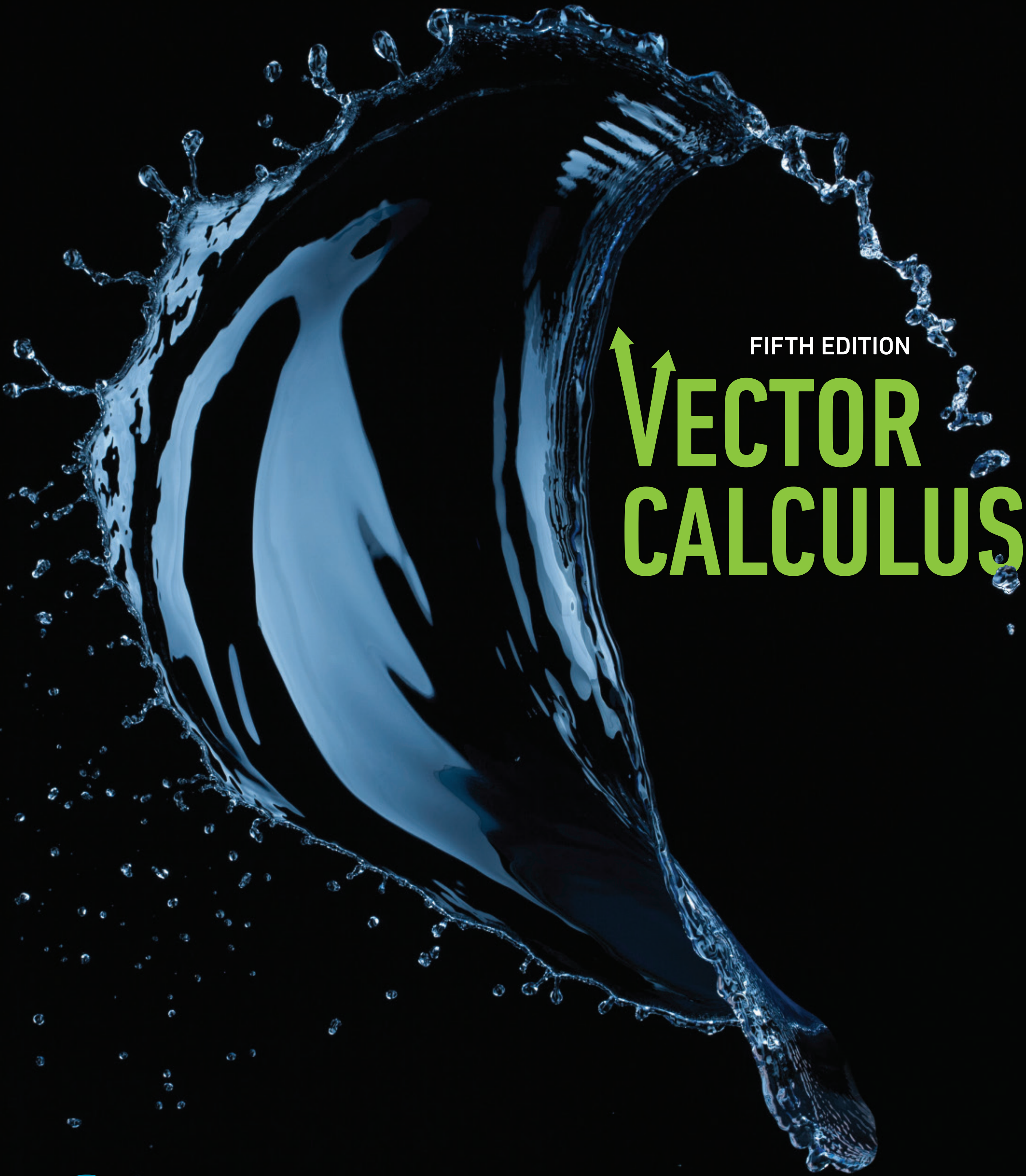
FIFTH EDITION

VECTOR CALCULUS



SUSAN JANE COLLEY • SANTIAGO CAÑEZ

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FIFTH EDITION

Vector Calculus

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Pearson

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To Will and Diane, with love

—SJC

To Adelina, Addy, Olivia, and Cecily

—SC

About the Authors

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William Colley

Susan Colley is the Andrew and Pauline Delaney Professor of Mathematics at Oberlin College. She has served as chair of the department as well as editor of *The American Mathematical Monthly*.

She received S.B. and Ph.D. degrees in mathematics from the Massachusetts Institute of Technology prior to joining the faculty at Oberlin in 1983.

Her research has primarily focused on algebraic geometry, particularly enumerative problems, multiple-point singularities, and higher-order data and contact of plane curves.

Professor Colley has published papers on algebraic geometry and commutative algebra as well as articles on other mathematical subjects. She has lectured internationally on her research and has taught a wide range of subjects in undergraduate mathematics.

Professor Colley is a member of several professional and honorary societies, including the American Mathematical Society, the Mathematical Association of America, Phi Beta Kappa, and Sigma Xi.

Santiago Cañez

Santiago Cañez is currently a Charles Deering McCormick Distinguished Professor of Instruction in the Department of Mathematics at Northwestern University and serves as its director of undergraduate studies. He received a B.S. degree in mathematics from the University of Arizona and a Ph.D. degree in mathematics from the University of California at Berkeley prior to joining the faculty at Northwestern in 2012.

Professor Cañez works in symplectic geometry and mathematical physics and is in particular interested in groupoids and higher-order structures. He has taught a wide range of courses and has supervised numerous student theses and research projects. He is a member of the American Mathematical Society and the Mathematical Association of America.

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Preface

Physical and natural phenomena depend on a complex array of factors. The sociologist or psychologist who studies group behavior, the economist who endeavors to understand the vagaries of a nation's employment cycles, the physicist who observes the trajectory of a particle or planet, or indeed anyone who seeks to understand geometry in two, three, or more dimensions recognizes the need to analyze changing quantities that depend on more than a single variable. Vector calculus is the essential mathematical tool for such analysis. Moreover, it is an exciting and beautiful subject in its own right, a true adventure in many dimensions.

The only technical prerequisite for this text, which is intended for a sophomore-level course in multivariable calculus, is a standard course in the calculus of functions of one variable. In particular, the necessary matrix arithmetic and algebra (not linear algebra) are developed as needed. Although the mathematical background assumed is not exceptional, the reader will still be challenged in places.

Our objectives in writing the book are simple ones: to develop in students a sound conceptual grasp of vector calculus and to help them begin the transition from first-year calculus to more advanced technical mathematics. We believe that the first goal can be met, at least in part, through the use of vector and matrix notation, so that many results, especially those of differential calculus, can be stated with reasonable levels of clarity and generality. Properly described, results in the calculus of several variables can look quite similar to those of the calculus of one variable. Reasoning by analogy will thus be an important pedagogical tool. We also believe that a conceptual understanding of mathematics can be obtained through the development of a good geometric intuition. Although many results are stated in the case of n variables (where n is arbitrary), we recognize that the most important and motivational examples usually arise for functions of two and three variables, so these concrete and visual situations are emphasized to explicate the general theory. Vector calculus is in many ways an ideal subject for students to begin exploration of the interrelations among analysis, geometry, and matrix algebra.

Multivariable calculus, for many students, represents the beginning of significant mathematical maturation. Consequently, we have written a rather expansive text so that they can see that there is a story behind the results, techniques, and examples—that the subject coheres and that this coherence is important for problem solving. To indicate some of the power of the methods introduced, a number of topics, not always discussed very fully in a first multivariable calculus course, are treated here in some detail:

- an early introduction of cylindrical and spherical coordinates (§1.7);
- the use of vector techniques to derive Kepler's laws of planetary motion (§3.1);
- the elementary differential geometry of curves in $\mathbf{R}^3$, including discussion of curvature, torsion, and the Frenet–Serret formulas for the moving frame (§3.2);
- Taylor's formula for functions of several variables (§4.1);
- the use of the Hessian matrix to determine the nature (as local extrema) of critical points of functions of n variables (§4.2 and §4.3);
- an extended discussion of the change of variables formula in double and triple integrals (§5.5);
- applications of vector analysis to physics (§7.4);
- an introduction to differential forms and the generalized Stokes's theorem (Chapter 8).

Included are a number of proofs of important results. The more technical proofs are collected as addenda at the ends of the appropriate sections so as not to disrupt the main conceptual flow and to allow for greater flexibility of use by the instructor and student. Nonetheless, some proofs (or sketches of proofs) embody such central ideas that they are included in the main body of the text.

New in the Fifth Edition

We have retained the overall structure and tone of prior editions. New features in this edition include the following:

- **NEW:** For the first time, this text is available as a Pearson eText, featuring a number of interactive GeoGebra applets.
- clarifications, new examples, and new exercises throughout the text;
- new derivations of the orthogonal projection formula (§1.3) and the Cauchy–Schwarz inequality (§1.6);
- a description of the geometric interpretation of second-order partial derivatives (§2.4);
- a description of the interpretation of the Lagrange multiplier (§4.3);
- new terminology in Chapter 5 to describe elementary regions of integration, and more examples of setting up double and triple integrals;
- a new subsection in §5.6 on probability as an application of multiple integrals, and new miscellaneous exercises in Chapter 5 on expected value;
- new examples illustrating interesting uses of Green's theorem (§6.2);
- new miscellaneous exercises in Chapters 1 and 4 for readers more familiar with linear algebra.
- **Authors' DEI statement:** We conducted an external review of the text's content to determine how it could be improved to address issues related to diversity, equity, and inclusion. The results of that review informed the revision.

How to Use This Book

There is more material in this book than can be covered comfortably during a single semester. Hence, the instructor will wish to eliminate some topics or subtopics—or to abbreviate the rather leisurely presentations of limits and differentiability. Since some instructors may find themselves without the time to treat surface integrals in detail, we have separated all material concerning parametrized surfaces, surface integrals, and Stokes's and Gauss's theorems (Chapter 7) from that concerning line integrals and Green's theorem (Chapter 6). In particular, in a one-semester course for students having little or no experience with vectors or matrices, instructors can probably expect to cover most of the material in Chapters 1–6, although no doubt it will be necessary to omit some of the optional subsections and to downplay many of the proofs of results. A rough outline for such a course, allowing for some instructor discretion, could be the following:

Chapter 1	8–9 lectures
Chapter 2	9 lectures
Chapter 3	4–5 lectures
Chapter 4	5–6 lectures
Chapter 5	8 lectures
Chapter 6	4 lectures
	38–41 lectures

If students have a richer background (so that much of the material in Chapter 1 can be left largely to them to read on their own), then it should be possible to treat a good portion of Chapter 7 as well. For a two-quarter or two-semester course, it should be possible to work through the entire book with reasonable care and rigor, although coverage of Chapter 8 should depend on students' exposure to introductory linear algebra, as somewhat more sophistication is assumed there.

The exercises vary from relatively routine computations to more challenging and provocative problems, generally (but not invariably) increasing in difficulty within each section. In a number of instances, groups of problems serve to introduce supplementary topics or new applications. Each chapter concludes with a set of miscellaneous exercises that both review and extend the ideas introduced in the chapter.

A word about the use of technology. The text was written without reference to any particular computer software or graphing calculator. Most of the exercises can be solved by hand, although there is no reason not to turn over some of the more tedious calculations to a computer. Those exercises that *require* a computer for computational or graphical purposes are marked with the symbol T and should be amenable to software such as *Mathematica*®, Maple®, or MATLAB.

Ancillary Materials

An **Instructor's Solutions Manual**, containing complete solutions to all of the exercises, is available to course instructors from the Pearson Instructor Resource Center (www.pearsonhighered.com/irc), as are many Microsoft® PowerPoint® files and Wolfram *Mathematica*® notebooks that can be adapted for classroom use. The reader can find errata for the text and accompanying solutions manuals at the following address: www.oberlin.edu/math/faculty/colley/VCErrata.html

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To the Student: Some Preliminary Notation

Here are the ideas that you need to keep in mind as you read this book and learn vector calculus.

Given two sets A and B, we assume that you are familiar with the notation $A \bigcup B$ for the **union** of A and B—those elements that are in either A or B (or both):

$$A \bigcup B = \{x | x \in A \text{ or } x \in B\}.$$

Similarly, $A \bigcap B$ is used to denote the **intersection** of A and B—those elements that are in both A and B:

$$A \bigcap B = \{x | x \in A \text{ and } x \in B\}.$$

The notation $A \subseteq B$, or $A \subset B$, indicates that A is a **subset** of B (possibly empty or equal to B).

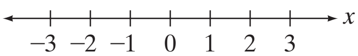


FIGURE 1 The coordinate line **R**.

One-dimensional space (also called the **real line** or **R**) is just a straight line. We put real number coordinates on this line by placing negative numbers on the left and positive numbers on the right. (See Figure 1.)

Two-dimensional space, denoted $\mathbf{R}^2$, is the familiar Cartesian plane. If we construct two perpendicular lines (the x- and y-**coordinate axes**), set the **origin** as the point of intersection of the axes, and establish numerical scales on these lines, then we may locate a point in $\mathbf{R}^2$ by giving an ordered pair of numbers (x, y), the **coordinates** of the point. Note that the coordinate axes divide the plane into four **quadrants**. (See Figure 2.)

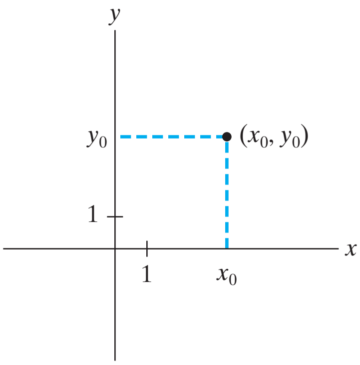


FIGURE 2 The coordinate plane $\mathbf{R}^2$.

Three-dimensional space, denoted $\mathbf{R}^3$, requires three mutually perpendicular coordinate axes (called the x-, y- and z-**axes**) that meet in a single point (called the **origin**) in order to locate an arbitrary point. Analogous to the case of $\mathbf{R}^2$, if we establish scales on the axes, then we can locate a point in $\mathbf{R}^3$ by giving an ordered triple of numbers (x, y, z). The coordinate axes divide three-dimensional space into eight **octants**. It takes some practice to get your sense of perspective correct when sketching points in $\mathbf{R}^3$. (See Figure 3.) Sometimes we draw the coordinate axes in $\mathbf{R}^3$ in different orientations in order to get a better view of things. However, we always maintain the axes in a **right-handed configuration**. This means that if you curl the fingers of your right hand from the positive x-axis to the positive y-axis, then your thumb will point along the positive z-axis. (See Figure 4.)

Although you need to recall particular techniques and methods from the calculus you have already learned, here are some of the more important concepts to keep in mind: Given a function $f(x)$, the **derivative** $f'(x)$ is the limit (if it exists) of the difference quotient of the function:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The significance of the derivative $f'(x_0)$ is that it measures the slope of the line tangent to the graph of f at the point $(x_0, f(x_0))$. (See Figure 5.) The derivative may also be considered to give the instantaneous rate of change of f at $x = x_0$. We also denote the derivative $f'(x)$ by df/dx.

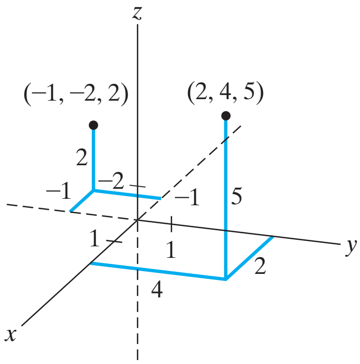


FIGURE 3 Three-dimensional space $\mathbf{R}^3$. Selected points are graphed.

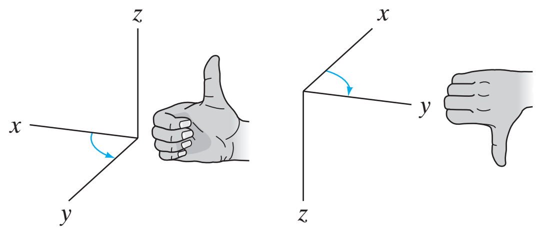


FIGURE 4 The x-, y-, and z-axes in $\mathbf{R}^3$ are always drawn in a right-handed configuration.

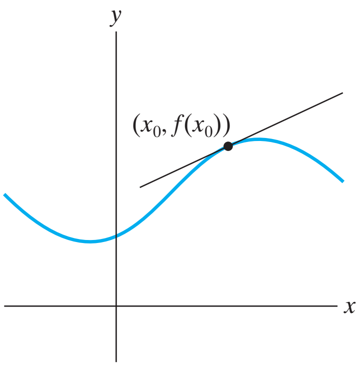


FIGURE 5 The derivative $f'(x_0)$ is the slope of the tangent line to $y = f(x)$ at $(x_0, f(x_0))$.

The **definite integral** $\int_a^b f(x)\,dx$ of f on the closed interval $[a, b]$ is the limit (provided it exists) of the so-called **Riemann sums** of f:

$$\int_a^b f(x)\,dx = \lim_{\text{all } \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

Here $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ denotes a **partition** of $[a, b]$ into subintervals $[x_{i-1}, x_i]$, the symbol $\Delta x_i = x_i - x_{i-1}$ (the length of the subinterval), and x_i^* denotes any point in $[x_{i-1}, x_i]$. If $f(x) \geq 0$ on $[a, b]$, then each term $f(x_i^*)\Delta x_i$ in the Riemann sum is the area of a rectangle related to the graph of f. The Riemann sum $\sum_{i=1}^{n} f(x_i^*)\Delta x_i$ thus approximates the total area under the graph of f between $x = a$ and $x = b$. (See Figure 6.)

The definite integral $\int_a^b f(x)\,dx$, if it exists, is taken to represent the area under $y = f(x)$ between $x = a$ and $x = b$. (See Figure 7.)

The derivative and the definite integral are connected by an elegant result known as **the fundamental theorem of calculus**. Let $f(x)$ be a continuous

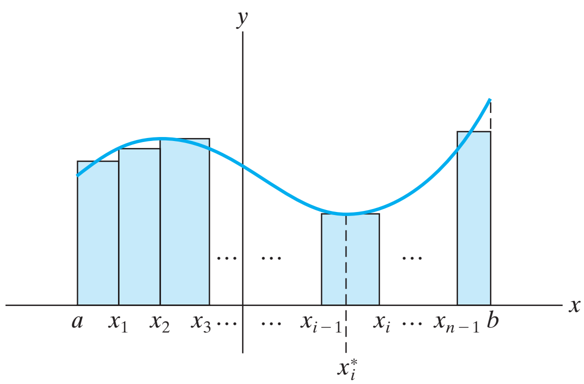


FIGURE 6 If $f(x) \geq 0$ on $[a, b]$, then the Riemann sum approximates the area under $y = f(x)$ by giving the sum of areas of rectangles.

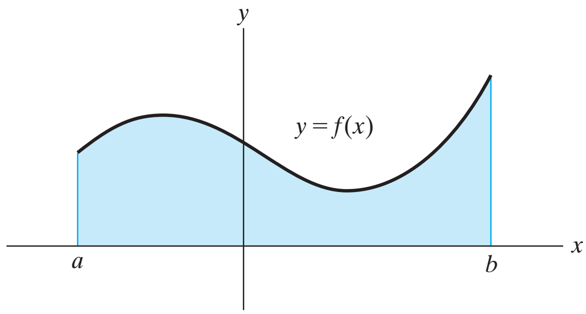


FIGURE 7 The area under the graph of $y = f(x)$ is $\int_a^b f(x)\,dx$.

function of one variable, and let $F(x)$ be such that $F'(x) = f(x)$. (The function F is called an **antiderivative** of f.) Then

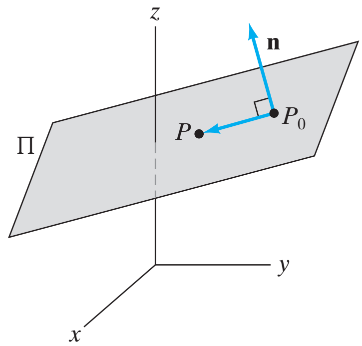
1. $\displaystyle\int_a^b f(x)dx = F(b) - F(a)$;

2. $\displaystyle\frac{d}{dx}\int_a^x f(t)dt = f(x)$.

Finally, the end of an example is denoted by the symbol ▭ and the beginning and end of a proof by the symbol ■.

1 Vectors

The idea of describing space in terms of coordinates played a major role in the development of mathematics and led to the ability to describe planes, spheres, and other geometric objects in terms of equations. In this chapter, we develop the tools necessary to formulate such equations, with the concept of a vector playing a key role. In the accompanying figure for example, we see that formulating an equation that characterizes points on a plane in space (such as a plane tangent to a sphere) requires only knowledge of a given point on the plane and a vector perpendicular to the plane.



1.1 Vectors in Two and Three Dimensions

For your study of the calculus of several variables, the notion of a vector is fundamental. As is the case for many of the concepts we shall explore, there are both *algebraic* and *geometric* points of view. You should become comfortable with both perspectives in order to solve problems effectively and to build on your basic understanding of the subject.

Vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$: The Algebraic Notion

DEFINITION 1.1 A **vector** in $\mathbf{R}^2$ is simply an ordered pair of real numbers. That is, a vector in $\mathbf{R}^2$ may be written as

$$(a_1, a_2) \quad (\text{e.g., } (1, 2) \text{ or } (\pi, 17)).$$

Similarly, a **vector** in $\mathbf{R}^3$ is simply an ordered triple of real numbers. That is, a vector in $\mathbf{R}^3$ may be written as

$$(a_1, a_2, a_3) \quad (\text{e.g., } (\pi, e, \sqrt{2})).$$

To emphasize that we want to consider the pair or triple of numbers as a single unit, we will use **boldface** letters; hence $\mathbf{a} = (a_1, a_2)$ or $\mathbf{a} = (a_1, a_2, a_3)$ will be our standard notation for vectors in $\mathbf{R}^2$ or $\mathbf{R}^3$. Whether we mean that $\mathbf{a}$ is a vector in $\mathbf{R}^2$ or in $\mathbf{R}^3$ will be clear from context (or else won't be important to the discussion). When doing handwritten work, it is difficult to "boldface" anything, so you'll want to put an arrow over the letter. Thus, $\vec{a}$ will mean the same thing as $\mathbf{a}$. Whatever notation you decide to use, it's important that you distinguish the *vector* $\mathbf{a}$ (or $\vec{a}$) from the *single real number* a. To contrast them with vectors, we will also refer to single real numbers as **scalars**.

In order to do anything interesting with vectors, it's necessary to develop some arithmetic operations for working with them. Before doing this, however, we need to know when two vectors are equal.

DEFINITION 1.2 Two vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ in $\mathbf{R}^2$ are **equal** if their corresponding components are equal, that is, if $a_1 = b_1$ and $a_2 = b_2$. The same definition holds for vectors in $\mathbf{R}^3$: $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are **equal** if their corresponding components are equal, that is, if $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$.

EXAMPLE 1 The vectors $\mathbf{a} = (1, 2)$ and $\mathbf{b} = (\frac{3}{3}, \frac{6}{3})$ are equal in $\mathbf{R}^2$, but $\mathbf{c} = (1, 2, 3)$ and $\mathbf{d} = (2, 3, 1)$ are *not* equal in $\mathbf{R}^3$. ◆

Next, we discuss the operations of vector addition and scalar multiplication. We'll do this by considering vectors in $\mathbf{R}^3$ only; exactly the same remarks will hold for vectors in $\mathbf{R}^2$ if we simply ignore the last component.

DEFINITION 1.3 (Vector addition) Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be two vectors in $\mathbf{R}^3$. Then the **vector sum** $\mathbf{a} + \mathbf{b}$ is the vector in $\mathbf{R}^3$ obtained via componentwise addition: $\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$.

EXAMPLE 2 We have $(0, 1, 3) + (7, -2, 10) = (7, -1, 13)$ and (in $\mathbf{R}^2$):

$$(1, 1) + (\pi, \sqrt{2}) = (1 + \pi, 1 + \sqrt{2}).$$ ◆

Properties of vector addition. We have

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ for all $\mathbf{a}$, $\mathbf{b}$ in $\mathbf{R}^3$ (commutativity);
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$ for all $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ in $\mathbf{R}^3$ (associativity);
3. a special vector, denoted $\mathbf{0}$ (and called the **zero vector**), with the property that $\mathbf{a} + \mathbf{0} = \mathbf{a}$ for all $\mathbf{a}$ in $\mathbf{R}^3$.

These three properties require proofs, which, like most facts involving the algebra of vectors, can be obtained by explicitly writing out the vector components. For example, for property 1, we have that if

$$\mathbf{a} = (a_1, a_2, a_3) \quad \text{and} \quad \mathbf{b} = (b_1, b_2, b_3),$$

then

$$\begin{aligned}\mathbf{a} + \mathbf{b} &= (a_1 + b_1, a_2 + b_2, a_3 + b_3)\\ &= (b_1 + a_1, b_2 + a_2, b_3 + a_3)\\ &= \mathbf{b} + \mathbf{a},\end{aligned}$$

since real number addition is commutative. For property 3, the "special vector" is just the vector whose components are all zero: $\mathbf{0} = (0, 0, 0)$. It's then easy to check that property 3 holds by writing out components. Similarly for property 2, so we leave the details as exercises.

DEFINITION 1.4 (Scalar multiplication) Let $\mathbf{a} = (a_1, a_2, a_3)$ be a vector in $\mathbf{R}^3$ and let $k \in \mathbf{R}$ be a scalar (real number). Then the **scalar product** $k\mathbf{a}$ is the vector in $\mathbf{R}^3$ given by multiplying each component of $\mathbf{a}$ by k: $k\mathbf{a} = (ka_1, ka_2, ka_3)$.

EXAMPLE 3 If $\mathbf{a} = (2, 0, \sqrt{2})$ and $k = 7$, then $k\mathbf{a} = (14, 0, 7\sqrt{2})$.

The results that follow are not difficult to check—just write out the vector components.

Properties of scalar multiplication. For all vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbf{R}^3$ (or $\mathbf{R}^2$) and scalars k and l in $\mathbf{R}$, we have

1. $(k + l)\mathbf{a} = k\mathbf{a} + l\mathbf{a}$ (distributivity);
2. $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$ (distributivity);
3. $k(l\mathbf{a}) = (kl)\mathbf{a} = l(k\mathbf{a})$.

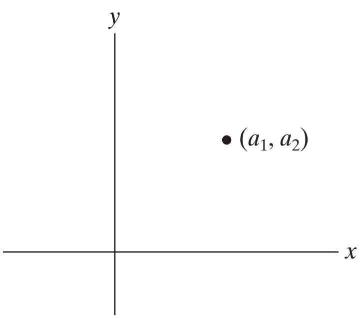


FIGURE 1.1 A vector $\mathbf{a} \in \mathbf{R}^2$ corresponds to a point in $\mathbf{R}^2$.

It is worth remarking that none of these definitions or properties really depends on dimension, that is, on the number of components. Therefore we could have introduced the algebraic concept of a vector in $\mathbf{R}^n$ as an **ordered n-tuple** $(a_1, a_2, \ldots, a_n)$ of real numbers and defined addition and scalar multiplication in a way analogous to what we did for $\mathbf{R}^2$ and $\mathbf{R}^3$. Think about what such a generalization means. We will discuss some of the technicalities involved in §1.6.

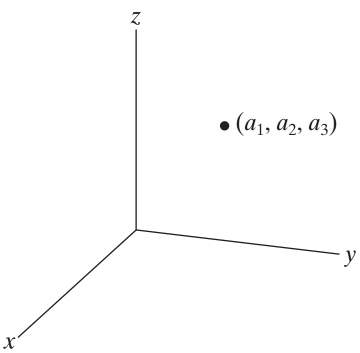


FIGURE 1.2 A vector $\mathbf{a} \in \mathbf{R}^3$ corresponds to a point in $\mathbf{R}^3$.

Vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$: The Geometric Notion

Although the algebra of vectors is certainly important and you should become adept at working algebraically, the formal definitions and properties tend to present a rather sterile picture of vectors. A better motivation for the definitions just given comes from geometry. We explore this geometry now. First of all, the fact that a vector $\mathbf{a}$ in $\mathbf{R}^2$ is a pair of real numbers (a_1, a_2) should make you think of the coordinates of a point in $\mathbf{R}^2$. (See Figure 1.1.) Similarly, if $\mathbf{a} \in \mathbf{R}^3$, then $\mathbf{a}$ may be written as (a_1, a_2, a_3), and this triple of numbers may be thought of as the coordinates of a point in $\mathbf{R}^3$. (See Figure 1.2.)

All of this is fine, but the results of performing vector addition or scalar multiplication don't have very interesting or meaningful geometric interpretations in terms of points. As we shall see, it is better to visualize a vector in $\mathbf{R}^2$ or $\mathbf{R}^3$ as an arrow that begins at the origin and ends at the point. (See Figure 1.3.)

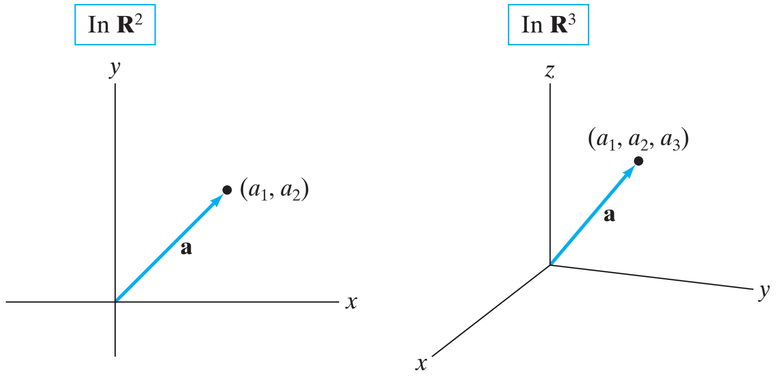


FIGURE 1.3 A vector **a** in $\mathbf{R}^2$ or $\mathbf{R}^3$ is represented by an arrow from the origin to **a**.

Such a depiction is often referred to as the **position vector** of the point (a_1, a_2) or (a_1, a_2, a_3).

If you've studied vectors in physics, you have heard them described as objects having "magnitude and direction." Figure 1.3 demonstrates this concept, provided that we take "magnitude" to mean "length of the arrow" and "direction" to be the orientation or sense of the arrow. (Note: There is an exception to this approach, namely, the zero vector. The zero vector just sits at the origin, like a point, and has no magnitude and, therefore, an indeterminate direction. This exception will not pose much difficulty.) However, in physics, one doesn't demand that all vectors be represented by arrows having their tails bound to the origin. One is free to "parallel translate" vectors throughout $\mathbf{R}^2$ and $\mathbf{R}^3$. That is, one may represent the vector $\mathbf{a} = (a_1, a_2, a_3)$ by an arrow with its tail at the origin (and its head at (a_1, a_2, a_3)) or with its tail at any other point, so long as the length and sense of the arrow are not disturbed. (See Figure 1.4.) For example, if we wish to represent **a** by an arrow with its tail at the point (x_1, x_2, x_3), then the head of the arrow would be at the point $(x_1 + a_1, x_2 + a_2, x_3 + a_3)$. (See Figure 1.5.)

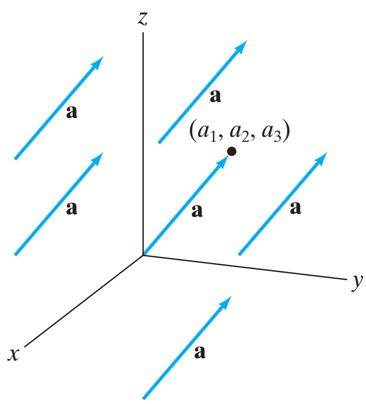


FIGURE 1.4 Each arrow is a parallel translate of the position vector of the point (a_1, a_2, a_3) and represents the same vector.

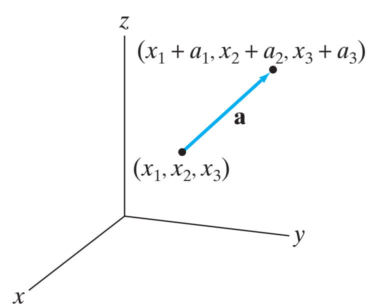


FIGURE 1.5 The vector $\mathbf{a} = (a_1, a_2, a_3)$ represented by an arrow with tail at the point (x_1, x_2, x_3).

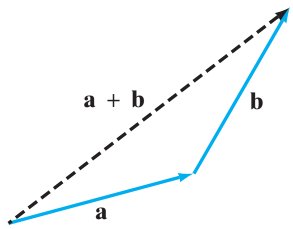


FIGURE 1.6 The vector **a** + **b** may be represented by an arrow whose tail is at the tail of **a** and whose head is at the head of **b**.

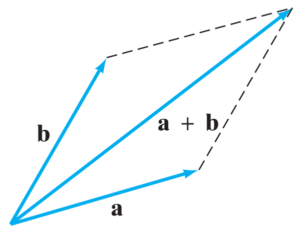


FIGURE 1.7 The vector **a** + **b** may be represented by the arrow that runs along the diagonal of the parallelogram determined by **a** and **b**.

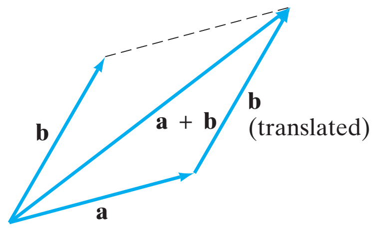


FIGURE 1.8 The equivalence of the parallelogram law and the head-to-tail methods of vector addition.

With this geometric description of vectors, vector addition can be visualized in two ways. The first is often referred to as the "head-to-tail" method for adding vectors. Draw the two vectors **a** and **b** to be added so that the tail of one of the vectors, say **b**, is at the head of the other. Then the vector sum **a** + **b** may be represented by an arrow whose tail is at the tail of **a** and whose head is at the head of **b**. (See Figure 1.6.) Note that it is *not* immediately obvious that **a** + **b** = **b** + **a** from this construction!

The second way to visualize vector addition is according to the so-called **parallelogram law**: If **a** and **b** are nonparallel vectors drawn with their tails emanating from the same point, then **a** + **b** may be represented by the arrow (with its tail at the common initial point of **a** and **b**) that runs along a diagonal of the parallelogram determined by **a** and **b** (Figure 1.7). The parallelogram law is completely consistent with the head-to-tail method. To see why, just parallel translate **b** to the opposite side of the parallelogram. Then the diagonal just described is the result of adding **a** and (the translate of) **b**, using the head-to-tail method. (See Figure 1.8.)

We still should check that these geometric constructions agree with our algebraic definition. For simplicity, we'll work in $\mathbf{R}^2$. Let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ as usual. Then the arrow obtained from the parallelogram law addition of **a** and **b** is the one whose tail is at the origin O and whose head is at the point P in Figure 1.9. If we parallel translate **b** so that its tail is at the head of **a**, then it is immediate that the coordinates of P must be $(a_1 + b_1, a_2 + b_2)$, as desired.

Scalar multiplication is easier to visualize: The vector $k\mathbf{a}$ may be represented by an arrow whose length is $|k|$ times the length of **a** and whose direction is the same as that of **a** when $k > 0$ and the opposite when $k < 0$. (See Figure 1.10.)

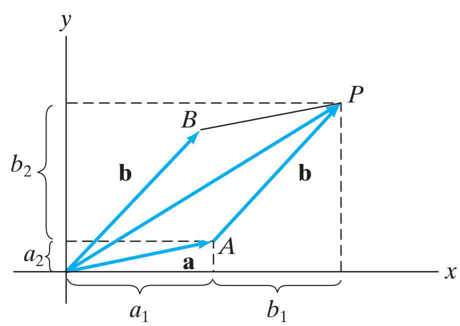


FIGURE 1.9 The point P has coordinates $(a_1 + b_1, a_2 + b_2)$.

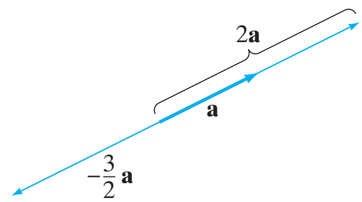


FIGURE 1.10 Visualization of scalar multiplication.

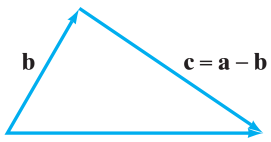


FIGURE 1.11 The geometry of vector subtraction. The vector $\mathbf{c}$ is such that $\mathbf{b} + \mathbf{c} = \mathbf{a}$. Hence, $\mathbf{c} = \mathbf{a} - \mathbf{b}$.

It is now a simple matter to obtain a geometric depiction of the **difference** between two vectors. (See Figure 1.11.) The difference $\mathbf{a} - \mathbf{b}$ is nothing more than $\mathbf{a} + (-\mathbf{b})$ (where $-\mathbf{b}$ means the scalar -1 times the vector $\mathbf{b}$). The vector $\mathbf{a} - \mathbf{b}$ may be represented by an arrow pointing from the head of $\mathbf{b}$ toward the head of $\mathbf{a}$; such an arrow is also a diagonal of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$. (As we have seen, the other diagonal can be used to represent $\mathbf{a} + \mathbf{b}$.) Note in Figure 1.11 that adding $\mathbf{b}$ to $\mathbf{c} = \mathbf{a} - \mathbf{b}$ using the "head-to-tail" method results in vector $\mathbf{a}$, precisely as one would expect of $\mathbf{b} + (\mathbf{a} - \mathbf{b})$.

Here is a construction that will be useful to us from time to time.

DEFINITION 1.5 Given two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in $\mathbf{R}^3$, the **displacement vector from P_1 to P_2** is

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1).$$

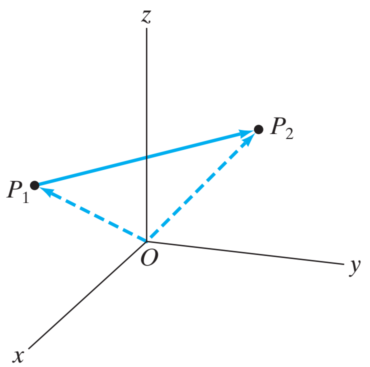


FIGURE 1.12 The displacement vector $\overrightarrow{P_1P_2}$, represented by the arrow from P_1 to P_2, is the difference between the position vectors of these two points.

This construction is not hard to understand if we consider Figure 1.12. Given the points P_1 and P_2, draw the corresponding position vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$. Then we see that $\overrightarrow{P_1P_2}$ is precisely $\overrightarrow{OP_2} - \overrightarrow{OP_1}$. An analogous definition may be made for $\mathbf{R}^2$.

In your study of the calculus of one variable, you no doubt used the notions of derivatives and integrals to look at such physical concepts as velocity, acceleration, force, etc. The main drawback of the work you did was that the techniques involved allowed you to study only *rectilinear,* or straight-line, activity. Intuitively, we all understand that motion in the plane or in space is more complicated than straight-line motion. Because vectors possess direction as well as magnitude, they are ideally suited for two- and three-dimensional dynamical problems.

For example, suppose a particle in space is at the point (a_1, a_2, a_3) (with respect to some appropriate coordinate system). Then it has position vector $\mathbf{a} = (a_1, a_2, a_3)$. If the particle travels with constant velocity $\mathbf{v} = (v_1, v_2, v_3)$ for t seconds, then the particle's displacement from its original position is $t\mathbf{v}$, and its new coordinate position is $\mathbf{a} + t\mathbf{v}$. (See Figure 1.13.)

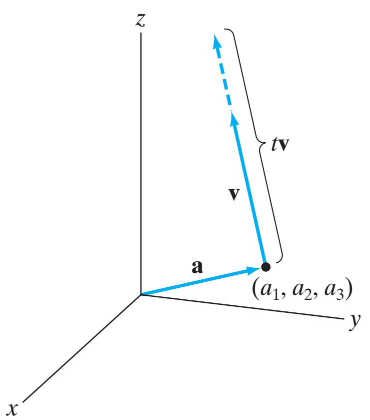


FIGURE 1.13 After t seconds, the point starting at $\mathbf{a}$, with velocity $\mathbf{v}$, moves to $\mathbf{a} + t\mathbf{v}$.

EXAMPLE 4 If a spaceship is at position $(100, 3, 700)$ and is traveling with velocity $(7, -10, 25)$ (meaning that the ship travels 7 mi/sec in the positive x-direction, 10 mi/sec in the negative y-direction, and 25 mi/sec in the positive z-direction), then after 20 seconds, the ship will be at position

$$(100, 3, 700) + 20(7, -10, 25) = (240, -197, 1200),$$

and the displacement from the initial position is $(140, -200, 500)$. ▭

EXAMPLE 5 The S.S. Calculus is cruising due south at a rate of 15 knots (nautical miles per hour) with respect to still water. However, there is also a current of $5\sqrt{2}$ knots southeast. What is the total velocity of the ship? If the ship is initially at the origin and a lobster pot is at position $(20, -79)$, will the ship collide with the lobster pot?

Since velocities are vectors, the total velocity of the ship is $\mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{v}_1$ is the velocity of the ship with respect to still water and $\mathbf{v}_2$ is the

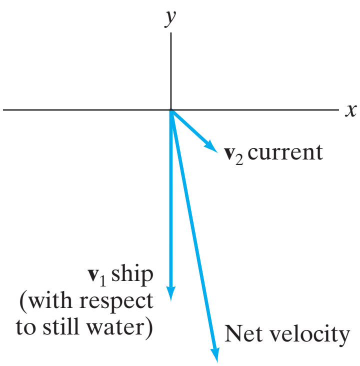


FIGURE 1.14 The length of $\mathbf{v}_1$ is 15, and the length of $\mathbf{v}_2$ is $5\sqrt{2}$.

southeast-pointing velocity of the current. Figure 1.14 makes it fairly straightforward to compute these velocities. We have that $\mathbf{v}_1 = (0, -15)$. Since $\mathbf{v}_2$ points southeastward, its direction must be along the line $y = -x$. Therefore, $\mathbf{v}_2$ can be written as $\mathbf{v}_2 = (v, -v)$, where v is a positive real number. By the Pythagorean theorem, if the length of $\mathbf{v}_2$ is $5\sqrt{2}$, then we must have $v^2 + (-v)^2 = (5\sqrt{2})^2$ or $2v^2 = 50$, so that $v = 5$. Thus, $\mathbf{v}_2 = (5, -5)$, and, hence, the net velocity is

$$(0, -15) + (5, -5) = (5, -20).$$

After 4 hours, therefore, the ship will be at position

$$(0, 0) + 4(5, -20) = (20, -80)$$

and thus will miss the lobster pot. ■

EXAMPLE 6 The theory behind the venerable martial art of judo is an excellent example of vector addition. If two people, one relatively strong and the other relatively weak, have a shoving match, it is clear who will prevail. For example, someone pushing one way with 200 lb of force will certainly succeed in overpowering another pushing the opposite way with 100 lb of force. Indeed, as Figure 1.15 shows, the net force will be 100 lb in the direction in which the stronger person is pushing.

FIGURE 1.15 A relatively strong person pushing with a force of 200 lb can quickly subdue a relatively weak one pushing with only 100 lb of force.

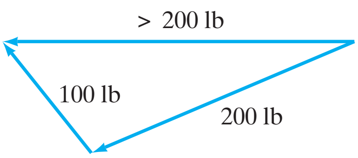


FIGURE 1.16 Vector addition in judo.

Dr. Jigoro Kano, the founder of judo, realized (though he never expressed his idea in these terms) that this sort of vector addition favors the strong over the weak. However, if weaker participants apply their 100 lb of force in a direction only slightly different from that of a stronger one, they will effect a vector sum of length large enough to surprise the opponent. (See Figure 1.16.) This is the basis for essentially all of the throws of judo and why judo is described as the art of "using a person's strength against oneself." In fact, the word "judo" means "the way of gentleness" or "the way of giving in." One "gives in" to the strength of another by attempting only to redirect his or her force rather than to oppose it. ■

1.1 Exercises

1. Sketch the following vectors in $\mathbf{R}^2$:

(a) $(2, 1)$

(b) $(3, 3)$

(c) $(-1, 2)$

2. Sketch the following vectors in $\mathbf{R}^3$:

(a) $(1, 2, 3)$

(b) $(-2, 0, 2)$

(c) $(2, -3, 1)$

3. Perform the indicated algebraic operations. Express your answers in the form of a single vector $\mathbf{a} = (a_1, a_2)$ in $\mathbf{R}^2$.

(a) $(3, 1) + (-1, 7)$

(b) $-2(8, 12)$

(c) $(8, 9) + 3(-1, 2)$

(d) $(1, 1) + 5(2, 6) - 3(10, 2)$

(e) $(8, 10) + 3((8, -2) - 2(4, 5))$

4. Perform the indicated algebraic operations. Express your answers in the form of a single vector $\mathbf{a} = (a_1, a_2, a_3)$ in $\mathbf{R}^3$.

(a) $(2, 1, 2) + (-3, 9, 7)$

(b) $\frac{1}{2}(8, 4, 1) + 2(5, -7, \frac{1}{4})$

(c) $-2((2, 0, 1) - 6(\frac{1}{2}, -4, 1))$

5. Graph the vectors $\mathbf{a} = (1, 2)$, $\mathbf{b} = (-2, 5)$, and $\mathbf{a} + \mathbf{b} = (1, 2) + (-2, 5)$, using both the parallelogram law and the head-to-tail method.

6. Graph the vectors $\mathbf{a} = (3, 2)$ and $\mathbf{b} = (-1, 1)$. Also calculate and graph $\mathbf{a} - \mathbf{b}$, $\frac{1}{2}\mathbf{a}$, and $\mathbf{a} + 2\mathbf{b}$.

7. Let A be the point with coordinates $(1, 0, 2)$, let B be the point with coordinates $(-3, 3, 1)$, and let C be the point with coordinates $(2, 1, 5)$.

(a) Describe the vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$.

(b) Describe the vectors $\overrightarrow{AC}$, $\overrightarrow{BC}$, and $\overrightarrow{AC} + \overrightarrow{CB}$.

(c) Explain, with pictures, why $\overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB}$.

8. Graph $(1, 2, 1)$ and $(0, -2, 3)$, and calculate and graph $(1, 2, 1) + (0, -2, 3)$, $-1(1, 2, 1)$, and $4(1, 2, 1)$.

9. If $(-12, 9, z) + (x, 7, -3) = (2, y, 5)$, what are x, y, and z?

10. What is the length (magnitude) of the vector $(3, 1)$? (Hint: A diagram will help.)

11. Sketch the vectors $\mathbf{a} = (1, 2)$ and $\mathbf{b} = (5, 10)$. Explain why $\mathbf{a}$ and $\mathbf{b}$ point in the same direction.

12. Sketch the vectors $\mathbf{a} = (2, -7, 8)$ and $\mathbf{b} = (-1, \frac{7}{2}, -4)$. Explain why $\mathbf{a}$ and $\mathbf{b}$ point in opposite directions.

13. How would you add the vectors $(1, 2, 3, 4)$ and $(5, -1, 2, 0)$ in $\mathbf{R}^4$? What should $2(7, 6, -3, 1)$ be? In general, suppose that

$$\mathbf{a} = (a_1, a_2, \ldots, a_n) \quad \text{and} \quad \mathbf{b} = (b_1, b_2, \ldots, b_n)$$

are two vectors in $\mathbf{R}^n$ and $k \in \mathbf{R}$ is a scalar. Then how would you define $\mathbf{a} + \mathbf{b}$ and $k\mathbf{a}$?

14. Find the displacement vectors from P_1 to P_2, where P_1 and P_2 are the points given. Sketch P_1, P_2, and $\overrightarrow{P_1P_2}$.

(a) $P_1(1, 0, 2)$, $P_2(2, 1, 7)$

(b) $P_1(1, 6, -1)$, $P_2(0, 4, 2)$

(c) $P_1(0, 4, 2)$, $P_2(1, 6, -1)$

(d) $P_1(3, 1)$, $P_2(2, -1)$

15. Let $P_1(2, 5, -1, 6)$ and $P_2(3, 1, -2, 7)$ be two points in $\mathbf{R}^4$. How would you define and calculate the displacement vector from P_1 to P_2? (See Exercise 13.)

16. If A is the point in $\mathbf{R}^3$ with coordinates $(2, 5, -6)$ and the displacement vector from A to a second point B is $(12, -3, 7)$, what are the coordinates of B?

17. Suppose that you and your friend are in New York talking on cellular phones. You inform each other of your own displacement vectors from the Empire State Building to your current position. Explain how you can use this information to determine the displacement vector from you to your friend.

18. Give the details of the proofs of properties 2 and 3 of vector addition given in this section.

19. Prove the properties of scalar multiplication given in this section.

20. (a) If $\mathbf{a}$ is a vector in $\mathbf{R}^2$ or $\mathbf{R}^3$, what is $0\mathbf{a}$? Prove your answer.

(b) If $\mathbf{a}$ is a vector in $\mathbf{R}^2$ or $\mathbf{R}^3$, what is $1\mathbf{a}$? Prove your answer.

21. (a) Let $\mathbf{a} = (2, 0)$ and $\mathbf{b} = (1, 1)$. For $0 \le s \le 1$ and $0 \le t \le 1$, consider the vector $\mathbf{x} = s\mathbf{a} + t\mathbf{b}$. Explain why the vector $\mathbf{x}$ lies in the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$. (Hint: It may help to draw a picture.)

(b) Now suppose that $\mathbf{a} = (2, 2, 1)$ and $\mathbf{b} = (0, 3, 2)$. Describe the set of vectors $\{\mathbf{x} = s\mathbf{a} + t\mathbf{b} \mid 0 \le s \le 1, 0 \le t \le 1\}$.

22. Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be two nonzero vectors such that $\mathbf{b} \ne k\mathbf{a}$. Use vectors to describe the set of points inside the parallelogram with vertex $P_0(x_0, y_0, z_0)$ and whose adjacent sides are parallel to $\mathbf{a}$ and $\mathbf{b}$ and have the same lengths as $\mathbf{a}$ and $\mathbf{b}$. (See Figure 1.17.) (Hint: If $P(x, y, z)$ is a point in the parallelogram, describe $\overrightarrow{OP}$, the position vector of P.)

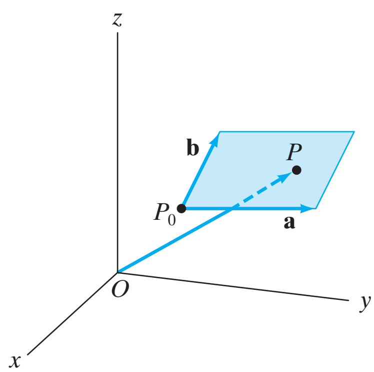


FIGURE 1.17 Figure for Exercise 22.

23. A flea falls onto marked graph paper at the point $(3, 2)$. She begins moving from that point with velocity vector $\mathbf{v} = (-1, -2)$ (i.e., she moves 1 graph paper unit per minute in the negative x-direction and 2 graph paper units per minute in the negative y-direction).

(a) What is the speed of the flea?

(b) Where is the flea after 3 minutes?

(c) How long does it take the flea to get to the point $(-4, -12)$?

(d) Does the flea reach the point $(-13, -27)$? Why or why not?

24. A plane takes off from an airport with velocity vector $(50, 100, 4)$. Assume that the units are miles per hour, that the positive x-axis points east, and that the positive y-axis points north.

(a) How fast is the plane climbing vertically at take-off?

(b) Suppose the airport is located at the origin and a skyscraper is located 5 miles east and 10 miles north of the airport. The skyscraper is 1,250 feet tall. When will the plane be directly over the building?

(c) When the plane is over the building, how much vertical clearance is there?

25. As mentioned in the text, physical forces (e.g., gravity) are quantities possessing both magnitude and direction and therefore can be represented by vectors. If an object has more than one force acting on it, then the **resultant** (or **net**) **force** can be represented by the sum of the individual force vectors. Suppose that two forces, $\mathbf{F}_1 = (2, 7, -1)$ and $\mathbf{F}_2 = (3, -2, 5)$, act on an object.

(a) What is the resultant force of $\mathbf{F}_1$ and $\mathbf{F}_2$?

(b) What force $\mathbf{F}_3$ is needed to counteract these forces (i.e., so that *no* net force results and the object remains at rest)?

26. A 50 lb sandbag is suspended by two ropes. Suppose that a three-dimensional coordinate system is introduced so that the sandbag is at the origin and the ropes are anchored at the points $(0, -2, 1)$ and $(0, 2, 1)$.

(a) Assuming that the force due to gravity points parallel to the vector $(0, 0, -1)$, give a vector $\mathbf{F}$ that describes this gravitational force.

(b) Now, use vectors to describe the forces along each of the two ropes. Use symmetry considerations and draw a figure of the situation.

27. A 10 lb weight is suspended in equilibrium by two ropes. Assume that the weight is at the point $(1, 2, 3)$ in a three-dimensional coordinate system, where the positive z-axis points straight up, perpendicular to the ground, and that the ropes are anchored at the points $(3, 0, 4)$ and $(0, 3, 5)$. Give vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ that describe the forces along the ropes.

1.2 More About Vectors

The Standard Basis Vectors

In $\mathbf{R}^2$, the vectors $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ play a special notational role. Any vector $\mathbf{a} = (a_1, a_2)$ may be written in terms of $\mathbf{i}$ and $\mathbf{j}$ via vector addition and scalar multiplication:

$$(a_1, a_2) = (a_1, 0) + (0, a_2) = a_1(1, 0) + a_2(0, 1) = a_1\mathbf{i} + a_2\mathbf{j}.$$

(It may be easier to follow this argument by reading it in reverse.) Insofar as notation goes, the preceding work simply establishes that one can write either (a_1, a_2) or $a_1\mathbf{i} + a_2\mathbf{j}$ to denote the vector $\mathbf{a}$. It's your choice which notation to use (as long as you're consistent), but the **ij**-notation is generally useful for emphasizing the "vector" nature of **a**, while the coordinate notation is more useful for emphasizing the "point" nature of **a** (in the sense of **a**'s role as a possible position vector of a point). Geometrically, the significance of the **standard basis vectors i** and **j** is that an arbitrary vector $\mathbf{a} \in \mathbf{R}^2$ can be decomposed pictorially into appropriate **vector components** along the x- and y-axes, as shown in Figure 1.18.

Exactly the same situation occurs in $\mathbf{R}^3$, except that we need three vectors, $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$, to form the standard basis. (See Figure 1.19.) The same argument as the one just given can be used to show that any vector $\mathbf{a} = (a_1, a_2, a_3)$ may also be written as $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$. We shall use both coordinate and standard basis notation throughout this text.

EXAMPLE 1 We may write the vector $(1, -2)$ as $\mathbf{i} - 2\mathbf{j}$ and the vector $(7, \pi, -3)$ as $7\mathbf{i} + \pi\mathbf{j} - 3\mathbf{k}$. ▭

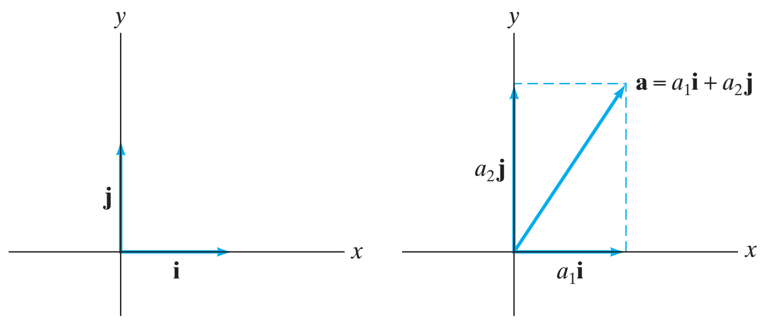


FIGURE 1.18 Any vector in $\mathbf{R}^2$ can be written in terms of **i** and **j**.

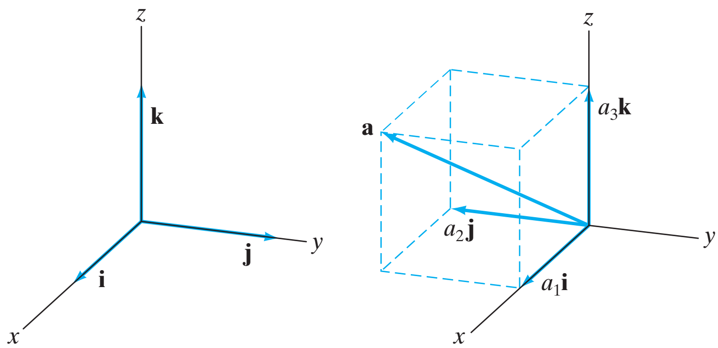


FIGURE 1.19 Any vector in $\mathbf{R}^3$ can be written in terms of **i**, **j**, and **k**.

Parametric Equations of Lines

In $\mathbf{R}^2$, we know that equations of the form $y = mx + b$ or $Ax + By = C$ describe straight lines. (See Figure 1.20.) Consequently, one might expect the same sort of equation to define a line in $\mathbf{R}^3$ as well. Consideration of a simple example or two (such as in Figure 1.21) should convince you that a single such linear equation describes a plane, not a line; in particular, one can always find three noncollinear points (x, y, z) in $\mathbf{R}^3$ satisfying the equation $Ax + By + Cz = D$, so such an equation certainly does not describe a line. A pair of simultaneous equations in x, y, and z is required to define a line.

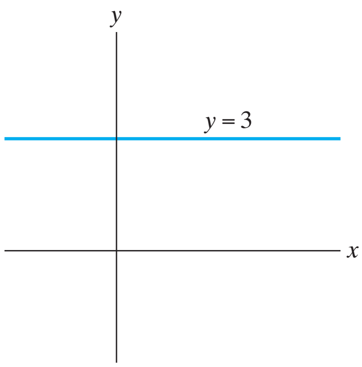


FIGURE 1.20 In $\mathbf{R}^2$, the equation $y = 3$ describes a line.

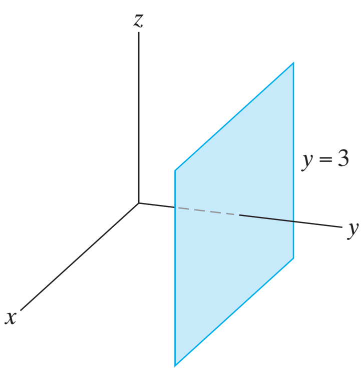


FIGURE 1.21 In $\mathbf{R}^3$, the equation $y = 3$ describes a plane.

We postpone discussing the derivation of equations for planes until §1.5 and concentrate here on using vectors to give sets of parametric equations for lines in $\mathbf{R}^2$ or $\mathbf{R}^3$ (or even $\mathbf{R}^n$).

First, we remark that a curve in the plane may be described analytically by points (x, y), where x and y are given as functions of a third variable (the **parameter**) t. These functions give rise to **parametric equations** for the curve:

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}.$$

As t varies, the points $(x, y) = (f(t), g(t))$ described by these equations trace out the curve in question.

EXAMPLE 2 The set of equations

$$\begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases} \quad 0 \le t < 2\pi$$

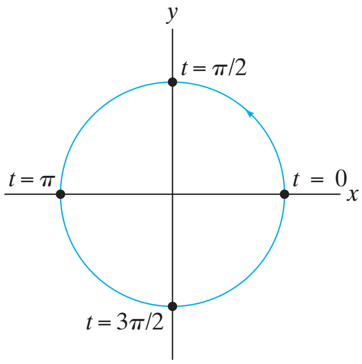


FIGURE 1.22 The graph of the parametric equations $x = 2\cos t$, $y = 2\sin t, 0 \leq t < 2\pi$.

describes a circle of radius 2, since we may check that

$$x^2 + y^2 = (2\cos t)^2 + (2\sin t)^2 = 4.$$

(See Figure 1.22.) In this case, the points $(2\cos t, 2\sin t)$ trace out the circle visually in a counterclockwise direction as t increases from 0 to 2π. ▭

Parametric equations may be used as readily to describe curves in $\mathbf{R}^3$; a curve in $\mathbf{R}^3$ is the set of points (x, y, z) whose coordinates x, y, and z are each given by a function of t:

$$\begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \end{cases}.$$

The advantages of using parametric equations are twofold. First, they offer a uniform way of describing curves in any number of dimensions. (How would you define parametric equations for a curve in $\mathbf{R}^4$? In $\mathbf{R}^{128}$?) Second, they allow you to get a dynamic sense of a curve if you consider the parameter variable t to represent time and imagine that a particle is traveling along the curve with time according to the given parametric equations. You can represent this geometrically by assigning a "direction" to the curve to signify increasing t. Notice the arrow in Figure 1.22.

Now, we see how to provide equations for lines. First, convince yourself that a line in $\mathbf{R}^2$ or $\mathbf{R}^3$ is uniquely determined by two pieces of geometric information: (1) a vector whose direction is parallel to that of the line and (2) any particular point lying on the line—see Figure 1.23. In Figure 1.24, we seek the vector

$$\mathbf{r} = \overrightarrow{OP}$$

between the origin O and an arbitrary point P on the line l (i.e., the position vector of $P(x, y, z)$). $\overrightarrow{OP}$ is the vector sum of the position vector $\mathbf{b}$ of the given point P_0 (i.e., $\overrightarrow{OP_0}$) and a vector parallel to $\mathbf{a}$. Any vector parallel to $\mathbf{a}$ must be a scalar multiple of $\mathbf{a}$. Letting this scalar be the parameter variable t, we have

$$\mathbf{r} = \overrightarrow{OP} = \overrightarrow{OP_0} + t\mathbf{a},$$

and we have established the following proposition:

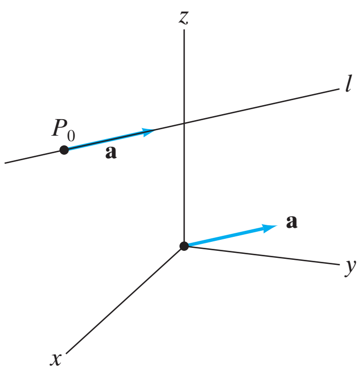


FIGURE 1.23 The line l is the unique line passing through P_0 and parallel to the vector $\mathbf{a}$.

PROPOSITION 2.1 The vector parametric equation for the line through the point $P_0(b_1, b_2, b_3)$, whose position vector is $\overrightarrow{OP_0} = \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and parallel to $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is

$$\mathbf{r}(t) = \mathbf{b} + t\mathbf{a}. \tag{1}$$

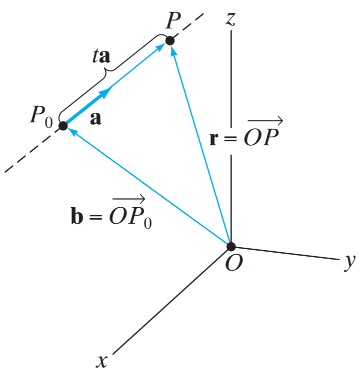


FIGURE 1.24 The graph of a line in $\mathbf{R}^3$.

Note that it is the endpoint of the vector $\mathbf{r}(t)$ that gives a point on the line being described.

Expanding formula (1),

$$\begin{aligned} \mathbf{r}(t) = \overrightarrow{OP} &= b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} + t(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \\ &= (a_1 t + b_1)\mathbf{i} + (a_2 t + b_2)\mathbf{j} + (a_3 t + b_3)\mathbf{k}. \end{aligned}$$

Next, write $\overrightarrow{OP}$ as $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ so that P has coordinates (x, y, z). Then, extracting components, we see that the coordinates of P are $(a_1t + b_1, a_2t + b_2, a_3t + b_3)$ and our parametric equations are

$$\begin{cases} x = a_1t + b_1 \\ y = a_2t + b_2 \\ z = a_3t + b_3 \end{cases}, \tag{2}$$

where t is any real number. Note that these equations are linear in t, in that t only appears with exponent at most one. Parametric equations of this form will always describe a line, but that is not to say that other types of parametric equations could not also describe a line. (See Exercises 32 and 33.)

These parametric equations work just as well in $\mathbf{R}^2$ (if we ignore the z-component) or in $\mathbf{R}^n$ where n is arbitrary. In $\mathbf{R}^n$, formula (1) remains valid, where we take $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$. The resulting parametric equations are

$$\begin{cases} x_1 = a_1t + b_1 \\ x_2 = a_2t + b_2 \\ \quad\vdots \\ x_n = a_nt + b_n \end{cases}.$$

EXAMPLE 3 To find the parametric equations of the line through $(1, -2, 3)$ and parallel to the vector $\pi\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, we have $\mathbf{a} = \pi\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ so that formula (1) yields

$$\begin{aligned} \mathbf{r}(t) &= \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + t(\pi\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \\ &= (1 + \pi t)\mathbf{i} + (-2 - 3t)\mathbf{j} + (3 + t)\mathbf{k}. \end{aligned}$$

The parametric equations may be read as

$$\begin{cases} x = \pi t + 1 \\ y = -3t - 2 \\ z = t + 3 \end{cases}.$$

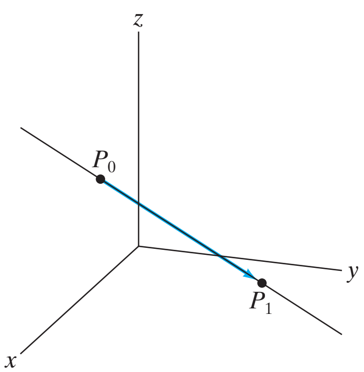


FIGURE 1.25 Finding equations for a line through two points in Example 4.

EXAMPLE 4 From Euclidean geometry, two distinct points determine a unique line in $\mathbf{R}^2$ or $\mathbf{R}^3$. Let's find the parametric equations of the line through the points $P_0(1, -2, 3)$ and $P_1(0, 5, -1)$. The situation is suggested by Figure 1.25. To use formula (1), we need to find a vector $\mathbf{a}$ parallel to the desired line. The vector with tail at P_0 and head at P_1 is such a vector. That is, we may use for $\mathbf{a}$ the vector

$$\overrightarrow{P_0P_1} = (0 - 1, 5 - (-2), -1 - 3) = -\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}.$$

For $\mathbf{b}$, the position vector of a particular point on the line, we have the choice of taking either $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ or $\mathbf{b} = 5\mathbf{j} - \mathbf{k}$. Hence, the equations in (2) yield parametric equations

$$\begin{cases} x = 1 - t \\ y = -2 + 7t \\ z = 3 - 4t \end{cases} \quad \text{or} \quad \begin{cases} x = -t \\ y = 5 + 7t \\ z = -1 - 4t \end{cases}.$$

In general, given two *arbitrary* points

$$P_0(a_1, a_2, a_3) \quad \text{and} \quad P_1(b_1, b_2, b_3),$$

the line joining them has vector parametric equation

$$\mathbf{r}(t) = \overrightarrow{OP_0} + t\overrightarrow{P_0P_1}. \tag{3}$$

Equation (3) gives parametric equations

$$\begin{cases} x = a_1 + (b_1 - a_1)t \\ y = a_2 + (b_2 - a_2)t. \\ z = a_3 + (b_3 - a_3)t \end{cases} \tag{4}$$

Alternatively, in place of equation (3), we could use the vector equation

$$\mathbf{r}(t) = \overrightarrow{OP_1} + t\overrightarrow{P_0P_1}, \tag{5}$$

or perhaps

$$\mathbf{r}(t) = \overrightarrow{OP_1} + t\overrightarrow{P_1P_0}, \tag{6}$$

each of which gives rise to somewhat different sets of parametric equations. Again, we refer you to Figure 1.25 for an understanding of the vector geometry involved.

Example 4 brings up an important point, namely, that parametric equations for a line (or, more generally, for any curve) are *never* unique. In fact, the two sets of equations calculated in Example 4 are by no means the only ones; we could have taken $\mathbf{a} = \overrightarrow{P_1P_0} = \mathbf{i} - 7\mathbf{j} + 4\mathbf{k}$ or any nonzero scalar multiple of $\overrightarrow{P_0P_1}$ for $\mathbf{a}$.

If parametric equations are not determined uniquely, then how can you check your work? In general, this is not so easy to do, but in the case of lines, there are two approaches to take. One is to produce two points that lie on the line specified by the first set of parametric equations and see that these points lie on the line given by the second set of parametric equations. The other approach is to use the parametric equations to find what is called the **symmetric form** of a line in $\mathbf{R}^3$. From the equations in (2), assuming that each a_i is nonzero, one can eliminate the parameter variable t in each equation to obtain:

$$\begin{cases} t = \dfrac{x - b_1}{a_1} \\[2ex] t = \dfrac{y - b_2}{a_2} \\[2ex] t = \dfrac{z - b_3}{a_3} \end{cases}.$$

The symmetric form is

$$\frac{x - b_1}{a_1} = \frac{y - b_2}{a_2} = \frac{z - b_3}{a_3}, \tag{7}$$

where points (x, y, z) on the line are those that satisfy these equations simultaneously.

In Example 4, the two sets of parametric equations give rise to corresponding symmetric forms

$$\frac{x-1}{-1}=\frac{y+2}{7}=\frac{z-3}{-4} \quad \text{and} \quad \frac{x}{-1}=\frac{y-5}{7}=\frac{z+1}{-4}.$$

It's not difficult to see that adding 1 to each "side" of the second symmetric form yields the first one. In general, symmetric forms for lines can differ only by a constant term or constant scalar multiples (or both).

The symmetric form is really a set of two simultaneous equations in $\mathbf{R}^3$. For example, the information in (7) can also be written as

$$\begin{cases} \dfrac{x-b_1}{a_1}=\dfrac{y-b_2}{a_2} \\ \dfrac{x-b_1}{a_1}=\dfrac{z-b_3}{a_3} \end{cases}.$$

This illustrates that we require two "scalar" equations in x, y, and z to describe a line in $\mathbf{R}^3$, although a single vector parametric equation, formula (1), is sufficient.

The next two examples illustrate how to use parametric equations for lines to identify the intersection of a line and a plane or of two lines.

EXAMPLE 5 We find where the line with parametric equations

$$\begin{cases} x=t+5 \\ y=-2t-4 \\ z=3t+7 \end{cases}$$

intersects the plane $3x+2y-7z=2$. (We will see precisely why this is a plane in §1.5.)

To locate the point of intersection, we must find what value of the parameter t gives a point on the line that also lies in the plane. This is readily accomplished by substituting the parametric values for x, y, and z from the line into the equation for the plane

$$3(t+5)+2(-2t-4)-7(3t+7)=2. \qquad (8)$$

Solving equation (8) for t, we find that $t=-2$. Setting t equal to -2 in the parametric equations for the line yields the point $(3, 0, 1)$, which, indeed, lies in the plane as well.

EXAMPLE 6 We determine whether and where the two lines

$$\begin{cases} x=t+1 \\ y=5t+6 \\ z=-2t \end{cases} \quad \text{and} \quad \begin{cases} x=3t-3 \\ y=t \\ z=t+1 \end{cases}$$

intersect.

The lines intersect provided that there is a specific value t_1 for the parameter of the first line and a value t_2 for the parameter of the second line that generate the same point. In other words, we must be able to find t_1 and t_2 so that, by equating the respective parametric expressions for x, y, and z, we have

$$\begin{cases} t_1+1=3t_2-3 \\ 5t_1+6=t_2 \\ -2t_1=t_2+1 \end{cases}. \qquad (9)$$

Let us emphasize here that the values of t_1 and t_2 are not required to be the same. If we think of t as representing a time parameter, the time at which a point on the first line reaches the intersection might be different from the time at which a point on the second line reaches it.

The last two equations of (9) yield

$$t_2 = 5t_1 + 6 = -2t_1 - 1 \quad \Rightarrow \quad t_1 = -1.$$

Using $t_1 = -1$ in the second equation of (9), we find that $t_2 = 1$. Note that the values $t_1 = -1$ and $t_2 = 1$ also satisfy the first equation of (9); therefore, we have solved the system. Setting $t = -1$ in the set of parametric equations for the first line gives the desired intersection point, namely, $(0, 1, 2)$.

Parametric Equations in General

Vector geometry makes it relatively easy to find parametric equations for a variety of curves. We provide two examples.

EXAMPLE 7 If a wheel rolls along a flat surface without slipping, a point on the rim of the wheel traces a curve called a **cycloid**, as shown in Figure 1.26.

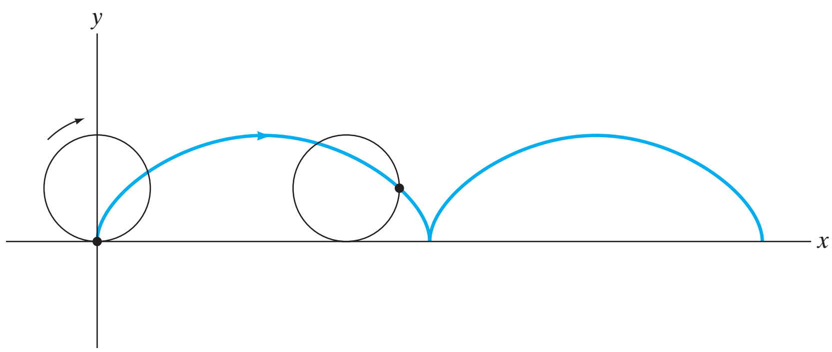


FIGURE 1.26 The graph of a cycloid.

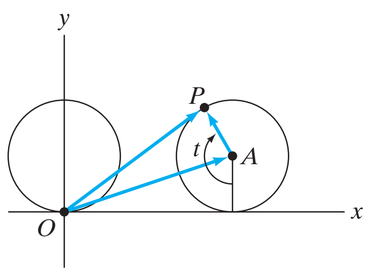


FIGURE 1.27 The result of the wheel in Figure 1.26 rolling through a central angle of t.

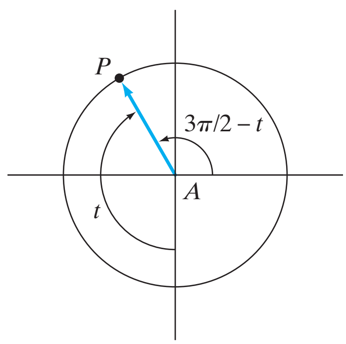


FIGURE 1.28 $\overrightarrow{AP}$ with its tail at the origin.

Suppose that the wheel has radius a and that coordinates in $\mathbf{R}^2$ are chosen so that the point of interest on the wheel is initially at the origin. After the wheel has rolled through a central angle of t radians, the situation is as shown in Figure 1.27. We seek the vector $\overrightarrow{OP}$, the position vector of P, in terms of the parameter t. Evidently, $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$, where the point A is the center of the wheel. The vector $\overrightarrow{OA}$ is not difficult to determine. Its $\mathbf{j}$-component must be a, since the center of the wheel does not vary vertically. Its $\mathbf{i}$-component must equal the distance the wheel has rolled; if t is measured in radians, then this distance is at, the length of the arc of the circle having central angle t. Hence, $\overrightarrow{OA} = at\mathbf{i} + a\mathbf{j}$.

The value of vector methods becomes apparent when we determine $\overrightarrow{AP}$. Parallel translate the picture so that $\overrightarrow{AP}$ has its tail at the origin, as in Figure 1.28. From the parametric equations of a circle of radius a,

$$\overrightarrow{AP} = a\cos\left(\frac{3\pi}{2} - t\right)\mathbf{i} + a\sin\left(\frac{3\pi}{2} - t\right)\mathbf{j} = -a\sin t\,\mathbf{i} - a\cos t\,\mathbf{j},$$

from the addition formulas for sine and cosine. We conclude that

$$\begin{aligned}\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} &= (at\mathbf{i} + a\mathbf{j}) + (-a\sin t\mathbf{i} - a\cos t\mathbf{j}) \\ &= a(t - \sin t)\mathbf{i} + a(1 - \cos t)\mathbf{j},\end{aligned}$$

so the parametric equations are

$$\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}.$$

EXAMPLE 8 If you unwind adhesive tape from a nonrotating circular tape dispenser so that the unwound tape is held taut and tangent to the dispenser roll, then the end of the tape traces a curve called the **involute** of the circle. Let's find the parametric equations for this curve, assuming that the dispensing roll has constant radius a and is centered at the origin. (As more and more tape is unwound, the radius of the roll will, of course, decrease. We'll assume that little enough tape is unwound so that the radius of the roll remains constant.)

Considering Figure 1.29, we see that the position vector $\overrightarrow{OP}$ of the desired point P is the vector sum $\overrightarrow{OB} + \overrightarrow{BP}$. To determine $\overrightarrow{OB}$ and $\overrightarrow{BP}$, we use the angle θ between the positive x-axis and $\overrightarrow{OB}$ as our parameter. Since B is a point on the circle,

$$\overrightarrow{OB} = a\cos\theta\,\mathbf{i} + a\sin\theta\,\mathbf{j}.$$

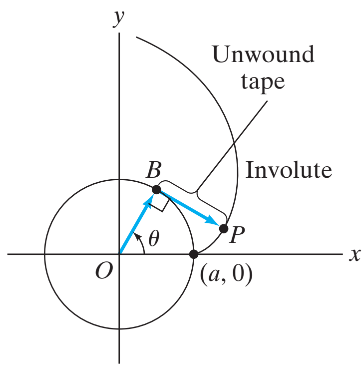


FIGURE 1.29 Unwinding tape, as in Example 8. The point P describes a curve known as the **involute** of the circle.

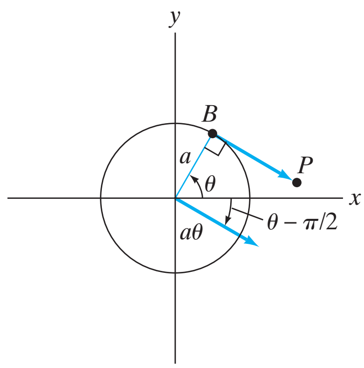


FIGURE 1.30 The vector $\overrightarrow{BP}$ must make an angle of $\theta - \pi/2$ with the positive x-axis.

To find the vector $\overrightarrow{BP}$, parallel translate it so that its tail is at the origin. Figure 1.30 shows that $\overrightarrow{BP}$'s length must be $a\theta$, the amount of unwound tape, and its direction must be such that it makes an angle of $\theta - \pi/2$ with the positive x-axis. From our experience with circular geometry and, perhaps, polar coordinates, we see that $\overrightarrow{BP}$ is described by

$$\overrightarrow{BP} = a\theta\cos\left(\theta - \frac{\pi}{2}\right)\mathbf{i} + a\theta\sin\left(\theta - \frac{\pi}{2}\right)\mathbf{j} = a\theta\sin\theta\,\mathbf{i} - a\theta\cos\theta\,\mathbf{j}.$$

Hence,

$$\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BP} = a(\cos\theta + \theta\sin\theta)\mathbf{i} + a(\sin\theta - \theta\cos\theta)\mathbf{j}.$$

So

$$\begin{cases} x = a(\cos\theta + \theta\sin\theta) \\ y = a(\sin\theta - \theta\cos\theta) \end{cases}$$

are the parametric equations of the involute, whose graph is pictured in Figure 1.31.

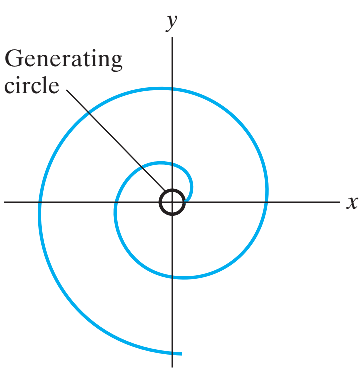


FIGURE 1.31 The involute.

1.2 Exercises

In Exercises 1–5, write the given vector by using the standard basis vectors for $\mathbf{R}^2$ *and* $\mathbf{R}^3$.

1. $(2, 4)$ **2.** $(9, -6)$ **3.** $(3, \pi, -7)$

4. $(-1, 2, 5)$ **5.** $(2, 4, 0)$

In Exercises 6–10, write the given vector without using the standard basis notation.

6. $\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ **7.** $9\mathbf{i} - 2\mathbf{j} + \sqrt{2}\mathbf{k}$

8. $-3(2\mathbf{i} - 7\mathbf{k})$

9. $\pi\mathbf{i} - \mathbf{j}$ (Consider this to be a vector in $\mathbf{R}^2$.)

10. $\pi\mathbf{i} - \mathbf{j}$ (Consider this to be a vector in $\mathbf{R}^3$.)

11. Let $\mathbf{a}_1 = (1, 1)$ and $\mathbf{a}_2 = (1, -1)$.

(a) Write the vector $\mathbf{b} = (3, 1)$ as $c_1\mathbf{a}_1 + c_2\mathbf{a}_2$, where c_1 and c_2 are appropriate scalars.

(b) Repeat part (a) for the vector $\mathbf{b} = (3, -5)$.

(c) Show that *any* vector $\mathbf{b} = (b_1, b_2)$ in $\mathbf{R}^2$ may be written in the form $c_1\mathbf{a}_1 + c_2\mathbf{a}_2$ for appropriate choices of the scalars c_1, c_2. (This shows that $\mathbf{a}_1$ and $\mathbf{a}_2$ form a basis for $\mathbf{R}^2$ that can be used instead of $\mathbf{i}$ and $\mathbf{j}$.)

12. Let $\mathbf{a}_1 = (1, 0, -1)$, $\mathbf{a}_2 = (0, 1, 0)$, and $\mathbf{a}_3 = (1, 1, -1)$.

(a) Find scalars c_1, c_2, c_3, so as to write the vector $\mathbf{b} = (5, 6, -5)$ as $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3$.

(b) Try to repeat part (a) for the vector $\mathbf{b} = (2, 3, 4)$. What happens?

(c) Can the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ be used as a basis for $\mathbf{R}^3$, instead of $\mathbf{i}, \mathbf{j}, \mathbf{k}$? Why or why not?

In Exercises 13–20, give a set of parametric equations for the lines so described.

13. The line in $\mathbf{R}^3$ through the point $(2, -1, 5)$ that is parallel to the vector $\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$.

14. The line in $\mathbf{R}^3$ through the point $(12, -2, 0)$ that is parallel to the vector $5\mathbf{i} - 12\mathbf{j} + \mathbf{k}$.

15. The line in $\mathbf{R}^2$ through the point $(2, -1)$ that is parallel to the vector $\mathbf{i} - 7\mathbf{j}$.

16. The line in $\mathbf{R}^3$ through the points $(2, 1, 2)$ and $(3, -1, 5)$.

17. The line in $\mathbf{R}^3$ through the points $(1, 4, 5)$ and $(2, 4, -1)$.

18. The line in $\mathbf{R}^2$ through the points $(8, 5)$ and $(1, 7)$.

19. The line in $\mathbf{R}^2$ through the point $(1, 3)$ and perpendicular to the line $y = 2x$.

20. The line in $\mathbf{R}^2$ through the point $(-1, 4)$ and perpendicular to the line with parametric equations $x = -3t + 2$, $y = t - 4$.

21. Write a set of parametric equations for the line in $\mathbf{R}^4$ through the point $(1, 2, 0, 4)$ and parallel to the vector $(-2, 5, 3, 7)$.

22. Write a set of parametric equations for the line in $\mathbf{R}^5$ through the points $(9, \pi, -1, 5, 2)$ and $(-1, 1, \sqrt{2}, 7, 1)$.

23. (a) Write a set of parametric equations for the line in $\mathbf{R}^3$ through the point $(-1, 7, 3)$ and parallel to the vector $2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$.

(b) Write a set of parametric equations for the line through the points $(5, -3, 4)$ and $(0, 1, 9)$.

(c) Write different (but equally correct) sets of equations for parts (a) and (b).

(d) Find the symmetric forms of your answers in (a)–(c).

24. Give a symmetric form for the line having parametric equations $x = 5 - 2t$, $y = 3t + 1$, $z = 6t - 4$.

25. Give a symmetric form for the line having parametric equations $x = t + 7$, $y = 3t - 9$, $z = 6 - 8t$.

26. A certain line in $\mathbf{R}^3$ has symmetric form

$$\frac{x-2}{5} = \frac{y-3}{-2} = \frac{z+1}{4}.$$

Write a set of parametric equations for this line.

27. Give a set of parametric equations for the line with symmetric form

$$\frac{x+5}{3} = \frac{y-1}{7} = \frac{z+10}{-2}.$$

28. Are the two lines with symmetric forms

$$\frac{x-1}{5} = \frac{y+2}{-3} = \frac{z+1}{4}$$

and

$$\frac{x-4}{10} = \frac{y-1}{-5} = \frac{z+5}{8}$$

the same? Why or why not?

29. Show that the two sets of equations

$$\frac{x-2}{3} = \frac{y-1}{7} = \frac{z}{5} \text{ and } \frac{x+1}{-6} = \frac{y+6}{-14} = \frac{z+5}{-10}$$

actually represent the same line in $\mathbf{R}^3$.

30. Determine whether the two lines l_1 and l_2 defined by the sets of parametric equations l_1: $x = 2t - 5$, $y = 3t + 2$, $z = 1 - 6t$, and l_2: $x = 1 - 2t$, $y = 11 - 3t$, $z = 6t - 17$ are the same. (Hint: First find two points on l_1 and then see if those points lie on l_2.)

31. Do the parametric equations l_1: $x = 3t + 2$, $y = t - 7$, $z = 5t + 1$, and l_2: $x = 6t - 1$, $y = 2t - 8$, $z = 10t - 3$ describe the same line? Why or why not?

32. Do the parametric equations $x = 3t^3 + 7$, $y = 2 - t^3$, $z = 5t^3 + 1$ determine a line? Why or why not?

33. Do the parametric equations $x = 5t^2 - 1$, $y = 2t^2 + 3$, $z = 1 - t^2$ determine a line? Explain.

34. A bird is flying along the straight-line path $x = 2t + 7$, $y = t - 2$, $z = 1 - 3t$, where t is measured in minutes.

(a) Where is the bird initially (at $t = 0$)? Where is the bird 3 minutes later?

(b) Give a vector that is parallel to the bird's path.

(c) When does the bird reach the point $\left(\frac{34}{3}, \frac{1}{6}, -\frac{11}{2}\right)$?

(d) Does the bird reach $(17, 4, -14)$?

35. Find where the line $x = 3t - 5$, $y = 2 - t$, $z = 6t$ intersects the plane $x + 3y - z = 19$.

36. Where does the line $x = 1 - 4t$, $y = t - 3/2$, $z = 2t + 1$ intersect the plane $5x - 2y + z = 1$?

37. Find the points of intersection of the line $x = 2t - 3$, $y = 3t + 2$, $z = 5 - t$ with each of the coordinate planes $x = 0$, $y = 0$, and $z = 0$.

38. Show that the line $x = 5 - t$, $y = 2t - 7$, $z = t - 3$ is contained in the plane having equation $2x - y + 4z = 5$.

39. Does the line $x = 5 - t$, $y = 2t - 3$, $z = 7t + 1$ intersect the plane $x - 3y + z = 1$? Why?

40. Find where the line having symmetric form

$$\frac{x-3}{6} = \frac{y+2}{3} = \frac{z}{5}$$

intersects the plane with equation $2x - 5y + 3z + 8 = 0$.

41. Show that the line with symmetric form

$$\frac{x-3}{-2} = y - 5 = \frac{z+2}{3}$$

lies entirely in the plane $3x + 3y + z = 22$.

42. Does the line with symmetric form

$$\frac{x+4}{3} = \frac{y-2}{-1} = \frac{z-1}{-9}$$

intersect the plane $2x - 3y + z = 7$?

43. Let a, b, c be nonzero constants. Show that the line with parametric equations $x = at + a$, $y = b$, $z = ct + c$ lies on the surface with equation $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$.

44. Find the point of intersection of the two lines l_1: $x = 2t + 3$, $y = 3t + 3$, $z = 2t + 1$ and l_2: $x = 15 - 7t$, $y = t - 2$, $z = 3t - 7$.

45. Do the lines l_1: $x = 2t + 1$, $y = -3t$, $z = t - 1$ and l_2: $x = 3t + 1$, $y = t + 5$, $z = 7 - t$ intersect? Explain your answer.

46. (a) Find the distance from the point $(-2, 1, 5)$ to any point on the line $x = 3t - 5$, $y = 1 - t$, $z = 4t + 7$. (Your answer should be in terms of the parameter t.)

(b) Now find the distance between the point $(-2, 1, 5)$ and the line $x = 3t - 5$, $y = 1 - t$, $z = 4t + 7$. (The distance between a point and a line is the distance between the given point and the *closest* point on the line.)

47. (a) Describe the curve given parametrically by

$$\begin{cases} x = 2\cos 3t \\ y = 2\sin 3t \end{cases} \quad 0 \le t < \frac{2\pi}{3}.$$

What happens if we allow t to vary between 0 and 2π?

(b) Describe the curve given parametrically by

$$\begin{cases} x = 5\cos 3t \\ y = 5\sin 3t \end{cases} \quad 0 \le t < \frac{2\pi}{3}.$$

(c) Describe the curve given parametrically by

$$\begin{cases} x = 5\sin 3t \\ y = 5\cos 3t \end{cases} \quad 0 \le t < \frac{2\pi}{3}.$$

(d) Describe the curve given parametrically by

$$\begin{cases} x = 5\cos 3t \\ y = 3\sin 3t \end{cases} \quad 0 \le t < \frac{2\pi}{3}.$$

48. Suppose that a bicycle wheel of radius a rolls along a flat surface without slipping. If a reflector is attached to a spoke of the wheel at a distance b from the center, the resulting curve traced by the reflector is called a **curtate cycloid**. One such cycloid appears in Figure 1.32, where $a = 3$ and $b = 2$.

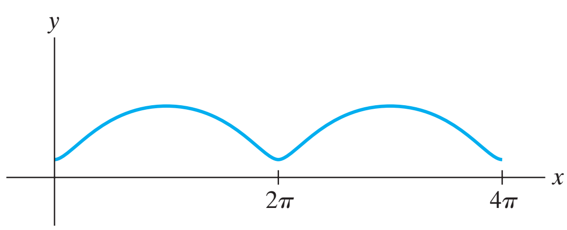


FIGURE 1.32 A curtate cycloid.

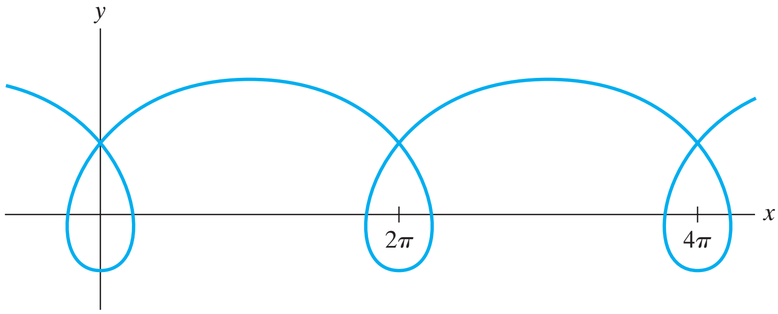


FIGURE 1.34 A prolate cycloid.

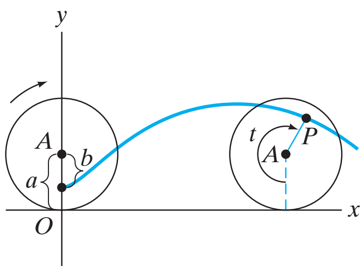


FIGURE 1.33 The point P traces a curtate cycloid.

Using vector methods or otherwise, find a set of parametric equations for the curtate cycloid. Figure 1.33 should help. (Take a low point of the cycloid to lie on the y-axis.) There is no theoretical reason that the cycloid just described cannot have $a < b$, although in such case the bicycle-wheel–reflector application is no longer relevant. (When $a < b$, the parametrized curve that results is called a **prolate cycloid**.) Your parametric equations should be such that the constants a and b can be chosen independently of one another. An example of a prolate cycloid, with $a = 2$ and $b = 4$, is shown in Figure 1.34. Try to think of a physical situation in which such a curve would arise.

49. Mac is unwinding tape from a circular dispenser of radius a by holding the tape taut and perpendicular to the dispenser. Find a set of parametric equations for the path traced by the end of the tape (the point P in Figure 1.35) as Mac unwinds the tape. Use the angle θ between $\overrightarrow{OP}$ and the positive x-axis for parameter. Assume that little enough tape is unwound so that the radius of the dispenser remains constant.

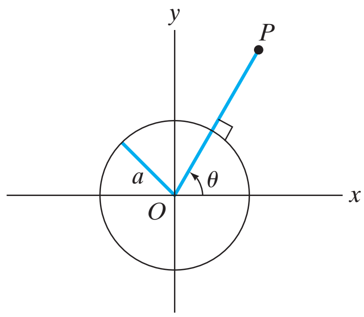


FIGURE 1.35 Figure for Exercise 49.

1.3 The Dot Product

When we introduced the arithmetic notions of vector addition and scalar multiplication, you may well have wondered why the product of two vectors was not defined. You might think that "vector multiplication" should be defined in a manner analogous to the way we defined vector addition (i.e., by componentwise multiplication). However, such a definition is not very useful. Instead, we shall define and use two different concepts of a product of two vectors: (1) the Euclidean inner product, or "dot" product, which may be defined for two vectors in $\mathbf{R}^n$ (where n is arbitrary) and (2) the "cross" or vector product, which is defined *only* for vectors in $\mathbf{R}^3$.

The Dot Product of Two Vectors

DEFINITION 3.1 Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be two vectors in $\mathbf{R}^3$. The **dot** (or **inner** or **scalar**) **product of a and b**, denoted $\mathbf{a} \cdot \mathbf{b}$, is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

In $\mathbf{R}^2$, the analogous definition is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2,$$

where $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$.

EXAMPLE 1 In $\mathbf{R}^3$, we have

$$(1, -2, 5) \cdot (2, 1, 3) = (1)(2) + (-2)(1) + (5)(3) = 15.$$

$$(3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot (\mathbf{i} - 2\mathbf{k}) = (3)(1) + (2)(0) + (-1)(-2) = 5.$$

In accordance with its name, the dot—or scalar—product takes two vectors and produces a *single real number* (*not* a vector).

The following facts are consequences of Definition 3.1:

Properties of dot products. If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are any vectors in $\mathbf{R}^3$ (or $\mathbf{R}^2$) and $k \in \mathbf{R}$ is any scalar, then

1. $\mathbf{a} \cdot \mathbf{a} \geq 0$, and $\mathbf{a} \cdot \mathbf{a} = 0$ if and only if $\mathbf{a} = \mathbf{0}$;
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$;
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$;
4. $(k\mathbf{a}) \cdot \mathbf{b} = k(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (k\mathbf{b})$.

PROOF OF PROPERTY 1 If $\mathbf{a} = (a_1, a_2, a_3)$, then we have

$$\mathbf{a} \cdot \mathbf{a} = a_1 a_1 + a_2 a_2 + a_3 a_3 = a_1^2 + a_2^2 + a_3^2.$$

This last expression is evidently nonnegative, since it is a sum of squares of real numbers. Moreover, such an expression is zero exactly when each of the terms is zero, that is, if and only if $a_1 = a_2 = a_3 = 0$. ■

We leave the proofs of properties 2, 3, and 4 as exercises.

Thus far, we have introduced the dot product of two vectors as a purely algebraic construction. It is the geometric interpretation of the definition that is really interesting. To establish this interpretation, we begin with the following:

DEFINITION 3.2 If $\mathbf{a} = (a_1, a_2, a_3)$, then the **length** of $\mathbf{a}$ (also called the **norm** or **magnitude**), denoted $\|\mathbf{a}\|$, is $\sqrt{a_1^2 + a_2^2 + a_3^2}$.

The motivation for this definition is evident if we draw **a** as the position vector of the point (a_1, a_2, a_3). Then the length of the arrow from the origin to (a_1, a_2, a_3) is

$$\sqrt{(a_1 - 0)^2 + (a_2 - 0)^2 + (a_3 - 0)^2},$$

as given by the distance formula, which is nothing more than an extension of the Pythagorean theorem in the plane. As we just saw, $\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2$, and we have

$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$$

or, equivalently,

$$\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2. \quad (1)$$

Now we're ready to state the main result concerning the geometry of the dot product. If **a** and **b** are two nonzero vectors in $\mathbf{R}^3$ (or $\mathbf{R}^2$) drawn with their tails at the same point, let θ, where $0 \le \theta \le \pi$, be the angle between **a** and **b**. If either **a** or **b** is the zero vector, then θ is indeterminate (i.e., can be any angle).

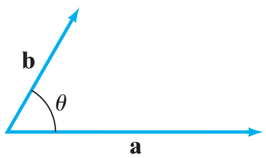


FIGURE 1.36 The dot product of **a** and **b** is $\|\mathbf{a}\|\,\|\mathbf{b}\|\cos\theta$.

THEOREM 3.3 If **a** and **b** are any two vectors in either $\mathbf{R}^2$ or $\mathbf{R}^3$, then

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\| \cos\theta.$$

(See Figure 1.36.)

■ **PROOF** If either **a** or **b** is the zero vector, say **a**, then $\mathbf{a} = (0, 0, 0)$ and so

$$\mathbf{a} \cdot \mathbf{b} = (0)(b_1) + (0)(b_2) + (0)(b_3) = 0.$$

Also, $\|\mathbf{a}\| = 0$ in this case, so the formula in Theorem 3.3 holds. In this case, the angle θ is indeterminate.

Now suppose that neither **a** nor **b** is the zero vector. Let $\mathbf{c} = \mathbf{b} - \mathbf{a}$. Then we may apply the law of cosines to the triangle whose sides are **a**, **b**, and **c** (Figure 1.37) to obtain

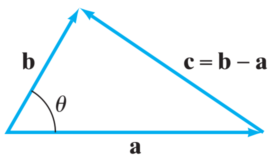


FIGURE 1.37 The vector triangle used in the proof of Theorem 3.3.

$$\|\mathbf{c}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\,\|\mathbf{b}\| \cos\theta.$$

Thus,

$$2\|\mathbf{a}\|\,\|\mathbf{b}\| \cos\theta = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - \|\mathbf{c}\|^2 = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - \mathbf{c} \cdot \mathbf{c}, \quad (2)$$

from equation (1). Now, use the properties of the dot product. Since $\mathbf{c} = \mathbf{b} - \mathbf{a}$,

$$\begin{aligned} \mathbf{c} \cdot \mathbf{c} &= (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) \\ &= (\mathbf{b} - \mathbf{a}) \cdot \mathbf{b} - (\mathbf{b} - \mathbf{a}) \cdot \mathbf{a} \\ &= \mathbf{b} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{a}, \end{aligned} \quad (3)$$

by properties 3 and 4 of the dot product. If we use equation (3) to substitute for $\mathbf{c} \cdot \mathbf{c}$ in equation (2), then

$$\begin{aligned} 2\|\mathbf{a}\|\,\|\mathbf{b}\| \cos\theta &= \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - (\mathbf{b} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{a}) \\ &= \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} \\ &= 2\mathbf{a} \cdot \mathbf{b}, \end{aligned}$$

by property 2 of the dot product. By canceling the factor of 2 on both sides, the desired result is obtained. ■

Angles Between Vectors

Theorem 3.3 may be used to find the angle between two nonzero vectors **a** and **b**—just solve for θ in the formula in Theorem 3.3 to obtain

$$\theta = \cos^{-1}\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}. \tag{4}$$

The use of the inverse cosine is unambiguous, since we take $0 \leq \theta \leq \pi$ when defining angles between vectors.

EXAMPLE 2 If $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{j} - \mathbf{k}$, then formula (4) gives

$$\theta = \cos^{-1}\frac{(\mathbf{i}+\mathbf{j})\cdot(\mathbf{j}-\mathbf{k})}{\|\mathbf{i}+\mathbf{j}\|\,\|\mathbf{j}-\mathbf{k}\|} = \cos^{-1}\frac{1}{(\sqrt{2}\cdot\sqrt{2})} = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}.$$

If **a** and **b** are nonzero, then Theorem 3.3 implies

$$\cos\theta = 0 \quad \text{if and only if} \quad \mathbf{a}\cdot\mathbf{b} = 0.$$

We have $\cos\theta = 0$ just in case $\theta = \pi/2$. (Remember our restriction on θ.) Hence, it makes sense for us to call **a** and **b** **perpendicular** (or **orthogonal**) when $\mathbf{a}\cdot\mathbf{b} = 0$. If either **a** or **b** is the zero vector, then we cannot use formula (4), and the angle θ is undefined. Nonetheless, since $\mathbf{a}\cdot\mathbf{b} = 0$ if **a** or **b** is **0**, we adopt the standard convention and say that **the zero vector is perpendicular to every vector**.

Since $\cos\theta > 0$ when $0 \leq \theta < \frac{\pi}{2}$ and $\cos\theta < 0$ when $\frac{\pi}{2} < \theta \leq \pi$, we also see that $\mathbf{a}\cdot\mathbf{b} > 0$ precisely when the angle between **a** and **b** is acute and $\mathbf{a}\cdot\mathbf{b} < 0$ when the angle is obtuse. Thus, even when $\mathbf{a}\cdot\mathbf{b} \neq \mathbf{0}$, we are also able to derive geometric information about the relation between **a** and **b**.

EXAMPLE 3 The vector $\mathbf{i} + \mathbf{j}$ is orthogonal to the vector $\mathbf{i} - \mathbf{j} + \mathbf{k}$, since

$$(\mathbf{i}+\mathbf{j})\cdot(\mathbf{i}-\mathbf{j}+\mathbf{k}) = (1)(1) + (1)(-1) + (0)(1) = 0.$$

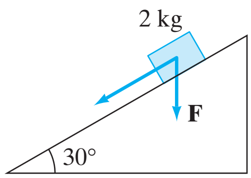


FIGURE 1.38 An object sliding down a ramp. The force due to gravity is downward, but the direction of travel of the object is inclined 30° to the horizontal.

Vector Projections

Suppose that a 2 kg object is sliding down a ramp having a 30° incline with the horizontal as in Figure 1.38. If we neglect friction, the only force acting on the object is gravity. What is the component of the gravitational force in the direction of motion of the object?

To answer questions of this nature, we need to find the projection of one vector on another. The general idea is as follows: Given two nonzero vectors **a** and **b**, imagine dropping a perpendicular line from the head of **b** to the line through **a**. Then the **projection of b onto a**, denoted $\text{proj}_{\mathbf{a}}\mathbf{b}$, is the vector represented by the arrow in Figure 1.39.

Given this intuitive understanding of the projection, we find a precise formula for it. Recall that a vector is determined by magnitude (length) and direction. It follows by definition that the direction of $\text{proj}_{\mathbf{a}}\mathbf{b}$ is either the same as that of **a**, or opposite to **a** if the angle θ between **a** and **b** is more than $\pi/2$. Either way, the vector given by $\text{proj}_{\mathbf{a}}\,\mathbf{b}$ must be some scalar multiple of **a**:

$$\text{proj}_{\mathbf{a}}\mathbf{b} = k\mathbf{a}$$

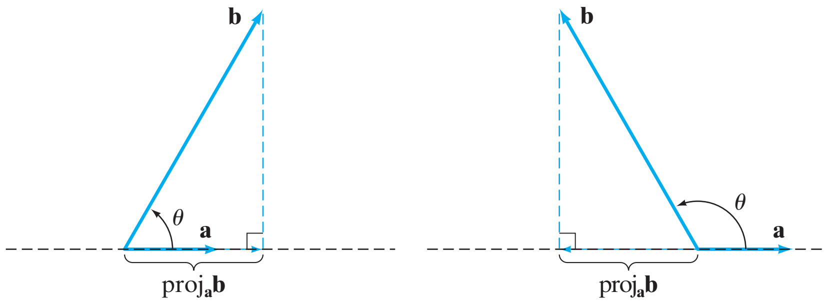


FIGURE 1.39 Projection of the vector **b** onto the vector **a**. Geometrically, the projection $\text{proj}_{\mathbf{a}}\mathbf{b}$ gives the point on the line through **a** that is closest to (the endpoint of) **b**.

for some k. By the defining property of projections, this specific scalar multiple is characterized by the property that $\mathbf{b} - k\mathbf{a}$ (which forms the missing edge in each right triangle drawn in Figure 1.39) should be perpendicular to **a**, meaning that

$$\mathbf{a} \cdot (\mathbf{b} - k\mathbf{a}) = 0.$$

Using properties of the dot product, we can rewrite this equation as

$$\mathbf{a} \cdot \mathbf{b} - k(\mathbf{a} \cdot \mathbf{a}) = 0,$$

which, after rearranging, gives

$$k = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}.$$

Thus, the specific scalar multiple of a that gives the desired projection is the one given by this scalar, so that

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\right)\mathbf{a}. \qquad (5)$$

Formula (5) is concise and not difficult to remember.

Let us give an alternative approach to deriving this formula—relying on the trigonometry of right triangles—so that we can introduce a few more concepts as well. Looking back at Figure 1.39, we see that

$$|\cos\theta| = \frac{\|\text{proj}_{\mathbf{a}}\mathbf{b}\|}{\|\mathbf{b}\|}.$$

(The absolute value sign around $\cos\theta$ is needed in case $\pi/2 \le \theta \le \pi$.) Hence, with a bit of algebra, we have

$$\|\text{proj}_{\mathbf{a}}\mathbf{b}\| = \|\mathbf{b}\|\,|\cos\theta| = \frac{\|\mathbf{a}\|\,\|\mathbf{b}\|\,|\cos\theta|}{\|\mathbf{a}\|} = \frac{|\mathbf{a} \cdot \mathbf{b}|}{\|\mathbf{a}\|}$$

by Theorem 3.3. Thus, we know the magnitude and direction of $\text{proj}_{\mathbf{a}}\mathbf{b}$. To obtain a compact formula for $\text{proj}_{\mathbf{a}}\mathbf{b}$, note the following:

PROPOSITION 3.4 Let k be any scalar and **a** any vector. Then

1. $\|k\mathbf{a}\| = |k|\,\|\mathbf{a}\|$.
2. A **unit vector** (i.e., a vector of length 1) in the direction of a nonzero vector **a** is given by $\mathbf{a}/\|\mathbf{a}\|$.

PROOF Part 1 is left as an exercise. (Write out $k\mathbf{a}$ and $\|k\mathbf{a}\|$ in terms of components.) For part 2, we must check that the length of $\mathbf{a}/\|\mathbf{a}\|$ is 1:

$$\left\|\frac{\mathbf{a}}{\|\mathbf{a}\|}\right\| = \left\|\frac{1}{\|\mathbf{a}\|}\mathbf{a}\right\| = \frac{1}{\|\mathbf{a}\|}\|\mathbf{a}\| = 1,$$

by part 1 (since $1/\|\mathbf{a}\|$ is a positive scalar). ■

Now $\text{proj}_\mathbf{a}\mathbf{b}$ is a vector of length $|\mathbf{a}\cdot\mathbf{b}|/\|\mathbf{a}\|$ in the "$\pm\mathbf{a}$-direction." That is,

$$\text{proj}_\mathbf{a}\mathbf{b} = \pm\underbrace{\left(\frac{|\mathbf{a}\cdot\mathbf{b}|}{\|\mathbf{a}\|}\right)}_{\substack{\text{length of}\\ \text{proj}_\mathbf{a}\mathbf{b}}} \times \underbrace{\frac{\mathbf{a}}{\|\mathbf{a}\|}}_{\substack{\text{unit vector}\\ \text{in direction of }\mathbf{a}}} = \pm\frac{\|\mathbf{a}\|\,\|\mathbf{b}\|\,|\cos\theta|}{\|\mathbf{a}\|}\frac{\mathbf{a}}{\|\mathbf{a}\|}.$$

Note that the angle θ keeps track of the appropriate sign of $\text{proj}_\mathbf{a}\mathbf{b}$; that is, when $0 \le \theta < \pi/2$, $\cos\theta$ is positive and $\text{proj}_\mathbf{a}\mathbf{b}$ points in the direction of $\mathbf{a}$, and when $\pi/2 < \theta \le \pi$, $\cos\theta$ is negative and $\text{proj}_\mathbf{a}\mathbf{b}$ points in the direction opposite to that of $\mathbf{a}$. Thus, we can eliminate *both* the $\pm$ sign and the absolute value, and we find that

$$\text{proj}_\mathbf{a}\mathbf{b} = \frac{\|\mathbf{a}\|\,\|\mathbf{b}\|\,\cos\theta}{\|\mathbf{a}\|}\frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a}$$

by Theorem 3.3, which is precisely formula (5) by equation (1).

EXAMPLE 4 Let us compute the projection of the vector $\mathbf{b} = (1, 0)$ onto the vector $\mathbf{a} = (1, 2)$. By formula (5), we have:

$$\text{proj}_\mathbf{a}\,\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} = \left(\frac{(1,0)\cdot(1,2)}{(1,2)\cdot(1,2)}\right)(1,2) = \frac{1}{5}(1,2) = \left(\frac{1}{5}, \frac{2}{5}\right).$$

As expected, $\mathbf{b} - \text{proj}_\mathbf{a}\mathbf{b} = \left(\frac{4}{5}, -\frac{2}{5}\right)$ is perpendicular to $\mathbf{a}$. The point $\left(\frac{1}{5}, \frac{2}{5}\right)$ is thus the point on the line $y = 2x$ through $\mathbf{a}$ that is closest to $(1, 0)$.

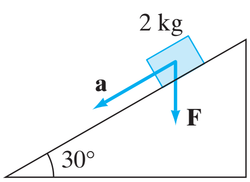


FIGURE 1.40 The 2 kg object sliding down a ramp in Example 5.

EXAMPLE 5 To answer the question posed at the beginning of this subsection, we need to calculate $\text{proj}_\mathbf{a}\mathbf{F}$, where $\mathbf{F}$ is the gravitational force vector and $\mathbf{a}$ points along the ramp as shown in Figure 1.40. We have a coordinate situation as shown in Figure 1.41. From trigonometric considerations, we must have $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ such that $a_1 = -\|\mathbf{a}\|\cos 30°$ and $a_2 = -\|\mathbf{a}\|\sin 30°$. Since we

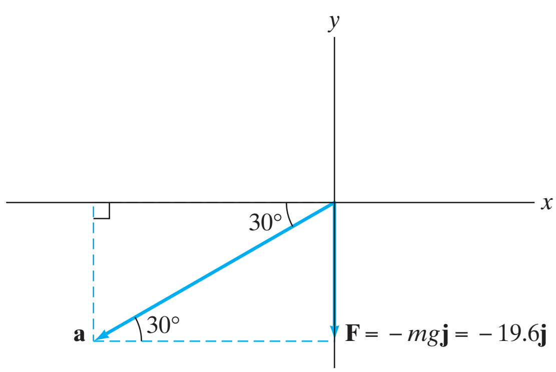


FIGURE 1.41 The vectors **a** and **F** in Example 5, realized in a coordinate system.

are really only interested in the *direction* of **a**, there is no loss in assuming that **a** is a unit vector. Thus,

$$\mathbf{a} = -\cos 30^\circ \mathbf{i} - \sin 30^\circ \mathbf{j} = -\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}.$$

Taking $g = 9.8$ m/sec^2, we have $\mathbf{F} = -2g\mathbf{j} = -19.6\mathbf{j}$. Therefore, formula (5) implies

$$\begin{aligned}\text{proj}_{\mathbf{a}}\mathbf{F} = \left(\frac{\mathbf{a}\cdot\mathbf{F}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} &= \frac{\left(-\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}\right)\cdot(-19.6\mathbf{j})}{1}\left(-\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}\right)\\ &= 9.8\left(-\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}\right)\\ &\approx -8.49\,\mathbf{i} - 4.9\,\mathbf{j},\end{aligned}$$

and the component of **F** in this direction is

$$\|\text{proj}_{\mathbf{a}}\mathbf{F}\| = \|-8.49\,\mathbf{i} - 4.9\,\mathbf{j}\| = 9.8 \text{ N}.$$

Unit vectors—that is, vectors of length 1—are important in that they capture the idea of direction (since they all have the same length). Part 2 of Proposition 3.4 shows that every nonzero vector **a** can have its length adjusted to give a unit vector $\mathbf{u} = \mathbf{a}/\|\mathbf{a}\|$ that points in the same direction as **a**. This operation is referred to as **normalization** of the vector **a**.

EXAMPLE 6 A fluid is flowing across a plane surface with uniform velocity vector **v**. If **n** is a unit vector perpendicular to the plane surface, let's find (in terms of **v** and **n**) the volume of the fluid that passes through a unit area of the plane in unit time. (See Figure 1.42.)

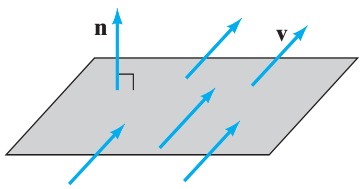


FIGURE 1.42 Fluid flowing across a plane surface.

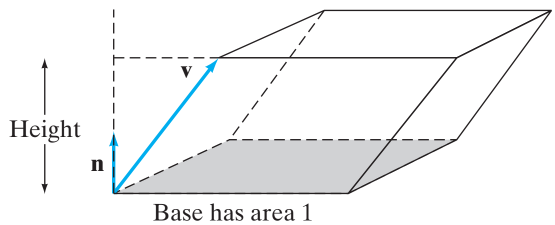


FIGURE 1.43 After one unit of time, the fluid passing across a square will have filled the box.

First, imagine one unit of time has elapsed. Then over a unit area of the plane (say over a unit square), the fluid will have filled a "box" as in Figure 1.43. The box may be represented by a parallelepiped (a three-dimensional analogue of a parallelogram). The volume we seek is the volume of this parallelepiped and is

$$\text{Volume} = (\text{area of base})(\text{height}).$$

The area of the base is 1 unit by construction. The height is given by $\|\text{proj}_{\mathbf{n}}\mathbf{v}\|$. From formula (5),

$$\text{proj}_{\mathbf{n}}\mathbf{v} = \left(\frac{\mathbf{n}\cdot\mathbf{v}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n} = (\mathbf{n}\cdot\mathbf{v})\,\mathbf{n},$$

since $\mathbf{n}\cdot\mathbf{n} = \|\mathbf{n}\|^2 = 1$. Hence,

$$\|\text{proj}_{\mathbf{n}}\mathbf{v}\| = \|(\mathbf{n}\cdot\mathbf{v})\mathbf{n}\| = |\mathbf{n}\cdot\mathbf{v}|\,\|\mathbf{n}\| = |\mathbf{n}\cdot\mathbf{v}|,$$

by part 1 of Proposition 3.4.

Vector Proofs

We conclude this section with two illustrations of how wonderfully well vectors are suited to providing elegant proofs of geometric results.

EXAMPLE 7 In an arbitrary triangle, show that the line segment joining the midpoints of two sides is parallel to and has half the length of the third side. (See Figure 1.44.) In other words, if M_1 is the midpoint of side $\overline{AB}$ and M_2 is the midpoint of side $\overline{AC}$, we wish to show that $\overline{M_1M_2}$ is parallel to $\overline{BC}$ and has half its length.

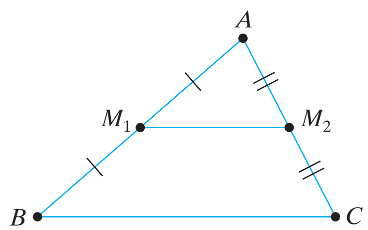


FIGURE 1.44 In triangle ABC, $\overline{M_1M_2}$ is parallel to $\overline{BC}$ and has half its length.

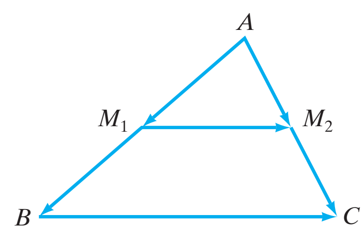


FIGURE 1.45 The vector version of triangle ABC in Example 7.

For a vector proof, we use the diagram in Figure 1.45, a slightly modified version of Figure 1.44. The midpoint conditions translate to the following statements about vectors:

$$\overrightarrow{AM_1} = \tfrac{1}{2}\overrightarrow{AB}, \qquad \overrightarrow{AM_2} = \tfrac{1}{2}\overrightarrow{AC}.$$

Now,

$$\overrightarrow{M_1M_2} = \overrightarrow{AM_2} - \overrightarrow{AM_1} = \tfrac{1}{2}\overrightarrow{AC} - \tfrac{1}{2}\overrightarrow{AB} = \tfrac{1}{2}(\overrightarrow{AC} - \overrightarrow{AB}) = \tfrac{1}{2}\overrightarrow{BC}.$$

But $\overrightarrow{M_1M_2} = \frac{1}{2}\overrightarrow{BC}$ is precisely what we wish to prove: To say $\overrightarrow{M_1M_2}$ is a scalar times $\overrightarrow{BC}$ means that the two vectors are parallel. Moreover, from part 1 of Proposition 3.4,

$$\|\overrightarrow{M_1M_2}\| = \|\tfrac{1}{2}\overrightarrow{BC}\| = \tfrac{1}{2}\|\overrightarrow{BC}\|,$$

so that the length condition also holds.

EXAMPLE 8 Show that every angle inscribed in a semicircle is a right angle, as suggested by Figure 1.46.

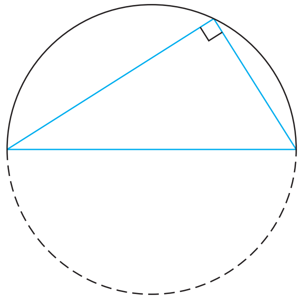


FIGURE 1.46 Every angle inscribed in a semicircle is a right angle.

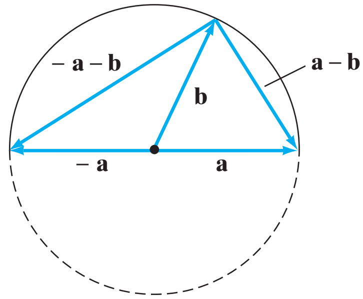


FIGURE 1.47 **a** and **b** are "radius vectors."

To prove this remark, we'll make use of Figure 1.47, where **a** and **b** are "radius vectors" with tails at the center of the circle. We need only show that $\mathbf{a} - \mathbf{b}$ (a vector along one ray of the angle in question) is perpendicular to $-\mathbf{a} - \mathbf{b}$ (a vector along the other ray). In other words, we wish to show that

$$(\mathbf{a} - \mathbf{b}) \cdot (-\mathbf{a} - \mathbf{b}) = 0.$$

We have

$$(\mathbf{a} - \mathbf{b}) \cdot (-\mathbf{a} - \mathbf{b}) = (-1)(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}),$$

by property 4 of dot products,

$$= (-1)((\mathbf{a} - \mathbf{b}) \cdot \mathbf{a} + (\mathbf{a} - \mathbf{b}) \cdot \mathbf{b})$$
$$= (-1)(\mathbf{a} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{b})$$
$$= (-1)(\|\mathbf{a}\|^2 - \|\mathbf{b}\|^2),$$

by properties 2 and 4,

$$= 0,$$

since both **a** and **b** are radius vectors (and therefore have the same length, namely, the radius of the circle). ▭

Vector proofs as in Examples 7 and 8 are elegant and sometimes allow you to write shorter and more direct proofs than those from your high school geometry days.

1.3 Exercises

Compute $\mathbf{a} \cdot \mathbf{b}$, $\|\mathbf{a}\|$, $\|\mathbf{b}\|$ *for the vectors listed in Exercises 1–6.*

1. $\mathbf{a} = (1, 5), \mathbf{b} = (-2, 3)$

2. $\mathbf{a} = (4, -1), \mathbf{b} = \left(\frac{1}{2}, 2\right)$

3. $\mathbf{a} = (-1, 0, 7), \mathbf{b} = (2, 4, -6)$

4. $\mathbf{a} = (2, 1, 0), \mathbf{b} = (1, -2, 3)$

5. $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k}, \mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

6. $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}, \mathbf{b} = -3\mathbf{j} + 2\mathbf{k}$

In Exercises 7–11, find the angle between each of the pairs of vectors.

7. $\mathbf{a} = \sqrt{3}\,\mathbf{i} + \mathbf{j}, \mathbf{b} = -\sqrt{3}\,\mathbf{i} + \mathbf{j}$

8. $\mathbf{a} = (-1, 2), \mathbf{b} = (3, 1)$

9. $\mathbf{a} = \mathbf{i} + \mathbf{j}, \mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

10. $\mathbf{a} = \mathbf{i} + \mathbf{j} - \mathbf{k}, \mathbf{b} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

11. $\mathbf{a} = (1, -2, 3), \mathbf{b} = (3, -6, -5)$

In Exercises 12–16, calculate $\text{proj}_{\mathbf{a}}\mathbf{b}$.

12. $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

13. $\mathbf{a} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$, $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

14. $\mathbf{a} = 5\mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$

15. $\mathbf{a} = -3\mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$

16. $\mathbf{a} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$

17. Find the point on the line $y = -4x$ that is closest to the point $(1, 5)$.

18. Find the point on the line $y = 5x - 2$ that is closest to the point $(-2, 3)$.

19. Find the point on the line with parametric equations $x = 3t$, $y = -2t$, $z = t$ that is closest to the point $(0, 1, -2)$.

20. Find the point on the line with parametric equations $x = 2t + 5$, $y = t - 3$, $z = 2t$ that is closest to the point $(1, 1, 1)$.

21. Give a unit vector that points in the same direction as the vector $2\mathbf{i} - \mathbf{j} + \mathbf{k}$.

22. Give a unit vector that points in the direction opposite to the vector $-\mathbf{i} + 2\mathbf{k}$.

23. Give a vector of length 3 that points in the same direction as the vector $\mathbf{i} + \mathbf{j} - \mathbf{k}$.

24. Find three nonparallel vectors that are perpendicular to $\mathbf{i} - \mathbf{j} + \mathbf{k}$.

25. Is it ever the case that $\text{proj}_{\mathbf{a}}\mathbf{b} = \text{proj}_{\mathbf{b}}\mathbf{a}$? If so, under what conditions?

26. Prove properties 2, 3, and 4 of dot products.

27. Prove part 1 of Proposition 3.4.

28. Suppose that a force $\mathbf{F} = \mathbf{i} - 2\mathbf{j}$ is acting on an object moving parallel to the vector $\mathbf{a} = 4\mathbf{i} + \mathbf{j}$. Decompose $\mathbf{F}$ into a sum of vectors $\mathbf{F}_1$ and $\mathbf{F}_2$, where $\mathbf{F}_1$ points along the direction of motion and $\mathbf{F}_2$ is perpendicular to the direction of motion. (Hint: A diagram may help.)

29. In physics, when a constant force acts on an object as the object is displaced, the **work** done by the force is the product of the length of the displacement and the component of the force in the direction of the displacement. Figure 1.48 depicts an object acted upon by a constant force $\mathbf{F}$, which displaces it from the point P to the point Q. Let θ denote the angle between $\mathbf{F}$ and the direction of displacement.

(a) Show that the work done by $\mathbf{F}$ is determinded by the formula $\mathbf{F} \cdot \overrightarrow{PQ}$.

(b) Find the work done by the (constant) force $\mathbf{F} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$ in moving a particle from the point $(1, -1, 1)$ to the point $(2, 0, -1)$.

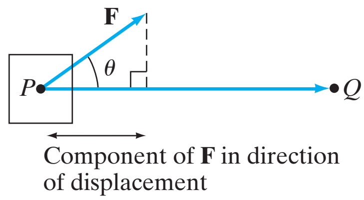


FIGURE 1.48 A constant force $\mathbf{F}$ displaces the object from P to Q. (See Exercise 25.)

30. A refrigerator is dragged 12 ft across a smooth floor using a rope and 60 lb of force directed along the rope. How much work is done if the rope makes a 20° angle with the horizontal?

31. How much work is done in pushing a handtruck loaded with 500 lb of bananas 40 ft up a ramp inclined 30° from horizontal?

Let $\mathbf{a}$ *be a nonzero vector in* $\mathbf{R}^3$. *The* ***direction cosines*** *of* $\mathbf{a}$ *are the three numbers* $\cos\alpha$, $\cos\beta$, $\cos\gamma$ *determined by the angles* α, β, γ *between* $\mathbf{a}$ *and, respectively, the positive x-, y-, and z-axes. In Exercises 32 and 33, find the direction cosines of the given vectors.*

32. $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$

33. $\mathbf{a} = 3\mathbf{i} + 4\mathbf{k}$

34. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, give expressions for the direction cosines of $\mathbf{a}$ in terms of the components of $\mathbf{a}$.

35. Let A, B, and C denote the vertices of a triangle. Let $0 < r < 1$. If P_1 is the point on $\overline{AB}$ located r times the distance from A to B and P_2 is the point on $\overline{AC}$ located r times the distance from A to C, use vectors to show that $\overline{P_1P_2}$ is parallel to $\overline{BC}$ and has r times the length of $\overline{BC}$. (This result generalizes that of Example 7 of this section.)

36. Let A, B, C, and D be four points in $\mathbf{R}^3$ such that no three of them lie on a line. Then $ABCD$ is a quadrilateral, though not necessarily one that lies in a plane. Denote the midpoints of the four sides of $ABCD$ by M_1, M_2, M_3, and M_4. Use vectors to show that, amazingly, $M_1M_2M_3M_4$ is always a parallelogram.

37. Use vectors to show that the diagonals of a parallelogram have the same length if and only if the parallelogram is a rectangle. (Hint: Let $\mathbf{a}$ and $\mathbf{b}$ be vectors along two sides of the parallelogram. Express vectors running along the diagonals in terms of **a** and **b**. See Figure 1.49.)

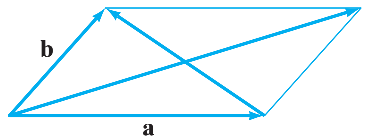


FIGURE 1.49 Diagram for Exercise 37.

38. Using vectors, prove that the diagonals of a parallelogram are perpendicular if and only if the parallelogram is a rhombus. (Note: A **rhombus** is a parallelogram whose four sides all have the same length.)

39. This problem concerns three circles of equal radius r that intersect in a single point O. (See Figure 1.50.)

(a) Let W_1, W_2, and W_3 denote the centers of the three circles and let $\mathbf{w}_i = \overrightarrow{OW_i}$ for $i = 1, 2, 3$. Similarly, let A, B, and C denote the remaining intersection points of the circles and set $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, and $\mathbf{c} = \overrightarrow{OC}$. By numbering the centers of the circles appropriately, write **a**, **b**, and **c** in terms of $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$.

(b) Show that A, B, and C lie on a circle of the same radius r as the three given circles. (Hint: The center of the circle is at the point P, where $\overrightarrow{OP} = \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3$.)

(c) Show that O is the orthocenter of triangle ABC. (The **orthocenter** of a triangle is the common intersection point of the altitudes perpendicular to the edges.)

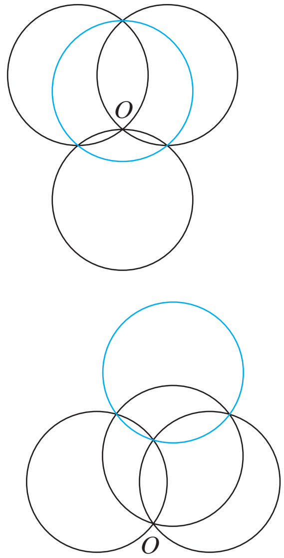


FIGURE 1.50 Two examples of three circles of equal radius intersecting in a single point O. (See Exercise 39.)

40. (a) Show that the vectors $\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b}$ and $\|\mathbf{b}\|\,\mathbf{a} - \|\mathbf{a}\|\mathbf{b}$ are orthogonal.

(b) Show that $\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b}$ bisects the angle between **a** and **b**.

1.4 The Cross Product

The cross product of two vectors in $\mathbf{R}^3$ is an "honest" product in the sense that it takes two vectors and produces a third one. However, the cross product possesses some curious properties (not the least of which is that it *cannot* be defined for vectors in $\mathbf{R}^2$ without first embedding them in $\mathbf{R}^3$ in some way) making it less "natural" than may at first seem to be the case.

When we defined the concepts of vector addition, scalar multiplication, and the dot product, we did so algebraically (i.e., by a formula in the vector components) and then saw what these definitions meant geometrically. In contrast, we will define the cross product first *geometrically,* and then deduce an algebraic formula for it. This technique is more convenient, since the coordinate formulation is fairly complicated (although we will find a way to organize it so as to make it easier to remember).

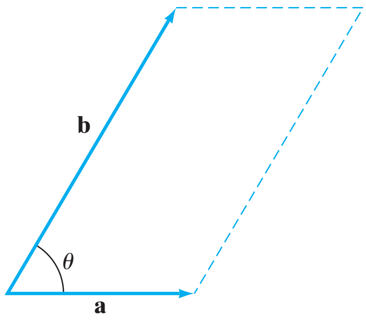


FIGURE 1.51 The area of this parallelogram is $\|\mathbf{a}\|\,\|\mathbf{b}\|\sin\theta$.

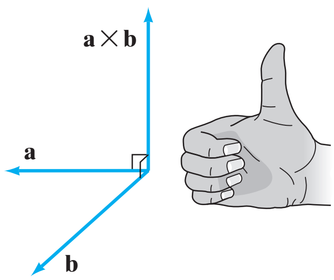


FIGURE 1.52 The right-hand rule for finding $\mathbf{a} \times \mathbf{b}$.

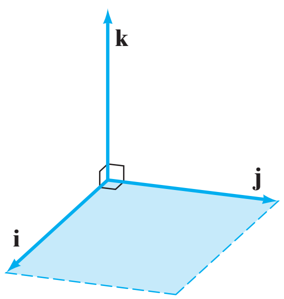


FIGURE 1.53 $\mathbf{i} \times \mathbf{j} = \mathbf{k}$.

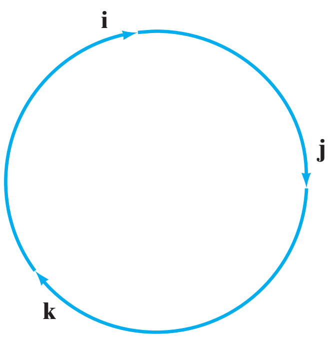


FIGURE 1.54 A mnemonic for finding the cross product of the unit basis vectors.

The Cross Product of Two Vectors in $\mathbf{R}^3$

DEFINITION 4.1 Let **a** and **b** be two vectors in $\mathbf{R}^3$ (*not* $\mathbf{R}^2$). The **cross product** (or **vector product**) of **a** and **b**, denoted $\mathbf{a} \times \mathbf{b}$, is the vector whose length and direction are given as follows:

- The length of $\mathbf{a} \times \mathbf{b}$ is the area of the parallelogram spanned by **a** and **b** or is zero if either **a** is parallel to **b** or if **a** or **b** is **0**. Alternatively, the following formula holds:

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|\sin\theta,$$

  where θ is the angle between **a** and **b**. (See Figure 1.51.) Note here that $\sin\theta \geq 0$ since $0 \leq \theta \leq \pi$ by how we have defined the angle between vectors.
- The direction of $\mathbf{a} \times \mathbf{b}$ is such that $\mathbf{a} \times \mathbf{b}$ is perpendicular to both **a** and **b** (when both **a** and **b** are nonzero) and is taken so that the ordered triple $(\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b})$ is a right-handed set of vectors, as shown in Figure 1.52. (If either **a** or **b** is **0**, or if **a** is parallel to **b**, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ from the aforementioned length condition.)

By saying that $(\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b})$ is right-handed, we mean that if you let the fingers of your right hand curl *from* **a** *toward* **b**, then your thumb will point in the direction of $\mathbf{a} \times \mathbf{b}$.

The second property in Definition 4.1 is often the one with most practical significance: If we are looking for a vector perpendicular to two given vectors, we should compute their cross product, using the coordinate formula we will soon derive.

EXAMPLE 1 Let's compute the cross product of the standard basis vectors for $\mathbf{R}^3$. First consider $\mathbf{i} \times \mathbf{j}$ as shown in Figure 1.53. The vectors **i** and **j** determine a square of unit area. Thus, $\|\mathbf{i} \times \mathbf{j}\| = 1$. Any vector perpendicular to *both* **i** and **j** must be perpendicular to the plane in which **i** and **j** lie. Hence, $\mathbf{i} \times \mathbf{j}$ must point in the direction of $\pm\mathbf{k}$. The "right-hand rule" implies that $\mathbf{i} \times \mathbf{j}$ must point in the positive **k** direction. Since $\|\mathbf{k}\| = 1$, we conclude that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$. The same argument establishes that $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$. To remember these basic equations, you can draw **i**, **j**, and **k** in a circle, as in Figure 1.54. Then the relations

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j} \tag{1}$$

may be read from the circle by beginning at any vector and then proceeding clockwise.

Properties of the Cross Product; Coordinate Formula

Example 1 demonstrates that the calculation of cross products from the geometric definition is not entirely routine. What we really need is a coordinate formula, analogous to that for the dot product or for vector projections, which is not difficult to obtain.

From our Definition 4.1, it is possible to establish the following:

Properties of the Cross Product. Let **a**, **b**, and **c** be any three vectors in $\mathbf{R}^3$ and let $k \in \mathbf{R}$ be any scalar. Then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ (anticommutativity);
2. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (distributivity);
3. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$ (distributivity);
4. $k(\mathbf{a} \times \mathbf{b}) = (k\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (k\mathbf{b})$.

We provide proofs of these properties at the end of the section, although you might give some thought now as to why they hold. It's worth remarking that these properties are entirely reasonable, ones that we would certainly want a product to have. However, you should be clear about the fact that the cross product *fails* to satisfy other properties that you might also consider to be eminently reasonable. In particular, since property 1 holds, we see that $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$ in general (i.e., the cross product is *not* commutative). Consequently, *be very careful about the order in which you write cross products*. Another property that the cross product does *not* possess is associativity. That is,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c},$$

in general. For example, let $\mathbf{a} = \mathbf{b} = \mathbf{i}$ and $\mathbf{c} = \mathbf{j}$. Then

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{k} \times \mathbf{i} = -\mathbf{j},$$

from properties 1 and 4, but $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0} \neq -\mathbf{j}$. (The equation $\mathbf{i} \times \mathbf{i} = \mathbf{0}$ holds because **i** is, of course, parallel to **i**.) Make sure that you do *not* try to use an associative law when working problems.

We now have the tools for producing a coordinate formula for the cross product. Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$. Then

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})\\ &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times b_1\mathbf{i} + (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times b_2\mathbf{j}\\ &\quad + (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times b_3\mathbf{k},\end{aligned}$$

by property 2,

$$\begin{aligned}&= a_1b_1\mathbf{i} \times \mathbf{i} + a_2b_1\mathbf{j} \times \mathbf{i} + a_3b_1\mathbf{k} \times \mathbf{i} + a_1b_2\mathbf{i} \times \mathbf{j} + a_2b_2\mathbf{j} \times \mathbf{j}\\ &\quad + a_3b_2\mathbf{k} \times \mathbf{j} + a_1b_3\mathbf{i} \times \mathbf{k} + a_2b_3\mathbf{j} \times \mathbf{k} + a_3b_3\mathbf{k} \times \mathbf{k},\end{aligned}$$

by properties 3 and 4. These nine terms may look rather formidable at first, but we can simplify by means of the formulas in (1), anticommutativity, and the fact that $\mathbf{c} \times \mathbf{c} = \mathbf{0}$ for any vector $\mathbf{c} \in \mathbf{R}^3$. (Why?) Thus,

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= -a_2b_1\mathbf{k} + a_3b_1\mathbf{j} + a_1b_2\mathbf{k} - a_3b_2\mathbf{i} - a_1b_3\mathbf{j} + a_2b_3\mathbf{i}\\ &= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}. \qquad (2)\end{aligned}$$

EXAMPLE 2 Formula (2) gives

$$\begin{aligned}(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) \times (2\mathbf{i} + 2\mathbf{k}) &= (3 \cdot 2 - (-2) \cdot 0)\mathbf{i} + (-2 \cdot 2 - 1 \cdot 2)\mathbf{j}\\ &\quad + (1 \cdot 0 - 3 \cdot 2)\mathbf{k}\\ &= 6\mathbf{i} - 6\mathbf{j} - 6\mathbf{k}.\end{aligned}$$

Formula (2) is more complicated than the corresponding formulas for all the other arithmetic operations of vectors that we've seen. Moreover, it is a rather difficult formula to remember. Fortunately, there is a more elegant way to understand formula (2). We explore this reformulation next.

Matrices and Determinants: A First Introduction

A **matrix** is a rectangular array of numbers. Examples of matrices are

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 \\ 2 & 7 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

If a matrix has m rows and n columns, we call it "$m \times n$" (read "m by n"). Thus, the three matrices just mentioned are, respectively, 2×3, 3×2, and 4×4. To some extent, matrices behave algebraically like vectors. We discuss some elementary matrix algebra in §1.6. For now, we are mainly interested in the notion of a **determinant**, which is a real number associated to an $n \times n$ (square) matrix. (There is no such thing as the determinant of a nonsquare matrix.) In fact, for the purposes of understanding the cross product, we need only study 2×2 and 3×3 determinants.

DEFINITION 4.2 Let A be a 2×2 or 3×3 matrix. Then the **determinant** of A, denoted det A or $|A|$, is the real number computed from the individual entries of A as follows:

- 2×2 case

  If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

- 3×3 case

  If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, then

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - afh - bdi$$

$$= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

in terms of 2×2 determinants.

Perhaps the easiest way to remember and compute 2×2 and 3×3 determinants (but **not** higher-order determinants) is by means of a "diagonal approach." We write (or imagine) diagonal lines running through the matrix entries. The determinant is the sum of the products of the entries that lie on the same diagonal, where negative signs are inserted in front of the products arising from diagonals going from lower left to upper right:

- 2×2 case

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

  and

$$|A| = ad - bc.$$

- 3×3 case (we need to repeat the first two columns for the method to work)

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

  Write

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{matrix} a & b \\ d & e \\ g & h \end{matrix} .$$

  Then

$$|A| = aei + bfg + cdh - gec - hfa - idb.$$

IMPORTANT WARNING This mnemonic device does *not* generalize beyond 3×3 determinants.

We now state the connection between determinants and cross products.

Key Fact. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \qquad (3)$$

The determinants arise from nothing more than rewriting formula (2). Note that the 3×3 determinant in formula (3) needs to be interpreted by using the 2×2 determinants that appear in formula (3). (The 3×3 determinant is sometimes referred to as a "symbolic determinant.")

EXAMPLE 3

$$(3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \times (\mathbf{i} - \mathbf{j} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix}\mathbf{k}$$

$$= \mathbf{i} - 4\mathbf{j} - 5\mathbf{k}.$$

We may also calculate the 3×3 determinant as

$$\left|\begin{array}{ccc|cc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ 3 & 2 & -1 & 3 & 2 \\ 1 & -1 & 1 & 1 & -1 \end{array}\right. = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k} - 2\mathbf{k} - \mathbf{i} - 3\mathbf{j} = \mathbf{i} - 4\mathbf{j} - 5\mathbf{k}.$$

Areas and Volumes

Cross products are used readily to calculate areas and volumes of certain objects. We illustrate the ideas involved with the next two examples.

EXAMPLE 4 Let's use vectors to calculate the area of the triangle whose vertices are $A(3, 1)$, $B(2, -1)$, and $C(0, 2)$ as shown in Figure 1.55.

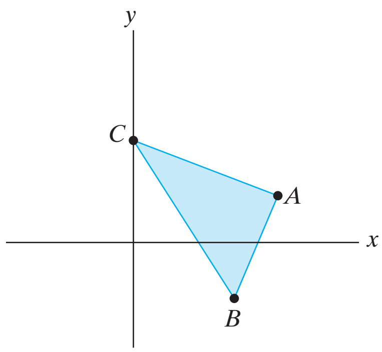


FIGURE 1.55 Triangle *ABC* in Example 4.

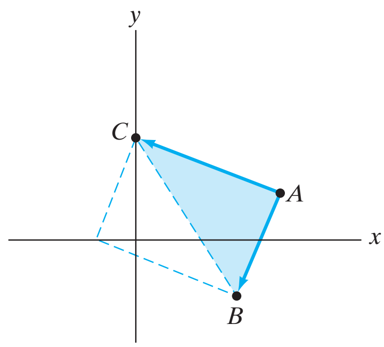


FIGURE 1.56 Any triangle may be considered to be half of a parallelogram.

The trick is to recognize that any triangle can be thought of as half of a parallelogram (see Figure 1.56) and that the area of a parallelogram is obtained from a cross product. In other words, $\overrightarrow{AB} \times \overrightarrow{AC}$ is a vector whose length measures the area of the parallelogram determined by $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and so

$$\text{Area of } \Delta ABC = \tfrac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\|.$$

To use the cross product, we must consider $\overrightarrow{AB}$ and $\overrightarrow{AC}$ to be vectors in $\mathbf{R}^3$. This is straightforward: We simply take the **k**-components to be zero. Thus,

$$\overrightarrow{AB} = -\mathbf{i} - 2\mathbf{j} = -\mathbf{i} - 2\mathbf{j} - 0\mathbf{k},$$

and

$$\overrightarrow{AC} = -3\mathbf{i} + \mathbf{j} = -3\mathbf{i} + \mathbf{j} + 0\mathbf{k}.$$

Therefore,

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -2 & 0 \\ -3 & 1 & 0 \end{vmatrix} = -7\mathbf{k}.$$

Hence,

$$\text{Area of } \Delta ABC = \tfrac{1}{2}\|-7\mathbf{k}\| = \tfrac{7}{2}.$$

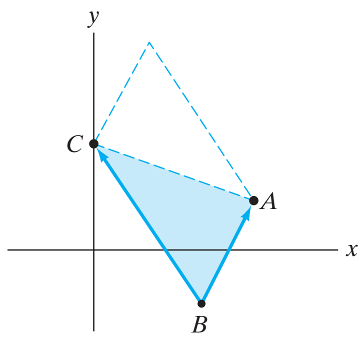


FIGURE 1.57 The area of ΔABC is $7/2$.

There is nothing sacred about using A as the common vertex. We could just as easily have used B or C, as shown in Figure 1.57. Then

$$\text{Area of } \Delta ABC = \tfrac{1}{2}\|\overrightarrow{BA} \times \overrightarrow{BC}\| = \tfrac{1}{2}\|(\mathbf{i} + 2\mathbf{j}) \times (-2\mathbf{i} + 3\mathbf{j})\| = \tfrac{1}{2}\|7\mathbf{k}\| = \tfrac{7}{2}.$$

EXAMPLE 5 Find a formula for the volume of the parallelepiped determined by the vectors **a**, **b**, and **c**. (See Figure 1.58.)

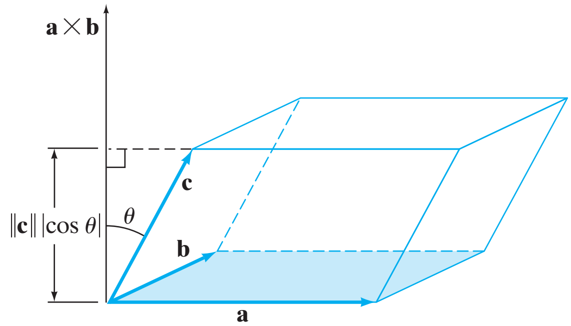


FIGURE 1.58 The parallelepiped determined by **a**, **b**, and **c**.

As explained in §1.3, the volume of a parallelepiped is equal to the product of the area of the base and the height. In Figure 1.58, the base is the parallelogram determined by **a** and **b**. Hence, its area is $\|\mathbf{a} \times \mathbf{b}\|$. The vector $\mathbf{a} \times \mathbf{b}$ is perpendicular to this parallelogram; the height of the parallelepiped is $\|\mathbf{c}\|\,|\cos\theta|$, where θ is the angle between $\mathbf{a} \times \mathbf{b}$ and **c**. (The absolute value is needed in case $\theta > \pi/2$.) Therefore,

$$\begin{aligned}\text{Volume of parallelepiped} &= (\text{area of base})(\text{height}) \\ &= \|\mathbf{a} \times \mathbf{b}\|\,\|\mathbf{c}\|\,|\cos\theta| \\ &= |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|.\end{aligned}$$

(The appearance of the $\cos\theta$ term should alert you to the fact that dot products are lurking somewhere.)

For example, the parallelepiped determined by the vectors

$$\mathbf{a} = \mathbf{i} + 5\mathbf{j}, \quad \mathbf{b} = -4\mathbf{i} + 2\mathbf{j}, \quad \text{and} \quad \mathbf{c} = \mathbf{i} + \mathbf{j} + 6\mathbf{k}$$

has volume equal to

$$\begin{aligned}|((\mathbf{i} + 5\mathbf{j}) \times (-4\mathbf{i} + 2\mathbf{j})) \cdot (\mathbf{i} + \mathbf{j} + 6\mathbf{k})| &= |22\mathbf{k} \cdot (\mathbf{i} + \mathbf{j} + 6\mathbf{k})| \\ &= |22(6)| \\ &= 132.\end{aligned}$$

The real number $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ appearing in Example 5 is known as the **triple scalar product** of the vectors **a**, **b**, and **c**. Since $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$ represents the volume of the parallelepiped determined by **a**, **b**, and **c**, it follows immediately that

$$|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = |(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}| = |(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}|.$$

In fact, if you are careful with the right-hand rule, you can convince yourself that the absolute value signs are not needed; that is,

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}. \tag{4}$$

This is a nice example of how the geometric significance of a quantity can provide an extremely brief proof of an algebraic property the quantity must satisfy. (Try proving it by writing out the expressions in terms of components to appreciate the value of geometric insight.)

We leave it to you to check the following beautiful (and convenient) formula for calculating triple scalar products:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$.

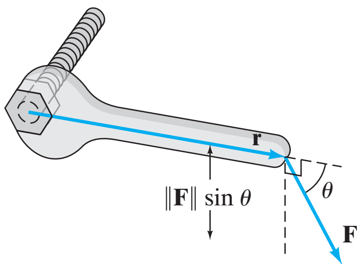


FIGURE 1.59 Turning a bolt with a wrench. The torque on the bolt is the vector $\mathbf{r} \times \mathbf{F}$.

Torque

Suppose you use a wrench to turn a bolt. What happens is the following: You apply some force to the end of the wrench handle farthest from the bolt and that causes the bolt to move in a direction perpendicular to the plane determined by the handle and the direction of your force (assuming such a plane exists). To measure exactly how much the bolt moves, we need the notion of **torque** (or twisting force).

In particular, letting $\mathbf{F}$ denote the force you apply to the wrench, we have

$$\text{Amount of torque} = (\text{length of wrench})(\text{component of } \mathbf{F} \perp \text{wrench}).$$

Let $\mathbf{r}$ be the vector from the center of the bolt head to the end of the wrench handle. Then

$$\text{Amount of torque} = \|\mathbf{r}\| \, \|\mathbf{F}\| \sin\theta,$$

where θ is the angle between $\mathbf{r}$ and $\mathbf{F}$. (See Figure 1.59.) That is, the amount of torque is $\|\mathbf{r} \times \mathbf{F}\|$, and it is easy to check that the direction of $\mathbf{r} \times \mathbf{F}$ is the same as the direction in which the bolt moves (assuming a right-handed thread on the bolt). Hence, it is quite natural to *define* the **torque vector T** to be $\mathbf{r} \times \mathbf{F}$. The torque vector $\mathbf{T}$ is a concise way to capture the physics of this situation.

Note that if $\mathbf{F}$ is parallel to $\mathbf{r}$, then $\mathbf{T} = \mathbf{0}$. This corresponds correctly to the fact that if you try to push or pull the wrench, the bolt does not turn.

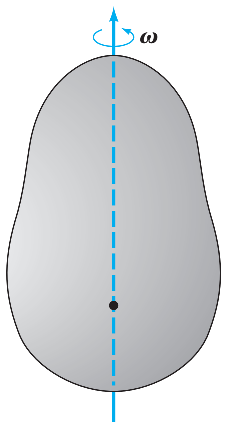


FIGURE 1.60 A potato spinning about an axis.

Rotation of a Rigid Body

Spin an object (a rigid body) about an axis as shown in Figure 1.60. What is the relation between the (linear) velocity of a point of the object and the rotational velocity? Vectors provide a good answer.

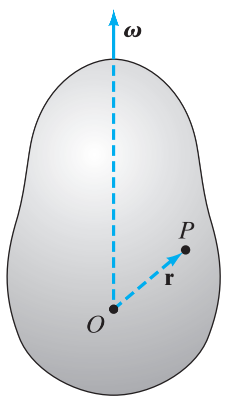


FIGURE 1.61 The angular velocity vector ω.

First we need to define a vector ω, the angular velocity vector of the rotation. This vector points along the axis of rotation, and its direction is determined by the right-hand rule. The magnitude of ω is the angular speed (measured in radians per unit time) at which the object spins. Assume that the angular speed is constant in this discussion. Next, fix a point O (the origin) on the axis of rotation, and let $\mathbf{r}(t) = \overrightarrow{OP}$ be the position vector of a point P of the body, measured as a function of time, as in Figure 1.61. The velocity $\mathbf{v}$ of P is defined by

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t},$$

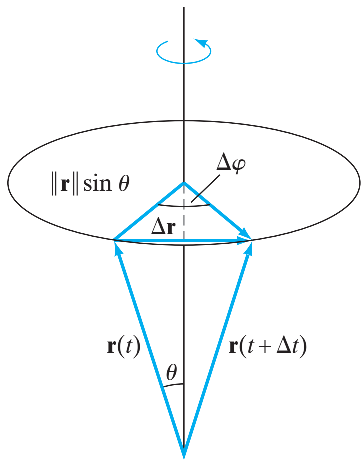


FIGURE 1.62 A spinning rigid body.

where $\Delta\mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$ (i.e., the vector change in position between times t and $t + \Delta t$). Our goal is to relate $\mathbf{v}$ and ω.

As the body rotates, the point P (at the tip of the vector $\mathbf{r}$) moves in a circle whose plane is perpendicular to ω. (See Figure 1.62, which depicts the motion of such a point of the body.) The radius of this circle is $\|\mathbf{r}(t)\| \sin\theta$, where θ is the angle between ω and $\mathbf{r}$. Both $\|\mathbf{r}(t)\|$ and θ must be constant for this rotation. (The direction of $\mathbf{r}(t)$ may change with t, however.) If $\Delta t \approx 0$, then $\|\Delta\mathbf{r}\|$ is approximately the length of the circular arc swept by P between t and $t + \Delta t$. That is,

$$\begin{aligned}\|\Delta\mathbf{r}\| &\approx (\text{radius of circle})(\text{angle swept through by } P)\\ &= (\|\mathbf{r}\| \sin\theta)(\Delta\phi)\end{aligned}$$

from the preceding remarks. Thus,

$$\left\|\frac{\Delta\mathbf{r}}{\Delta t}\right\| \approx \|\mathbf{r}\| \sin\theta \frac{\Delta\varphi}{\Delta t}.$$

Now, let $\Delta t \to 0$. Then $\Delta\mathbf{r}/\Delta t \to \mathbf{v}$ and $\Delta\varphi/\Delta t \to \|\omega\|$ by definition of the angular velocity vector ω, and we have

$$\|\mathbf{v}\| = \|\omega\| \, \|\mathbf{r}\| \sin\theta = \|\omega \times \mathbf{r}\|. \tag{5}$$

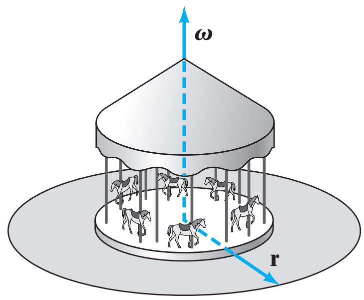


FIGURE 1.63 A carousel wheel.

It's not difficult to see intuitively that $\mathbf{v}$ must be perpendicular to both ω and $\mathbf{r}$. A moment's thought about the right-hand rule should enable you to establish the vector equation

$$\mathbf{v} = \omega \times \mathbf{r}. \tag{6}$$

If we apply formula (5) to a bicycle wheel, it tells us that the speed of a point on the edge of the wheel is equal to the product of the radius of the wheel and the angular speed (θ is $\pi/2$ in this case). Hence, if the rate of rotation is kept constant, a point on the rim of a large wheel goes faster than a point on the rim of a small one. In the case of a carousel wheel, this result tells you to sit on an outside horse if you want a more exciting ride. (See Figure 1.63.)

Summary of Products Involving Vectors

Following is a collection of some basic information concerning scalar multiplication of vectors, the dot product, and the cross product:

Scalar Multiplication: $k\mathbf{a}$

Result is a vector in the direction of $\mathbf{a}$.

Magnitude is $\|k\mathbf{a}\| = |k| \, \|\mathbf{a}\|$.

Zero if $k = 0$ or $\mathbf{a} = \mathbf{0}$.

Commutative: $k\mathbf{a} = \mathbf{a}k$.

Associative: $k(l\mathbf{a}) = (kl)\mathbf{a}$.

Distributive: $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$; $(k + l)\mathbf{a} = k\mathbf{a} + l\mathbf{a}$.