 7$

13. $3a^2 + 10a + 7$

14. $15y^2 + y - 2$

15. $21m^2 + 13mn + 2n^2$

16. $6a^2 - 48a - 120$

17. $3m^3 + 12m^2 + 9m$

18. $4a^2 + 10a + 6$

19. $24a^4 + 10a^3b - 4a^2b^2$

20. $24x^4 + 36x^3y - 60x^2y^2$

21. $x^2 - 64$

22. $9m^2 - 25$

23. $10x^2 - 160$

24. $9x^2 + 64$

25. $z^2 + 14zy + 49y^2$

26. $s^2 - 10st + 25t^2$

27. $9p^2 - 24p + 16$

28. $a^3 - 216$

29. $27r^3 - 64s^3$

30. $3m^3 + 375$

31. $x^4 - y^4$

32. $16a^4 - 81b^4$

YOUR TURN Answers

1. $2z^2(2z^2 + 2z + 9)$

2. $(x + 2)(x - 5)$

3. $(2a - b)(3a + 4b)$

R.3 Rational Expressions

Many algebraic fractions are **rational expressions**, which are quotients of polynomials with nonzero denominators. Examples include

$$\frac{8}{x-1}, \quad \frac{3x^2+4x}{5x-6}, \quad \text{and} \quad \frac{2y+1}{y^2}.$$

Next, we summarize properties for working with rational expressions.

Properties of Rational Expressions

For all mathematical expressions P, Q, R, and S, with $Q \neq 0$ and $S \neq 0$:

$$\frac{P}{Q} = \frac{PS}{QS}$$ **Fundamental property**

$$\frac{P}{Q} + \frac{R}{Q} = \frac{P+R}{Q}$$ **Addition**

$$\frac{P}{Q} - \frac{R}{Q} = \frac{P-R}{Q}$$ **Subtraction**

$$\frac{P}{Q} \cdot \frac{R}{S} = \frac{PR}{QS}$$ **Multiplication**

$$\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \cdot \frac{S}{R} \quad (R \neq 0)$$ **Division**

When writing a rational expression in lowest terms, we may need to use the fact that $\frac{a^m}{a^n} = a^{m-n}$. For example,

$$\frac{x^4}{3x} = \frac{1x^4}{3x} = \frac{1}{3} \cdot \frac{x^4}{x} = \frac{1}{3} \cdot x^{4-1} = \frac{1}{3}x^3 = \frac{x^3}{3}.$$

EXAMPLE 1 Reducing Rational Expressions

Write each rational expression in lowest terms, that is, reduce the expression as much as possible.

(a) $\dfrac{8x + 16}{4} = \dfrac{8(x + 2)}{4} = \dfrac{4 \cdot 2(x + 2)}{4} = 2(x + 2)$

Factor both the numerator and denominator in order to identify any common factors, which have a quotient of 1. The answer could also be written as $2x + 4$.

(b) $\dfrac{k^2 + 7k + 12}{k^2 + 2k - 3} = \dfrac{(k + 4)(k + 3)}{(k - 1)(k + 3)} = \dfrac{k + 4}{k - 1}$

The answer cannot be further reduced. Try YOUR TURN 1

YOUR TURN 1 Write in lowest terms $\dfrac{z^2 + 5z + 6}{2z^2 + 7z + 3}$.

CAUTION One of the most common errors in algebra involves incorrect use of the fundamental property of rational expressions. Only common *factors* may be divided or "canceled." It is essential to factor rational expressions before writing them in lowest terms. In Example 1(b), for instance, it is not correct to "cancel" k^2 (or cancel k, or divide 12 by -3) because the additions and subtraction must be performed first. Here they cannot be performed, so it is not possible to divide. After factoring, however, the fundamental property can be used to write the expression in lowest terms.

EXAMPLE 2 Combining Rational Expressions

Perform each operation.

(a) $\dfrac{3y + 9}{6} \cdot \dfrac{18}{5y + 15}$

SOLUTION Factor where possible, then multiply numerators and denominators and reduce to lowest terms.

$$\frac{3y + 9}{6} \cdot \frac{18}{5y + 15} = \frac{3(y + 3)}{6} \cdot \frac{18}{5(y + 3)} \qquad \textbf{Factor.}$$

$$= \frac{3 \cdot 18(y + 3)}{6 \cdot 5(y + 3)} \qquad \textbf{Multiply.}$$

$$= \frac{3 \cdot \cancel{6} \cdot 3\cancel{(y + 3)}}{\cancel{6} \cdot 5\cancel{(y + 3)}} = \frac{3 \cdot 3}{5} = \frac{9}{5} \qquad \textbf{Reduce to lowest terms.}$$

(b) $\dfrac{m^2 + 5m + 6}{m + 3} \cdot \dfrac{m}{m^2 + 3m + 2}$

SOLUTION Factor where possible.

$$\frac{(m + 2)(m + 3)}{m + 3} \cdot \frac{m}{(m + 2)(m + 1)} \qquad \textbf{Factor.}$$

$$= \frac{m\cancel{(m + 2)}\cancel{(m + 3)}}{\cancel{(m + 3)}\cancel{(m + 2)}(m + 1)} = \frac{m}{m + 1} \qquad \textbf{Reduce to lowest terms.}$$

(c) $\dfrac{9p - 36}{12} \div \dfrac{5(p - 4)}{18}$

SOLUTION Use the division property of rational expressions.

$$\frac{9p - 36}{12} \div \frac{5(p - 4)}{18} = \frac{9p - 36}{12} \cdot \frac{18}{5(p - 4)} \qquad \textbf{Invert and multiply.}$$

$$= \frac{9\cancel{(p - 4)}}{\cancel{6} \cdot 2} \cdot \frac{\cancel{6} \cdot 3}{5\cancel{(p - 4)}} = \frac{27}{10} \qquad \textbf{Factor and reduce to lowest terms.}$$

(d) $\dfrac{4}{5k} - \dfrac{11}{5k}$

SOLUTION As shown in the list of properties, to subtract two rational expressions that have the same denominators, subtract the numerators while keeping the same denominator.

$$\frac{4}{5k} - \frac{11}{5k} = \frac{4 - 11}{5k} = -\frac{7}{5k}$$

(e) $\dfrac{7}{p} + \dfrac{9}{2p} + \dfrac{1}{3p}$

SOLUTION These three fractions cannot be added until their denominators are the same. A **common denominator** into which p, $2p$, and $3p$ all divide is $6p$. Note that $12p$ is also a common denominator, but $6p$ is the **least common denominator**. Use the fundamental property to rewrite each rational expression with a denominator of $6p$.

$$\frac{7}{p} + \frac{9}{2p} + \frac{1}{3p} = \frac{6 \cdot 7}{6 \cdot p} + \frac{3 \cdot 9}{3 \cdot 2p} + \frac{2 \cdot 1}{2 \cdot 3p}$$ **Rewrite with common denominator $6p$.**

$$= \frac{42}{6p} + \frac{27}{6p} + \frac{2}{6p}$$

$$= \frac{42 + 27 + 2}{6p}$$

$$= \frac{71}{6p}$$

(f) $\dfrac{x + 1}{x^2 + 5x + 6} - \dfrac{5x - 1}{x^2 - x - 12}$

SOLUTION To find the least common denominator, we first factor each denominator. Then we change each fraction so they all have the same denominator, being careful to multiply only by quotients that equal 1.

$$\frac{x + 1}{x^2 + 5x + 6} - \frac{5x - 1}{x^2 - x - 12}$$

$$= \frac{x + 1}{(x + 2)(x + 3)} - \frac{5x - 1}{(x + 3)(x - 4)}$$ **Factor denominators.**

$$= \frac{x + 1}{(x + 2)(x + 3)} \cdot \frac{(x - 4)}{(x - 4)} - \frac{5x - 1}{(x + 3)(x - 4)} \cdot \frac{(x + 2)}{(x + 2)}$$ **Rewrite with common denominators.**

$$= \frac{(x^2 - 3x - 4) - (5x^2 + 9x - 2)}{(x + 2)(x + 3)(x - 4)}$$ **Multiply numerators.**

$$= \frac{-4x^2 - 12x - 2}{(x + 2)(x + 3)(x - 4)}$$ **Subtract.**

$$= \frac{-2(2x^2 + 6x + 1)}{(x + 2)(x + 3)(x - 4)}$$ **Factor numerator.**

Because the numerator cannot be factored further, we leave our answer in this form. We could also multiply out the denominator, but factored form is usually more useful.

Try **YOUR TURN 2**

YOUR TURN 2 Perform each of the following operations.

(a) $\dfrac{z^2 + 5z + 6}{2z^2 - 5z - 3} \cdot \dfrac{2z^2 - z - 1}{z^2 + 2z - 3}$.

(b) $\dfrac{a - 3}{a^2 + 3a + 2} + \dfrac{5a}{a^2 - 4}$.

Notice in Example 2(f) that we get a common denominator by multiplying the numerator and denominator of each term by whatever factor is missing in the denominator, but is present in the denominator of another term. In this example, the factor $(x - 4)$ is in the denominator of the second term but not in the first, and the factor $(x + 2)$ is present in the denominator of the first term but not in the second.

R.3 Exercises

Write each rational expression in lowest terms.

1. $\frac{5v^2}{35v}$

2. $\frac{25p^3}{10p^2}$

3. $\frac{8k + 16}{9k + 18}$

4. $\frac{2(t - 15)}{(t - 15)(t + 2)}$

5. $\frac{4x^3 - 8x^2}{4x^2}$

6. $\frac{36y^2 + 72y}{9y}$

7. $\frac{m^2 - 4m + 4}{m^2 + m - 6}$

8. $\frac{r^2 - r - 6}{r^2 + r - 12}$

9. $\frac{3x^2 + 3x - 6}{x^2 - 4}$

10. $\frac{z^2 - 5z + 6}{z^2 - 4}$

11. $\frac{m^4 - 16}{4m^2 - 16}$

12. $\frac{6y^2 + 11y + 4}{3y^2 + 7y + 4}$

Perform the indicated operations.

13. $\frac{9k^2}{25} \cdot \frac{5}{3k}$

14. $\frac{15p^3}{9p^2} \div \frac{6p}{10p^2}$

15. $\frac{3a + 3b}{4c} \cdot \frac{12}{5(a + b)}$

16. $\frac{a - 3}{16} \div \frac{a - 3}{32}$

17. $\frac{2k - 16}{6} \div \frac{4k - 32}{3}$

18. $\frac{9y - 18}{6y + 12} \cdot \frac{3y + 6}{15y - 30}$

19. $\frac{4a + 12}{2a - 10} \div \frac{a^2 - 9}{a^2 - a - 20}$

20. $\frac{6r - 18}{9r^2 + 6r - 24} \cdot \frac{12r - 16}{4r - 12}$

21. $\frac{k^2 + 4k - 12}{k^2 + 10k + 24} \cdot \frac{k^2 + k - 12}{k^2 - 9}$

22. $\frac{m^2 + 3m + 2}{m^2 + 5m + 4} \div \frac{m^2 + 5m + 6}{m^2 + 10m + 24}$

23. $\frac{2m^2 - 5m - 12}{m^2 - 10m + 24} \div \frac{4m^2 - 9}{m^2 - 9m + 18}$

24. $\frac{4n^2 + 4n - 3}{6n^2 - n - 15} \cdot \frac{8n^2 + 32n + 30}{4n^2 + 16n + 15}$

25. $\frac{a + 1}{2} - \frac{a - 1}{2}$

26. $\frac{3}{p} + \frac{1}{2}$

27. $\frac{6}{5y} - \frac{3}{2}$

28. $\frac{1}{6m} + \frac{2}{5m} + \frac{4}{m}$

29. $\frac{1}{m - 1} + \frac{2}{m}$

30. $\frac{5}{2r + 3} - \frac{2}{r}$

31. $\frac{8}{3(a - 1)} + \frac{2}{a - 1}$

32. $\frac{2}{5(k - 2)} + \frac{3}{4(k - 2)}$

33. $\frac{4}{x^2 + 4x + 3} + \frac{3}{x^2 - x - 2}$

34. $\frac{y}{y^2 + 2y - 3} - \frac{1}{y^2 + 4y + 3}$

35. $\frac{3k}{2k^2 + 3k - 2} - \frac{2k}{2k^2 - 7k + 3}$

36. $\frac{4m}{3m^2 + 7m - 6} - \frac{m}{3m^2 - 14m + 8}$

37. $\frac{2}{a + 2} + \frac{1}{a} + \frac{a - 1}{a^2 + 2a}$

38. $\frac{5x + 2}{x^2 - 1} + \frac{3}{x^2 + x} - \frac{1}{x^2 - x}$

YOUR TURN Answers

1. $(z + 2)/(2z + 1)$
2. **(a)** $(z + 2)/(z - 3)$
   (b) $6(a^2 + 1)/[(a - 2)(a + 2)(a + 1)]$

R.4 Equations

Linear Equations

Equations that can be written in the form $ax + b = 0$, where a and b are real numbers, with $a \neq 0$, are **linear equations**. Examples of linear equations include $5y + 9 = 16$, $8x = 4$, and $-3p + 5 = -8$. Equations that are *not* linear include absolute value equations such as $|x| = 4$. The following properties are used to solve linear equations.

Properties of Equality

For all real numbers a, b, and c:

1. **If $a = b$, then $a + c = b + c$.** (The same number may be added to both sides of an equation.) — **Addition property of equality**

2. **If $a = b$, then $ac = bc$.** (Both sides of an equation may be multiplied by the same number.) — **Multiplication property of equality**

EXAMPLE 1 Solving Linear Equations

Solve the following equations.

(a) $x - 2 = 3$

SOLUTION The goal is to isolate the variable. Using the addition property of equality yields

$$x - 2 + 2 = 3 + 2, \qquad \text{or} \qquad x = 5.$$

(b) $\frac{x}{2} = 3$

SOLUTION Using the multiplication property of equality yields

$$2 \cdot \frac{x}{2} = 2 \cdot 3, \qquad \text{or} \qquad x = 6.$$

■

The following example shows how these properties are used to solve linear equations. The goal is to isolate the variable. The solutions should always be checked by substitution into the original equation.

EXAMPLE 2 Solving a Linear Equation

Solve $2x - 5 + 8 = 3x + 2(2 - 3x)$.

SOLUTION

$2x - 5 + 8 = 3x + 4 - 6x$	Distributive property
$2x + 3 = -3x + 4$	Combine like terms.
$5x + 3 = 4$	Add $3x$ to both sides.
$5x = 1$	Add -3 to both sides.
$x = \frac{1}{5}$	Multiply both sides by $\frac{1}{5}$.

YOUR TURN 1 Solve $3x - 7 = 4(5x + 2) - 7x$.

Check by substituting into the original equation. The left side becomes $2(1/5) - 5 + 8$ and the right side becomes $3(1/5) + 2[2 - 3(1/5)]$. Verify that both of these expressions simplify to $17/5$.

Try YOUR TURN 1 ■

Quadratic Equations

An equation with 2 as the greatest exponent of the variable is a *quadratic equation*. A **quadratic equation** has the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$. A quadratic equation written in the form $ax^2 + bx + c = 0$ is said to be in **standard form**.

The simplest way to solve a quadratic equation, but one that is not always applicable, is by factoring. This method depends on the **zero-factor property**.

Zero-Factor Property

If a and b are real numbers, with $ab = 0$, then either

$$a = 0 \text{ or } b = 0 \quad (\text{or both}).$$

EXAMPLE 3 Solving a Quadratic Equation

Solve $6r^2 + 7r = 3$.

SOLUTION First write the equation in standard form.

$$6r^2 + 7r - 3 = 0$$

Now factor $6r^2 + 7r - 3$ to get

$$(3r - 1)(2r + 3) = 0.$$

By the zero-factor property, the product $(3r - 1)(2r + 3)$ can equal 0 if and only if

$$3r - 1 = 0 \quad \text{or} \quad 2r + 3 = 0.$$

Solve each of these equations separately to find that the solutions are 1/3 and −3/2. Check these solutions by substituting them into the original equation. Try **YOUR TURN 2**

YOUR TURN 2 Solve $2m^2 + 7m = 15$.

CAUTION Remember, the zero-factor property requires that the product of two (or more) factors be equal to *zero*, not some other quantity. It would be incorrect to use the zero-factor property with an equation in the form $(x + 3)(x - 1) = 4$, for example.

If a quadratic equation cannot be solved easily by factoring, use the *quadratic formula*. (The derivation of the quadratic formula is given in most algebra books.)

Quadratic Formula

The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

EXAMPLE 4 Quadratic Formula

Solve $x^2 - 4x - 5 = 0$ by the quadratic formula.

SOLUTION The equation is already in standard form (it has 0 alone on one side of the equal sign), so the values of a, b, and c from the quadratic formula are easily identified. The coefficient of the squared term gives the value of a; here, $a = 1$. Also, $b = -4$ and $c = -5$, where b is the coefficient of the linear term and c is the constant coefficient. (Be careful to use the correct signs.) Substitute these values into the quadratic formula.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-5)}}{2(1)} \qquad a = 1, b = -4, c = -5$$

$$x = \frac{4 \pm \sqrt{16 + 20}}{2} \qquad (-4)^2 = (-4)(-4) = 16$$

$$x = \frac{4 \pm 6}{2} \qquad \sqrt{16 + 20} = \sqrt{36} = 6$$

The ± sign represents the two solutions of the equation. To find both of the solutions, first use + and then use −.

$$x = \frac{4 + 6}{2} = \frac{10}{2} = 5 \quad \text{or} \quad x = \frac{4 - 6}{2} = \frac{-2}{2} = -1$$

The two solutions are 5 and −1.

CAUTION Notice in the quadratic formula that the square root is added to or subtracted from the value of $-b$ *before* dividing by $2a$.

EXAMPLE 5 Quadratic Formula

Solve $x^2 + 1 = 4x$.

SOLUTION First, add $-4x$ on both sides of the equal sign in order to get the equation in standard form.

$$x^2 - 4x + 1 = 0$$

Now identify the values of a, b, and c. Here $a = 1$, $b = -4$, and $c = 1$. Substitute these numbers into the quadratic formula.

$$\begin{aligned} x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)} \qquad a = 1, b = -4, c = 1 \\ &= \frac{4 \pm \sqrt{16 - 4}}{2} \\ &= \frac{4 \pm \sqrt{12}}{2} \end{aligned}$$

Simplify the solutions by writing $\sqrt{12}$ as $\sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}$. Substituting $2\sqrt{3}$ for $\sqrt{12}$ gives

$$\begin{aligned} x &= \frac{4 \pm 2\sqrt{3}}{2} \qquad \text{Note that the two 2's cannot cancel at this step.} \\ &= \frac{2(2 \pm \sqrt{3})}{2} \qquad \text{Factor } 4 \pm 2\sqrt{3}. \\ &= 2 \pm \sqrt{3}. \qquad \text{Reduce to lowest terms.} \end{aligned}$$

The two solutions are $2 + \sqrt{3}$ and $2 - \sqrt{3}$.

The exact values of the solutions are $2 + \sqrt{3}$ and $2 - \sqrt{3}$. The $\sqrt{\ }$ key on a calculator gives decimal approximations of these solutions (to the nearest thousandth):

$$2 + \sqrt{3} \approx 2 + 1.732 = 3.732^*$$
$$2 - \sqrt{3} \approx 2 - 1.732 = 0.268$$

Try **YOUR TURN 3** ■

YOUR TURN 3 Solve $z^2 + 6 = 8z$.

NOTE

Sometimes the quadratic formula will give a result with a negative number under the radical sign, such as $3 \pm \sqrt{-5}$. A solution of this type is a complex number. Since this text deals only with real numbers, such solutions cannot be used.

Equations with Fractions

When an equation includes fractions, first eliminate all denominators by multiplying both sides of the equation by a common denominator, a number that can be divided (with no remainder) by each denominator in the equation. When an equation involves fractions with variable denominators, it is *necessary* to check all solutions in the original equation to be sure that no solution will lead to a zero denominator.

EXAMPLE 6 Solving Rational Equations

Solve each equation.

(a) $\dfrac{r}{10} - \dfrac{2}{15} = \dfrac{3r}{20} - \dfrac{1}{5}$

SOLUTION The denominators are 10, 15, 20, and 5. Each of these numbers can be divided into 60, so 60 is a common denominator. Multiply both sides of the equation by

*The symbol $\approx$ means "is approximately equal to."

60 and use the distributive property. (If a common denominator cannot be found easily, all the denominators in the problem can be multiplied together to produce one.)

$$\frac{r}{10} - \frac{2}{15} = \frac{3r}{20} - \frac{1}{5}$$

$$60\left(\frac{r}{10} - \frac{2}{15}\right) = 60\left(\frac{3r}{20} - \frac{1}{5}\right) \qquad \text{Multiply by the common denominator.}$$

$$60\left(\frac{r}{10}\right) - 60\left(\frac{2}{15}\right) = 60\left(\frac{3r}{20}\right) - 60\left(\frac{1}{5}\right) \qquad \text{Distributive property}$$

$$6r - 8 = 9r - 12$$

Add $-9r$ and 8 to both sides.

$$6r - 8 + (-9r) + 8 = 9r - 12 + (-9r) + 8$$

$$-3r = -4$$

$$r = \frac{4}{3} \qquad \text{Multiply each side by } -\tfrac{1}{3}.$$

Check by substituting into the original equation.

(b) $\dfrac{3}{x^2} - 12 = 0$

SOLUTION Begin by multiplying both sides of the equation by x^2 to get $3 - 12x^2 = 0$. This equation could be solved by using the quadratic formula with $a = -12$, $b = 0$, and $c = 3$. Another method that works well for the type of quadratic equation in which $b = 0$ is shown below.

$$3 - 12x^2 = 0$$

$$3 = 12x^2 \qquad \text{Add } 12x^2.$$

$$\frac{1}{4} = x^2 \qquad \text{Multiply by } \tfrac{1}{12}.$$

$$\pm\frac{1}{2} = x \qquad \text{Take square roots.}$$

Verify that there are two solutions, $-1/2$ and $1/2$.

(c) $\dfrac{2}{k} - \dfrac{3k}{k+2} = \dfrac{k}{k^2 + 2k}$

SOLUTION Factor $k^2 + 2k$ as $k(k + 2)$. The least common denominator for all the fractions is $k(k + 2)$. Multiplying both sides by $k(k + 2)$ gives the following:

$$k(k+2) \cdot \left(\frac{2}{k} - \frac{3k}{k+2}\right) = k(k+2) \cdot \frac{k}{k^2 + 2k}$$

$$2(k + 2) - 3k(k) = k$$

$$2k + 4 - 3k^2 = k \qquad \text{Distributive property}$$

$$-3k^2 + k + 4 = 0 \qquad \text{Add } -k\text{; rearrange terms.}$$

$$3k^2 - k - 4 = 0 \qquad \text{Multiply by } -1.$$

$$(3k - 4)(k + 1) = 0 \qquad \text{Factor.}$$

$$3k - 4 = 0 \quad \text{or} \quad k + 1 = 0$$

$$k = \frac{4}{3} \qquad k = -1$$

Verify that the solutions are $4/3$ and -1.

Try YOUR TURN 4

YOUR TURN 4 Solve $\dfrac{1}{x^2 - 4} + \dfrac{2}{x - 2} = \dfrac{1}{x}$.

CAUTION It is possible to get, as a solution of a rational equation, a number that makes one or more of the denominators in the original equation equal to zero. That number is not a solution, so it is *necessary* to check all potential solutions of rational equations. These introduced solutions are called **extraneous solutions**.

EXAMPLE 7 Solving a Rational Equation

Solve $\frac{2}{x-3}+\frac{1}{x}=\frac{6}{x(x-3)}$.

SOLUTION The common denominator is $x(x-3)$. Multiply both sides by $x(x-3)$ and solve the resulting equation.

$$x(x-3)\cdot\left(\frac{2}{x-3}+\frac{1}{x}\right)=x(x-3)\cdot\left[\frac{6}{x(x-3)}\right]$$

$$2x+x-3=6$$

$$3x=9$$

$$x=3$$

Checking this potential solution by substitution into the original equation shows that 3 makes two denominators 0. Thus, 3 cannot be a solution, so there is no solution for this equation.

R.4 Exercises

Solve each equation.

1. $2x+8=x-4$
2. $5x+2=8-3x$
3. $0.2m-0.5=0.1m+0.7$
4. $\frac{2}{3}k-k+\frac{3}{8}=\frac{1}{2}$
5. $3r+2-5(r+1)=6r+4$
6. $5(a+3)+4a-5=-(2a-4)$
7. $2[3m-2(3-m)-4]=6m-4$
8. $4[2p-(3-p)+5]=-7p-2$

Solve each equation by factoring or by using the quadratic formula. If the solutions involve square roots, give both the exact solutions and the approximate solutions to three decimal places.

9. $x^2+5x+6=0$
10. $x^2=3+2x$
11. $m^2=14m-49$
12. $2k^2-k=10$
13. $12x^2-5x=2$
14. $m(m-7)=-10$
15. $4x^2-36=0$
16. $z(2z+7)=4$
17. $12y^2-48y=0$
18. $3x^2-5x+1=0$
19. $2m^2-4m=3$
20. $p^2+p-1=0$
21. $k^2-10k=-20$
22. $5x^2-8x+2=0$
23. $2r^2-7r+5=0$
24. $2x^2-7x+30=0$
25. $3k^2+k=6$
26. $5m^2+5m=0$

Solve each equation.

27. $\frac{3x-2}{7}=\frac{x+2}{5}$
28. $\frac{x}{3}-7=6-\frac{3x}{4}$
29. $\frac{4}{x-3}-\frac{8}{2x+5}+\frac{3}{x-3}=0$
30. $\frac{5}{p-2}-\frac{7}{p+2}=\frac{12}{p^2-4}$
31. $\frac{2m}{m-2}-\frac{6}{m}=\frac{12}{m^2-2m}$
32. $\frac{2y}{y-1}=\frac{5}{y}+\frac{10-8y}{y^2-y}$
33. $\frac{1}{x-2}-\frac{3x}{x-1}=\frac{2x+1}{x^2-3x+2}$
34. $\frac{5}{a}+\frac{-7}{a+1}=\frac{a^2-2a+4}{a^2+a}$
35. $\frac{5}{b+5}-\frac{4}{b^2+2b}=\frac{6}{b^2+7b+10}$
36. $\frac{2}{x^2-2x-3}+\frac{5}{x^2-x-6}=\frac{1}{x^2+3x+2}$
37. $\frac{4}{2x^2+3x-9}+\frac{2}{2x^2-x-3}=\frac{3}{x^2+4x+3}$

YOUR TURN Answers

1. $-3/2$
2. $3/2, -5$
3. $4\pm\sqrt{10}$
4. $-1, -4$

R.5 Inequalities

To write that one number is greater than or less than another number, we use the following symbols.

Inequality Symbols

$<$ means *is less than* $\leq$ means *is less than or equal to*
$>$ means *is greater than* $\geq$ means *is greater than or equal to*

Linear Inequalities

An equation states that two expressions are equal; an **inequality** states that they are unequal. A **linear inequality** is an inequality that can be simplified to the form $ax < b$. (Properties introduced in this section are given only for $<$, but they are equally valid for $>$, $\leq$, or $\geq$.) Linear inequalities are solved with the following properties.

Properties of Inequality

For all real numbers a, b, and c:

1. If $a < b$, then $a + c < b + c$.
2. If $a < b$ and if $c > 0$, then $ac < bc$.
3. If $a < b$ and if $c < 0$, then $ac > bc$.

Pay careful attention to property 3; it says that if both sides of an inequality are multiplied by a negative number, the direction of the inequality symbol must be reversed.

EXAMPLE 1 Solving a Linear Inequality

Solve $4 - 3y \leq 7 + 2y$.

SOLUTION Use the properties of inequality.

$$4 - 3y + (-4) \leq 7 + 2y + (-4) \qquad \text{Add } -4 \text{ to both sides.}$$
$$-3y \leq 3 + 2y$$

Remember that *adding* the same number to both sides never changes the direction of the inequality symbol.

$$-3y + (-2y) \leq 3 + 2y + (-2y) \qquad \text{Add } -2y \text{ to both sides.}$$
$$-5y \leq 3$$

Multiply both sides by $-1/5$. Since $-1/5$ is negative, change the direction of the inequality symbol.

$$-\frac{1}{5}(-5y) \geq -\frac{1}{5}(3)$$
$$y \geq -\frac{3}{5}$$

Try YOUR TURN 1

YOUR TURN 1 Solve $3z - 2 > 5z + 7$.

CAUTION It is a common error to forget to reverse the direction of the inequality sign when multiplying or dividing by a negative number. For example, to solve $-4x \leq 12$, we must multiply by $-1/4$ on both sides *and* reverse the inequality symbol to get $x \geq -3$.

The solution $y \geq -3/5$ in Example 1 represents an interval on the number line. **Interval notation** often is used for writing intervals. With interval notation, $y \geq -3/5$ is written as $[-3/5, \infty)$. This is an example of a **half-open interval**, since one endpoint, $-3/5$, is included. The **open interval** $(2, 5)$ corresponds to $2 < x < 5$, with neither endpoint included. The **closed interval** $[2, 5]$ includes both endpoints and corresponds to $2 \leq x \leq 5$.

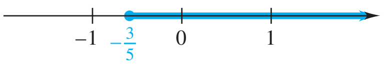


Figure 1

The **graph** of an interval shows all points on a number line that correspond to the numbers in the interval. To graph the interval $[-3/5, \infty)$, for example, use a solid circle at $-3/5$, since $-3/5$ is part of the solution. To show that the solution includes all real numbers greater than or equal to $-3/5$, draw a heavy arrow pointing to the right (the positive direction). See Figure 1.

EXAMPLE 2 Graphing a Linear Inequality

Solve $-2 < 5 + 3m < 20$. Graph the solution.

SOLUTION The inequality $-2 < 5 + 3m < 20$ says that $5 + 3m$ is *between* -2 and 20. Solve this inequality with an extension of the properties given above. Work as follows, first adding -5 to each part.

$$-2 + (-5) < 5 + 3m + (-5) < 20 + (-5)$$
$$-7 < 3m < 15$$

Now multiply each part by $1/3$.

$$-\frac{7}{3} < m < 5$$

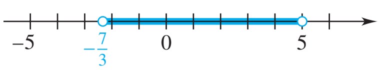


Figure 2

A graph of the solution is given in Figure 2; here open circles are used to show that $-7/3$ and 5 are *not* part of the graph.* ■

Quadratic Inequalities

A **quadratic inequality** has the form $ax^2 + bx + c > 0$ (or $<$, or $\leq$, or $\geq$). The greatest exponent is 2. The next few examples show how to solve quadratic inequalities.

EXAMPLE 3 Solving a Quadratic Inequality

Solve the quadratic inequality $x^2 - x < 12$.

SOLUTION Write the inequality with 0 on one side, as $x^2 - x - 12 < 0$. This inequality is solved with values of x that make $x^2 - x - 12$ negative (< 0). The quantity $x^2 - x - 12$ changes from positive to negative or from negative to positive at the points where it equals 0. For this reason, first solve the *equation* $x^2 - x - 12 = 0$.

$$x^2 - x - 12 = 0$$
$$(x - 4)(x + 3) = 0$$
$$x = 4 \quad \text{or} \quad x = -3$$

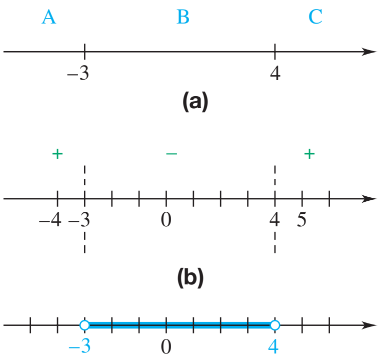


Figure 3

Locating -3 and 4 on a number line, as shown in Figure 3(a), determines three intervals A, B, and C. Decide which intervals include numbers that make $x^2 - x - 12$ negative by substituting any number from each interval into the polynomial. For example,

$$\text{choose } -4 \text{ from interval A: } (-4)^2 - (-4) - 12 = 8 > 0;$$
$$\text{choose 0 from interval B: } 0^2 - 0 - 12 = -12 < 0;$$
$$\text{choose 5 from interval C: } 5^2 - 5 - 12 = 8 > 0.$$

YOUR TURN 2 Solve $3y^2 \leq 16y + 12$. ■

Only numbers in interval B satisfy the given inequality, as shown in Figure 3(b), so the solution is $(-3, 4)$. A graph of this solution is shown in Figure 3(c). **Try YOUR TURN 2** ■

*Some textbooks use brackets in place of solid circles for the graph of a closed interval, and parentheses in place of open circles for the graph of an open interval.

EXAMPLE 4 Solving a Polynomial Inequality

Solve the inequality $x^3 + 2x^2 - 3x \geq 0$.

SOLUTION This is not a quadratic inequality because of the x^3 term, but we solve it in a similar way by first factoring the polynomial.

$$x^3 + 2x^2 - 3x = x(x^2 + 2x - 3) \quad \text{Factor out the common factor.}$$
$$= x(x - 1)(x + 3) \quad \text{Factor the quadratic.}$$

Now solve the corresponding equation.

$$x(x - 1)(x + 3) = 0$$
$$x = 0 \quad \text{or} \quad x - 1 = 0 \quad \text{or} \quad x + 3 = 0$$
$$x = 1 \qquad x = -3$$

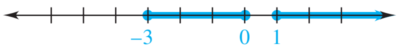


Figure 4

These three solutions determine four intervals on the number line: $(-\infty, -3)$, $(-3, 0)$, $(0, 1)$, and $(1, \infty)$. Substitute a number from each interval into the original inequality to determine that the solution consists of the numbers between -3 and 0 (including the endpoints) and all numbers that are greater than or equal to 1. See Figure 4. In interval notation, the solution is

$$[-3, 0] \cup [1, \infty).^*$$

■

Inequalities with Fractions

Inequalities with fractions are solved in a similar manner as quadratic inequalities.

EXAMPLE 5 Solving a Rational Inequality

Solve $\dfrac{2x - 3}{x} \geq 1$.

SOLUTION First solve the corresponding equation.

$$\frac{2x - 3}{x} = 1$$
$$2x - 3 = x \quad \text{Multiply both sides by } x.$$
$$x = 3 \quad \text{Solve for } x.$$

The solution, $x = 3$, determines the intervals on the number line where the fraction may change from greater than 1 to less than 1. This change also may occur on either side of a number that makes the denominator equal 0. Here, the x-value that makes the denominator 0 is $x = 0$. Test each of the three intervals determined by the numbers 0 and 3.

$$\text{For } (-\infty, 0), \text{ choose } -1: \frac{2(-1) - 3}{-1} = 5 \geq 1.$$
$$\text{For } (0, 3), \text{ choose } 1: \frac{2(1) - 3}{1} = -1 \not\geq 1.$$
$$\text{For } (3, \infty), \text{ choose } 4: \frac{2(4) - 3}{4} = \frac{5}{4} \geq 1.$$

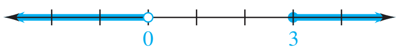


Figure 5

The symbol $\not\geq$ means "is *not* greater than or equal to." Testing the endpoints 0 and 3 shows that the solution is $(-\infty, 0) \cup [3, \infty)$, as shown in Figure 5. ■

CAUTION A common error is to try to solve the inequality in Example 5 by multiplying both sides by x. The reason this is wrong is that we don't know in the beginning whether x is positive, negative, or 0. If x is negative, the $\geq$ would change to $\leq$ according to the third property of inequality listed at the beginning of this section. ■

*The symbol $\cup$ indicates the *union* of two sets, which includes all elements in either set.

EXAMPLE 6 Solving a Rational Inequality

Solve $\dfrac{(x-1)(x+1)}{x} \leq 0.$

SOLUTION We first solve the corresponding equation.

$$\frac{(x-1)(x+1)}{x} = 0$$

$$(x-1)(x+1) = 0 \qquad \text{Multiply both sides by } x.$$

$$x = 1 \quad \text{or} \quad x = -1 \qquad \text{Use the zero-factor property.}$$

Setting the denominator equal to 0 gives $x = 0$, so the intervals of interest are $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$. Testing a number from each region in the original inequality and checking the endpoints, we find the solution is

$$(-\infty, -1] \cup (0, 1],$$

Figure 6

as shown in Figure 6. ■

CAUTION Remember to solve the equation formed by setting the *denominator* equal to zero. Any number that makes the denominator zero always creates two intervals on the number line. For instance, in Example 6, substituting $x = 0$ makes the denominator of the rational inequality equal to 0, so we know that there may be a sign change from one side of 0 to the other (as was indeed the case). ■

EXAMPLE 7 Solving a Rational Inequality

Solve $\dfrac{x^2 - 3x}{x^2 - 9} < 4.$

SOLUTION Solve the corresponding equation.

$$\frac{x^2 - 3x}{x^2 - 9} = 4$$

$$x^2 - 3x = 4x^2 - 36 \qquad \text{Multiply by } x^2 - 9.$$

$$0 = 3x^2 + 3x - 36 \qquad \text{Get 0 on one side.}$$

$$0 = x^2 + x - 12 \qquad \text{Multiply by } \tfrac{1}{3}.$$

$$0 = (x+4)(x-3) \qquad \text{Factor.}$$

$$x = -4 \quad \text{or} \quad x = 3$$

Now set the denominator equal to 0 and solve that equation.

$$x^2 - 9 = 0$$

$$(x-3)(x+3) = 0$$

$$x = 3 \quad \text{or} \quad x = -3$$

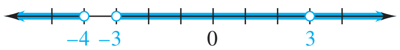


Figure 7

The intervals determined by the three (different) solutions are $(-\infty, -4)$, $(-4, -3)$, $(-3, 3)$, and $(3, \infty)$. Testing a number from each interval in the given inequality shows that the solution is

$$(-\infty, -4) \cup (-3, 3) \cup (3, \infty),$$

as shown in Figure 7. For this example, none of the endpoints are part of the solution because $x = 3$ and $x = -3$ make the denominator zero and $x = -4$ produces an equality.

Try **YOUR TURN 3** ■

YOUR TURN 3 Solve $\dfrac{k^2 - 35}{k} \geq 2.$

R.5 Exercises

Write each expression in interval notation. Graph each interval.

1. $x < 4$

2. $x \geq -3$

3. $1 \leq x < 2$

4. $-2 \leq x \leq 3$

5. $-9 > x$

6. $6 \leq x$

Using the variable x, write each interval as an inequality.

7. $[-7, -3]$

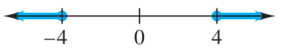
8. $[4, 10)$

9. $(-\infty, -1]$

10. $(3, \infty)$

11.


12.


13.


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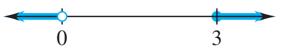

Solve each inequality and graph the solution.

15. $6p + 7 \leq 19$

16. $6k - 4 < 3k - 1$

17. $m - (3m - 2) + 6 < 7m - 19$

18. $-2(3y - 8) \geq 5(4y - 2)$

19. $3p - 1 < 6p + 2(p - 1)$

20. $x + 5(x + 1) > 4(2 - x) + x$

21. $-11 < y - 7 < -1$

22. $8 \leq 3r + 1 \leq 13$

23. $-2 < \dfrac{1 - 3k}{4} \leq 4$

24. $-1 \leq \dfrac{5y + 2}{3} \leq 4$

25. $\dfrac{3}{5}(2p + 3) \geq \dfrac{1}{10}(5p + 1)$

26. $\dfrac{8}{3}(z - 4) \leq \dfrac{2}{9}(3z + 2)$

Solve each inequality. Graph each solution.

27. $(m - 3)(m + 5) < 0$

28. $(t + 6)(t - 1) \geq 0$

29. $y^2 - 3y + 2 < 0$

30. $2k^2 + 7k - 4 > 0$

31. $x^2 - 16 > 0$

32. $2k^2 - 7k - 15 \leq 0$

33. $x^2 - 4x \geq 5$

34. $10r^2 + r \leq 2$

35. $3x^2 + 2x > 1$

36. $3a^2 + a > 10$

37. $9 - x^2 \leq 0$

38. $p^2 - 16p > 0$

39. $x^3 - 4x \geq 0$

40. $x^3 + 7x^2 + 12x \leq 0$

41. $2x^3 - 14x^2 + 12x < 0$

42. $3x^3 - 9x^2 - 12x > 0$

Solve each inequality.

43. $\dfrac{m - 3}{m + 5} \leq 0$

44. $\dfrac{r + 1}{r - 1} > 0$

45. $\dfrac{k - 1}{k + 2} > 1$

46. $\dfrac{a - 5}{a + 2} < -1$

47. $\dfrac{2y + 3}{y - 5} \leq 1$

48. $\dfrac{a + 2}{3 + 2a} \leq 5$

49. $\dfrac{2k}{k - 3} \leq \dfrac{4}{k - 3}$

50. $\dfrac{5}{p + 1} > \dfrac{12}{p + 1}$

51. $\dfrac{2x}{x^2 - x - 6} \geq 0$

52. $\dfrac{8}{p^2 + 2p} > 1$

53. $\dfrac{z^2 + z}{z^2 - 1} \geq 3$

54. $\dfrac{a^2 + 2a}{a^2 - 4} \leq 2$

YOUR TURN Answers

1. $z < -9/2$ **2.** $[-2/3, 6]$ **3.** $[-5, 0) \cup [7, \infty)$

R.6 Exponents

Integer Exponents

Recall that $a^2 = a \cdot a$, while $a^3 = a \cdot a \cdot a$, and so on. In this section, a more general meaning is given to the symbol a^n.

Definition of Exponent

If n is a natural number, then

$$a^n = a \cdot a \cdot a \cdot \cdots \cdot a,$$

where a appears as a factor n times.

In the expression a^n, the power n is the **exponent** and a is the **base**. This definition can be extended by defining a^n for zero and negative integer values of n.

Zero and Negative Exponents

If a is any nonzero real number, and if n is a positive integer, then

$$a^0 = 1 \quad \text{and} \quad a^{-n} = \frac{1}{a^n}.$$

(The symbol 0^0 is meaningless.)

EXAMPLE 1 Exponents

(a) $6^0 = 1$

(b) $(-9)^0 = 1$

(c) $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

(d) $9^{-1} = \frac{1}{9^1} = \frac{1}{9}$

(e) $\left(\frac{3}{4}\right)^{-1} = \frac{1}{(3/4)^1} = \frac{1}{3/4} = \frac{4}{3}$

(f) $(-2)^{-2} = \frac{1}{(-2)^2} = \frac{1}{4}$

(g) $-2^{-2} = \frac{-1}{2^2} = -\frac{1}{4}$

Try YOUR TURN 1

YOUR TURN 1 Find $\left(\frac{2}{3}\right)^{-3}$.

The following properties follow from the definitions of exponents given above.

Properties of Exponents

For any integers m and n, and any real numbers a and b for which the following exist:

1. $a^m \cdot a^n = a^{m+n}$
2. $\frac{a^m}{a^n} = a^{m-n}$
3. $(a^m)^n = a^{mn}$
4. $(ab)^m = a^m \cdot b^m$
5. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Note that $(-a)^n = a^n$ if n is an even integer, but $(-a)^n = -a^n$ if n is an odd integer.

EXAMPLE 2 Simplifying Exponential Expressions

Use the properties of exponents to simplify each expression. Leave answers with positive exponents. Assume that all variables represent positive real numbers.

(a) $7^4 \cdot 7^6 = 7^{4+6} = 7^{10}$ (or 282,475,249) Property 1

(b) $\frac{9^{14}}{9^6} = 9^{14-6} = 9^8$ (or 43,046,721) Property 2

(c) $\frac{r^9}{r^{17}} = r^{9-17} = r^{-8} = \frac{1}{r^8}$ Property 2

(d) $(2m^3)^4 = 2^4 \cdot (m^3)^4 = 16m^{12}$ Properties 3 and 4

(e) $(3x)^4 = 3^4 \cdot x^4 = 81x^4$ Property 4

(f) $\left(\frac{x^2}{y^3}\right)^6 = \frac{(x^2)^6}{(y^3)^6} = \frac{x^{2\cdot 6}}{y^{3\cdot 6}} = \frac{x^{12}}{y^{18}}$ Properties 3 and 5

(g) $\frac{a^{-3}b^5}{a^4b^{-7}} = \frac{b^{5-(-7)}}{a^{4-(-3)}} = \frac{b^{5+7}}{a^{4+3}} = \frac{b^{12}}{a^7}$ Property 2

(h) $p^{-1} + q^{-1} = \frac{1}{p} + \frac{1}{q}$ **Definition of a^{-n}.**

$= \frac{1}{p} \cdot \frac{q}{q} + \frac{1}{q} \cdot \frac{p}{p}$ **Get common denominator.**

$= \frac{q}{pq} + \frac{p}{pq} = \frac{p + q}{pq}$ **Add.**

(i) $\frac{x^{-2} - y^{-2}}{x^{-1} - y^{-1}} = \frac{\frac{1}{x^2} - \frac{1}{y^2}}{\frac{1}{x} - \frac{1}{y}}$ **Definition of a^{-n}**

$= \frac{\frac{y^2 - x^2}{x^2y^2}}{\frac{y - x}{xy}}$ **Get common denominators and combine terms.**

$= \frac{y^2 - x^2}{x^2y^2} \cdot \frac{xy}{y - x}$ **Invert and multiply.**

$= \frac{(y - x)(y + x)}{x^2y^2} \cdot \frac{xy}{y - x}$ **Factor.**

$= \frac{x + y}{xy}$ **Simplify.**

Try **YOUR TURN 2**

YOUR TURN 2 Simplify $\left(\frac{y^2z^{-4}}{y^{-3}z^4}\right)^{-2}$.

CAUTION

1. If Example 2(e) were written $3x^4$, the properties of exponents would not apply. When no parentheses are used, the exponent refers only to the factor closest to it. Also notice in Examples 2(c), 2(g), 2(h), and 2(i) that a negative exponent does *not* indicate a negative number.
2. Notice that factors in the denominator with negative exponents move to the numerator, while those in the numerator move to the denominator. Therefore, x^{-4}/y^{-2} simplifies to y^2/x^4. But this applies only to factors, not to terms that are added or subtracted. For example, it would be a serious error to simplify the expression in Example 2(i) to $(x - y)/(x^2 - y^2)$.

Roots

For *even* values of n and nonnegative values of a, the expression $a^{1/n}$ is defined to be the **positive *n*th root** of a or the **principal *n*th root** of a. For example, $a^{1/2}$ denotes the positive second root, or **square root**, of a, while $a^{1/4}$ is the positive fourth root of a. When n is *odd*, there is only one nth root, which has the same sign as a. For example, $a^{1/3}$, the **cube root** of a, has the same sign as a. By definition, if $b = a^{1/n}$, then $b^n = a$. On a calculator, a number is raised to a power using a key labeled x^y, y^x, or $\wedge$. For example, to take the fourth root of 6 on a TI-84 calculator, enter $6 \wedge (1/4)$ to get the result 1.56508458.

EXAMPLE 3 Calculations with Exponents

(a) $121^{1/2} = 11$, since 11 is positive and $11^2 = 121$.

(b) $625^{1/4} = 5$, since $5^4 = 625$.

(c) $256^{1/4} = 4$

(d) $64^{1/6} = 2$

(e) $27^{1/3} = 3$

(f) $(-32)^{1/5} = -2$

(g) $128^{1/7} = 2$

(h) $(-49)^{1/2}$ is not a real number.

Try **YOUR TURN 3**

YOUR TURN 3 Find $125^{1/3}$.

Rational Exponents

In the following definition, the domain of an exponent is extended to include all rational numbers.

Definition of $a^{m/n}$

For all real numbers a for which the indicated roots exist, and for any rational number m/n,

$$a^{m/n} = (a^{1/n})^m.$$

EXAMPLE 4 Calculations with Exponents

YOUR TURN 4 Find $16^{-3/4}$.

(a) $27^{2/3} = (27^{1/3})^2 = 3^2 = 9$

(b) $32^{2/5} = (32^{1/5})^2 = 2^2 = 4$

(c) $64^{4/3} = (64^{1/3})^4 = 4^4 = 256$

(d) $25^{3/2} = (25^{1/2})^3 = 5^3 = 125$

Try YOUR TURN 4

NOTE

$27^{2/3}$ could also be evaluated as $(27^2)^{1/3}$, but this is more difficult to perform without a calculator because it involves squaring 27 and then taking the cube root of this large number. On the other hand, when we evaluate it as $(27^{1/3})^2$, we know that the cube root of 27 is 3 without using a calculator, and squaring 3 is easy.

All the properties for integer exponents given in this section also apply to any rational exponent on a nonnegative real-number base.

EXAMPLE 5 Simplifying Exponential Expressions

YOUR TURN 5 Simplify $\left(\dfrac{x^{1/2}x^4}{x^{3/2}}\right)^{1/3}$.

(a) $\dfrac{y^{1/3}y^{5/3}}{y^3} = \dfrac{y^{1/3+5/3}}{y^3} = \dfrac{y^2}{y^3} = y^{2-3} = y^{-1} = \dfrac{1}{y}$

(b) $m^{2/3}(m^{7/3} + 2m^{1/3}) = m^{2/3+7/3} + 2m^{2/3+1/3} = m^3 + 2m$

(c) $\left(\dfrac{m^7n^{-2}}{m^{-5}n^2}\right)^{1/4} = \left(\dfrac{m^{7-(-5)}}{n^{2-(-2)}}\right)^{1/4} = \left(\dfrac{m^{12}}{n^4}\right)^{1/4} = \dfrac{(m^{12})^{1/4}}{(n^4)^{1/4}} = \dfrac{m^{12/4}}{n^{4/4}} = \dfrac{m^3}{n}$

Try YOUR TURN 5

In calculus, it is often necessary to factor expressions involving fractional exponents.

EXAMPLE 6 Simplifying Exponential Expressions

Factor out the smallest power of the variable, assuming all variables represent positive real numbers.

(a) $4m^{1/2} + 3m^{3/2}$

SOLUTION The smallest exponent is $1/2$. Factoring out $m^{1/2}$ yields

$$\begin{aligned} 4m^{1/2} + 3m^{3/2} &= m^{1/2}(4m^{1/2-1/2} + 3m^{3/2-1/2}) \\ &= m^{1/2}(4 + 3m). \end{aligned}$$

Check this result by multiplying $m^{1/2}$ by $4 + 3m$.

(b) $9x^{-2} - 6x^{-3}$

SOLUTION The smallest exponent here is -3. Since 3 is a common numerical factor, factor out $3x^{-3}$.

$$9x^{-2} - 6x^{-3} = 3x^{-3}(3x^{-2-(-3)} - 2x^{-3-(-3)}) = 3x^{-3}(3x - 2)$$

Check by multiplying. The factored form can be written without negative exponents as

$$\frac{3(3x - 2)}{x^3}.$$

(c) $(x^2+5)(3x-1)^{-1/2}(2)+(3x-1)^{1/2}(2x)$

SOLUTION There is a common factor of 2. Also, $(3x-1)^{-1/2}$ and $(3x-1)^{1/2}$ have a common factor. Always factor out the quantity to the *smallest* exponent. Here $-1/2 < 1/2$, so the common factor is $2(3x-1)^{-1/2}$ and the factored form is

$$2(3x-1)^{-1/2}[(x^2+5)+(3x-1)x] = 2(3x-1)^{-1/2}(4x^2-x+5).$$

Try **YOUR TURN 6**

YOUR TURN 6 Factor $5z^{1/3}+4z^{-2/3}$.

R.6 Exercises

Evaluate each expression. Write all answers without exponents.

1. 8^{-2}

2. 3^{-4}

3. 5^0

4. $\left(-\frac{3}{4}\right)^0$

5. $-(-3)^{-2}$

6. $-(-3^{-2})$

7. $\left(\frac{1}{6}\right)^{-2}$

8. $\left(\frac{4}{3}\right)^{-3}$

Simplify each expression. Assume that all variables represent positive real numbers. Write answers with only positive exponents.

9. $\frac{4^{-2}}{4}$

10. $\frac{8^9 \cdot 8^{-7}}{8^{-3}}$

11. $\frac{10^8 \cdot 10^{-10}}{10^4 \cdot 10^2}$

12. $\left(\frac{7^{-12} \cdot 7^3}{7^{-8}}\right)^{-1}$

13. $\frac{x^4 \cdot x^3}{x^5}$

14. $\frac{y^{10} \cdot y^{-4}}{y^6}$

15. $\frac{(4k^{-1})^2}{2k^{-5}}$

16. $\frac{(3z^2)^{-1}}{z^5}$

17. $\frac{3^{-1} \cdot x \cdot y^2}{x^{-4} \cdot y^5}$

18. $\frac{5^{-2}m^2y^{-2}}{5^2m^{-1}y^{-2}}$

19. $\left(\frac{a^{-1}}{b^2}\right)^{-3}$

20. $\left(\frac{c^3}{7d^{-2}}\right)^{-2}$

Simplify each expression, writing the answer as a single term without negative exponents.

21. $a^{-1}+b^{-1}$

22. $b^{-2}-a$

23. $\frac{2n^{-1}-2m^{-1}}{m+n^2}$

24. $\left(\frac{m}{3}\right)^{-1}+\left(\frac{n}{2}\right)^{-2}$

25. $(x^{-1}-y^{-1})^{-1}$

26. $(x \cdot y^{-1} - y^{-2})^{-2}$

Write each number without exponents.

27. $121^{1/2}$

28. $27^{1/3}$

29. $32^{2/5}$

30. $-125^{2/3}$

31. $\left(\frac{36}{144}\right)^{1/2}$

32. $\left(\frac{64}{27}\right)^{1/3}$

33. $8^{-4/3}$

34. $625^{-1/4}$

35. $\left(\frac{27}{64}\right)^{-1/3}$

36. $\left(\frac{121}{100}\right)^{-3/2}$

Simplify each expression. Write all answers with only positive exponents. Assume that all variables represent positive real numbers.

37. $3^{2/3} \cdot 3^{4/3}$

38. $27^{2/3} \cdot 27^{-1/3}$

39. $\frac{4^{9/4} \cdot 4^{-7/4}}{4^{-10/4}}$

40. $\frac{3^{-5/2} \cdot 3^{3/2}}{3^{7/2} \cdot 3^{-9/2}}$

41. $\left(\frac{x^6y^{-3}}{x^{-2}y^5}\right)^{1/2}$

42. $\left(\frac{a^{-7}b^{-1}}{b^{-4}a^2}\right)^{1/3}$

43. $\frac{7^{-1/3} \cdot 7r^{-3}}{7^{2/3} \cdot (r^{-2})^2}$

44. $\frac{12^{3/4} \cdot 12^{5/4} \cdot y^{-2}}{12^{-1} \cdot (y^{-3})^{-2}}$

45. $\frac{3k^2 \cdot (4k^{-3})^{-1}}{4^{1/2} \cdot k^{7/2}}$

46. $\frac{8p^{-3} \cdot (4p^2)^{-2}}{p^{-5}}$

47. $\frac{a^{4/3} \cdot b^{1/2}}{a^{2/3} \cdot b^{-3/2}}$

48. $\frac{x^{3/2} \cdot y^{4/5} \cdot z^{-3/4}}{x^{5/3} \cdot y^{-6/5} \cdot z^{1/2}}$

49. $\frac{k^{-3/5} \cdot h^{-1/3} \cdot t^{2/5}}{k^{-1/5} \cdot h^{-2/3} \cdot t^{1/5}}$

50. $\frac{m^{7/3} \cdot n^{-2/5} \cdot p^{3/8}}{m^{-2/3} \cdot n^{3/5} \cdot p^{-5/8}}$

Factor each expression.

51. $3x^3(x^2+3x)^2-15x(x^2+3x)^2$

52. $6x(x^3+7)^2-6x^2(3x^2+5)(x^3+7)$

53. $10x^3(x^2-1)^{-1/2}-5x(x^2-1)^{1/2}$

54. $9(6x+2)^{1/2}+3(9x-1)(6x+2)^{-1/2}$

55. $x(2x+5)^2(x^2-4)^{-1/2}+2(x^2-4)^{1/2}(2x+5)$

56. $(4x^2+1)^2(2x-1)^{-1/2}+16x(4x^2+1)(2x-1)^{1/2}$

YOUR TURN Answers

1. 27/8

2. z^{16}/y^{10}

3. 5

4. 1/8

5. x

6. $z^{-2/3}(5z+4)$

R.7 Radicals

We have defined $a^{1/n}$ as the positive or principal nth root of a for appropriate values of a and n. An alternative notation for $a^{1/n}$ uses radicals.

Radicals

If n is an even natural number and $a > 0$, or n is an odd natural number, then

$$a^{1/n} = \sqrt[n]{a}.$$

The symbol $\sqrt[n]{\ }$ is a **radical sign**, the number a is the **radicand**, and n is the **index** of the radical. The familiar symbol $\sqrt{a}$ is used instead of $\sqrt[2]{a}$.

EXAMPLE 1 Radical Calculations

(a) $\sqrt[4]{16} = 16^{1/4} = 2$

(b) $\sqrt[5]{-32} = -2$

(c) $\sqrt[3]{1000} = 10$

(d) $\sqrt[6]{\frac{64}{729}} = \frac{2}{3}$ ■

With $a^{1/n}$ written as $\sqrt[n]{a}$, the expression $a^{m/n}$ also can be written using radicals.

$$a^{m/n} = (\sqrt[n]{a})^m \quad \text{or} \quad a^{m/n} = \sqrt[n]{a^m}$$

The following properties of radicals depend on the definitions and properties of exponents.

Properties of Radicals

For all real numbers a and b and natural numbers m and n such that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers:

1. $(\sqrt[n]{a})^n = a$
2. $\sqrt[n]{a^n} = \begin{cases} |a| & \text{if } n \text{ is even} \\ a & \text{if } n \text{ is odd} \end{cases}$
3. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$
4. $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}} \quad (b \neq 0)$
5. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

Property 3 can be used to simplify certain radicals. For example, since $48 = 16 \cdot 3$,

$$\sqrt{48} = \sqrt{16 \cdot 3} = \sqrt{16} \cdot \sqrt{3} = 4\sqrt{3}.$$

To some extent, simplification is in the eye of the beholder, and $\sqrt{48}$ might be considered as simple as $4\sqrt{3}$. In this textbook, we will consider an expression to be simpler when we have removed as many factors as possible from under the radical.

EXAMPLE 2 Radical Calculations

(a) $\sqrt{1000} = \sqrt{\mathbf{100} \cdot 10} = \sqrt{\mathbf{100}} \cdot \sqrt{10} = \mathbf{10}\sqrt{10}$

(b) $\sqrt{128} = \sqrt{\mathbf{64} \cdot 2} = \mathbf{8}\sqrt{2}$

(c) $\sqrt{2} \cdot \sqrt{18} = \sqrt{2 \cdot 18} = \sqrt{36} = 6$

(d) $\sqrt[3]{54} = \sqrt[3]{\mathbf{27} \cdot 2} = \sqrt[3]{\mathbf{27}} \cdot \sqrt[3]{2} = \mathbf{3}\sqrt[3]{2}$

(e) $\sqrt{288m^5} = \sqrt{\mathbf{144} \cdot \boldsymbol{m^4} \cdot 2m} = \mathbf{12}\boldsymbol{m^2}\sqrt{2m}$

(f) $2\sqrt{18} - 5\sqrt{32} = 2\sqrt{9 \cdot 2} - 5\sqrt{16 \cdot 2}$

$= 2\sqrt{9} \cdot \sqrt{2} - 5\sqrt{16} \cdot \sqrt{2}$

$= 2(3)\sqrt{2} - 5(4)\sqrt{2} = -14\sqrt{2}$

(g) $\sqrt{x^5} \cdot \sqrt[3]{x^5} = x^{5/2} \cdot x^{5/3} = x^{5/2+5/3} = x^{25/6} = \sqrt[6]{x^{25}} = x^4\sqrt[6]{x}$

Try YOUR TURN 1 ■

YOUR TURN 1 Simplify $\sqrt{28x^9y^5}$.

When simplifying a square root, keep in mind that $\sqrt{x}$ is nonnegative by definition. Also, $\sqrt{x^2}$ is not x, but $|x|$, the **absolute value of x**, defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

For example, $\sqrt{(-5)^2} = |-5| = 5$. It is correct, however, to simplify $\sqrt{x^4} = x^2$. We need not write $|x^2|$ because x^2 is always nonnegative.

EXAMPLE 3 Simplifying by Factoring

Simplify $\sqrt{m^2 - 4m + 4}$.

SOLUTION Factor the polynomial as $m^2 - 4m + 4 = (m - 2)^2$. Then by property 2 of radicals and the definition of absolute value,

$$\sqrt{(m-2)^2} = |m - 2| = \begin{cases} m - 2 & \text{if } m - 2 \geq 0 \\ -(m - 2) = 2 - m & \text{if } m - 2 < 0. \end{cases}$$ ■

CAUTION Avoid the common error of writing $\sqrt{a^2 + b^2}$ as $\sqrt{a^2} + \sqrt{b^2}$. We must add a^2 and b^2 *before* taking the square root. For example, $\sqrt{16 + 9} = \sqrt{25} = 5$, *not* $\sqrt{16} + \sqrt{9} = 4 + 3 = 7$. This idea applies as well to higher roots. For example, in general,

$$\sqrt[3]{a^3 + b^3} \neq \sqrt[3]{a^3} + \sqrt[3]{b^3},$$

$$\sqrt[4]{a^4 + b^4} \neq \sqrt[4]{a^4} + \sqrt[4]{b^4}.$$

Also, $$\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}.$$ ■

Rationalizing Denominators The next example shows how to *rationalize* (remove all radicals from) the denominator in an expression containing radicals.

EXAMPLE 4 Rationalizing Denominators

Simplify each expression by rationalizing the denominator.

(a) $\dfrac{4}{\sqrt{3}}$

SOLUTION To rationalize the denominator, multiply by $\sqrt{3}/\sqrt{3}$ (or 1) so the denominator of the product is a rational number.

$$\frac{4}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{4\sqrt{3}}{3} \qquad \sqrt{3} \cdot \sqrt{3} = \sqrt{9} = 3$$

(b) $\dfrac{2}{\sqrt[3]{x}}$

SOLUTION Here, we need a perfect cube under the radical sign to rationalize the denominator. Multiplying by $\sqrt[3]{x^2}/\sqrt[3]{x^2}$ gives

$$\frac{2}{\sqrt[3]{x}} \cdot \frac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}} = \frac{2\sqrt[3]{x^2}}{\sqrt[3]{x^3}} = \frac{2\sqrt[3]{x^2}}{x}.$$

(c) $\dfrac{1}{1 - \sqrt{2}}$

SOLUTION The best approach here is to multiply both numerator and denominator by the number $1 + \sqrt{2}$. The expressions $1 + \sqrt{2}$ and $1 - \sqrt{2}$ are conjugates,* and their product is $1^2 - (\sqrt{2})^2 = 1 - 2 = -1$. Thus,

$$\frac{1}{1 - \sqrt{2}} = \frac{1(1 + \sqrt{2})}{(1 - \sqrt{2})(1 + \sqrt{2})} = \frac{1 + \sqrt{2}}{1 - 2} = -1 - \sqrt{2}.$$

YOUR TURN 2 Rationalize the denominator in $\dfrac{5}{\sqrt{x} - \sqrt{y}}$.

Try YOUR TURN 2 ■

Sometimes it is advantageous to rationalize the *numerator* of a rational expression. The following example arises in calculus when evaluating a *limit.*

EXAMPLE 5 Rationalizing Numerators

Rationalize each numerator.

(a) $\dfrac{\sqrt{x} - 3}{x - 9}$.

SOLUTION Multiply the numerator and denominator by the conjugate of the numerator, $\sqrt{x} + 3$.

$$\frac{\sqrt{x} - 3}{x - 9} \cdot \frac{\sqrt{x} + 3}{\sqrt{x} + 3} = \frac{(\sqrt{x})^2 - 3^2}{(x - 9)(\sqrt{x} + 3)} \qquad (a - b)(a + b) = a^2 - b^2$$

$$= \frac{x - 9}{(x - 9)(\sqrt{x} + 3)}$$

$$= \frac{1}{\sqrt{x} + 3}$$

*If a and b are real numbers, the *conjugate* of $a + b$ is $a - b$.

(b) $\dfrac{\sqrt{3}+\sqrt{x+3}}{\sqrt{3}-\sqrt{x+3}}$

SOLUTION Multiply the numerator and denominator by the conjugate of the numerator, $\sqrt{3}-\sqrt{x+3}$.

$$\frac{\sqrt{3}+\sqrt{x+3}}{\sqrt{3}-\sqrt{x+3}}\cdot\frac{\sqrt{3}-\sqrt{x+3}}{\sqrt{3}-\sqrt{x+3}}=\frac{3-(x+3)}{3-2\sqrt{3}\sqrt{x+3}+(x+3)}$$
$$=\frac{-x}{6+x-2\sqrt{3(x+3)}}$$

Try **YOUR TURN 3**

YOUR TURN 3 Rationalize the numerator in $\dfrac{4+\sqrt{x}}{16-x}$.

R.7 Exercises

Simplify each expression by removing as many factors as possible from under the radical. Assume that all variables represent positive real numbers.

1. $\sqrt[3]{125}$
2. $\sqrt[4]{1296}$
3. $\sqrt[5]{-3125}$
4. $\sqrt{50}$
5. $\sqrt{2000}$
6. $\sqrt{32y^5}$
7. $\sqrt{27}\cdot\sqrt{3}$
8. $\sqrt{2}\cdot\sqrt{32}$
9. $7\sqrt{2}-8\sqrt{18}+4\sqrt{72}$
10. $4\sqrt{3}-5\sqrt{12}+3\sqrt{75}$
11. $4\sqrt{7}-\sqrt{28}+\sqrt{343}$
12. $3\sqrt{28}-4\sqrt{63}+\sqrt{112}$
13. $\sqrt[3]{2}-\sqrt[3]{16}+2\sqrt[3]{54}$
14. $2\sqrt[3]{5}-4\sqrt[3]{40}+3\sqrt[3]{135}$
15. $\sqrt{2x^3y^2z^4}$
16. $\sqrt{160r^7s^9t^{12}}$
17. $\sqrt[3]{128x^3y^8z^9}$
18. $\sqrt[4]{x^8y^7z^{11}}$
19. $\sqrt{a^3b^5}-2\sqrt{a^7b^3}+\sqrt{a^3b^9}$
20. $\sqrt{p^7q^3}-\sqrt{p^5q^9}+\sqrt{p^9q}$
21. $\sqrt{a}\cdot\sqrt[3]{a}$
22. $\sqrt{b^3}\cdot\sqrt[4]{b^3}$

Simplify each root, if possible.

23. $\sqrt{16-8x+x^2}$
24. $\sqrt{9y^2+30y+25}$
25. $\sqrt{4-25z^2}$
26. $\sqrt{9k^2+h^2}$

Rationalize each denominator. Assume that all radicands represent positive real numbers.

27. $\dfrac{5}{\sqrt{7}}$
28. $\dfrac{5}{\sqrt{10}}$
29. $\dfrac{-3}{\sqrt{12}}$
30. $\dfrac{4}{\sqrt{8}}$
31. $\dfrac{3}{1-\sqrt{2}}$
32. $\dfrac{5}{2-\sqrt{6}}$
33. $\dfrac{6}{2+\sqrt{2}}$
34. $\dfrac{\sqrt{5}}{\sqrt{5}+\sqrt{2}}$
35. $\dfrac{1}{\sqrt{r}-\sqrt{3}}$
36. $\dfrac{5}{\sqrt{m}-\sqrt{5}}$
37. $\dfrac{y-5}{\sqrt{y}-\sqrt{5}}$
38. $\dfrac{\sqrt{z}-1}{\sqrt{z}-\sqrt{5}}$
39. $\dfrac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}}$
40. $\dfrac{\sqrt{p}+\sqrt{p^2-1}}{\sqrt{p}-\sqrt{p^2-1}}$

Rationalize each numerator. Assume that all radicands represent positive real numbers.

41. $\dfrac{1+\sqrt{2}}{2}$
42. $\dfrac{3-\sqrt{3}}{6}$
43. $\dfrac{\sqrt{x}+\sqrt{x+1}}{\sqrt{x}-\sqrt{x+1}}$
44. $\dfrac{\sqrt{p}-\sqrt{p-2}}{\sqrt{p}}$

YOUR TURN Answers

1. $2x^4y^2\sqrt{7xy}$ **2.** $5(\sqrt{x}+\sqrt{y})/(x-y)$ **3.** $1/(4-\sqrt{x})$

1 Linear Functions

Before using mathematics to solve a real-world problem, we must usually set up a **mathematical model**, a mathematical description of the situation. In this chapter we look at some mathematics of *linear* models, which are used for data whose graphs can be approximated by straight lines. Linear models have an immense number of applications, because even when the underlying phenomenon is not linear, a linear model often provides an approximation that is sufficiently accurate and simpler to use than a nonlinear model.

Over short time intervals, many changes in the economy are well modeled by linear functions. In an exercise in the first section of this chapter, we will examine a linear model that predicts the number of cellular telephone users in the United States. Such predictions are important tools for cellular telephone company executives and planners.

1.1 Slopes and Equations of Lines

Apply It

How fast has tuition at public colleges been increasing in recent years, and how well can we predict tuition in the future?

In Example 13 of this section, we will answer these questions using the equation of a line.

There are many everyday situations in which two quantities are related. For example, if a bank account pays 6% simple interest per year, then the interest I that a deposit of P dollars would earn in one year is given by

$$I = 0.06 \cdot P, \qquad \text{or} \qquad I = 0.06P.$$

The formula $I = 0.06P$ describes the relationship between interest and the amount of money deposited.

Using this formula, we see, for example, that if $P = \$100$, then $I = \$6$, and if $P = \$200$, then $I = \$12$. These corresponding pairs of numbers can be written as **ordered pairs**, $(100, 6)$ and $(200, 12)$, whose order is important. The first number denotes the value of P and the second number the value of I.

Ordered pairs are graphed with the perpendicular number lines of a **Cartesian coordinate system**, shown in Figure 1.* The horizontal number line, or ***x*-axis**, represents the first components of the ordered pairs, while the vertical or ***y*-axis** represents the second components. The point where the number lines cross is the zero point on both lines; this point is called the **origin**.

Each point on the xy-plane corresponds to an ordered pair of numbers, where the x-value is written first. From now on, we will refer to the point corresponding to the ordered pair (x, y) as "the point (x, y)."

Locate the point $(-2, 4)$ on the coordinate system by starting at the origin and counting 2 units to the left on the horizontal axis and 4 units upward, parallel to the vertical axis. This point is shown in Figure 1, along with several other sample points. The number -2 is the ***x*-coordinate** and the number 4 is the ***y*-coordinate** of the point $(-2, 4)$.

The x-axis and y-axis divide the plane into four parts, or **quadrants**. For example, quadrant I includes all those points whose x- and y-coordinates are both positive. The quadrants are numbered as shown in Figure 1. The points on the axes themselves belong to no quadrant. The set of points corresponding to the ordered pairs of an equation is the **graph** of the equation.

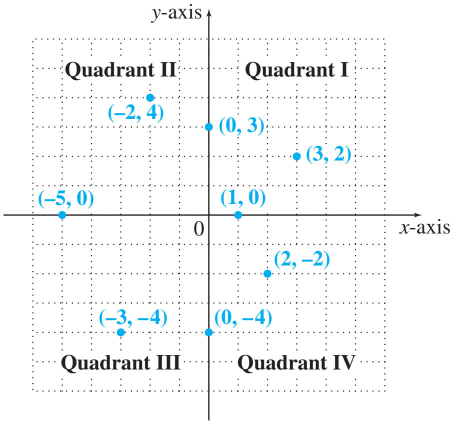

Figure 1

*The name "Cartesian" honors René Descartes (1596–1650), one of the greatest mathematicians of the 17th century. According to legend, Descartes was lying in bed when he noticed an insect crawling on the ceiling and realized that if he could determine the distance from the bug to each of two perpendicular walls, he could describe its position at any given moment. The same idea can be used to locate a point in a plane.

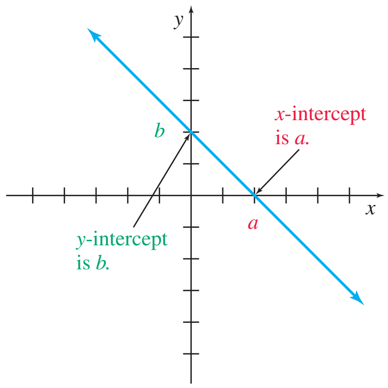

Figure 2

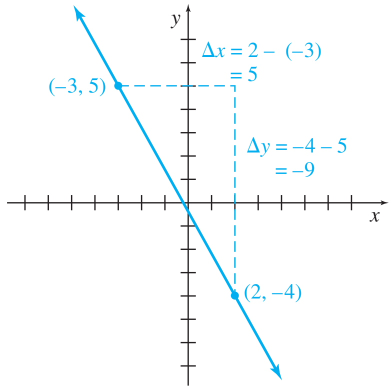

Figure 3

The x- and y-values of the points where the graph of an equation crosses the axes are called the ***x*-intercept** and ***y*-intercept**, respectively.* See Figure 2.

Slope of a Line

An important characteristic of a straight line is its *slope*, a number that represents the "steepness" of the line. To see how slope is defined, look at the line in Figure 3. The line passes through the points $(x_1, y_1) = (-3, 5)$ and $(x_2, y_2) = (2, -4)$. The difference in the two x-values,

$$x_2 - x_1 = 2 - (-3) = 5$$

in this example, is called the **change in x**. The symbol Δx (read "delta x") is used to represent the change in x. In the same way, Δy represents the **change in y**. In our example,

$$\begin{aligned}\Delta y &= y_2 - y_1\\ &= -4 - 5\\ &= -9.\end{aligned}$$

These symbols, Δx and Δy, are used in the following definition of slope.

Slope of a Nonvertical Line

The **slope** of a nonvertical line is defined as the vertical change (the "rise") over the horizontal change (the "run") as one travels along the line. In symbols, taking two different points (x_1, y_1) and (x_2, y_2) on the line, the slope is

$$m = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1},$$

where $x_1 \neq x_2$.

By this definition, the slope of the line in Figure 3 is

$$m = \frac{\Delta y}{\Delta x} = \frac{-4 - 5}{2 - (-3)} = -\frac{9}{5}.$$

The slope of a line tells how fast y changes for each unit of change in x.

NOTE

Using similar triangles, it can be shown that the slope of a line is independent of the choice of points on the line. That is, the same slope will be obtained for *any* choice of two different points on the line.

*Some people prefer to define the intercepts as ordered pairs, rather than as numbers.