DANIEL J. INMAN

ENGINEERING VIBRATION

Fifth Edition

Engineering Vibration

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Pearson

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Preface

This book is intended for use in a first course in vibrations or structural dynamics for undergraduates in mechanical, civil, and aerospace engineering or engineering mechanics. The text contains the topics normally found in such courses in accredited engineering departments as set out initially by Den Hartog and refined by Thompson. In addition, topics on design, measurement, and computation are addressed.

Pedagogy

Originally, a major difference between the pedagogy of this text and competing texts is the use of high level computing codes. Since then, the other authors of vibrations texts have started to embrace use of these codes. While the book is written so that the codes do not have to be used, I strongly encourage their use. MATLAB®, is very easy to use, at the level of a programmable calculator, and hence does not require any prerequisite courses or training. In fact, the MATLAB® codes can be copied directly and will run as listed. The use of these codes greatly enhances the student's understanding of the fundamentals of vibration. Just as a picture is worth a thousand words, a numerical simulation or plot can enable a completely dynamic understanding of vibration phenomena. Computer calculations and simulations are presented at the end of each of the first four chapters. After that, many of the problems assume that codes are second nature in solving vibration problems.

Another unique feature of this text is the use of "windows," which are distributed throughout the book and provide reminders of essential information pertinent to the text material at hand. The windows are placed in the text at points where such prior information is required. The windows are also used to summarize essential information. The book attempts to make strong connections to previous course work in a typical engineering curriculum. In particular, reference is made to calculus, differential equations, statics, dynamics, and strength of materials course work.

WHAT'S NEW IN THIS EDITION

Most of the changes made in this edition are the result of comments sent to me by students, faculty and practicing engineers who have used the 4th edition. These changes consist of improved clarity in explanations, the addition of some new examples that clarify concepts, and enhanced and revised problem statements. In addition, some text material deemed outdated and not useful has been removed. The MATLAB codes have also been updated. However, software companies update their codes much faster than the publishers can update their texts, so users should consult the web for updates in syntax, commands, etc. One consistent request from students has been not to reference data appearing previously in other examples or problems. This has been addressed by providing all of the relevant data in the problem statements. A number of students and instructors have written with suggestions for improvement. Their suggestions prompted us to make the following changes in order to improve readability from the student's perspective:

- Improved clarity in explanations added in 27 different passages in the text. In addition, a new subsection and one new section (5.8) have been added. One section has been completely redone (5.3).
- Thirty new examples that clarify concepts and enhanced problem statements have been added, and ten examples have been modified to improve clarity.
- Text material deemed outdated and not useful has been removed. In particular the Toolbox and codes in Mathematica and Mathcad have been removed.
- All MATLAB codes have been updated to 2020 standards and several codes added to example problems.
- Thirty-eight new problems have been added and 20 problems have been modified for clarity and numerical changes.
- Twenty-nine new figures have been added and several previous figures have been modified.
- Six new equations have been added two have been deleted.

Chapter 1: Changes include new examples, equations, and problems. New textual explanations have been added and/or modified to improve clarity based on student suggestions. Some photographs have been included. Modifications have been made to problems to make the problem statement clear by not referring to data from previous problems or examples. All of the codes have been updated to current syntax, and older, obsolete commands have been replaced.

Chapter 2: New examples and figures have been added, while previous examples and figures have been modified for clarity. New textual explanations have also been added and/or modified. Some photographs have been included. New problems have been added and older problems modified to make the problem statement clear by not referring to data from previous problems or examples. All of the codes have been updated to current syntax, and older, obsolete commands have been replaced.

Chapter 3: New examples and equations have been added, as well as new problems. In particular, the explanation of impulse has been expanded. In addition, previous problems have been rewritten for clarity and precision. All examples and problems that referred to prior information in the text have been modified to present a more self-contained statement. All of the codes have been updated to current syntax, and older, obsolete commands have been replaced.

Chapter 4: Along with the addition of an entirely new example, many of the examples have been changed and modified for clarity and to include improved information. Three new figures have been added. Problems have been modified with the goal of making all problems and examples more self-contained. All of the codes have been updated to current syntax, and older, obsolete commands have been replaced. Several new plots intermixed in the codes have been redone to reflect MATLAB's latest format. Several explanations have been modified according to useres suggestions.

Chapter 5: Section 5.3 has been changed, a figure added, and an example added for clarity correcting a long standing miss representation existing in all current and past vibration texts. The problems are largely the same but many have been changed or modified with different details and to make the problems more self-contained. A new section (5.8) has been added on approximation and scaling.

Chapter 6: One new figure has been added, one change in textual clarity and one new example has been included. A number of small additions have been made to the text for clarity.

Chapters 7 and 8: These chapters were not changed much, except to make minor corrections and additions as suggested by users to tighten up clarity.

Units

This book uses SI units. The 1st edition used a mixture of US Customary and SI, but at the insistence of the editor all units were changed to SI. A new appendix is added to comment on US Customary vs SI units to recognize the importance of being able to switch between units as the globalization of engineering increases. Our engineers need to work in SI to be competitive in this increasingly international work place and non US engineers need to learn to communicate and translate between systems of units.

Instructor Support

This text comes with a bit of support. In particular, MS PowerPoint presentations are available for each chapter. The solutions manual is available in both MS Word and PDF format (sorry, instructors only). Sample tests are available. The MS Word solutions manual can be cut and pasted into presentation slides, tests, or other class enhancements. These resources can be found at www.pearsonhighered.com/irc and will be updated often. Please also email me at daninman@umich.edu with corrections, typos, questions, and suggestions. The book is reprinted often, and at each reprint I have the option to fix typos, so please report any you find to me, as others as well as I will appreciate it.

Student Support

The best place to get help in studying this material is from your instructor, as there is nothing more educational than a verbal exchange. However, the book was written as much as possible from a student's perspective. Many students critiqued the original manuscript, and many of the changes in text have been the result of suggestions from students trying to learn from the material, so please feel free to email me (daninman@umich.edu) should you have questions about explanations. Also I would appreciate knowing about any corrections or typos and, in particular, if you find an explanation hard to follow. My goal in writing this was to provide a useful resource for students learning vibration for the first time.

ACKNOWLEDGEMENTS

Each chapter starts with two photos of different systems that vibrate to remind the reader that the material in this text has broad application across numerous sectors of human activity. These photographs were taken by friends, students, colleagues, relatives, and some by me. I am greatly appreciative of Robert Hargreaves (guitar), P. Timothy Wade (wind mill, Presidential helicopter), Roy Trifilio (bridge), and Catherine Little (damper).

I have implemented many of their suggestions, and I believe the book's explanations are much clearer due to their input. Numerous people have helped make this possible and I thank them all. First, I appreciate the editors at Pearson for suggesting a fifth edition and this new format. I also thank the anonymous reviewers engaged by Pearson and their helpful comments and suggestions. I also think the numerous faculty and students who have been kind enough to write with suggestions. I would like to thank Professor Nejat Olgac of the University of Connecticut for his many helpful suggestions and for inviting me to give a virtual lecture to his class and pointed out the miss conception regarding the presentation of undamped vibration absorbers published in all prior can current vibration texts leading to the correction illustrated in Example 5.3.3.

Special thanks to Dr. Charles Roche of Western New England University. Chuck served as a reviewer of the fourth edition and a collaborator for this edition. His background as a former jet engine engineer and current faculty at a four-year institution helped me keep a good balance between research one-university students and those attending four-year institutions. In addition, his years of working

as a practicing engineer helped tremendously in building relevant examples and problems.

I believe that all of these suggestions have made the book more precise, more readable, and hence more usable.

I have also had the good fortune of being sponsored by numerous companies and federal agencies over the last 40 years to study, design, test, and analyze a large variety of vibrating structures and machines. Without these projects, I would not have been able to write this book nor revise it with the appreciation for the practice of vibration, which I hope permeates the text.

Lastly, thanks to my wife Catherine Little for her constant support through five editions and two university moves.

Daniel J. Inman
Ann Arbor, Michigan

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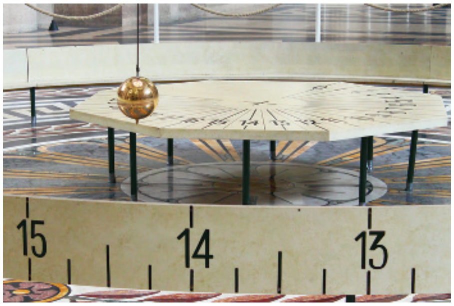
1 Introduction to Vibration and the Free Response



(Photo courtesy of Marco Cannizzaro/ Shutterstock.)

The pendulum often displayed in museums is one of the key sources of mankind's understanding of vibration, how to measure it, how to model its behavior with a mathematical equation to be able to design with, and to be able to predict future behavior. In fact, the pendulum is used here to introduce the topic of vibration and the free response in this chapter. Here "free response" is used to mean that no driving force is applied to the structure and only a restoring force is present. In case of the pendulum, the restoring force is that due to gravity. Vibration is the subdiscipline of dynamics that deals with repetitive motion. Most of the examples in this text are mechanical or structural elements. However, vibration is prevalent in biological systems and is in fact at the source of communication (the ear vibrates to hear and the tongue and vocal cords vibrate to speak). In the case of music, vibrations, say of a stringed instrument are desired. On the other hand, in most mechanical systems and structures, vibration is unwanted and even destructive. For example, vibration in an aircraft frame causes fatigue and can eventually lead to failure. Everyday experiences are full of vibration and usually ways of mitigating vibration. Automobiles, airplanes, trains, and even some bicycles have devices to reduce the vibration induced by motion and transmitted to the driver.

The basic concepts of understanding vibration, analyzing vibration, and predicting the behavior of vibrating systems form the topics of this text. The concepts and formulations presented in the following chapters are intended to provide the skills needed for designing vibrating systems with desired properties that enhance vibration when it is wanted and reduce vibration when it is not. This chapter introduces both the important concept of natural frequency and how to model vibration mathematically.

The Internet is a great source for examples of vibration, and the reader is encouraged to search for movies of vibrating systems and other examples that can be found there.

1.1 INTRODUCTION TO FREE VIBRATION

Vibration is the study of the repetitive motion of objects relative to a stationary frame of reference or nominal position (usually equilibrium). Vibration is evident everywhere and in many cases greatly affects the nature of engineering designs. The vibrational properties of engineering devices are often limiting factors in their performance. When harmful, vibration should be avoided, but it can also be extremely useful. In either case, knowledge about vibration—how to analyze, measure, and control it—is beneficial and forms the topic of this book.

Typical examples of vibration familiar to most include the motion of a guitar string, the ride quality of an automobile or motorcycle, the motion of an airplane's wings, and the swaying of a large building due to wind or an earthquake. In the chapters that follow, vibration is modeled mathematically based on fundamental principles, such as Newton's laws, and analyzed using results from calculus and differential equations. Techniques used to measure the vibration of a system are then developed. In addition, information and methods are given that are useful for designing particular systems to have specific vibrational responses.

The physical explanation of the phenomena of vibration concerns the interplay between potential energy and kinetic energy. A vibrating system must have a component that stores potential energy and releases it as kinetic energy in the form of motion (vibration) of a mass. The motion of the mass then gives up kinetic energy to the potential-energy storing device.

Engineering is built on a foundation of previous knowledge and the subject of vibration is no exception. In particular, the topic of vibration builds on previous courses in dynamics, system dynamics, strength of materials, differential equations, and some matrix analysis. In most accredited engineering programs, these courses are prerequisites for a course in vibration. Thus, the material that follows draws information and methods from these courses. Vibration analysis is based on a coalescence of mathematics and physical observation.

The science of vibration likely started with Galileo's book (1590) about the oscillation of pendulums and strings. In 1602, Galileo measured the period of a pendulum against his pulse, setting the stage for today's vibration analysis. You may have seen a pendulum in a science museum, in a grandfather clock, or you might make a simple one with a string and a marble. As the pendulum swings back and forth, observe that its motion as a function of time can be described very nicely by the sine function from trigonometry. Even more interesting, if you make a free-body diagram of the pendulum and apply Newtonian mechanics to get the equation of motion (summing moments in this case), the resulting equation of motion has the sine function as its solution. Further, the equation of motion predicts the time it takes for the pendulum to repeat its motion. In this example, dynamics, observation, and mathematics all come into agreement to produce a predictive model of the motion of a pendulum, which is easily verified by experiment (physical observation).

This pendulum example tells the essence of this text and the study of vibrations. We propose a series of steps to build on the modeling skills developed in your first courses in statics, dynamics, and strength of materials combined with system dynamics to find equations of motion of successively more complicated systems. Then we will use the techniques of differential equations and numerical integration to solve these equations of motion to predict how various mechanical systems and structures vibrate. The following example illustrates the importance of recalling the methods learned in the first course in dynamics.

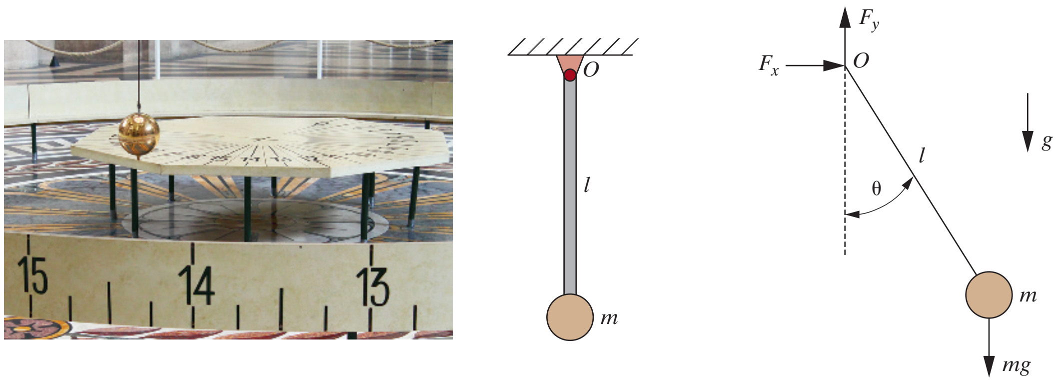


Figure 1.1 (a) Photograph of a pendulum in a museum. (Photo courtesy of Marco Cannizzaro/Shutterstock.) (b) A schematic of a pendulum. (c) The free-body diagram of (b).

Example 1.1.1

Derive the equation of motion of the pendulum in Figure 1.1.

Solution Consider the schematic of a pendulum in Figure 1.1. In this case, the mass of the rod will be ignored as well as any friction in the hinge. Typically, one starts with a photograph or sketch of the part or structure of interest and is immediately faced with having to make assumptions. This is the "art" or experience side of vibration analysis and modeling. The general philosophy is to start with the simplest model possible (hence, here we ignore friction and the mass of the rod and assume the motion remains in a plane) and try to answer the relevant engineering questions. If the simple model doesn't agree with the experiment, then make it more complex by relaxing the assumptions until the model successfully predicts physical observation. With the assumptions in mind, the next step is to create a free-body diagram of the system, as indicated in Figure 1.1(c), in order to identify all of the relevant forces. With all the modeled forces identified, Newton's second law and Euler's second law are used to derive the equations of motion. The resulting equation of motion and its solution can then be used to predict future motions and to design with.

In this example Euler's second law takes the form of summing moments about point O. This yields

$$\Sigma \mathbf{M}_O = J\alpha$$

where $\mathbf{M}_O$ denotes moments about the point $O, J = ml^2$ is the mass moment of inertia of the mass m about the point O, l is the length of the massless rod, and α is the angular acceleration vector. Since the problem is really in one dimension, the vector sum of moments equation becomes the single scalar equation

$$J\alpha(t) = -mgl\sin\theta(t) \quad \text{or} \quad ml^2\ddot{\theta}(t) + mgl\,\sin\theta(t) = 0$$

Here the moment arm for the force mg is the horizontal distance $l \sin\theta$, and the two overdots indicate two differentiations with respect to the time, t. This is a second-order ordinary differential equation, which governs the time response of the pendulum. This is exactly the procedure used in the first course in dynamics to obtain equations of motion.

The equation of motion is nonlinear because of the appearance of the $\sin(\theta)$ and hence difficult to solve. The nonlinear term can be made linear by approximating the sine for small values of $\theta(t)$ as $\sin\theta \approx \theta$. Then the equation of motion becomes

$$\ddot{\theta}(t) + \frac{g}{l}\theta(t) = 0$$

This is a linear, second-order ordinary differential equation with constant coefficients and is commonly solved in the first course of differential equations (usually the third course in the calculus sequence). As we will see later in this chapter, this linear equation of motion and its solution predict the period of oscillation for a simple pendulum quite accurately. The last section of this chapter revisits the nonlinear version of the pendulum equation.

■

Since Newton's second law for a constant mass system is stated in terms of force, which is equated to the mass multiplied by acceleration, an equation of motion with two time derivatives will always result. Such equations require two constants of integration to solve. Euler's second law for constant mass systems also yields two time derivatives. Hence the initial position for $\theta(0)$ and velocity of $\dot{\theta}(0)$ must be specified in order to solve for $\theta(t)$ in Example 1.1.1. The term $mgl\sin\theta$ is called the *restoring force*. In Example 1.1.1, the restoring force is gravity, which provides a potential-energy storing mechanism. However, in most structures and machine parts the restoring force is elastic. This establishes the need for background in strength of materials when studying vibrations of structures and machines.

As mentioned in the example, when modeling a structure or machine it is best to start with the simplest possible model. In this chapter, we model only systems that can be described by a single degree of freedom, that is, systems for which Newtonian mechanics result in a single scalar equation with one displacement coordinate. The degree of freedom of a system is the minimum number of displacement coordinates needed to represent the position of the system's mass at any instant of time. For instance, if the mass of the pendulum in Example 1.1.1 were a rigid body, free to rotate about the end of the pendulum as the pendulum swings, the angle of rotation of the mass would define an additional degree of freedom. The problem would then require two coordinates to determine the position of the mass in space, hence two degrees of freedom. On the other hand, if the rod in Figure 1.1 is flexible, its distributed mass must be considered, effectively resulting in an infinite number of degrees of freedom. Systems with more than one degree of freedom are discussed in Chapter 4, and systems with distributed mass and flexibility are discussed in Chapter 6.

The next important classification of vibration problems after degree of freedom is the nature of the input or stimulus to the system. In this chapter, only the free response of the system is considered. Free response refers to analyzing the vibration of a system resulting from a nonzero initial displacement and/or velocity of the system with no external force or moment applied. In Chapter 2, the response of a single-degree-of-freedom system to a harmonic input (i.e., a sinusoidal applied force) is discussed. Chapter 3 examines the response of a system to a general forcing function (impulse or shock loads, step functions, random inputs, etc.), building on information learned in a course in system dynamics. In the remaining chapters, the models of vibration and methods of analysis become more complex.

The following sections analyze equations similar to the linear version of the pendulum equation given in Example 1.1.1. In addition, energy dissipation is introduced, and details of elastic restoring forces are presented. Introductions to design, measurement, and simulation are also presented. The chapter ends with the introduction to MATLAB® as a means to visualize the response of a vibrating system and for making the calculations required to solve vibration problems more efficiently. In addition, numerical integration is introduced in order to solve nonlinear vibration problems that typically do not have analytical solutions.

1.1.1 The Spring–Mass Model

From introductory physics and dynamics, the fundamental kinematical quantities used to describe the motion of a particle are displacement, velocity, and acceleration vectors. In addition, the laws of physics state that the motion of a mass with changing velocity is determined by the net force acting on the mass. An easy device to use in thinking about vibration is a spring (such as the one used to pull a storm door shut, or an automobile suspension spring) with one end attached to a fixed object and a mass attached to the other end. A schematic of this arrangement is given in Figure 1.2.

Ignoring the mass of the spring itself, the forces acting on the mass consist of the force of gravity pulling down (mg) and the elastic-restoring force of the spring pulling back up (f_k). Note that in this case the force vectors are collinear, reducing the static equilibrium equation to one dimension easily treated as a scalar. The

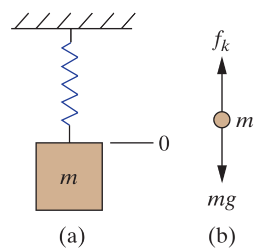


Figure 1.2 A schematic of (a) a single-degree-of-freedom spring–mass oscillator and (b) its free-body diagram.

nature of the spring force can be deduced by performing a simple static experiment. With no mass attached, the spring stretches to the position labeled $x_0 = 0$ in Figure 1.3. As successively more mass is attached to the spring, the force of gravity causes the spring to stretch further. If the value of the mass is recorded, along with the value of the displacement of the end of the spring each time more mass is added, the plot of the force (mass, denoted by m, times the acceleration due to gravity, denoted by g) versus this displacement, denoted by x, yields a curve similar to that illustrated in Figure 1.4. Note that in the region of values for x between 0 and about 20 mm (millimeters), the curve is a straight line. This indicates that for deflections less than 20 mm and forces less than 1000 N (newtons), the force that is applied by the spring to the mass is proportional to the stretch of the spring. The constant of proportionality is the slope of the straight line between 0 and 20 mm. For the particular spring of Figure 1.4, the constant is $50\,\text{N/mm}$, or 5×10^4 N/m. Thus, the equation that describes the force applied by the spring, denoted by f_k, to the mass is the linear relationship

$$f_k = kx \tag{1.1}$$

The value of the slope, denoted by k, is called the *stiffness* of the spring and is a property that characterizes the spring for all situations for which the displacement is less than 20 mm. From strength-of-materials considerations, a linear spring of stiffness k stores potential energy of the amount $\frac{1}{2}kx^2$.

The spring in its linear region behaves the same in both compression and in tension, meaning that if the experiment in Figure 1.3 were performed upside down,

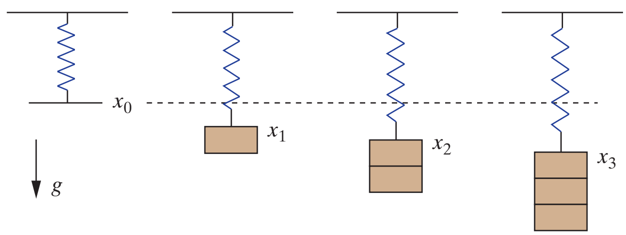


Figure 1.3 A schematic of a massless spring with no mass attached showing its static equilibrium position, followed by increments of increasing added mass illustrating the corresponding deflections.

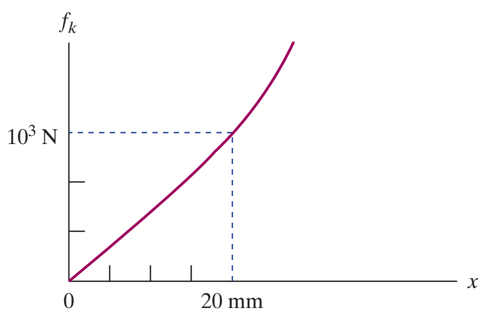


Figure 1.4 The static deflection curve for the spring of Figure 1.3.

that is if the spring were supported at the ground and masses were successively added to the top of the spring, the same curve in Figure 1.4 would result. The spring in Figure 1.3 is in tension whereas in the upside down experiment the spring is in compression. In addition, once the spring has one mass attached, it is considered to be preloaded. Preloaded springs operating in a nonlinear region of the curve in Figure 1.4 can behave differently in compression and in tension. Here, however, we mostly focus on linear springs operated in a linear fashion.

Note that the relationship between f_k and x of equation (1.1) is *linear* (i.e., the curve is linear and f_k depends linearly on x). If the displacement of the spring is larger than 20 mm, the relationship between f_k and x becomes *nonlinear*, as indicated in Figure 1.4. Nonlinear systems are much more difficult to analyze and form the topic of Section 1.10. In this and all other chapters, it is assumed that displacements (and forces) are limited to be in the linear range unless specified otherwise.

Next, consider a free-body diagram of the mass in Figure 1.5, with the massless spring elongated from its rest (equilibrium or unstretched) position. As in the earlier figures, the mass of the object is taken to be m and the stiffness of the spring is taken to be k. Assuming that the mass moves on a frictionless surface along the x direction, the only force acting on the mass in the x direction is the spring force. From Newton's law, the sum of the forces in the x direction must equal the product of mass and acceleration.

Summing the forces on the free-body diagram in Figure 1.5 along the x direction yields

$$m\ddot{x}(t) = -kx(t) \quad \text{or} \quad m\ddot{x}(t) + kx(t) = 0 \tag{1.2}$$

where $\ddot{x}(t)$ denotes the second time derivative of the displacement (i.e., the acceleration). Note that the direction of the spring force is opposite that of the deflection (+ is marked to the right in the figure). As in Example 1.1.1, the displacement vector and acceleration vector are reduced to scalars, since the net force in the y direction is zero ($N = mg$) and the force in the x direction is collinear with the inertial force. Both the displacement and acceleration are functions of the elapsed time t, as

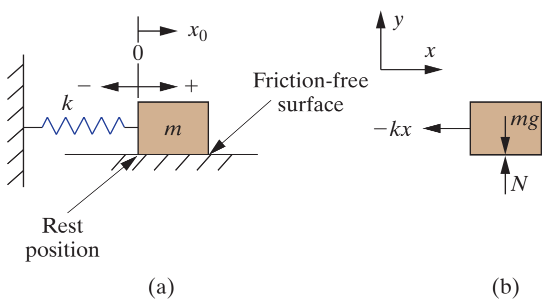


Figure 1.5 (a) A model of a single spring–mass system given an initial displacement of x_0 from its rest, or equilibrium, position and zero initial velocity. (b) The system's free-body diagram.

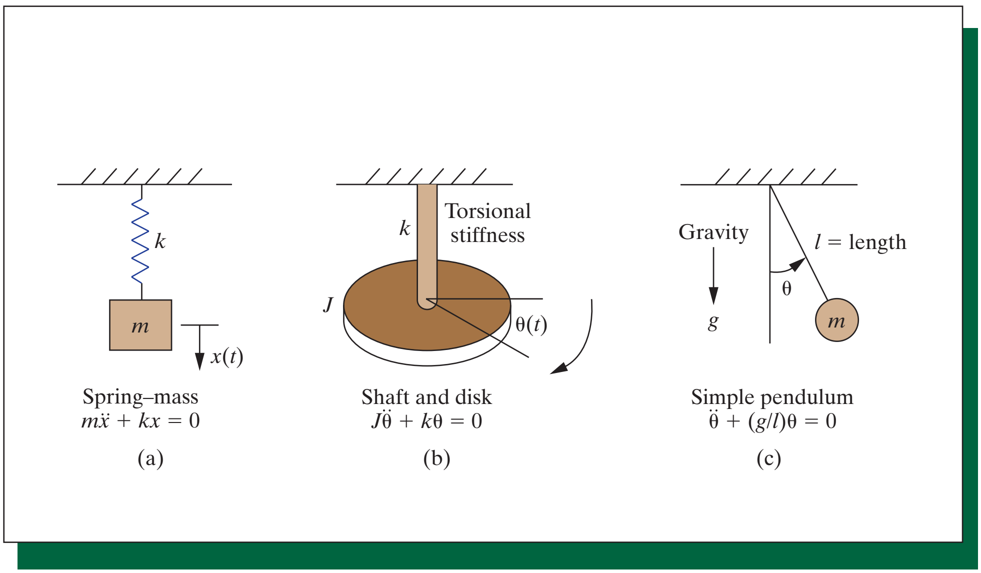
denoted in equation (1.2). Window 1.1 illustrates three types of mechanical systems, which for small oscillations can be described by equation (1.2): a spring–mass system, a rotating shaft, and a swinging pendulum (Example 1.1.1). Other examples are given in Section 1.4 and throughout the book.

One of the major goals of vibration analysis is to be able to predict the response, or motion, of a vibrating system. Thus it is desirable to calculate the solution to equation (1.2). Fortunately, the differential equation of (1.2) is well known and is covered extensively in introductory calculus and physics texts, as well as in texts on differential equations. In fact, there are a variety of ways to calculate this solution. These are all discussed in some detail in the next section. For now, it is sufficient to present a solution based on physical observation. From experience watching a spring, such as the one in Figure 1.5 (or a pendulum), it is guessed that the motion is periodic, of the form

$$x(t) = A \sin(\omega_n t + \phi) \tag{1.3}$$

This choice is made because the sine function repeats itself and hence nicely describes oscillation. Equation (1.3) is the sine function in its most general form, where the constant A is the *amplitude*, or maximum value, of the displacement; ω_n, the *angular natural frequency*, determines the interval in time during which the function repeats itself; and ϕ, called the *phase*, determines the initial value of the sine function. As will be discussed in the following sections, the phase and amplitude are determined by the initial

Window 1.1
Examples of Single-Degree-of-Freedom Systems (for small displacements)



state of the system (see Figure 1.7). It is standard to measure the time t in seconds (s). The phase is measured in radians (rad), and the frequency is measured in radians per second (rad/s). As derived in the following equation, the frequency ω_n is determined by the physical properties of mass and stiffness (m and k), and the constants A and ϕ are determined by the initial position and velocity as well as the frequency.

To see if equation (1.3) is in fact a solution of the equation of motion, it is substituted into equation (1.2). Successive differentiation of the displacement, $x(t)$ in the form of equation (1.3), yields the velocity, $\dot{x}(t)$ given by

$$\dot{x}(t) = \omega_n A \cos(\omega_n t + \phi) \tag{1.4}$$

and the acceleration, $\ddot{x}(t)$, given by

$$\ddot{x}(t) = -\omega_n^2 A \sin(\omega_n t + \phi) \tag{1.5}$$

Substitution of equations (1.5) and (1.3) into (1.2) yields

$$-m\omega_n^2 A \sin(\omega_n t + \phi) = -kA \sin(\omega_n t + \phi)$$

Dividing by A and m yields the fact that this last equation is satisfied if

$$\omega_n^2 = \frac{k}{m}, \quad \text{or} \quad \omega_n = \sqrt{\frac{k}{m}} \tag{1.6}$$

Hence, equation (1.3) is a solution of the equation of motion. The constant ω_n characterizes the spring–mass system, as well as the frequency at which the motion repeats itself, and hence is called the system's *natural frequency*. A plot of the solution $x(t)$ versus time t is given in Figure 1.6. It remains to interpret the constants A and ϕ.

The units associated with the notation ω_n are rad/s and in older texts natural frequency in these units is often referred to as the *circular natural frequency* or *circular frequency* to emphasize that the units are consistent with trigonometric

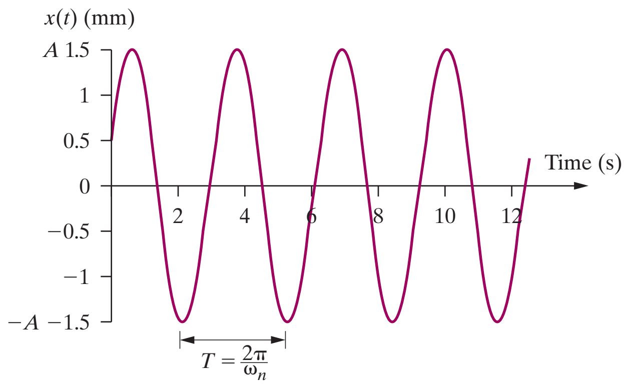


Figure 1.6 The response of a simple spring–mass system to an initial displacement of $x_0 = 0.5$ mm and an initial velocity of $v_0 = 2\sqrt{2}$ mm/s. The natural frequency is 2 rad/s and the amplitude is 1.5 mm. The period is $T = 2\pi/\omega_n = 2\pi/2 = \pi$ s.

functions and to distinguish this from frequency stated in units of hertz (Hz) or cycles per second, denoted by f_n, and commonly used in discussing frequency. The two are related by $f_n = \omega_n/2\pi$ as discussed in Section 1.2. In practice, the phrase *natural frequency* is used to refer to either f_n or ω_n, and the units are stated explicitly to avoid confusion. For example, a common statement is: the natural frequency is 10 Hz, or the natural frequency is 20π rad/s.

Recall from differential equations that because the equation of motion is of second order, solving equation (1.2) involves integrating twice. Thus there are two constants of integration to evaluate. These are the constants A and ϕ. The physical significance, or interpretation, of these constants is that they are determined by the initial state of motion of the spring–mass system. Again, recall Newton's laws, if no force is imparted to the mass, it will stay at rest. If, however, the mass is displaced to a position of x_0 at time $t = 0$, the force kx_0 in the spring will result in motion. Also, if the mass is given an initial velocity of v_0 at time $t = 0$, motion will result because of the induced change in momentum. These are called *initial conditions* and when substituted into the solution (1.3) yield

$$x_0 = x(0) = A \sin(\omega_n 0 + \phi) = A \sin\phi \tag{1.7}$$

and

$$v_0 = \dot{x}(0) = \omega_n A \cos(\omega_n 0 + \phi) = \omega_n A \cos\phi \tag{1.8}$$

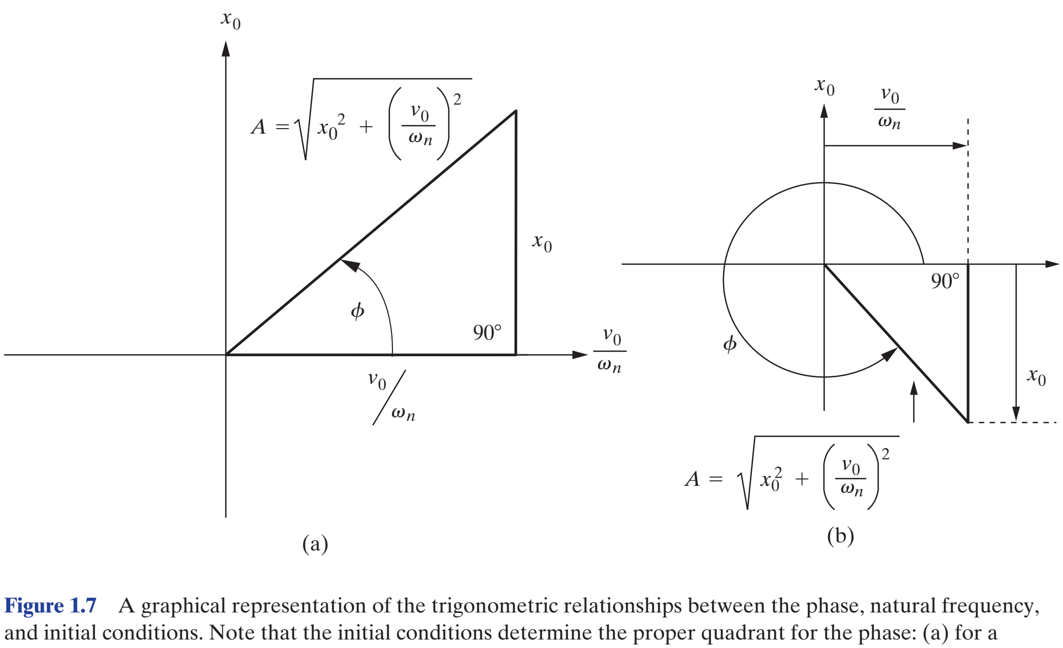


Figure 1.7 A graphical representation of the trigonometric relationships between the phase, natural frequency, and initial conditions. Note that the initial conditions determine the proper quadrant for the phase: (a) for a positive initial position and velocity, (b) for a negative initial position and a positive initial velocity.

Solving these two simultaneous equations for the two unknowns A and ϕ yields

$$A = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} \quad \text{and} \quad \phi = \tan^{-1}\frac{\omega_n x_0}{v_0} \tag{1.9}$$

as illustrated in Figure 1.7. Here the phase ϕ must lie in the proper quadrant, so care must be taken in evaluating the arctangent. Thus, the solution of the equation of motion for the spring–mass system is given by

$$x(t) = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n}\sin\left(\omega_n t + \tan^{-1}\frac{\omega_n x_0}{v_0}\right) \tag{1.10}$$

and is plotted in Figure 1.6. This solution is called the *free response* of the system, because no force external to the system is applied after $t = 0$. The motion of the spring–mass system is called *simple harmonic motion* or *oscillatory motion* and is discussed in detail in the following section. The spring–mass system is also referred to as a *simple harmonic oscillator*, as well as an *undamped single-degree-of-freedom system*.

Example 1.1.2

The phase angle ϕ describes the relative shift in the sinusoidal vibration of the spring–mass system resulting from the initial displacement, x_0. Verify that equation (1.10) satisfies the initial condition $x(0) = x_0$.

Solution Substitution of $t = 0$ in equation (1.10) yields

$$x(0) = A\sin\phi = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n}\sin\left(\tan^{-1}\frac{\omega_n x_0}{v_0}\right)$$

Figure 1.7 illustrates the phase angle ϕ defined by equation (1.9). This right triangle is used to define the sine and tangent of the angle ϕ. From the geometry of a right triangle, and the definitions of the sine and tangent functions, the value of $x(0)$ is computed to be

$$x(0) = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n}\,\frac{\omega_n x_0}{\sqrt{\omega_n^2 x_0^2 + v_0^2}} = x_0$$

which verifies that the solution given by equation (1.10) is consistent with the initial displacement condition.

■

Example 1.1.3

A vehicle wheel, tire, and suspension assembly can be modeled crudely as a single-degree-of-freedom spring–mass system. The (unsprung) mass of the assembly is measured to be about 30 kilograms (kg). Its frequency of oscillation is observed to be 10 Hz. What is the approximate stiffness of the suspension assembly? Figure 1.8 is a close up of a vehicle suspension system showing the spring mechanism.

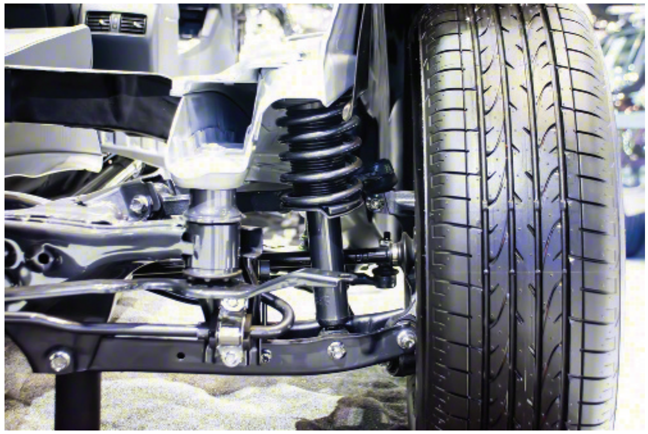


Figure 1.8 Close up of a vehicle suspension system showing the spring mechanism. (Photo courtesy of ScofieldZa/Shutterstock.)

Solution The relationship between frequency, mass, and stiffness is $\omega_n = \sqrt{k/m}$, so that

$$k = m\omega_n^2 = (30\,\text{kg})\left(10\,\frac{\text{cycle}}{\text{s}} \cdot \frac{2\pi\,\text{rad}}{\text{cycle}}\right)^2 = 1.184 \times 10^5\ \text{N/m}$$

This provides one simple way to estimate the stiffness of a complicated device. This stiffness could also be estimated by using a static deflection experiment similar to that suggested by Figures 1.3 and 1.4.

■

Example 1.1.4

Compute the amplitude and phase of the response of a system with a mass of 2 kg and a stiffness of 200 N/m, to the following initial conditions:

a) $x_0 = 2\,\text{mm}$ and $v_0 = 1\,\text{mm/s}$

b) $x_0 = -2\,\text{mm}$ and $v_0 = 1\,\text{mm/s}$

c) $x_0 = 2\,\text{mm}$ and $v_0 = -1\,\text{mm/s}$

Compare the results of these calculations.

Solution First, compute the natural frequency, as this does not depend on the initial conditions and will be the same in each case. From equation (1.6):

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{200\ \text{N/m}}{2\ \text{kg}}} = 10\ \text{rad/s}$$

Next, compute the amplitude, as it depends on the squares of the initial conditions and will be the same in each case. From equation (1.9):

$$A = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} = \frac{\sqrt{10^2 \cdot 2^2 + 1^2}}{10} = 2.0025 \text{ mm}$$

Thus the difference between the three responses in this example is determined only by the phase. Using equation (1.9) and referring to Figure 1.7 to determine the proper quadrant, the following yields the phase information for each case:

a) $\phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right) = \tan^{-1}\left(\frac{(10\,\text{rad/s})(2\text{mm})}{1\,\text{mm/s}}\right) = 1.521\,\text{rad}\left(\text{or } 87.147^\circ\right)$

which is in the first quadrant.

b) $\phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right) = \tan^{-1}\left(\frac{(10\,\text{rad/s})(-2\text{mm})}{1\,\text{mm/s}}\right) = -1.521\,\text{rad}\left(\text{or } -87.147^\circ\right)$

which is in the fourth quadrant.

c) $\phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right) = \tan^{-1}\left(\frac{(10\,\text{rad/s})(2\text{mm})}{-1\,\text{mm/s}}\right) = (-1.521 + \pi)\,\text{rad}\left(\text{or } 92.85^\circ\right)$

which is in the second quadrant (position positive, velocity negative places the angle in the second quadrant in Figure 1.7 requiring that the raw calculation be shifted 180°).

Note that if equation (1.9) is used without regard to Figure 1.7, parts b and c would result in the same answer (which makes no sense physically as the responses each have different starting points). Thus in computing the phase it is important to consider which quadrant the angle should lie in. Fortunately, some calculators and some codes use an arc tangent function, which corrects for the quadrant (for instance, MATLAB uses the `atan2(w0*´x0, v0)` command).

The $\tan(\phi)$ can be positive or negative. If the tangent is positive, the phase angle is in the first or third quadrant. If the sign of the initial displacement is positive, the phase angle is in the first quadrant. If the sign is negative or the initial displacement is negative, the phase angle is in the third quadrant. If on the other hand the tangent is negative, the phase angle is in the second or fourth quadrant. As in the previous case, by examining the sign of the initial displacement, the proper quadrant can be determined. That is, if the sign is positive, the phase angle is in the second quadrant, and if the sign is negative, the phase angle is in the fourth quadrant. The remaining possibility is that the tangent is equal to zero. In this case, the phase angle is either zero or 180°. The initial velocity determines which quadrant is correct. If the initial displacement is zero and if the initial velocity is zero, then the phase angle is zero. If on the other hand the initial velocity is negative, the phase angle is 180°.

■

Example 1.1.5

Plot the response of the spring–mass system of Example 1.1.4a for two periods.

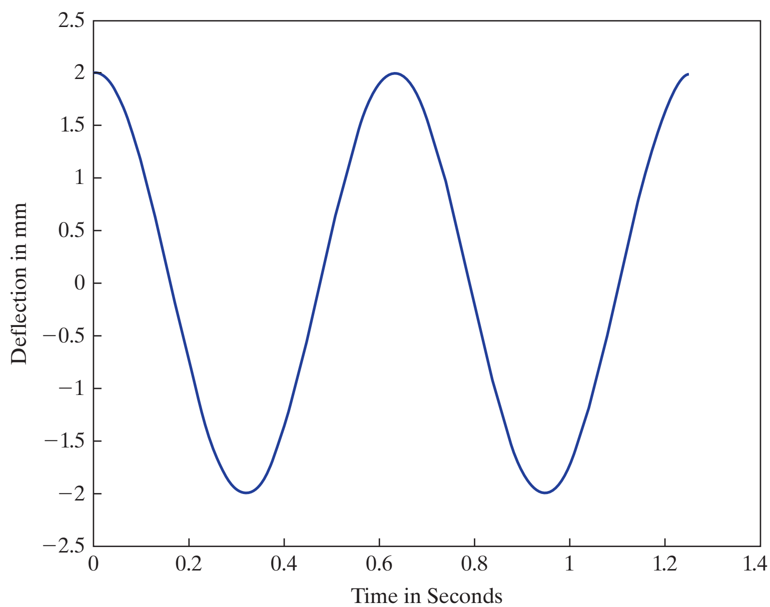
Solution The following MATLAB code is used to plot the response. First calculate the period

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{200 \text{ N/m}}{2 \text{ kg}}} = 10 \text{ rad/s} \Rightarrow T = \frac{2\pi}{\omega_n} = \frac{2\pi}{10} = 0.62 \text{ s}$$

Thus the plot should go from 0 to 1.25 s. From Example 1.1.4 it can be seen that the amplitude is $A = 2.0025$ mm and the phase is $\phi = 1.521$ rad. Type the following in the command window:

```
>> t=0:1.25/100:1.25;
>> x=2.0025*sin(10*t+1.521);
>> plot(t,x)
>> xlabel('time in seconds')
>> ylabel('deflection in mm')
```

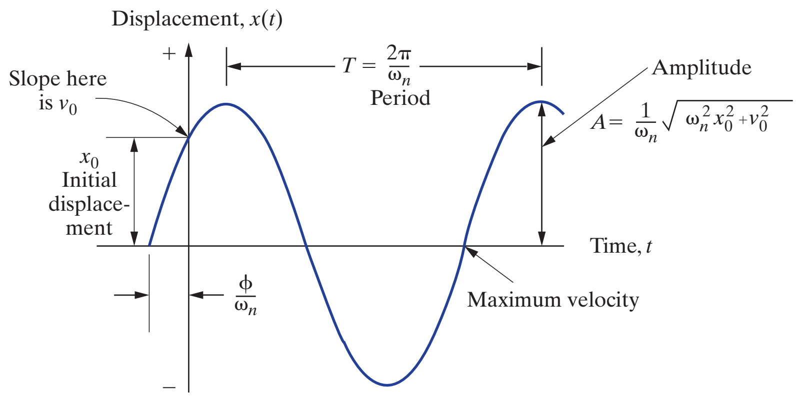
The first line starts the time at zero, uses time steps of 1.25/100, and ends at 1.25 s. The second line is the function of time and the third line is the plot command followed by two labels. This all results in the following plot:



■

The main point of this section is summarized in Window 1.2. This illustrates harmonic motion and how the initial conditions determine the response of such a system.

Window 1.2
Summary of the Description of Simple Harmonic Motion



$$x(t) = \frac{1}{\omega_n}\sqrt{\omega_n^2 x_0^2 + v_0^2}\ \sin(\omega_n t + \phi)$$

$$\phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right) \qquad v_0 = \text{initial velocity}$$

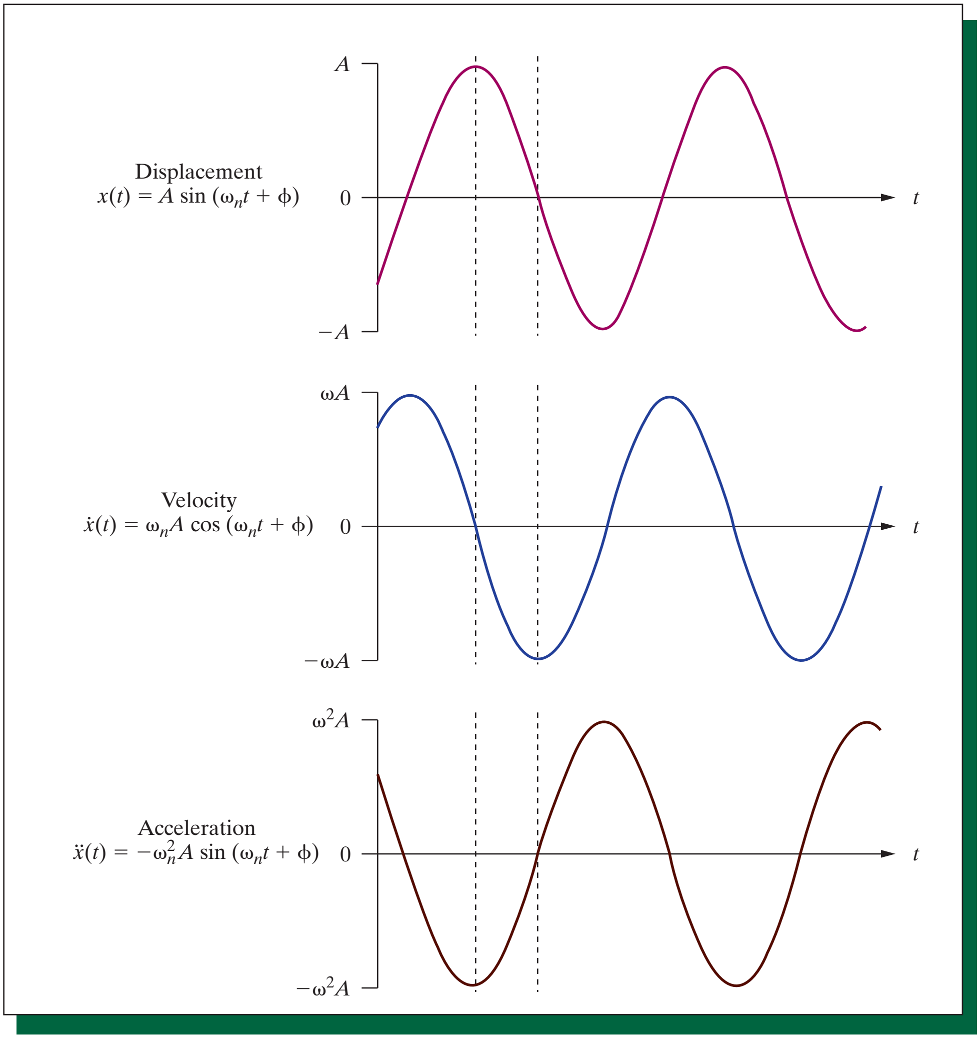
1.2 HARMONIC MOTION

The fundamental kinematic properties of a particle moving in one dimension are displacement, velocity, and acceleration. For the harmonic motion of a simple spring–mass system, these are given by equations (1.3), (1.4), and (1.5), respectively. These equations reveal the different relative amplitudes of each quantity. For systems with natural frequency larger than 1 rad/s, the relative amplitude of the velocity response is larger than that of the displacement response by a multiple of ω_n, and the acceleration response is larger by a multiple of ω_n^2. For systems with frequency less than 1, the velocity and acceleration have smaller relative amplitudes than the displacement. Also note that the velocity is 90° (or $\pi/2$ radians) out of phase with the position [i.e., $\sin(\omega_n t + \pi/2 + \phi) = \cos(\omega_n t + \phi)$], while the acceleration is 180° out of phase with the position and 90° out of phase with the velocity. This is summarized and illustrated in Window 1.3.

The angular natural frequency, ω_n, used in equations (1.3) and (1.10), is measured in radians per second and describes the repetitiveness of the oscillation. As indicated in Window 1.2, the time the cycle takes to repeat itself is the *period, T*, which is related to the natural frequency by

$$T = \frac{2\pi \text{ rad}}{\omega_n \text{ rad/s}} = \frac{2\pi}{\omega_n} \text{ s} \tag{1.11}$$

Window 1.3
The Relationship between Displacement, Velocity, and Acceleration for Simple Harmonic Motion



Here the non italic s denotes seconds. This results from the elementary definition of the period of a sine function. The frequency in hertz (Hz), denoted by f_n, is related to the frequency in radians per second, denoted by ω_n:

$$f_n = \frac{\omega_n}{2\pi} = \frac{\omega_n \text{ rad/s}}{2\pi \text{ rad/cycle}} = \frac{\omega_n \text{ cycles}}{2\pi \text{ s}} = \frac{\omega_n}{2\pi} \text{ (Hz)} \tag{1.12}$$

Equation (1.2) is exactly the same form of differential equation as the linear pendulum equation of Example 1.1.1 and of the shaft and disk of Window 1.1(b). As such, the pendulum will have exactly the same form of solution as equation (1.3), with frequency

$$\omega_n = \sqrt{\frac{g}{l}} \text{ rad/s}$$

The solution of the pendulum equation thus predicts that the period of oscillation of the pendulum is

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{l}{g}} \text{ s}$$

This analytical value of the period can be checked by measuring the period of oscillation of a pendulum with a simple stopwatch. The period of the disk and shaft system of Window 1.1 will have a frequency and period of

$$\omega_n = \sqrt{\frac{k}{J}} \text{ rad/s} \qquad \text{and} \qquad T = 2\pi\sqrt{\frac{J}{k}} \text{ s}$$

respectively. The concept of frequency of vibration of a mechanical system is the single most important physical concept (and number) in vibration analysis. Measurement of either the period or the frequency allows validation of the analytical model. (If you made a 1-meter pendulum, the period would be about 2 s. This is something you could try at home.)

As long as the only disturbance to these systems is a set of nonzero initial conditions, the system will respond by oscillating with frequency ω_n and period T. For the case of the pendulum, the longer the pendulum, the smaller the frequency and the longer the period. That's why in museum demonstrations of a pendulum, the length is usually very large so that T is large and one can easily see the period (also a pendulum is usually used to illustrate the earth's precession; Google the phrase Foucault Pendulum).

Example 1.2.1

Consider a small spring about 30 mm (or 1.18 in) long, welded to a stationary table (ground) so that it is fixed at the point of contact, with a 12-mm (or 0.47-in) bolt welded to the other end, which is free to move. The mass of this system is about 49.2×10^{-3} kg (equivalent to about 1.73 ounces). The spring stiffness can be measured using the method suggested in Figure 1.4 and yields a spring constant of $k = 857.8$ N/m. Calculate the natural frequency and period. Also determine the maximum amplitude of the response if the spring is initially deflected 10 mm. Assume that the spring is oriented along the direction of gravity as in Window 1.1. (Ignore the effect of gravity; see below.)

Solution From equation (1.6) the natural frequency is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{857.8 \text{ N/m}}{49.2 \times 10^{-3} \text{kg}}} = 132 \text{ rad/s}$$

In hertz, this becomes

$$f_n = \frac{\omega_n}{2\pi} = 21 \text{ Hz}$$

The period is

$$T = \frac{2\pi}{\omega_n} = \frac{1}{f_n} = 0.0476 \text{ s}$$

To determine the maximum value of the displacement response, note from Figure 1.6 that this corresponds to the value of the constant A. Assuming that no initial velocity is given to the spring ($v_0 = 0$), equation (1.9) yields

$$x(t)_{\max} = A = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} = x_0 = 10 \text{ mm}$$

Note that the maximum value of the velocity response is $\omega_n A$ or $\omega_n x_0 = 1320\,\text{mm/s}$ and the acceleration response has maximum value

$$\omega_n^2 A = \omega_n^2 x_0 = 174.24 \times 10^3\,\text{mm/s}^2$$

Since $v_0 = 0$, the phase is $\phi = \tan^{-1}(\omega_n x_0/0) = \pi/2$, or 90°. Hence, in this case, the response is $x(t) = 10\sin(132t + \pi/2) = 10\cos(132t)$ mm.

■

Does gravity matter in spring problems? The answer is no, if the system oscillates in the linear region. Consider the spring of Figure 1.3 and let a mass of value m extend the spring. Let Δ denote the distance deflected in this static experiment (Δ is called the static deflection); then the force acting upon the mass is $k\Delta$. From static equilibrium the forces acting on the mass must be zero so that (taking positive down in the figure)

$$mg - k\Delta = 0$$

Next, sum the forces along the vertical for the mass at some point x and apply Newton's law to get

$$m\ddot{x}(t) = -k(x + \Delta) + mg = -kx + mg - \Delta k$$

Note the sign on the spring term is negative because the spring force opposes the motion, which is taken here as positive down. The last two terms add to zero $(mg - k\Delta = 0)$ because of the static equilibrium condition, and the equation of motion becomes

$$m\ddot{x}(t) + kx(t) = 0$$

Thus gravity does not affect the dynamic response. Note $x(t)$ is measured from the elongated (or compressed if upside down) position of the spring–mass system, that is, from its rest position. This is discussed again using energy methods in Figure 1.14.

Example 1.2.2

(a) A pendulum in Brussels swings with a period of 3 seconds. Compute the length of the pendulum. (b) At another location, assume the length of the pendulum is known to be 2 meters and suppose the period is measured to be 2.839 seconds. What is the acceleration due to gravity at that location?

Solution The relationship between period and natural frequency is given in equation (1.11). (a) Substitution of the value of natural frequency for a pendulum and solving for the length of the pendulum yields

$$T = \frac{2\pi}{\omega_n} \Rightarrow \omega_n^2 = \frac{g}{l} = \frac{4\pi^2}{T^2} \Rightarrow l = \frac{gT^2}{4\pi^2} = \frac{(9.811\ \text{m/s}^2)(3)^2\ \text{s}^2}{4\pi^2} = 2.237\ \text{m}$$

Here the value of $g = 9.811\ \text{m/s}^2$ is used, as that is the value it has in Brussels (at 51° latitude and an altitude of 102 m). (b) Next, manipulate the pendulum period equation to solve for g. This yields

$$\frac{g}{l} = \frac{4\pi^2}{T^2} \Rightarrow g = \frac{4\pi^2}{T^2} l = \frac{4\pi^2}{(2.839)^2\ \text{s}^2}(2)\text{m} = 9.796\,\text{m/s}^2$$

This is the value of the acceleration due to gravity in Denver, Colorado, United States (at an altitude 1638 m and latitude 40°).

These sorts of calculations are usually done in high school science classes but are repeated here to underscore the usefulness of the concept of natural frequency and period in terms of providing information about the vibration system's physical properties. In addition, this example serves to remind the reader of a familiar vibration phenomenon. ■

The solution given by equation (1.10) was developed assuming that the response should be harmonic based on physical observation. The form of the response can also be derived by a more analytical approach following the theory of elementary differential equations (see, e.g., Boyce et al., 2017). This approach is reviewed here and will be generalized in later sections and chapters to solve for the response of more complicated systems.

Assume that the solution $x(t)$ is of the form

$$x(t) = ae^{\lambda t} \tag{1.13}$$

where a and λ are nonzero constants to be determined. Upon successive differentiation, equation (1.13) becomes $\dot{x}(t) = \lambda a e^{\lambda t}$ and $\ddot{x}(t) = \lambda^2 a e^{\lambda t}$. Substitution of the assumed exponential form into equation (1.2) yields

$$m\lambda^2 a e^{\lambda t} + k a e^{\lambda t} = 0 \tag{1.14}$$

Since the term $ae^{\lambda t}$ is never zero, expression (1.14) can be divided by $ae^{\lambda t}$ to yield

$$m\lambda^2 + k = 0 \tag{1.15}$$

Solving this algebraically results in

$$\lambda = \pm\sqrt{-\frac{k}{m}} = \pm\sqrt{\frac{k}{m}}\,j = \pm\omega_n j \tag{1.16}$$

where $j = \sqrt{-1}$ is the imaginary number and $\omega_n = \sqrt{k/m}$ is the natural frequency as before. Note that there are two values for $\lambda, \lambda = +\omega_n j$ and $\lambda = -\omega_n j$, because the equation for λ is of second order. This implies that there must be two solutions of equation (1.2) as well. Substitution of equation (1.16) into equation (1.13) yields that the two solutions for $x(t)$ are

$$x(t) = a_1 e^{+j\omega_n t} \quad \text{and} \quad x(t) = a_2 e^{-j\omega_n t} \tag{1.17}$$

Since equation (1.2) is linear, the sum of two solutions is also a solution; hence, the response $x(t)$ is of the form

$$x(t) = a_1 e^{+j\omega_n t} + a_2 e^{-j\omega_n t} \tag{1.18}$$

Window 1.4
Three Equivalent Representations of Harmonic Motion

The solution of $m\ddot{x} + kx = 0$ subject to nonzero initial conditions can be written in three equivalent ways. First, the solution can be written as

$$x(t) = a_1 e^{j\omega_n t} + a_2 e^{-j\omega_n t}, \quad \omega_n = \sqrt{\frac{k}{m}}, \quad j = \sqrt{-1}$$

where a_1 and a_2 are complex-valued constants. Second, the solution can be written as

$$x(t) = A \sin(\omega_n t + \phi)$$

where A and ϕ are real-valued constants. Last, the solution can be written as

$$x(t) = A_1 \sin \omega_n t + A_2 \cos \omega_n t$$

where A_1 and A_2 are real-valued constants. Each set of two constants is determined by the initial conditions, x_0 and v_0. The various constants are related by the following:

$$A = \sqrt{A_1^2 + A_2^2} \qquad \phi = \tan^{-1}\left(\frac{A_2}{A_1}\right)$$

$$A_1 = (a_1 - a_2)j \qquad A_2 = a_1 + a_2$$

$$a_1 = \frac{A_2 - A_1 j}{2} \qquad a_2 = \frac{A_2 + A_1 j}{2}$$

all of which follow from trigonometric identities and Euler's formulas. Note that a_1 and a_2 are a complex conjugate pair, so that A_1 and A_2 are both real numbers provided that the initial conditions are real valued, as is normally the case.

where a_1 and a_2 are complex-valued constants of integration. The Euler relations for trigonometric functions state that $2j\sin\theta = \left(e^{\theta j} - e^{-\theta j}\right)$ and $2\cos\theta = \left(e^{\theta j} + e^{-\theta j}\right)$, where $j = \sqrt{-1}$. [See Appendix A, equations (A.18), (A.19), and (A.20), as well as Window 1.5.] Using the Euler relations, equation (1.18) can be written as

$$x(t) = A\sin(\omega_n t + \phi) \tag{1.19}$$

where A and ϕ are real-valued constants of integration. Note that equation (1.19) is in agreement with the physically intuitive solution given by equation (1.3). The relationships among the various constants in equations (1.18) and (1.19) are given in Window 1.4. Window 1.5 illustrates the use of Euler relations for deriving harmonic functions from exponentials for the underdamped case.

Often when computing frequencies from equation (1.16) such as $\lambda^2 = -4$, there is a temptation to write that the frequency is $\omega_n = \pm 2$. This is incorrect because the $\pm$ sign is used up when the Euler relation is used to obtain the function $\sin\omega_n t$ from the exponential form. The concept of frequency is not defined until it appears in the argument of the sine function and, as such, is always positive.

Precise terminology is useful in discussing an engineering problem, and the subject of vibration is no exception. Since the position, velocity, and acceleration change continually with time, several other quantities are used to discuss vibration. The *peak value*, defined as the maximum displacement, or magnitude A of equation (1.9), is often used to indicate the region in space in which the object vibrates. Another quantity useful in describing vibration is the *average value*, denoted by $\bar{x}$ and defined by

$$\bar{x} = \lim_{T\to\infty}\frac{1}{T}\int_0^T x(t)\,dt \tag{1.20}$$

Note that the average value of $x(t) = A\sin\omega_n t$ over one period of oscillation is zero.

Since the square of displacement is associated with a system's potential energy, the average of the displacement squared is sometimes a useful vibration property to discuss. The mean-square value (or variance) of the displacement $x(t)$, denoted by $\bar{x}^2$ is defined by

$$\bar{x}^2 = \lim_{T\to\infty}\frac{1}{T}\int_0^T x^2(t)\,dt \tag{1.21}$$

The square root of this value, called the *root mean-square* (rms) value, is commonly used in specifying vibration. Because the peak value of the velocity and acceleration are multiples of the natural frequency times the displacement amplitude [i.e., equations (1.3)–(1.5)], these three peak values often differ in value by an order of magnitude or more. Hence, logarithmic scales are often used. A common unit of measurement for vibration amplitudes and rms values is the *decibel* (dB). The decibel was originally defined in terms of the base 10 logarithm of the power ratio of two electrical signals, or as the ratio of the square of the amplitudes of two signals. Following this idea, the decibel is defined as

$$\text{dB} \equiv 10 \log_{10}\left(\frac{x_1}{x_0}\right)^2 = 20 \log_{10} \frac{x_1}{x_0} \tag{1.22}$$

Here the signal x_0 is a reference signal. The decibel is used to quantify how far the measured signal x_1 is above the reference signal x_0. Note that if the measured signal is equal to the reference signal, then this corresponds to 0 dB. The decibel is used extensively in acoustics to compare sound levels. Using a dB scale expands or compresses vibration response information for convenience in graphical representation.

Example 1.2.3

Consider a 2-meter long pendulum placed on the moon and given an initial angular displacement of 0.2 rad and zero initial velocity. Calculate the maximum angular velocity and the maximum angular acceleration of the swinging pendulum (note that gravity on the earth's moon is $g_m = g/6$, where g is the acceleration due to gravity on earth).

Solution From Example 1.1.1 the equation of motion of a pendulum is

$$\ddot{\theta}(t) + \frac{g_m}{l}\theta(t) = 0$$

This equation is of the same form as equation (1.2) and hence has a solution of the form

$$\theta(t) = A\sin(\omega_n t + \phi), \quad \omega_n = \sqrt{\frac{g_m}{l}}$$

From equation (1.9) the amplitude is given by

$$A = \sqrt{\frac{\omega_n^2 x_0^2 + v_0^2}{\omega_n^2}} = x_0 = 0.2 \text{ rad}$$

From Window 1.3 the maximum velocity is just $\omega_n A$ or

$$v_{\max} = \omega_n A = \sqrt{\frac{g_m}{l}}(0.2) = (0.2)\sqrt{\frac{9.8/6}{2}} = 0.18 \text{ rad/s}$$

The maximum acceleration is

$$a_{\max} = \omega_n^2 A = \frac{g_m}{l}A = \frac{9.8/6}{2}(0.2) = 0.163 \text{ rad/s}^2$$

■

Frequencies of concern in mechanical vibration range from fractions of a hertz to several thousand hertz. Amplitudes range from micrometers up to meters (for systems such as tall buildings). According to Mansfield (2005), human beings are more sensitive to acceleration than displacement and easily perceive vibration around 5 Hz at about $0.01\,\text{m/s}^2$ (about 0.01 mm). Horizontal vibration is easy to experience near 2 Hz. Work attempting to characterize comfort levels for human vibrations is still ongoing.

Example 1.2.4

Consider a spring–mass system subject to an arbitrary initial velocity, v_0 initially at rest ($x_0 = 0$). Determine the mean square value of the displacement.

Solution From Window 1.2, the form of the solution for the initial conditions given is

$$x(t) = \frac{v_0}{\omega_n}\sin(\omega_n t)$$

Substitution of this form of the solution into equation (1.21) for the mean square value yields

$$\bar{x}^2 = \lim_{T\to\infty}\frac{1}{T}\int_0^T\left(\frac{v_0}{\omega_n}\right)^2\sin^2(\omega_n t)\,dt$$

Recalling equation (1.11) for the period T the mean square value becomes

$$\bar{x}^2 = \left(\frac{v_0}{\omega_n}\right)^2\lim_{T\to\infty}\frac{1}{T}\int_0^T\sin^2(\omega_n t)\,dt$$

From a table of integrals and letting $x = \omega_n t$ yields

$$\int_0^T\sin^2(x)\,dx = \left(\frac{x}{2} - \frac{\sin 2x}{4}\right)_0^T \Rightarrow \frac{T}{2} - \frac{\sin 2T}{4} = \frac{T}{2}$$

Thus,

$$\bar{x}^2 = \left(\frac{v_0}{\omega_n}\right)^2\lim_{T\to\infty}\frac{1}{T}\frac{T}{2} = \frac{1}{2}\left(\frac{v_0}{\omega_n}\right)^2$$

■

1.3 VISCOUS DAMPING

The response of the spring–mass model (Section 1.1) predicts that the system will oscillate indefinitely. However, everyday observation indicates that freely oscillating systems eventually die out and reduce to zero motion. This observation suggests that the model sketched in Figure 1.5 and the corresponding mathematical model given by equation (1.2) need to be modified to account for this decaying motion. The choice of a representative model for the observed decay in an oscillating system is based partially on physical observation and partially on mathematical convenience. The theory of differential equations suggests that adding a term to equation (1.2) of the form $c\dot{x}(t)$ where c is a constant, will result in a solution $x(t)$ that dies out. Physical observation agrees fairly well with this model and is used successfully to model the damping, or decay, in a variety of mechanical systems. This type of damping, called *viscous damping*, is described in detail in this section.

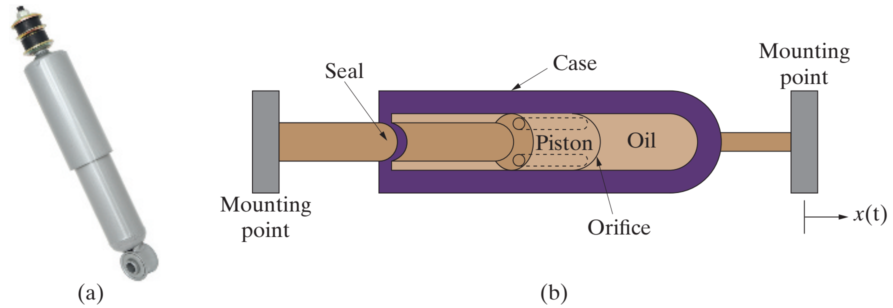


Figure 1.9 (a) A comercial damper. (Photo courtesy of Winai Tepsuttinun/Shutterstock.) (b) A schematic of a dashpot that produces a damping force $f_c(t) = c\dot{x}(t)$ where $x(t)$ is the motion of the case relative to the piston.

While the spring forms a physical model for storing potential energy and hence causing vibration, the *dashpot*, or *damper*, forms the physical model for dissipating energy and thus damping the response of a mechanical system. An example dashpot consists of a piston fit into a cylinder filled with oil as indicated in Figure 1.9. This piston is perforated with holes so that motion of the piston in the oil is possible. The laminar flow of the oil through the perforations as the piston moves causes a damping force on this piston. The force is proportional to the velocity of the piston in a direction opposite that of the piston motion. This damping force, denoted by f_c, has the form

$$f_c = c\dot{x}(t) \tag{1.23}$$

where c is a constant of proportionality related to the oil viscosity. The constant c, called the *damping coefficient*, has units of force per velocity, or N·s/m, as it is customarily written. However, following the strict rules of SI units, the units on damping can be reduced to kg/s, which states the units on damping in terms of the fundamental (also called basic) SI units (mass, time, and length).

In the case of the oil-filled dashpot, the constant c can be determined by fluid principles. However, in most cases, f_c is provided by equivalent effects occurring in the material forming the device. A good example is a block of rubber (which also provides stiffness f_k) such as an automobile motor mount, or the effects of air flowing around an oscillating mass. In all cases in which the damping force f_c is proportional to velocity, the schematic of a dashpot is used to indicate the presence of this force. The schematic is illustrated in Figure 1.10. Unfortunately, the damping coefficient of a system cannot be measured as simply as the mass or stiffness of a system can be. This is pointed out in Section 1.6.

Using a simple force balance on the mass of Figure 1.10 in the x direction, the equation of motion for $x(t)$ becomes

$$m\ddot{x} = -f_c - f_k \tag{1.24}$$

or

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0 \tag{1.25}$$

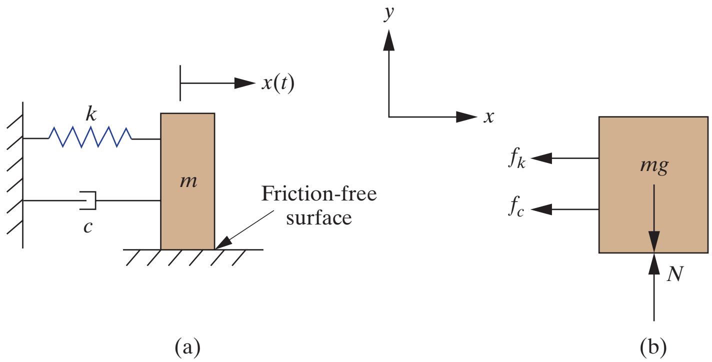


Figure 1.10 (a) The schematic of a single-degree-of-freedom system with viscous damping indicated by a dashpot and (b) the corresponding free-body diagram.

subject to the initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$. The forces f_c and f_k are negative in equation (1.24) because they oppose the motion (positive to the right). Equation (1.25) and Figure 1.10, referred to as a *damped single-degree-of-freedom system*, form the topic of Chapters 1 through 3.

To solve the damped system of equation (1.25), the same method used for solving equation (1.2) is used. In fact, this provides an additional reason to choose f_c to be of the form $c\dot{x}$. Let $x(t)$ have the form given in equation (1.13), $x(t) = ae^{\lambda t}$. Substitution of this form into equation (1.25) yields

$$\left(m\lambda^2 + c\lambda + k\right) ae^{\lambda t} = 0 \tag{1.26}$$

Again, $ae^{\lambda t} \neq 0$, so that this reduces to a quadratic equation in λ of the form

$$m\lambda^2 + c\lambda + k = 0 \tag{1.27}$$

called the *characteristic equation*. This is solved using the quadratic formula to yield the two solutions

$$\lambda_{1,\,2} = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4km} \tag{1.28}$$

Examination of this expression indicates that the roots λ will be real or complex, depending on the value of the discriminant, $c^2 - 4km$. As long as m, c, and k are positive real numbers, λ_1 and λ_2 will be distinct negative real numbers if $c^2 - 4km > 0$. On the other hand, if this discriminant is negative, the roots will be a complex conjugate pair with a negative real part. If the discriminant is zero, the two roots λ_1 and λ_2 are equal negative real numbers. Note that equation (1.15) represents the characteristic equation for the special undamped case (i.e., $c = 0$).

In examining these three cases, it is both convenient and useful to define the *critical damping coefficient*, c_{cr}, by

$$c_{cr} = 2m\omega_n = 2\sqrt{km} \tag{1.29}$$

where ω_n is the undamped natural frequency in rad/s. Furthermore, the nondimensional number ζ, called the *damping ratio*, defined by

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{km}} \tag{1.30}$$

can be used to characterize the three types of solutions to the characteristic equation. Rewriting the roots given by equation (1.28) yields

$$\lambda_{1,\,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{1.31}$$

where it is now clear that the damping ratio ζ determines whether the roots are complex or real. This in turn determines the nature of the response of the damped single-degree-of-freedom system. For positive mass, damping, and stiffness coefficients, there are three cases, which are delineated next.

1.3.1 Underdamped Motion

The most common case is when the damping ratio ζ is less than 1 ($0 < \zeta < 1$) and the discriminant of equation (1.31) is negative, resulting in a complex conjugate pair of roots. Factoring out (–1) from the discriminant in order to clearly distinguish that the second term is imaginary yields

$$\sqrt{\zeta^2 - 1} = \sqrt{(1 - \zeta^2)(-1)} = \sqrt{1 - \zeta^2}\, j \tag{1.32}$$

where $j = \sqrt{-1}$. Thus the two roots become

$$\lambda_1 = -\zeta\omega_n - \omega_n\sqrt{1 - \zeta^2}\, j \tag{1.33}$$

and

$$\lambda_2 = -\zeta\omega_n + \omega_n\sqrt{1 - \zeta^2}\, j \tag{1.34}$$

Following the same argument as that made for the undamped response of equation (1.18), the solution of (1.25) is then of the form

$$x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{j\sqrt{1-\zeta^2}\omega_n t} + a_2 e^{-j\sqrt{1-\zeta^2}\omega_n t}\right) \tag{1.35}$$

where a_1 and a_2 are arbitrary complex-valued constants of integration to be determined by the initial conditions. Using the Euler relations (see Window 1.5), this can be written as

Window 1.5
Euler Relations and the Underdamped Solution

An underdamped solution of $m\ddot{x} + c\dot{x} + kx = 0$ to nonzero initial conditions is of the form

$$x(t) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}$$

where λ_1 and λ_2 are complex numbers of the form

$$\lambda_1 = -\zeta\omega_n + \omega_d j \qquad \text{and} \qquad \lambda_2 = -\zeta\omega_n - \omega_d j$$

where $\omega_n = \sqrt{k/m}$, $\zeta = c/(2m\omega_n)$, $\omega_d = \omega_n\sqrt{1-\zeta^2}$, and $j = \sqrt{-1}$. The two constants a_1 and a_2 are complex numbers and hence represent four unknown constants rather than the two constants of integration required to solve a second-order differential equation. This demands that the two complex numbers a_1 and a_2 be conjugate pairs so that $x(t)$ depends only on two undetermined constants. Substitution of the foregoing values of λ_i into the solution $x(t)$ yields

$$x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{\omega_d jt} + a_2 e^{-\omega_d jt}\right)$$

Using the Euler relations $e^{\phi j} = \cos\phi + j\sin\phi$ and $e^{-\phi j} = \cos\phi - j\sin\phi$, $x(t)$ becomes

$$x(t) = e^{-\zeta\omega_n t}\left[(a_1 + a_2)\cos\omega_d t + j(a_1 - a_2)\sin\omega_d t\right]$$

Choosing the real numbers $A_2 = a_1 + a_2$ and $A_1 = (a_1 - a_2)j$, this becomes

$$x(t) = e^{-\zeta\omega_n t}(A_1 \sin\omega_d t + A_2 \cos\omega_d t)$$

which is real valued. Defining the constant $A = \sqrt{A_1^2 + A_2^2}$ and the angle $\phi = \tan^{-1}(A_2/A_1)$ so that $A_1 = A\cos\phi$ and $A_2 = A\sin\phi$, the form of $x(t)$ becomes [recall that $\sin a\cos b + \cos a\sin b = \sin(a+b)$]

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

where A and ϕ are the constants of integration to be determined from the initial conditions. Complex numbers are reviewed in Appendix A.

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) \tag{1.36}$$

where A and ϕ are constants of integration and ω_d, called the *damped natural frequency*, is given by

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{1.37}$$

in units of rad/s.

The constants A and ϕ are evaluated using the initial conditions in exactly the same fashion as they were for the undamped system as indicated in equations (1.7) and (1.8). Set $t = 0$ in equation (1.36) to get $x_0 = A \sin \phi$. Differentiating (1.36) yields

$$\dot{x}(t) = -\zeta\omega_n A e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) + \omega_d A e^{-\zeta\omega_n t} \cos(\omega_d t + \phi)$$

Let $t = 0$ and $A = x_0/\sin\phi$ in this last expression to get

$$\dot{x}(0) = v_0 = -\zeta\omega_n x_0 + x_0\omega_d \cot\phi$$

Solving this last expression for ϕ yields

$$\tan\phi = \frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}$$

With this value of ϕ, the sine becomes

$$\sin\phi = \frac{x_0\omega_d}{\sqrt{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}}$$

Thus the value of A and ϕ are determined to be

$$A = \sqrt{\frac{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}{\omega_d^2}}, \quad \phi = \tan^{-1}\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0} \tag{1.38}$$

where x_0 and v_0 are the initial displacement and velocity. A plot of $x(t)$ versus t for this underdamped case is given in Figure 1.11. Note that the motion is oscillatory with exponentially decaying amplitude. The damping ratio ζ determines the rate of decay. The response illustrated in Figure 1.11 is exhibited in many mechanical systems and constitutes the most common case. As a check to see that equation (1.38) is reasonable, note that if $\zeta = 0$ in the expressions for A and ϕ, the undamped relations of equation (1.9) result.

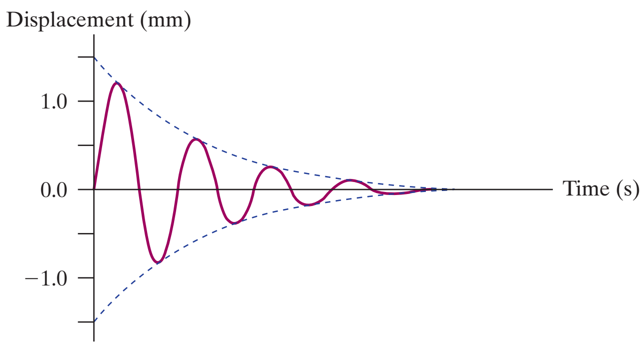


Figure 1.11 The response of an underdamped system: $0 < \zeta < 1$.

1.3.2 Overdamped Motion

In this case, the damping ratio is greater than 1 ($\zeta > 1$). The discriminant of equation (1.31) is positive, resulting in a pair of distinct real roots. These are

$$\lambda_1 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \tag{1.39}$$

and

$$\lambda_2 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \tag{1.40}$$

The solution of equation (1.25) then becomes

$$x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{-\omega_n\sqrt{\zeta^2-1}t} + a_2 e^{+\omega_n\sqrt{\zeta^2-1}t}\right) \tag{1.41}$$

which represents a nonoscillatory response. Again, the constants of integration a_1 and a_2 are determined by the initial conditions indicated in equations (1.7) and (1.8). In this nonoscillatory case, the constants of integration are real valued and are given by

$$a_1 = \frac{-v_0 + \left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}} \tag{1.42}$$

and

$$a_2 = \frac{v_0 + \left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}} \tag{1.43}$$

Typical responses are plotted in Figure 1.12, where it is clear that motion does not involve oscillation. An overdamped system does not oscillate but rather returns to its rest position exponentially.

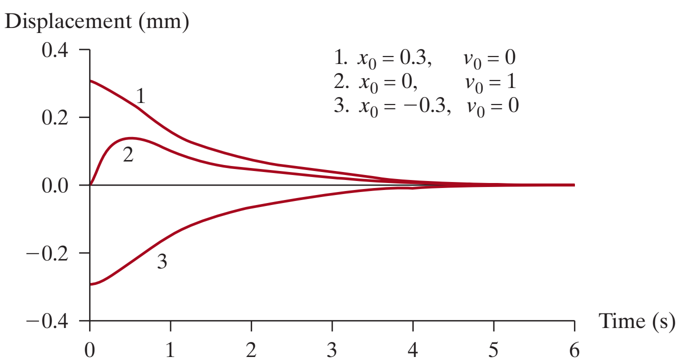


Figure 1.12 The response of an overdamped system, $\zeta > 1$, for two different values of initial displacement (in mm) both with the initial velocity set to zero and one case with $x_0 = 0$ and $v_0 = 1\text{mm/s}$.

1.3.3 Critically Damped Motion

In this last case, the damping ratio is exactly one ($\zeta = 1$) and the discriminant of equation (1.31) is equal to zero. This corresponds to the value of ζ that separates oscillatory motion from nonoscillatory motion. Since the roots are repeated, they have the value

$$\lambda_1 = \lambda_2 = -\omega_n \tag{1.44}$$

The solution takes the form

$$x(t) = (a_1 + a_2 t)e^{-\omega_n t} \tag{1.45}$$

where, again, the constants a_1 and a_2 are determined by the initial conditions. Substituting the initial displacement into equation (1.45) and the initial velocity into the derivative of equation (1.45) yields

$$a_1 = x_0, \quad a_2 = v_0 + \omega_n x_0 \tag{1.46}$$

Critically damped motion is plotted in Figure 1.13 for two different values of initial conditions. It should be noted that critically damped systems can be thought of in several ways. They represent systems with the smallest value of damping rate that yields nonoscillatory motion. Critical damping can also be thought of as the case that separates nonoscillation from oscillation, or the value of damping that provides the fastest return to zero without oscillation.

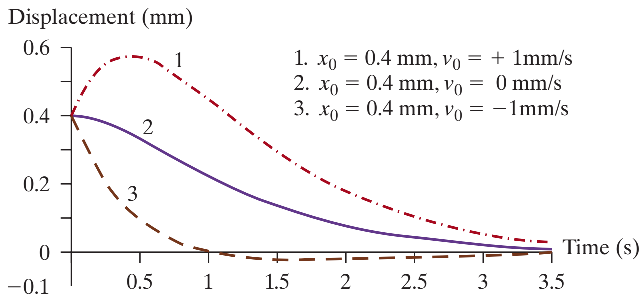


Figure 1.13 The response of a critically damped system for three different initial velocities. The physical properties are $m = 100\,\text{kg}$, $k = 225\,\text{N/m}$, and $\zeta = 1$.

Example 1.3.1

Recall the small spring of Example 1.2.1 (i.e., $\omega_n = 132\,\text{rad/s}$). The damping rate of the spring is measured to be 0.11 kg/s. Calculate the damping ratio and determine if the free motion of the spring–bolt system is overdamped, underdamped, or critically damped.

Solution From Example 1.2.1, $m = 49.2 \times 10^{-3}\,\text{kg}$ and $k = 857.8$ N/m. Using the definition of the critical damping coefficient of equation (1.29) and these values for m and k yields

$$c_{cr} = 2\sqrt{km} = 2\sqrt{(857.8\,N/m)(49.2\times10^{-3}\,kg)}$$
$$= 12.993\,\text{kg/s}$$

If c is measured to be 0.11 kg/s, the critical damping ratio becomes

$$\zeta = \frac{c}{c_{cr}} = \frac{0.11(\text{kg/s})}{12.993(\text{kg/s})} = 0.0085$$

or 0.85% damping. Since ζ is less than 1, the system is underdamped. The motion resulting from giving the spring–bolt system a small displacement will be oscillatory. ■

The single-degree-of-freedom damped system of equation (1.25) is often written in a standard form. This is obtained by dividing equation (1.25) by the mass, m. This yields

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0 \tag{1.47}$$

The coefficient of $x(t)$ is ω_n^2, the undamped natural frequency squared. A little manipulation illustrates that the coefficient of the velocity $\dot{x}$ is $2\zeta\omega_n$. Thus equation (1.47) can be written as

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = 0 \tag{1.48}$$

In this standard form, the values of the natural frequency and the damping ratio are clear. In differential equations, equation (1.48) is said to be in monic form, meaning that the leading coefficient (coefficient of the highest derivative) is one.

Example 1.3.2

The human leg has a measured natural frequency of around 20 Hz when in its rigid (knee-locked) position in the longitudinal direction (i.e., along the length of the bone) with a damping ratio of $\zeta = 0.224$. Calculate the response of the tip of the leg bone to an initial velocity of $v_0 = 0.6\,\text{m/s}$ and zero initial displacement (this would correspond to the vibration induced while landing on your feet, with your knees locked from a height of 18 mm) and plot the response. Last, calculate the maximum acceleration experienced by the leg assuming no damping.

Solution The damping ratio is $\zeta = 0.224 < 1$, so the system is clearly underdamped. The natural frequency is $\omega_n = \frac{20}{1}\,\frac{\text{cycles}}{\text{s}}\,\frac{2\pi\ \text{rad}}{\text{cycles}} = 125.66\ \text{rad/s}$. The damped natural frequency is $\omega_d = 125.66\sqrt{1-(0.224)^2} = 122.467\,\text{rad/s}$. Using equation (1.38) with $v_0 = 0.6\,\text{m/s}$ and $x_0 = 0$ yields

$$A = \frac{\sqrt{[0.6 + (0.224)(125.66)(0)]^2 + [(0)(122.467)]^2}}{122.467} = 0.005 \text{ m} = 5 \text{ mm}$$

$$\phi = \tan^{-1}\left(\frac{(0)(\omega_d)}{v_0 + \zeta\omega_n(0)}\right) = 0$$

The response as given by equation (1.36) is

$$x(t) = 5e^{-28.148t} \sin(122.467t) \text{mm}$$

This is plotted in Figure 1.14. To find the maximum acceleration rate that the leg experiences for zero damping, use the undamped case of equation (1.9):

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}, \ \omega_n = 125.66, \ v_0 = 0.6, \ x_0 = 0$$

$$A = \frac{v_0}{\omega_n} \text{ m} = \frac{0.6}{\omega_n} \text{ m}$$

$$\max(\ddot{x}) = \left|-\omega_n^2 A\right| = \left|-\omega_n^2\left(\frac{0.6}{\omega_n}\right)\right| = (0.6)\left(125.66 \text{ m/s}^2\right) = 75.396 \text{ m/s}^2$$

In terms of $g = 9.81 \text{ m/s}^2$, this becomes

$$\text{maximum acceleration} = \frac{75.396 \text{ m/s}^2}{9.81 \text{ m/s}^2} g = 7.69 \ g\text{'s}$$

■

Example 1.3.3

Compute the form of the response of an underdamped system using the Cartesian form of the solution given in Window 1.5.

Solution From basic trigonometry $\sin(x + y) = \sin x \cos y + \cos x \sin y$. Applying this to equation (1.36) with $x = \omega_d t$ and $y = \phi$ yields

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) = e^{-\zeta\omega_n t}(A_1 \sin\omega_d t + A_2 \cos\omega_d t)$$

where $A_1 = A\cos\phi$ and $A_2 = A\sin\phi$, as indicated in Window 1.5. Evaluating the initial conditions yields

$$x(0) = x_0 = e^0(A_1 \sin 0 + A_2 \cos 0)$$

Solving yields $A_2 = x_0$. Next, differentiate $x(t)$ to get

$$\dot{x} = -\zeta\omega_n e^{-\zeta\omega_n t}(A_1 \sin\omega_d t + A_2 \cos\omega_d t) + \omega_d e^{-\zeta\omega_n t}(A_1 \cos\omega_d t - A_2 \sin\omega_d t)$$

Applying the initial velocity condition yields

$$v_0 = \dot{x}(0) = -\zeta\omega_n(A_1 \sin 0 + x_0 \cos 0) + \omega_d(A_1 \cos 0 - x_0 \sin 0)$$

Solving this last expression yields

$$A_1 = \frac{v_0 + \zeta\omega_n x_0}{\omega_d}$$

Thus the free response in Cartesian form becomes

$$x(t) = e^{-\zeta\omega_n t}\left(\frac{v_0 + \zeta\omega_n x_0}{\omega_d}\sin\omega_d t + x_0\cos\omega_d t\right)$$

■

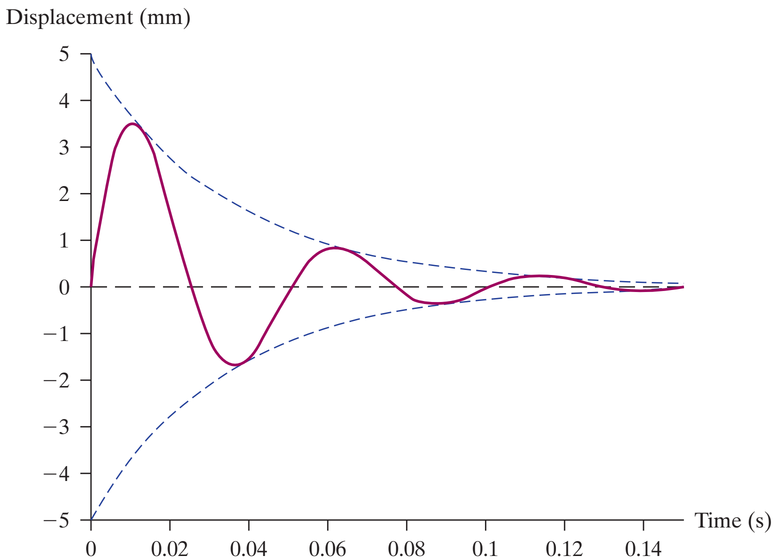


Figure 1.14 A plot of displacement versus time for the leg bone of Example 1.3.2.

Example 1.3.4

Consider a spring–mass damper system, like the one in Figure 1.10, with the following values: $m = 10\ kg, c = 3$ N/s, and $k = 1000$ N/m. (a) Is the system overdamped, underdamped, or critically damped? (b) Compute the solution if the system is given initial conditions $x_0 = 0.01$ m and $v_0 = 0$.

Solution (a) Using equation 1.30 the damping ratio is

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{3}{2\sqrt{10 \cdot 1000}} = 0.015 < 1$$

Thus the system is underdamped.

(b) Using equation (1.38) the amplitude and phase can be calculated from the initial conditions:

$$A = \sqrt{\frac{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}{\omega_d^2}} =$$

$$\frac{1}{9.999}\sqrt{(0.015 \cdot 10 \cdot 0.01)^2 + (0.01 \cdot 9.999)^2} = 0.01 \text{ m}$$

$$\phi = \tan^{-1}\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0} = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = 1.556 \text{ rad}$$

So the solution is $Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) = \underline{0.01e^{-0.15t}\sin(9.999t + 1.556)}$ m.

Note that for any system with $v_0 = 0$, the phase is strictly a function of the damping ratio. For system with a zero initial displacement, $x_0 = 0$, the phase is zero.

■

In all three viscously damped cases the displacement, $x(t)$ eventually *dies out*. Physically this means at some time the motion stops, and $x(t)$ is zero. Mathematically the solution approaches zero exponentially, which means the displacement only reaches zero in the limit. Controls engineers define this more precisely by defining the settling time, the time it takes to die out, as time required for the displacement to reach and stay within 2% of zero.

Example 1.3.5

The time it takes for a damped system to die out, or the settling time, is often taken to be

$$T_S = \frac{4}{\zeta\omega_{nn}}$$

(a) Calculate the settling time for the system of example 1.3.3. (b) Compare the answer in part (a) to the settling time if the system is critically damped.

Solution (a) The natural frequency is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10 \text{ rad/s}$$

Using the value for the damping ratio and frequency from Example 1.3.3, the settling time is

$$T_S = \frac{4}{\zeta\omega_n} = \frac{4}{(0.015)(10)} = 26.67 \text{ s}$$

(b) If the system is crucially damped, $\zeta = 1$ and the settling time becomes

$$T_S = \frac{4}{\zeta\omega_n} = \frac{4}{10} = 0.4 \text{ s}$$

This is a fraction of the time illustrating the significance of large damping in a vibrating system.

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1.4 MODELING AND ENERGY METHODS

Modeling is the art or process of writing down an equation, or system of equations, to describe the motion of a physical device. For example, equation (1.2) was obtained by modeling the spring–mass system of Figure 1.5. By summing the forces acting on the mass along the x direction and employing the experimental evidence of the mathematical model of the force in a spring given by Figure 1.4, equation (1.2) can be obtained. The success of this model is determined by how well the solution of equation (1.2) predicts the observed and measured behavior of the actual system. This comparison between the vibration response of a device and the response predicted by the analytical model is discussed in Section 1.6. The majority of this book is devoted to the analysis of vibration models. However, two methods of modeling—force balance and energy methods—are presented in this section. Newton's three laws form the basis of dynamics. Fifty years after Newton, Euler published his laws of motion. Newton's second law states: the sum of forces acting on a body is equal to the body's mass times its acceleration, and Euler's second law states: the rate of change of angular momentum is equal to the sum of external moments acting on the mass. Euler's second law can be manipulated to reveal that the sum of moments acting on a mass is equal to its rotational inertia times its angular acceleration. These two laws require the use of free-body diagrams and the proper identification of forces and moments acting on a body, forming most of the activity in the study of dynamics.

An alternative approach, studied in dynamics, is to examine the energy in the system, giving rise to what is referred to as energy methods for determining the equations of motion. The energy methods do not require free-body diagrams but rather require an understanding of the energy in a system, providing a useful alternative when forces are not easy to determine. More comprehensive treatments of modeling can be found in Doebelin (1980), Shames (1980, 1989), and Cannon (1967), for example. The best reference for modeling is the text you used to study dynamics. There are also many excellent descriptions on the Internet which can be found using Google or other search engines.

The force summation method is used in the previous sections and should be familiar to the reader from introductory dynamics. For systems with constant mass (such as those considered here) moving in only one direction, the rate of change of momentum becomes the scalar relation

$$\frac{d}{dt}(m\dot{x}) = m\ddot{x}$$

which is often called the inertia force. The physical device of interest is examined by noting the forces acting on the device. The forces are then summed (as vectors) to produce a dynamic equation following Newton's second law. For motion along the x direction only, this becomes the scalar equation

$$\sum_i f_{xi} = m\ddot{x} \tag{1.49}$$

where f_{xi} denotes the ith force acting on the mass m along the x direction and the summation is over the number of such forces. In the first three chapters, only single-degree-of-freedom systems moving in one direction are considered; thus, Newton's law takes on a scalar nature. In more practical problems with many degrees of freedom, energy considerations can be combined with the concepts of virtual work to produce Lagrange's equations, as discussed in Section 4.7. Lagrange's equations also provide an energy-based alternative to summing forces to derive equations of motion.

For rigid bodies in plane motion (i.e., rigid bodies for which all the forces acting on them are coplanar in a plane perpendicular to a principal axis) and free to rotate, Euler's second law states that the sum of the applied torques is equal to the rate of change of angular momentum of the mass. This is expressed as

$$\sum_i M_{0i} = J\ddot{\theta} \tag{1.50}$$

where M_{0i} are the torques acting on the object about the point 0, J is the moment of inertia (also denoted by I_0) about the rotation axis, and θ is the angle of rotation. The sum of moments method was used in Example 1.1.1 to find the equation of motion of a pendulum and is discussed in more detail in Example 1.5.1.

If the forces or torques acting on an object or mechanical part are difficult to determine, an energy approach may be more efficient. In this method, the differential equation of motion is established by using the principle of energy conservation. This principle is equivalent to Newton's law for conservative systems and states that the sum of the potential energy and kinetic energy of a particle remains constant at each instant of time throughout the particle's motion:

$$T + U = \text{constant} \tag{1.51}$$

where T and U denote the total kinetic and potential energy, respectively. Conservation of energy also implies that the change in kinetic energy must equal the change in potential energy:

$$U_1 - U_2 = T_2 - T_1 \tag{1.52}$$

where U_1 and U_2 represent the particle's potential energy at the times t_1 and t_2, respectively, and T_1 and T_2 represent the particle's kinetic energy at times t_1 and t_2, respectively. For undamped periodic motion, energy conservation also implies that

$$T_{\max} = U_{\max} \tag{1.53}$$

Since energy is a scalar quantity, using the conservation of energy principle yields a possibility of obtaining the equation of motion of a system without using force or moment summations.

Equations (1.51), (1.52), and (1.53) are three statements of the conservation of energy. Each of these can be used to determine the equation of motion of a spring–mass system. As an illustration, consider the energy of the spring–mass system of Figure 1.15 hanging in a gravitational field of strength g. The effect of adding the mass m to the massless spring of stiffness k is to stretch the spring from its rest position at 0 to the static equilibrium position Δ. The total potential energy of the spring–mass system is the sum of the potential energy of the spring (or strain energy; see, e.g., Shames, 1989) and the gravitational potential energy. The potential energy of the spring is given by

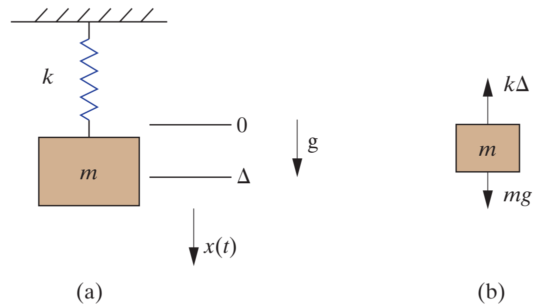

Figure 1.15 (a) A spring–mass system hanging in a gravitational field. Here Δ is the static equilibrium position and x is the displacement from equilibrium. (b) The free-body diagram for static equilibrium.

$$U_{\text{spring}} = \frac{1}{2}k(\Delta + x)^2 \tag{1.54}$$

The gravitational potential energy is

$$U_{\text{grav}} = \; -mgx \tag{1.55}$$

where the negative sign indicates that $x = 0$ is the reference for zero potential energy. The kinetic energy of the system is

$$T = \frac{1}{2}m\dot{x}^2 \tag{1.56}$$

Substituting these energy expressions into equation (1.51) yields

$$\frac{1}{2}m\dot{x}^2 - mgx + \frac{1}{2}k(\Delta + x)^2 = \text{constant} \tag{1.57}$$

Differentiating this expression with respect to time yields

$$\dot{x}(m\ddot{x} + kx) + \dot{x}(k\Delta - mg) = 0 \tag{1.58}$$

Since the static force balance on the mass from Figure 1.14(b) yields the fact that $k\Delta = mg$, equation (1.58) becomes

$$\dot{x}(m\ddot{x} + kx) = 0 \tag{1.59}$$

The velocity $\dot{x}$ cannot be zero for all time; otherwise, $x(t) =$ constant and no vibration would be possible. Hence equation (1.59) yields the standard equation of motion

$$m\ddot{x} + kx = 0 \tag{1.60}$$

This procedure is called the *energy method* of obtaining the equation of motion.

The gravitational force effectively adds a tensile preload to the spring. The entire analysis also holds if Figure 1.15a is turned upside down, putting a compressive preload on the spring. In either case, the mass oscillates around the equilibrium position defined by the static deflection. The difference between compression and tension of the spring only matters if the spring is forced into a nonlinear deflection, then the equilibrium changes and the spring will behave differently in compression than in tension.

The energy method can also be used to obtain the frequency of vibration directly for conservative systems that are oscillatory. The maximum value of sine (and cosine) is one. Hence, from equations (1.3) and (1.4), the maximum displacement is A and the maximum velocity is $\omega_n A$ (recall Window 1.3). Substitution of these maximum values into the expression for $U_{\max}$ and $T_{\max}$ and using the energy equation (1.53) yields

$$\frac{1}{2}m(\omega_n A)^2 = \frac{1}{2}kA^2 \tag{1.61}$$

Solving equation (1.61) for ω_n yields the standard natural frequency relation $\omega_n = \sqrt{k/m}$.

Example 1.4.1

Figure 1.16 is a simple single-degree-of-freedom model of a wheel mounted on a spring. The friction in the system is such that the wheel rolls without slipping. Calculate the natural frequency of oscillation using the energy method. Assume that no energy is lost during the contact.

Solution From introductory dynamics, the rotational kinetic energy of the wheel is $T_{\text{rot}} = \frac{1}{2}J\dot{\theta}^2$, where J is the mass moment of inertia of the wheel and $\theta = \theta(t)$ is the angle of rotation of the wheel. This assumes that the wheel moves relative to the surface without slipping (so that no energy is lost at contact). The translational kinetic energy of the wheel is $T_T = \frac{1}{2}m\dot{x}^2$.

The rotation θ and the translation x are related by $x = r\theta$. Thus $\dot{x} = r\dot{\theta}$ and $T_{\text{rot}} = \frac{1}{2}J\dot{x}^2/r^2$. At maximum energy $x = A$ and $\dot{x} = \omega_n A$ so that

$$T_{\max} = \frac{1}{2}m\dot{x}^2_{\max} + \frac{1}{2}\frac{J}{r^2}\dot{x}^2_{\max} = \frac{1}{2}(m + J/r^2)\omega_n^2 A^2$$

and

$$U_{\max} = \frac{1}{2}\,kx^2_{\max} = \frac{1}{2}\,kA^2$$

Using conservation of energy in the form of equation (1.53) yields $T_{\max} = U_{\max}$, or

$$\frac{1}{2}\left(m + \frac{J}{r^2}\right)\omega_n^2 = \frac{1}{2}\,k$$

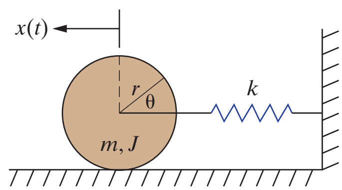

Figure 1.16 The rotational displacement of the wheel of radius r is given by $\theta(t)$ and the linear displacement is denoted by $x(t)$. The wheel has a mass m and a moment of inertia J. The spring has a stiffness k.

Solving this last expression for ω_n yields

$$\omega_n = \sqrt{\frac{k}{m + J/r^2}}$$

the desired frequency of oscillation of the suspension system.

The denominator in the frequency expression derived in this example is called the *effective mass* because the term $(m + J/r^2)$ has the same effect on the natural frequency as does a mass of value $(m + J/r^2)$.

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Example 1.4.2

Use the energy method to determine the equation of motion of the simple pendulum (the rod l is assumed massless) shown in Example 1.1.1 and repeated in Figure 1.17.

Solution Several assumptions must first be made to ensure simple behavior (a more complicated version is considered in Example 1.4.6). Using the same assumptions given in Example 1.1.1 (massless rod, no friction in the hinge), the mass moment of inertia about point 0 is

$$J = ml^2$$

The angular displacement $\theta(t)$ is measured from the static equilibrium or rest position of the pendulum. The kinetic energy of the system is

$$T = \frac{1}{2}J\dot{\theta}^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

The potential energy of the system is determined by the distance h in the figure so that

$$U = mgl\,(1 - \cos\theta)$$

since $h = l(1 - \cos\theta)$ is the geometric change in elevation of the pendulum mass. Substitution of these expressions for the kinetic and potential energy into equation (1.51) and differentiating yields

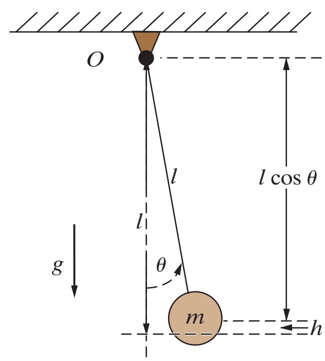

Figure 1.17 The geometry of the pendulum for Example 1.4.2.

$$\frac{d}{dt}\left[\frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta)\right] = 0$$

or

$$ml^2\dot{\theta}\ddot{\theta} + mgl(\sin\theta)\dot{\theta} = 0$$

Factoring out $\dot{\theta}$ yields

$$\dot{\theta}\left(ml^2\ddot{\theta} + mgl\sin\theta\right) = 0$$

Since $\dot{\theta}(t)$ cannot be zero for all time, this becomes

$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

or

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

This is a nonlinear equation in θ and is discussed in Section 1.10 and is derived from summing moments on a free-body diagram in Example 1.1.1. However, since $\sin\theta$ can be approximated by θ for small angles, the linear equation of motion for the pendulum becomes

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

This corresponds to an oscillation with natural frequency $\omega_n = \sqrt{g/l}$ for initial conditions such that θ remains small, as defined by the approximation $\sin\theta \approx \theta$, as discussed in Example 1.1.1.

In Example 1.4.2, it is important to not invoke the small-angle approximation before the final equation of motion is derived. For instance, if the small-angle approximation is used in the potential energy term, then $U = mgl(1-\cos\theta) = 0$, since the small-angle approximation for $\cos\theta$ is 1. This would yield an incorrect equation of motion.

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Example 1.4.3

Determine the equation of motion of the shaft and disk illustrated in Window 1.1 using the energy method.

Solution The shaft and disk of Window 1.1 are modeled as a rod stiffness in twisting, resulting in torsional motion. The shaft, or rod, exhibits a torque in twisting proportional to the angle of twist $\theta(t)$. The potential energy associated with the torsional spring stiffness is $U = \frac{1}{2}k\theta^2$, where the stiffness coefficient k is determined much like the method used to determine the spring stiffness in translation, as discussed in Section 1.1. The angle $\theta(t)$ is measured from the static equilibrium, or rest, position. The kinetic energy associated with the disk of mass moment of inertia J is $T = \frac{1}{2}J\dot{\theta}^2$. This assumes that the inertia of the rod is much smaller than that of the disk and can be neglected.

Substitution of these expressions for the kinetic and potential energy into equation (1.51) and differentiating yields

$$\frac{d}{dt}\left(\tfrac{1}{2}J\dot{\theta}^2 + \tfrac{1}{2}k\dot{\theta}^2\right) = \left(J\ddot{\theta} + k\theta\right)\dot{\theta} = 0$$

so that the equation of motion becomes (because $\dot{\theta} \neq 0$)

$$J\ddot{\theta} + k\theta = 0$$

This is the equation of motion for torsional vibration of a disk on a shaft. The natural frequency of vibration is $\omega_n = \sqrt{k/J}$.

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Example 1.4.4

Model the mass of the spring in the system shown in Figure 1.18 and determine the effect of including the mass of the spring on the value of the natural frequency.

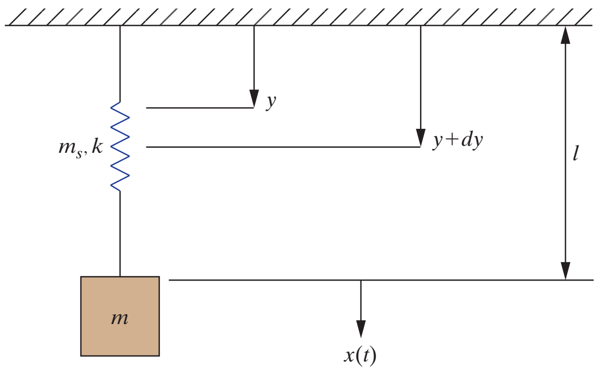

Figure 1.18 A spring–mass system with a spring of mass m_s that is too large to neglect.

Solution One approach to considering the mass of the spring in analyzing the system vibration response is to calculate the kinetic energy of the spring. Consider the kinetic energy of the element dy of the spring. If m_s is the total mass of the spring, then $\frac{m_s}{l}dy$, is the mass of the element dy. The velocity of this element, denoted by v_{dy}, may be approximated by assuming that the velocity at any point varies linearly over the length of the spring:

$$v_{dy} = \frac{y}{l}\dot{x}(t)$$

The total kinetic energy of the spring is the kinetic energy of the element dy integrated over the length of the spring:

$$T_{\text{spring}} = \frac{1}{2}\int_0^l \frac{m_s}{l}\left[\frac{y}{l}\dot{x}\right]^2 dy$$

$$= \frac{1}{2}\left(\frac{m_s}{3}\right)\dot{x}^2$$

From the form of this expression, the effective mass of the spring is $\frac{m_s}{3}$, or one-third of that of the spring. Following the energy method, the maximum kinetic energy of the system is thus

$$T_{\max} = \frac{1}{2}\left(m + \frac{m_s}{3}\right)\omega_n^2 A^2$$

Equating this to the maximum potential energy, $\frac{1}{2}kA^2$ yields the fact that the natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m + m_s/3}}$$

Thus, including the effects of the mass of the spring in the system decreases the natural frequency. Note that if the mass of the spring is much smaller than the system mass m, the effect of the spring's mass on the natural frequency is negligible.

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Example 1.4.5

Fluid systems, as well as solid systems, exhibit vibration. Calculate the natural frequency of oscillation of the fluid in the U-shaped manometer illustrated in Figure 1.19 using the energy method.

Solution The fluid has weight density γ (i.e., the specific weight). The restoring force is provided by gravity. The potential energy of the fluid [(weight)(displacement of c.g.)] is $0.5(\gamma Ax)x$ in each column, so that the total change in potential energy is

$$U = U_2 - U_1 = \frac{1}{2}\gamma A x^2 - \left(-\frac{1}{2}\gamma A x^2\right) = \gamma A x^2$$

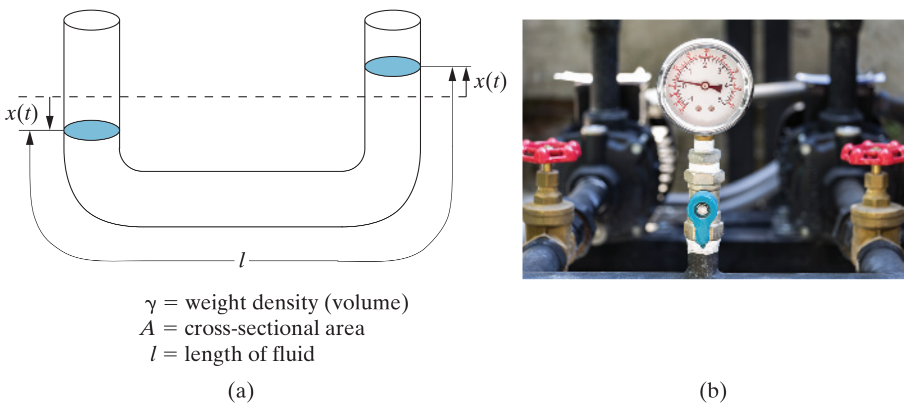

Figure 1.19 (a) The schematic of a U-shaped manometer consisting of a fluid moving in a tube. (b) Close-up of an industrial manometer used to monitor gas pressure. (Photo courtesy of Nicemyphoto/Shutterstock.)