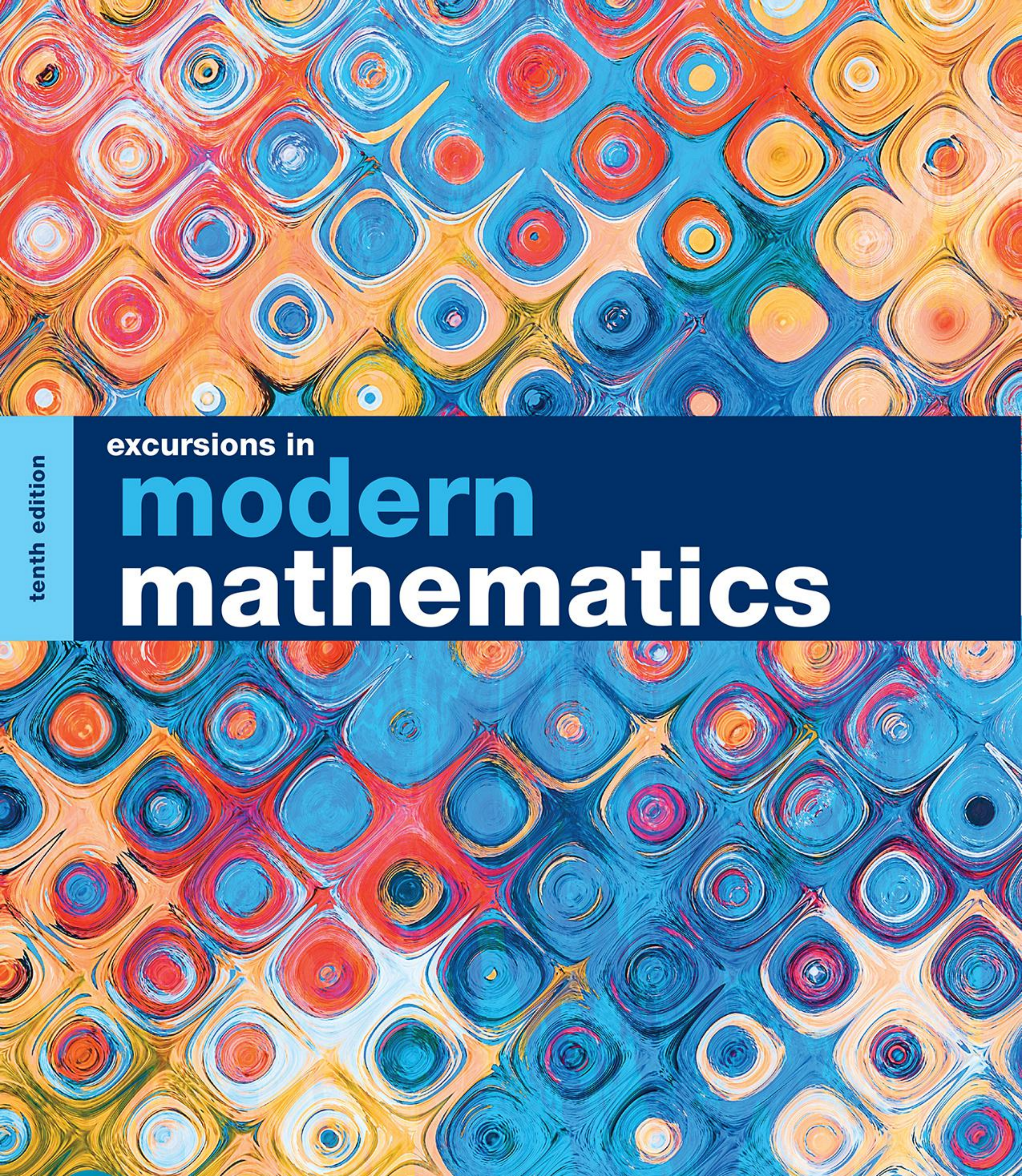

tenth edition

excursions in modern mathematics

peter tannenbaum

Excursions in Modern Mathematics

TENTH EDITION

Excursions in Modern Mathematics

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California State University—Fresno

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To the members of the board of Last Tango

Contents

PART 1 Social Choice

1 The Mathematics of Elections *2*

The Paradoxes of Democracy

2 The Mathematics of Power *38*

Weighted Voting

3 The Mathematics of Sharing *68*

Fair-Division Games

4 The Mathematics of Apportionment *102*

Making the Rounds

PART 2 Management Science

5 The Mathematics of Getting Around *136*

Euler Paths and Circuits

6 The Mathematics of Touring *172*

Traveling Salesman Problems

7 The Mathematics of Networks *204*

The Cost of Being Connected

PART 4 Shape and Form

Preface

> “To most outsiders, modern mathematics is unknown territory. Its borders are protected by dense thickets of technical terms; its landscapes are a mass of indecipherable equations and incomprehensible concepts. Few realize that the world of modern mathematics is rich with vivid images and provocative ideas.”
>
> – *Ivars Peterson,*
> The Mathematical Tourist

This text started many years ago as a set of lecture notes for a new, experimental “math appreciation” course (these types of courses are described, sometimes a bit derisively, as “math for poets”). Over time, the lecture notes grew into a text and the “poets” turned out to be social scientists, political scientists, economists, psychologists, environmentalists, and many other “ists.” Over time, and with the input of many users, the contents have been expanded and improved, but the underlying philosophy of the text has remained the same since those handwritten lecture notes were handed out to my first group of students.

Excursions in Modern Mathematics is a travelogue into that vast and alien frontier that many people perceive mathematics to be. My goal is to show the open-minded reader that mathematics is a lively, interesting, useful, and surprisingly rich human activity.

The “excursions” in *Excursions* represent a collection of topics chosen to meet the following simple criteria.

- **Applicability.** There is no need to worry here about that great existential question of college mathematics: What is this stuff good for? The connection between the mathematics presented in these excursions and down-to-earth, concrete real-life problems is transparent and immediate.
- **Accessibility.** As a general rule, the excursions in this text do not presume a background beyond standard high school mathematics—by and large, intermediate algebra and a little Euclidean geometry are appropriate and sufficient prerequisites. (In the few instances in which more advanced concepts are unavoidable, an effort has been made to provide enough background to make the material self-contained.) A word of caution—this does not mean that the excursions in this book are easy! In mathematics, as in many other walks of life, simple and basic are not synonymous with easy and superficial.
- **Modernity.** Unlike much of traditional mathematics, which is often hundreds of years old, most of the mathematics in this text has been discovered within the last 100 years, and in some cases, within the last couple of decades. Modern mathematical discoveries do not have to be the exclusive province of professional mathematicians.
- **Aesthetics.** The notion that there is such a thing as beauty in mathematics is surprising to most casual observers. There is an important aesthetic component in mathematics, and just as in art and music (which mathematics very much resembles), it often surfaces in the simplest ideas. A fundamental objective of this text is to develop an appreciation of the aesthetic elements of mathematics.

Outline of Contents

The excursions are organized into five independent parts, each touching on a different area where mathematics and the real-world interface.

PART 1 Social Choice. This part deals with mathematical applications to politics, social science, and government. How are *elections* decided? (Chapter 1); How can the power of individuals, groups, or voting blocs be measured? (Chapter 2); How can assets commonly owned be *divided* in a *fair* and equitable manner? (Chapter 3); How are seats *apportioned* in a legislative body? (Chapter 4).

PART 2 Management Science. This part deals with questions of efficiency—how to manage some valuable resource (time, money, energy) so that utility is maximized. How do we sweep over a network with the least amount of backtracking? (Chapter 5); How do we find the shortest or least expensive route that *visits* a specified set of locations? (Chapter 6); How do we create efficient networks that *connect* people or things? (Chapter 7); How do we schedule a project so that it is completed as early as possible? (Chapter 8).

PART 3 Growth. In this part, we discuss, in very broad terms, the mathematics of growth and decay, profit and loss. In Chapter 9, we cover mathematical models of *population growth*, mostly biological and human populations but also populations of inanimate "things" such as garbage and pollution. Since money plays such an important role in our lives, it deserves a chapter of its own. In Chapter 10, we discuss a few of the key concepts of *financial mathematics*: interest, investments, retirement savings, and consumer debt.

PART 4 Shape and Form. In this part, we cover a few connections between mathematics and the shape and form of objects—natural or human-made. What is *symmetry*? What *types* of symmetries exist in nature and art? (Chapter 11); What kind of geometry lies hidden behind the *kinkiness* of the many irregular shapes we find in nature? (Chapter 12); What is the connection between the *Fibonacci numbers* and the *golden ratio* (two abstract mathematical constructs) and the *spiral* forms that we regularly find in nature? (Chapter 13).

PART 5 Statistics. In one way or another, statistics affects all our lives. Government policy, insurance rates, our health, our diet, and our political lives are all governed by statistical information. This part deals with how the statistical information that affects our lives is collected, organized, and interpreted. What are the purposes and strategies of *data collection*? (Chapter 14); How is data *organized, presented,* and *summarized*? (Chapter 15); How do we use mathematics to measure *uncertainty* and *risk*? (Chapter 16); How do we use mathematics to model, analyze, and make predictions about *real-life, bell-shaped* data sets? (Chapter 17).

Exercise Sets

An important goal for this book is that it be flexible enough to appeal to a wide range of readers in a variety of settings. The exercise sets at the end of each chapter have been designed to convey the depth of the subject matter and are organized by level of difficulty:

- **Walking.** These exercises are meant to test a basic understanding of the main concepts, and they are intended to be within the capabilities of students at all levels.
- **Jogging.** These are exercises that can no longer be considered as routine–either because they use basic concepts at a higher level of complexity or they require slightly higher-order critical thinking skills, or both.
- **Running.** This is an umbrella category for problems that range from slightly unusual or slightly above average in difficulty to problems that can be a real challenge to even the most talented of students.
- **Applet Bytes.** Some chapters include at the end of the exercise set a set of *Applet Byte* exercises. These are exercises that involve the use of one of the applets that are available to accompany this text. The applets are available to all students, either through the MyLab course or by following the link *bit.ly/2NcwKFn*.

New to This Edition

- New and updated examples from pop culture, sports, politics, and science keep the discussion current and relevant for today's students. Examples include discussion of the COVID-19 pandemic, the role of vaccines, and election polling.
- New and updated exercises have been informed by MyLab Math data analytics.
- Many MyLab exercises have been redesigned to more closely match the text's pedagogy. In many cases, this includes updating the learning aids, such as "Help Me Solve This" and "View an Example," to provide students a consistent experience between the text and the MyLab materials.
- New and updated videos have been added to the MyLab course to pair with the examples in the text, featuring expanded example video coverage new to this edition. When needed, videos in the MyLab have been updated to match any updates made to examples in the text.
- New StatCrunch data sets have been added, which allow users to see the full data behind some examples and exercises used in the book. These data sets are identified in the text with a StatCrunch icon **StatCrunch**. These data sets can be used and manipulated by students to better understand the relevant concepts and ideas and answer questions about them.
- **Integrated Review** content and assessments are now available in the MyLab course. Integrated Review assessments, provided for each chapter, allow the user to diagnose gaps in prerequisite skills that would impede progress on course-level objectives. Users can then use *personalized* homework assignments to address any gaps in skills identified. With personalized assignments, each user works on only those skills that they have not mastered.
- **Personal Inventory Assessments**, located in the Skills for Success module in the MyLab, are a collection of exercises designed to promote self-reflection and engagement in students. These 33 assessments include topics such as a Stress Management Assessment, Diagnosing Poor Performance and Enhancing Motivation, and Time Management Assessment.
- A list of the MyLab resources available for each section can now be found at the end of each chapter in the Annotated Instructor's Edition.

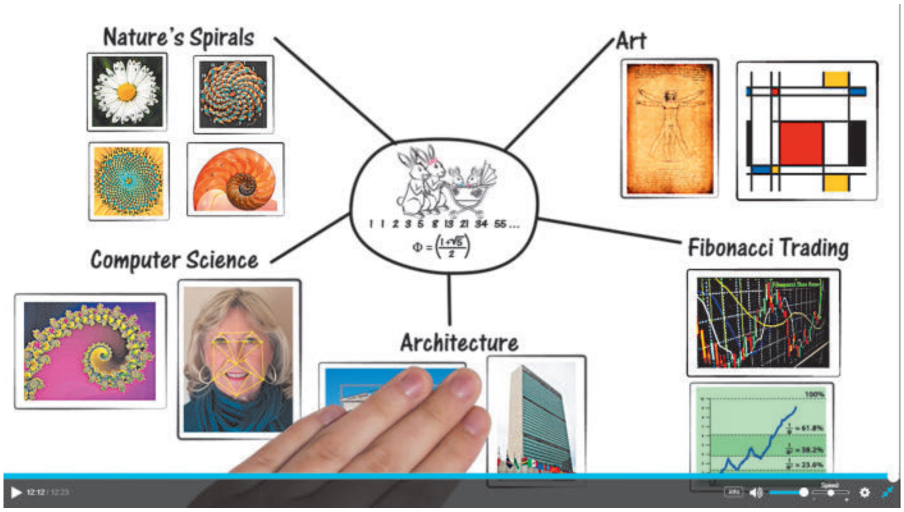
MyLab® Math

MyLab Math is an online course delivery and course management platform that is integrated with this text. The MyLab resources can be used either to complement the text or for a stand-alone course, and include the following:

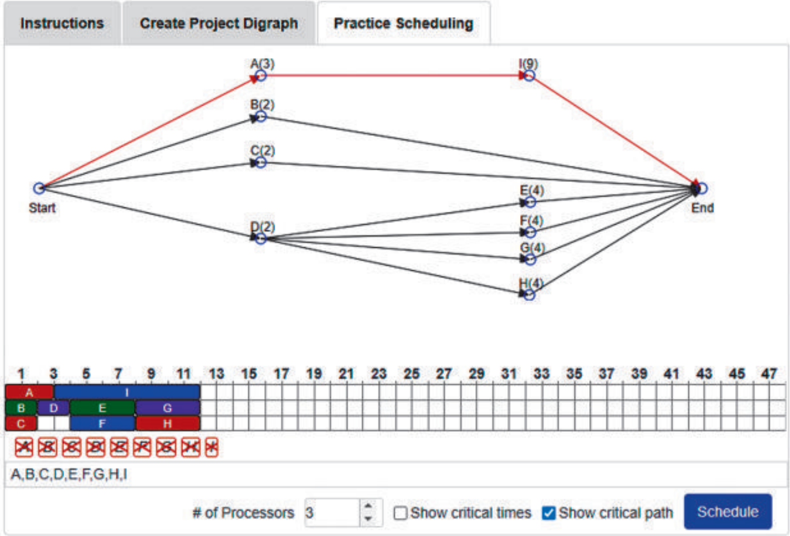
Student Resources

- **eText**—Available in two formats: one that matches the textbook page-for-page, and another that is "reflowable" for use on tablets and smartphones. The latter eText is also fully accessible using screen-readers.
- **UPDATED! Exercises with Immediate Feedback**—The exercises in MyLab Math reflect the approach and learning style of this text and regenerate algorithmically to give students unlimited opportunity for practice and mastery. Most exercises include learning aids, such as guided solutions and sample problems, and they offer helpful feedback when students enter incorrect answers. The exercises are parallel to the exercises in the text, cover all levels of difficulty and include some Applet-based exercises.

- **UPDATED! Example Videos**—Example videos cover many of the examples in the text to demonstrate the concepts through the voice of an instructor. All videos have closed captioning available.
- **Animated Whiteboard Concept Videos**—These videos use narration and animated drawing to bring concepts to life in an engaging manner making the concepts easier to comprehend. Videos cover topics such as Fair Division, Eulerizing Graphs, Self-Similarity, The Golden Ratio, and Normal Curves. These videos are also linked in the eText.



- **Personalized Homework**—With Personalized Homework activated for an assignment, students taking the quiz or test receive a subsequent homework assignment that is personalized based on their performance. This way, students can focus on just the topics they have not yet mastered.
- **Applets**—These applets found in the Video & Resource Library and Learning Tools help students explore concepts more deeply, encouraging them to visualize and interact with concepts such as apportionment, methods, Hamilton paths and circuits, priority list scheduling, and geometric fractals. Applets are also linked in the eText.



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- **Student's Solutions Manual** provides detailed worked out solutions to odd-numbered walking and jogging exercises. Instructors can choose to make this available to their class in the MyLab.

- **Projects & Papers**—The Projects & Papers included in earlier editions of the text are included as a MyLab Math resource for use as discussion material or project ideas.
- **Profiles**—The biographical profiles included in earlier editions of the text are also included as a MyLab Math for use as discussion material or project ideas.
- **NEW! StatCrunch** data sets have been added, which allow users to see the full data behind some examples and exercises used in the book.
- **Mindset videos** and assignable, open-ended exercises foster a growth mindset in students. This material encourages students to maintain a positive attitude about learning, value their own ability to grow, and view mistakes as learning opportunities–so often a hurdle for math students. These videos are one of many **Study Skills and Career-Readiness Resources** that address the nonmath-related issues that can affect student success.

Instructor Resources

- **NEW! Integrated Review** in MyLab Math provides embedded and personalized review of prerequisite topics within relevant chapters. Integrated Review assignments, noted below, are premade and can be edited and assigned by instructors.
    - Students begin relevant chapters with a premade, assignable Skills Check to check each student's understanding of prerequisite skills needed to be successful in that chapter.
    - For any gaps in skill that are identified, a personalized review homework is populated. Students practice on the topics they need to focus on—no more, no less.
    - A suite of resources is available to help students understand the objectives they missed on the Skills Check quiz, including worksheets and videos to help remediate.
- Integrated Review in the MyLab is ideal for corequisite courses, where students are enrolled in a college-level course while receiving just-in-time remediation. But it can also be used simply to get underprepared students up to speed on prerequisite skills in order to be more successful in the main course content.
- **TestGen**® enables instructors to build, edit, print, and administer tests using a computerized bank of questions developed to cover all the objectives of the text.
- **Learning Catalytics**—Integrated into MyLab Math, Learning Catalytics (LC) uses students' mobile devices for an engagement, assessment, and classroom intelligence system that gives instructors real-time feedback on student learning. **LC** annotations in the Annotated Instructor's Edition provide a corresponding tag to search for when a LC question is relevant to the topic at hand. For more information on how to use these tags, go to bit.ly/3m0FYEB.
- **Instructor's Testing Manual** includes two alternative multiple-choice tests per chapter.
- **Instructor's Solutions Manual** contains detailed, worked out solutions to all exercises in the text.
- **Image Resources Library** contains all art from the text for instructors to use in their own presentations and handouts.
- **PowerPoint**® editable slides present key concepts and definitions from the text. You can add art from the Image Resource Library or slides that you develop on your own.
- **NEW! Early Alerts**—Now included with Performance Analytics, Early Alerts use predictive analytics to identify struggling students—even if their assignment scores are not a cause for concern. In both Performance Analytics and Early Alerts, instructors can e-mail students individually or by group to provide feedback.

Available in print for instructors:

- **REVISED! Annotated Instructor's Edition** (ISBN: 978-0-13-696895-5) provides annotations for instructors, including suggestions about where media resources like Applets and Animated Whiteboard Videos apply, as well as Learning Catalytics questions, discussion ideas, and teaching tips.

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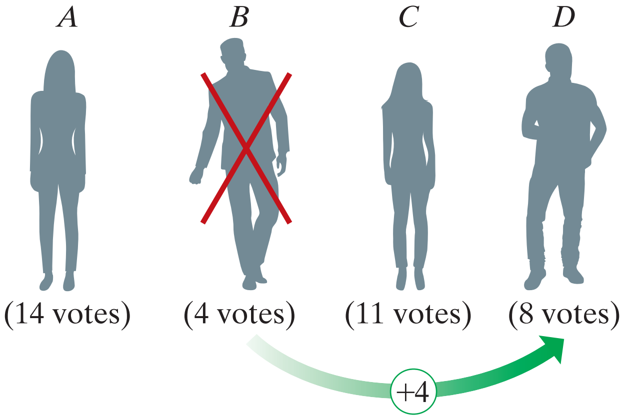
PART

1

Social Choice

1 The Mathematics of Elections



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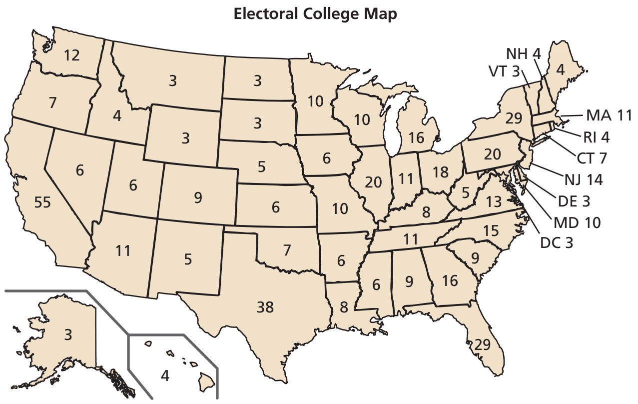
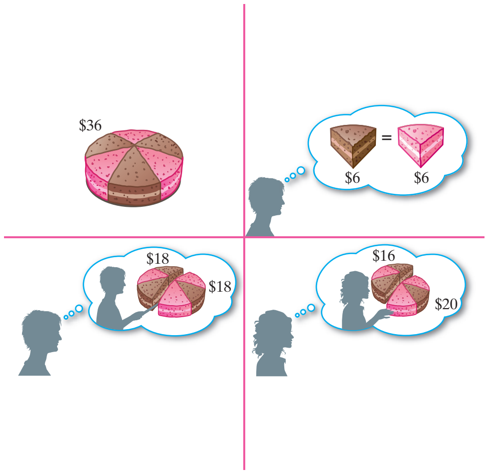
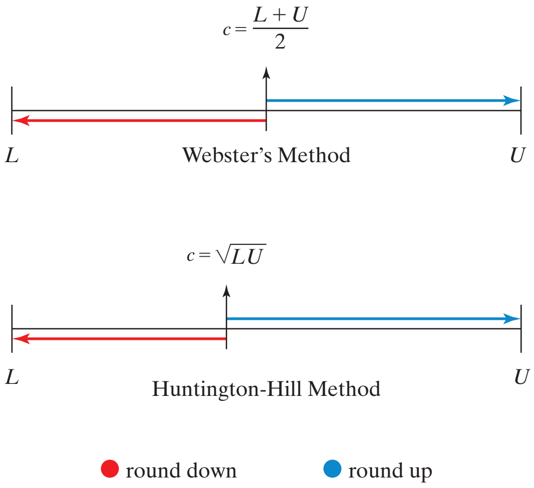


3 The Mathematics of Sharing



4 The Mathematics of Apportionment



1

△ 2019 *American Idol* top ten finalists (see Examples 1.5 and 1.16 for more).

The Mathematics of Elections

The Paradoxes of Democracy

Whether we like it or not, we are all affected by the outcomes of elections. Our president, senators, governors, and mayors make decisions that impact our lives in significant ways, and they all get to be in that position because an election made it possible. But elections touch our lives not just in politics. The Academy Awards, Heisman trophies, NCAA football rankings, *American Idol*—they are all decided by some sort of election. Even something as simple as deciding where to go for dinner might require a little family election.

We have elections because we don't all think alike. Since we cannot all have things our way, we vote. But *voting* is only the first half of the story, the one we are most familiar with. As playwright Tom Stoppard suggests, it's the second half of the story—the *counting*—that is at the heart of the democratic process. How do we sift through the many choices of individual voters to find the collective choice of the group? More important, how well does the process work? Is the process always fair? Answering these questions and explaining a few of the many intricacies and subtleties of *voting theory* are the purpose of this chapter.

But wait just a second! Voting theory? Why do we need a fancy theory to figure out how to count the votes? It all sounds pretty simple: We have an election; we count the ballots. Based on that count, we decide the outcome of the election in a consistent and fair manner. Surely, there must be a reasonable way to accomplish this. Surprisingly, there isn't!

In the late 1940s the American economist Kenneth Arrow discovered a remarkable fact: For elections involving three or more candidates, there is no consistently fair democratic method for choosing a winner. In fact, Arrow demonstrated that *a method for determining election results that is always fair is a mathematical impossibility*. This fact, the most famous in voting theory, is known as *Arrow's Impossibility Theorem*.

> “It's not the voting that's democracy; it's the counting.”
>
> – *Tom Stoppard*

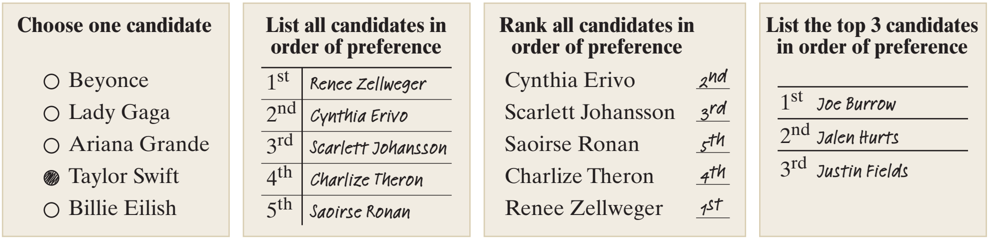
This chapter is organized as follows. We will start with a general discussion of *elections* and *ballots* in Section 1.1. This discussion provides the backdrop for the remaining sections, which are the heart of the chapter. In Sections 1.2 through 1.5 we will explore four of the most commonly used *voting methods*—how they work and how they are used in real-life applications. In Section 1.6 we will introduce some basic principles of fairness for voting methods and apply these *fairness criteria* to the voting methods discussed in Sections 1.2 through 1.5. The section concludes with a discussion of the meaning and significance of Arrow's Impossibility Theorem.

1.1 The Basic Elements of an Election

Big or small, important or trivial, *all* elections share a common set of elements.

- **The candidates.** The purpose of an election is to choose from a set of *candidates* or *alternatives* (at least two—otherwise it is not a real election). Typically, the word *candidate* is used for people and the word *alternative* is used for other things (movies, football teams, pizza toppings, etc.), but it is acceptable to use the two terms interchangeably. In the case of a generic choice (when we don't know if we are referring to a person or a thing), we will use the term *candidate*. While in theory there is no upper limit on the number of candidates, for most elections (in particular the ones we will discuss in this chapter) the number of candidates is small.
- **The voters.** These are the people who get a say in the outcome of the election. In most democratic elections the presumption is that all voters have an equal say, and we will assume this to be the case in this chapter. (This is not always true, as we will see in great detail in Chapter 2.) The number of voters in an election can range from very small (as few as 3 or 4) to very large (hundreds of millions). In this section we will see examples of both.
- **The ballots.** A ballot is the device by means of which a voter gets to express his or her opinion of the candidates. The most common type is a paper ballot, but a

voice vote, a text message, or an online vote can also serve as a “ballot” (see Example 1.5 *American Idol*). There are many different forms of ballots that can be used in an election, and Fig. 1-1 shows a few common examples. The simplest form is the **single-choice ballot**, shown in Fig. 1-1(a). Here very little is being asked of the voter (“pick the candidate you like best, and keep the rest of your opinions to yourself!”). At the other end of the spectrum is the **preference ballot**, where the voter is asked to rank *all* the candidates in order of preference. Figure 1-1(b) shows a typical preference ballot in an election with five candidates. In this ballot, the voter has entered the candidates’ names in order of preference. An alternative version of the same preference ballot is shown in Fig. 1-1(c). Here the names of the candidates are already printed on the ballot and the voter simply has to mark first, second, third, etc. In elections where there are a large number of candidates, a **truncated preference ballot** is often used. In a truncated preference ballot the voter is asked to rank some, but not all, of the candidates. Figure 1-1(d) shows a truncated preference ballot for an election with dozens of candidates.



Choose one candidate

○ Beyonce
○ Lady Gaga
○ Ariana Grande
● Taylor Swift
○ Billie Eilish

List all candidates in order of preference

1st	Renee Zellweger
2nd	Cynthia Erivo
3rd	Scarlett Johansson
4th	Charlize Theron
5th	Saoirse Ronan

Rank all candidates in order of preference

Cynthia Erivo	2nd
Scarlett Johansson	3rd
Saoirse Ronan	5th
Charlize Theron	4th
Renee Zellweger	1st

List the top 3 candidates in order of preference

1st	Joe Burrow
2nd	Jalen Hurts
3rd	Justin Fields

(a) (b) (c) (d)

Figure 1-1 (a) Single-choice ballot, (b) preference ballot, (c) a different version of the same preference ballot, and (d) truncated preference ballot.

- **The outcome.** The purpose of an election is to use the information provided by the ballots to produce some type of outcome. But what types of outcomes are possible? The most common is **winner-only**. As the name indicates, in a winner-only election all we want is to find a winner. We don’t distinguish among the nonwinners. There are, however, situations where we want a broader outcome than just a winner—say we want to determine a first-place, second-place, and third-place candidate from a set of many candidates (but we don’t care about fourth place, fifth place, etc.). We call this type of outcome a **partial ranking**. Finally, there are some situations where we want to rank *all* the candidates in order: first, second, third, . . . , last. We call this type of outcome a **full ranking**, or just a **ranking** for short.
- **The voting method.** The final piece of the puzzle is the method that we use to tabulate the ballots and produce the outcome. This is the most interesting (and complicated) part of the story, but we will not dwell on the topic here, as we will discuss voting methods throughout the rest of the chapter.

It is now time to illustrate and clarify the above concepts with some examples. We start with a simple example of a fictitious election. This is an important example, and we will revisit it many times throughout the chapter. You may want to think of Example 1.1 as a mathematical parable, its importance being not in the story itself but in what lies hidden behind it. (As you will soon see, there is a lot more to Example 1.1 than first meets the eye.)

EXAMPLE 1.1 The Math Club Election (Winner-Only)

The Math Appreciation Society (MAS) is a student club dedicated to an unsung but worthy cause: that of fostering the enjoyment and appreciation of mathematics among college students. The MAS chapter at Tasmania State University is holding its annual election for club president, and there are four *candidates* running: Alisha, Boris, Carmen, and Dave (A, B, C, and D for short).

Every member of the club is eligible to vote, and the vote takes the form of a *preference ballot*. Each voter is asked to rank each of the four candidates in order of preference. There are 37 *voters* who submit their ballots, and the 37 *preference ballots* submitted are shown in Fig. 1-2.

Ballot	**Ballot**	**Ballot**	**Ballot**	**Ballot**	**Ballot**	**Ballot**	**Ballot**	**Ballot**	**Ballot**	**Ballot**	**Ballot**	**Ballot**	**Ballot**	**Ballot**	**Ballot**	**Ballot**	**Ballot**
1st *A*	1st *B*	1st *A*	1st *C*	1st *B*	1st *C*	1st *A*	1st *B*	1st *C*	1st *A*	1st *C*	1st *D*	1st *A*	1st *A*	1st *C*	1st *A*	1st *C*	1st *D*
2nd *B*	2nd *D*	2nd *B*	2nd *B*	2nd *D*	2nd *B*	2nd *B*	2nd *D*	2nd *B*	2nd *B*	2nd *B*	2nd *C*	2nd *B*	2nd *B*	2nd *B*	2nd *B*	2nd *B*	2nd *C*
3rd *C*	3rd *C*	3rd *C*	3rd *D*	3rd *C*	3rd *D*	3rd *C*	3rd *C*	3rd *D*	3rd *C*	3rd *D*	3rd *B*	3rd *C*	3rd *C*	3rd *D*	3rd *C*	3rd *D*	3rd *B*
4th *D*	4th *A*	4th *D*	4th *A*	4th *A*	4th *A*	4th *D*	4th *A*	4th *A*	4th *D*	4th *A*	4th *A*	4th *D*	4th *D*	4th *A*	4th *D*	4th *A*	4th *A*

Ballot	**Ballot**	**Ballot**	**Ballot**	**Ballot**	**Ballot**	**Ballot**	**Ballot**	**Ballot**	**Ballot**	**Ballot**	**Ballot**	**Ballot**	**Ballot**	**Ballot**	**Ballot**	**Ballot**	**Ballot**	**Ballot**
1st *C*	1st *A*	1st *D*	1st *D*	1st *C*	1st *C*	1st *D*	1st *A*	1st *D*	1st *C*	1st *A*	1st *D*	1st *B*	1st *A*	1st *C*	1st *A*	1st *A*	1st *D*	1st *A*
2nd *B*	2nd *B*	2nd *C*	2nd *C*	2nd *B*	2nd *B*	2nd *C*	2nd *B*	2nd *C*	2nd *B*	2nd *B*	2nd *C*	2nd *D*	2nd *B*	2nd *D*	2nd *B*	2nd *B*	2nd *C*	2nd *B*
3rd *D*	3rd *C*	3rd *B*	3rd *B*	3rd *D*	3rd *D*	3rd *B*	3rd *C*	3rd *B*	3rd *D*	3rd *C*	3rd *B*	3rd *C*	3rd *C*	3rd *B*	3rd *C*	3rd *C*	3rd *B*	3rd *C*
4th *A*	4th *D*	4th *A*	4th *A*	4th *A*	4th *A*	4th *A*	4th *D*	4th *A*	4th *A*	4th *D*	4th *A*	4th *A*	4th *D*	4th *A*	4th *D*	4th *D*	4th *A*	4th *D*

Figure 1-2 The 37 preference ballots for the Math Club election.

Last but not least, what about the *outcome* of the election? Since the purpose of the election is to choose a club president, it is pointless to discuss or consider which candidate comes in second place, third place, etc. This is a *winner-only* election.

EXAMPLE 1.2 The Math Club Election (Full Ranking)

Suppose now that we have pretty much the same situation as in Example 1.1 (same candidates, same voters, same preference ballots), but in this election we have to choose not only a president but also a vice-president, a treasurer, and a secretary. According to the club bylaws, the president is the candidate who comes in first, the vice-president is the candidate who comes in second, the treasurer is the candidate who comes in third, and the secretary is the candidate who comes in fourth. Given that there are four candidates, each candidate will get to be an officer, but there is a big difference between being elected president and being elected treasurer (the president gets status and perks; the treasurer gets to collect the dues and balance the budget). In this version how you place matters, and the outcome should be a full *ranking* of the candidates.

EXAMPLE 1.3 The Academy Awards

2020 Academy Award winner (Best Actress) Renee Zellweger.

The Academy Awards (also known as the Oscars) are given out each year by the Academy of Motion Picture Arts and Sciences for Best Picture, Best Actress, Best Actor, Best Director, and many other categories (Sound Mixing, Makeup, etc.). The election process is not the same for all awards, so for the sake of simplicity we will just discuss the selection of Best Picture.

The *voters* in this election are all the eligible members of the Academy (approximately 8500 voting members in 2020). After a complicated preliminary round (a process that we won't discuss here), somewhere between eight and ten films are selected as the nominees—these are our *candidates*. (For most other awards there are only five nominees.) Each voter is asked to submit a preference ballot ranking all the candidates. There is only a winner (the other candidates are not ranked), with the winner determined by a voting method called plurality-with-elimination that we will discuss in detail in Section 1.4.

The part with which people are most familiar comes after the ballots are submitted and tabulated—the annual Academy Awards ceremony, held each year in late February. How many movie fans realize that behind one of the most extravagant and glamorous events in pop culture lies an election?

EXAMPLE 1.4 The Heisman Trophy

The Heisman Memorial Trophy Award is given annually to the "most outstanding player in collegiate football." The Heisman, as it is usually known, is not only a very prestigious award but also a very controversial award. With so many players playing so many different positions, how do you determine who is the most "outstanding"?


2019 Heisman Trophy finalists Joe Burrow, Justin Fields, Jalen Hurts, and Chase Young.

In theory, any football player in any division of college football is a potential *candidate* for the award. In practice, the real candidates are players from Division I programs and are almost always in the glamour positions—quarterback or running back. (Since its inception in 1935, only once has the award gone to a defensive player—Charles Woodson of Michigan.)

The *voters* are members of the media plus all past Heisman award winners still living, plus one vote from the public (as determined by a survey conducted by ESPN). There are approximately 930 *voters* (the exact number of voters varies each year). Each voter submits a *truncated preference ballot* consisting of a first, second, and third choice (see Fig. 1-1[d]). A first-place vote is worth 3 points, a second-place vote 2 points, and a third-place vote 1 point. The candidate with the most total points from all the ballots is awarded the Heisman trophy in a televised ceremony held each December at the Downtown Athletic Club in New York. (We will discuss this voting method in more detail in Section 1.3.)

While only one player gets the award, the finalists are ranked by the number of total points received, in effect making the *outcome* of the Heisman trophy a *partial ranking* of the top candidates. (For the 2019 season, the winner was Joe Burrow of Louisiana State University, second place went to Jalen Hurts of Oklahoma, third place went to Justin Fields of Ohio State, and fourth place went to Chase Young, also of Ohio State.)

EXAMPLE 1.5 American Idol

American Idol is a popular reality TV singing competition for individuals. Each year, the winner of *American Idol* gets a big recording contract, and many past winners have gone on to become famous recording artists (Kelly Clarkson, Carrie Underwood, Taylor Hicks). While there is a lot at stake and a big reward for winning, *American Idol* is not a winner-only competition, and there is indeed a ranking of all the finalists. In fact, some nonwinners (Clay Aiken, Jennifer Hudson) have gone on to become great recording artists in their own right.

The 10 candidates who reach the final rounds of the competition compete in a weekly televised show. During and immediately after each weekly show the voters cast their votes. The two candidates with the fewest number of votes are eliminated from the competition (sometimes when the voting is close only one candidate is eliminated), and the following week the process starts all over again with the remaining candidates. And who are the *voters* responsible for deciding the fate of these candidates? Anyone and everyone—you, me, Aunt Betsie—we are all potential voters. All one has to do to vote for a particular candidate is to go online, text or use an app. *American Idol* voting is an example of democracy run amok—you can vote for a candidate even if you never heard her sing, and you can vote as many times as you want.

By the final week of the competition the race is narrowed to the last three candidates, and after one last frenzied round of singing (followed by two more elimination rounds), the winner is determined. (See Example 1.16 for the full details on how it all played out in the 2019 American Idol finals.)

Examples 1.1 through 1.5 represent just a small sample of how elections can be structured, both in terms of the ballots (think of these as the *inputs* to the election) and the types of outcomes we look for (the *outputs* of the election). We will revisit some of these examples and many others as we wind our way through the chapter.

Preference Ballots and Preference Schedules

Let's focus now on elections where the balloting is done by means of preference ballots, as in Examples 1.1 and 1.2. The great advantage of preference ballots (compared with, for example, single-choice ballots) is that they provide a great deal of useful information about an individual voter's preferences—in both direct and indirect ways.

To illustrate what we mean, consider the preference ballot shown in Fig. 1-3. This ballot directly tells us that the voter likes candidate C best, B second best, D third best, and A least. But, in fact, this ballot tells us a lot more—it tells us unequivocally which candidate the voter would choose if it came down to a choice between just two candidates. For example, if it came down to a choice between, say, A and B, which one would this voter choose? Of course she would choose B—she has B above A in her ranking. Thus, a preference ballot allows us to make relative comparisons between any two candidates—*the candidate higher on the ballot is always preferred over the candidate in the lower position*. Please take note of this simple but important idea, as we will use it repeatedly later in the chapter.

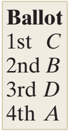
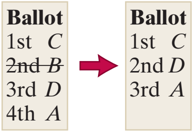

Ballot
1st C
2nd B
3rd D
4th A

Figure 1-3

The second important idea we will use later is the assumption that the relative preferences in a preference ballot do not change if one of the candidates withdraws or is eliminated. Once again, we can illustrate this using Fig. 1-3. What would happen if for some unforeseen reason candidate B drops out of the race right before the ballots are tabulated? Do we have to have a new election? Absolutely not—the old ballot simply becomes the ballot shown on the right side of Fig. 1-4. The candidates above B stay put and each of the candidates below B moves up a spot.


Ballot
1st C
~~2nd B~~
3rd D
4th A

Ballot
1st C
2nd D
3rd A

Figure 1-4

In an election with many voters, some voters will vote exactly the same way—for the same candidates in the same order of preference. If we take a careful look at the 37 ballots submitted for the Math Club election shown in Fig. 1-2, we see that 14 ballots look exactly the same (A first, B second, C third, D fourth), another 10 ballots look the same, and so on. So, if you were going to tabulate the 37 ballots, it might make sense to put all the A-B-C-D ballots in one pile, all the C-B-D-A ballots in another pile, and so on. If you were to do this you would get the five piles shown in Fig. 1-5 (the order in which you list the piles from left to right is irrelevant). Better yet, you can make the whole idea a little more formal by putting all the ballot information in a table such as Table 1-1, called the **preference schedule** for the election.

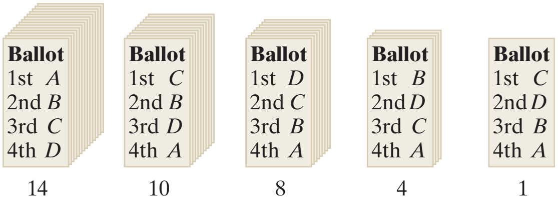

Ballot	**Ballot**	**Ballot**	**Ballot**	**Ballot**
1st A	1st C	1st D	1st B	1st C
2nd B	2nd B	2nd C	2nd D	2nd D
3rd C	3rd D	3rd B	3rd C	3rd B
4th D	4th A	4th A	4th A	4th A
14	10	8	4	1

Figure 1-5 The 37 Math Club election ballots organized into piles.

Number of voters	14	10	8	4	1
1st	A	C	D	B	C
2nd	B	B	C	D	D
3rd	C	D	B	C	B
4th	D	A	A	A	A

Table 1-1 ■ Preference schedule for the Math Club election

We will be working with preference schedules throughout the chapter, so it is important to emphasize that a preference schedule is nothing more than a convenient bookkeeping tool—it summarizes all the elements that constitute the input to an election: the candidates, the voters, and the balloting. Just to make sure this is clear, we conclude this section with a quick example of how to read a preference schedule.

EXAMPLE 1.6 The City of Kingsburg Mayoral Election

Number of voters	93	44	10	30	42	81
1st	*A*	*B*	*C*	*C*	*D*	*E*
2nd	*B*	*D*	*A*	*E*	*C*	*D*
3rd	*C*	*E*	*E*	*B*	*E*	*C*
4th	*D*	*C*	*B*	*A*	*A*	*B*
5th	*E*	*A*	*D*	*D*	*B*	*A*

Table 1-2 ■ Preference schedule for the Kingsburg mayoral election

Table 1-2 shows the preference schedule summarizing the results of the most recent election for mayor of the city of Kingsburg (there actually is a city by that name, but the election is fictitious). Just by looking at the preference schedule we can answer all of the relevant input questions:

- *Candidates*: there were five candidates (A, B, C, D, and E, which are just abbreviations for their real names).
- *Voters*: there were 300 voters that submitted ballots (add the numbers at the head of each column: $93 + 44 + 10 + 30 + 42 + 81 = 300$).
- *Balloting*: the 300 preference ballots were organized into six piles as shown in Table 1-2.

The question that still remains unanswered: Who is the winner of the election? In the next four sections we will discuss different ways in which such output questions can be answered.

Ties

In any election, be it a *winner-only* election or a *ranking* of the candidates, ties can occur. What happens then?

In some elections (for example, Academy Awards, many sports awards, and reality TV competitions) ties are allowed to stand and need not be broken. Here are a few interesting examples:

- **Academy Awards:** In 1932, Frederic March and Wallace Beery tied for Best Actor; in 1969, Katharine Hepburn and Barbra Streisand tied for Best Actress. A few more ties for lesser awards have occurred over the years. When ties occur, both winners receive the Oscar.
- **Grammys:** Over the years, there have been several ties for Grammy Awards. The most recent: A tie for Best Rap Performance at the 2019 Grammys between Anderson .Paak (for "Bubblin") and Jay Rock, Kendrick Lamar, Future, and James Blake (for "King's Dead").
- **NFL Most Valuable Player:** In 2004, Peyton Manning and Steve McNair shared the MVP award. (So did Brett Favre and Barry Sanders in 1997, as well as Norm van Brocklin and Joe Schmidt in 1960.)
- **Cy Young Award:** There has only been one tie in the history of the Cy Young Award. In 1969 Mike Cuellar of the Baltimore Orioles and Denny McLain of the Detroit Tigers tied for the American League award in 1969. There have been no ties since.

2019 Grammy tie for Best Rap Performance: Jay Rock (left); Anderson .Paak (right).

In other situations, especially in elections for political office (president, senator, mayor, city council, etc.), ties cannot be allowed (can you imagine having co-presidents or co-mayors?), and then a tie-breaking rule must be specified. The Constitution, for example, stipulates how a tie in the Electoral College is broken, and most elections have a set rule for breaking ties. The most common method for breaking a tie for political office is through a runoff election, but runoff elections are expensive and take time, so many other tie-breaking procedures are used. Here are a few interesting examples:

- In the 2018 election for a seat in the Virginia House of Delegates' 94th District, incumbent Republican David Yancey received 11,608 votes. Problem was that his opponent, Democrat Shelly Simonds also received 11,608 votes and there can only be one winner, so what to do? A 1705 Virginia law to the rescue: ties must be broken by drawing names "out of a hat." (The "hat" turned out to be a figure of speech—each of the names was placed inside a small canister and the canisters put inside a ceramic bowl.) When all was said and done, Mr. Yancey retained his seat as his name was drawn first. The moral of this story is that "every vote counts" is not just a cliche.
- In the 2009 election for a seat in the Cave Creek, Arizona, city council, Thomas McGuire and Adam Trenk tied with 660 votes each. The winner was decided by drawing from a deck of cards. Mr. McGuire drew first—a six of hearts. Mr. Trenk (the young man with the silver belt buckle) drew next and drew a king of hearts. This is how Mr. Trenk became a city councilman.

Ties and tie-breaking procedures add another layer of complexity to an already rich subject. To simplify our presentation, in this chapter we will stay away from ties as much as possible.

1.2 The Plurality Method

The **plurality method** is arguably the simplest and most commonly used method for determining the outcome of an election. With the plurality method, all that matters is how many first-place votes each candidate gets: In a *winner-only* election the candidate with the most first-place votes is the winner; in a *ranked* election the candidate with the most first-place votes is first, the candidate with the second most is second, and so on.

For an election decided under the plurality method, *preference ballots* are not needed, since the voter's second, third, etc. choices are not used. But, since we already have the preference schedule for the Math Club election (Examples 1.1 and 1.2) let's use it to determine the outcome under the plurality method.

EXAMPLE 1.7 The Math Club Election Under the Plurality Method

Number of voters	14	10	8	4	1
1st	*A*	*C*	*D*	*B*	*C*
2nd	*B*	*B*	*C*	*D*	*D*
3rd	*C*	*D*	*B*	*C*	*B*
4th	*D*	*A*	*A*	*A*	*A*

Table 1-3 ■ Preference schedule for the Math Club election

We discussed the Math Club election in Section 1.1. Table 1-3 shows once again the preference schedule for the election. Counting only first-place votes, we can see that *A* gets 14, *B* gets 4, *C* gets 11, and *D* gets 8. So there you have it: In the case of a *winner-only* election (see Example 1.1) the winner is *A* (Headline: "Alisha wins presidency of the Math Club"); in the case of a *ranked election* (see Example 1.2) the results are: *A* first (14 votes); *C* second (11 votes); *D* third (8 votes); *B* fourth (4 votes). (Headline "New board of MAS elected! President: Alisha; VP: Carmen; Treasurer: Dave; Secretary: Boris.")

The vast majority of elections for political office in the United States are decided using the plurality method. The main appeal of the plurality method is its simplicity, but as we will see in our next example, the plurality method has many drawbacks.

EXAMPLE 1.8 The 2010 Maine Governor's Election

Up until 2016, the governor of Maine was elected using the plurality method, but because of the many problems raised in the 2010 election (as well as several other previous elections), in 2016 the citizens of Maine passed a referendum that changed the voting method for statewide elections from plurality to ranked-choice voting (no worries, we will discuss ranked-choice voting in Section 1.4). In the 2010 Maine gubernatorial election there were five candidates: Eliot Cutler (Independent), Paul LePage (Republican), Libby Mitchell (Democrat), Shawn Moody (Independent), and Kevin Scott (Independent). Table 1-4 shows the results of the election. Before reading on, take a close look at the numbers in Table 1-4 and draw your own conclusions.

Candidate	Votes	Percent
Eliot Cutler (Independent)	208,270	36.5%
Paul LePage (Republican)	218,065	38.2%
Libby Mitchell (Democrat)	109,937	19.3%
Shawn Moody (Independent)	28,756	5.0%
Kevin Scott (Independent)	5,664	1.0%

Table 1-4 ■ Results of 2010 Maine gubernatorial election. (*Source:* The New York Times, www.elections.nytimes.com/2010/results/governor.)

As Table 1-4 shows, Paul LePage became governor with the support of only 38.2% of the voters (which means, of course, that 61.8% of the voters in Maine wanted someone else). A few days after the election, an editorial piece in the *Portland Press Herald* expressed the public concern about the outcome.

> *The election of Paul LePage with 38 percent of the vote means Maine's next governor won't take office with the support of the majority of voters—a situation that has occurred in six of the last seven gubernatorial elections . . . . Some people . . . say it's time to reform the system so Maine's next governor can better represent the consensus of voters.* (From *Is Winning An Election Enough?* by Tom Bell in Portland Press Herald. Copyright © 2010 by MaineToday Media, Inc. Used by permission of MaineToday Media, Inc.)

The second problem with the Maine governor's election is the closeness of the election: Out of roughly 571,000 votes cast, less than 10,000 votes separated the winner and the runner-up. This is not the plurality method's fault, but it does raise the possibility that the results of the election could have been *manipulated* by a small number of voters. Imagine for a minute being inside the mind of a voter we call Mr. Insincere: "Of all these candidates, I like Kevin Scott the best. But if I vote for Scott I'm just wasting my vote—he doesn't have a chance. All the polls say that it really is a tight race between LePage and Cutler. I don't much care for either one, but LePage is the better of two evils. I'd better vote for LePage." The same thinking, of course, can be applied in the other direction—voters afraid to "waste" their vote on Scott (or Moody, or Mitchell) and insincerely voting for Cutler over Le

Page. The problem is that we don't know how many *insincere votes* went one way or the other, and the possibility that there were enough insincere votes to tip the results of the election cannot be ruled out.

While all voting methods can be manipulated by insincere voters, the plurality method is the one that can be most easily manipulated, and insincere voting is quite common in real-world elections. For Americans, the most significant cases of insincere voting occur in close presidential or gubernatorial races between the two major party candidates and a third candidate ("the spoiler") who has little or no chance of winning. Insincere voting not only hurts small parties and fringe candidates, it has unintended and often negative consequences for the political system itself. The history of American political elections is littered with examples of independent candidates and small parties that never get a fair voice or a fair level of funding (it takes 5% of the vote to qualify for federal funds for the next election) because of the "let's not waste our vote" philosophy of insincere voters. The ultimate consequence of the plurality method is an entrenched two-party system that often gives the voters little real choice.

The last, but not least, of the problems with the plurality method is that a candidate may be preferred by the voters over all other candidates and yet not win the election. We will illustrate how this can happen with the example of the fabulous Tasmania State University marching band.

EXAMPLE 1.9 The Fabulous TSU Band Goes Bowling

Tasmania State University has a superb marching band. They are so good that this coming bowl season they have invitations to perform at five different bowl games: the Rose Bowl (R), the Hula Bowl (H), the Fiesta Bowl (F), the Orange Bowl (O), and the Sugar Bowl (S). An election is held among the 100 band members to decide in which of the five bowl games they will perform. Each band member submits a preference ballot ranking the five choices. The results of the election are shown in Table 1-5.

Number of voters	49	48	3
1st	R	H	F
2nd	H	S	H
3rd	F	O	S
4th	O	F	O
5th	S	R	R

Table 1-5 ■ Preference schedule for the band election

Under the plurality method the winner of the election is the Rose Bowl (R), with 49 first-place votes. It's hard not to notice that this is a rather bad outcome, as there are 51 voters that have the Rose Bowl as their last choice. By contrast, the Hula Bowl (H) has 48 first-place votes and 52 second-place votes. Simple common sense tells us that the Hula Bowl is a far better choice to represent the wishes of the entire band. In fact, we can make the following persuasive argument in favor of the Hula Bowl: If we compare the Hula Bowl with any other bowl on a *head-to-head* basis, the Hula Bowl is always the preferred choice. Take, for example, a comparison between the Hula Bowl and the Rose Bowl. There are 51 votes for the Hula Bowl (48 from the second column plus the 3 votes in the last column) versus 49 votes for the Rose Bowl. Likewise, a comparison between the Hula Bowl and the Fiesta Bowl would result in 97 votes for the Hula Bowl (first and second columns) and 3 votes for the Fiesta Bowl. And when the Hula Bowl is compared with either the Orange Bowl or the Sugar Bowl, it gets all 100 votes. Thus, no matter with which bowl we compare the Hula Bowl, there is always a majority of the band that prefers the Hula Bowl.

Marie Jean Antoine Nicolas Caritat, Marquis de Condorcet (1743–1794)

A candidate preferred by a majority of the voters over every other candidate when the candidates are compared in head-to-head comparisons is called a **Condorcet candidate** (named after the Marquis de Condorcet, an eighteenth-century French mathematician and philosopher). Not every election has a Condorcet candidate, but if there is one, it is a good sign that this candidate represents the voice of the voters better than any other candidate. In Example 1.9 the Hula Bowl is the Condorcet candidate—it is not unreasonable to expect that it should be the winner of the election. We will return to this topic in Section 1.6.

1.3 The Borda Count Method

The second most commonly used method for determining the winner of an election is the **Borda count method**, named after the Frenchman Jean-Charles de Borda. In this method each place on a ballot is assigned points as follows: 1 point for *last* place, 2 points for *second from last* place, and so on. At the top of the ballot, a *first-place* vote is worth N points, where N represents the number of candidates. The points are tallied for each candidate separately, and the candidate with the highest total is the winner. If we are ranking the candidates, the candidate with the second-most points comes in second, the candidate with the third-most points comes in third, and so on. We will start our discussion of the Borda count method by revisiting the Math Club election.

EXAMPLE 1.10 The Math Club Election (Borda Count)

Table 1-6 shows the preference schedule for the Math Club election with the Borda points for the candidates shown in parentheses to the right of their names. For example, the 14 voters in the first column ranked A first (giving A $14 \times 4 = 56$ points), B second $(14 \times 3 = 42$ points$)$, and so on.

Number of voters	14	10	8	4	1
1st (4 points)	A (56)	C (40)	D (32)	B (16)	C (4)
2nd (3 points)	B (42)	B (30)	C (24)	D (12)	D (3)
3rd (2 points)	C (28)	D (20)	B (16)	C (8)	B (2)
4th (1 point)	D (14)	A (10)	A (8)	A (4)	A (1)

Table 1-6 ■ Borda points for the Math Club election

When we tally the points,

A gets 56 + 10 + 8 + 4 + 1 = 79 points,
B gets 42 + 30 + 16 + 16 + 2 = 106 points,
C gets 28 + 40 + 24 + 8 + 4 = 104 points,
D gets 14 + 20 + 32 + 12 + 3 = 81 points.

The Borda winner of this election is Boris! (Wasn't Alisha the winner of this election under the plurality method?)

If we have to rank the candidates, B is first, C second, D third, and A fourth. To see what a difference the voting method makes, compare this ranking with the ranking obtained under the plurality method (Example 1.7).

EXAMPLE 1.11 The 2019 Heisman Award

2019 Heisman Trophy winner Joe Burrow.

For general details on the Heisman Award, see Example 1.4. The Heisman is determined using a Borda count, but with *truncated preference ballots*: each voter chooses a first, second, and third choice out of a large list of candidates, with a first-place vote worth 3 points, a second-place vote worth 2 points, and a third-place vote worth 1 point.

Table 1-7 shows a summary of the ballots cast for the four finalists in the 2019 race. The table shows the number of first-, second-, and third-place votes for each of the finalists; the last column shows the total point tally for each. Notice that Table 1-7 is not a preference schedule. Because the Heisman uses truncated preference ballots and many candidates get votes, it is easier and more convenient to summarize the balloting this way.

Player	1st (3 pts.)	2nd (2 pts.)	3rd (1 pt.)	Total points
Joe Burrow	841	41	3	2608
Jalen Hurts	12	231	264	762
Justin Fields	6	271	187	747
Chase Young	20	205	173	643

Table 1-7 ■ 2019 Heisman Award: top four finalists (*Source:* Heisman Award, www.heisman.com)

The last column of Table 1-7 shows the total number of points received by each finalist: Joe Burrow of Louisiana State was the overwhelming winner with 2608 total points (a Heisman record), Jalen Hurts of Oklahoma came in second with 762 points, Justin Fields of Ohio State was a very close third with 747 points, and Chase Young of Ohio State came in fourth with 643 points.

Many variations of the standard Borda count method are possible, the most common being a change in the values assigned to the various positions on the ballot. We will call these **modified Borda count** methods. Example 1.12 illustrates one situation where a modified Borda count is used.

EXAMPLE 1.12 The 2019 National League Cy Young Award

2019 National League Cy Young Award winner Jacob deGrom.

The Cy Young Award is an annual award given by Major League baseball for "the best pitcher" in each league (one award for the American League and one for the National League). For each league, the Cy Young award is determined by the votes of 30 baseball writers where each writer submits a truncated preference ballot with a first, second, third, fourth, and fifth choice. The modification in the Cy Young calculations (in effect for the first time with the 2010 award) is that first place is worth 7 points (rather than 5). The other places in the ballot count just as in a regular Borda count: 4 points for second, 3 points for third, 2 points for fourth, and 1 point for fifth. The idea here is to give extra weight to first-place votes—the gap between a first and a second place is bigger than the gap between a second and a third place, a third and fourth place, and so on.

Table 1-8 shows the votes for the top five finalists for the 2019 National League Cy Young award.

Pitcher	1st (7 pts.)	2nd (4 pts.)	3rd (3 pts.)	4th (2 pts.)	5th (1 pt.)	Total points
Jacob deGrom (Mets)	29	1	0	0	0	207
Hyun-Jin Ryu (Dodgers)	1	10	8	7	3	88
Max Scherzer (Nationals)	0	8	8	6	4	72
Jack Flaherty (Cardinals)	0	5	11	6	4	69
Stephen Strasburg (Nationals)	0	6	1	9	8	53

Table 1-8 ■ 2019 National League Cy Young Award: top five finalists

In real life, the Borda count method (or some variation of it) is widely used in a variety of settings, from individual sport awards to music industry awards to the hiring of school principals, university presidents, and corporate executives. It is generally considered to be a much better method for determining the outcome of an election than the plurality method. In contrast to the plurality method, it takes into account the voter's preferences not just for first place but also for second, third, etc., and then chooses as the winner the candidate with the best average ranking—the best compromise candidate, if you will.

1.4 The Plurality-with-Elimination Method

In the United States most municipal and local elections have a majority requirement—a candidate needs a majority of the votes to get elected. With only two candidates this is rarely a problem (unless they tie, one of the two candidates must have a majority of the votes). When there are three or more candidates running, it can easily happen that no candidate has a majority. Typically, the candidate or candidates with the fewest first-place votes are eliminated, and a runoff election is held. But runoff elections are expensive, and in these times of tight budgets more efficient ways to accomplish the "runoff" are highly desirable.

A very efficient way to implement the runoff process without needing runoff elections is to use preference ballots, since a preference ballot tells us not only which candidate the voter wants to win but also which candidate the voter would choose in a runoff (with one important caveat—we assume the voters are consistent in their preferences and would stick with their original ranking of the candidates). The idea is to use the information in the preference schedule to eliminate the candidates with the *fewest* first-place votes one at a time until some candidate gets a majority. This method has become increasingly popular and is now known under several other names, including, *instant-runoff voting* (IRV), *ranked-choice voting* (RCV), and the *Hare method*. For the sake of clarity, we will call it the *plurality-with-elimination* method—it is the most descriptive of all the names.

Here is a formal description of the **plurality-with-elimination method**:

- **Round 1.** Count the first-place votes for each candidate, just as you would in the plurality method. If a candidate has a majority of the first-place votes, then that candidate is automatically declared the winner. If no candidate has a majority of the first-place votes, eliminate the candidate (or candidates if there is a tie) with the *fewest* first-place votes and transfer (pass down) those first-place votes to the next eligible candidate(s) on those ballots. Cross out the name(s) of the eliminated candidate(s) from the preference schedule.
- **Round 2.** Recount the votes. If a candidate now has a majority of the first-place votes, declare that candidate the winner. Otherwise, eliminate the candidate(s) with the fewest votes and transfer (pass down) those first-place votes to the next eligible candidate(s) on those ballots. Cross out the name(s) of the eliminated candidate(s) from the preference schedule.
- **Rounds 3, 4, . . .** Repeat the process, each time eliminating the candidate with the fewest first-place votes and transferring those first-place votes to the next eligible candidates on those ballots. Continue until there is a candidate with a majority of the first-place votes. That candidate is the winner of the election.

In a ranked election the candidates should be ranked in reverse order of elimination: the candidate eliminated in the last round gets second place, the candidate eliminated in the second-to-last round gets third place, and so on.

EXAMPLE 1.13 The Math Club Election (Plurality-with-Elimination)

Let's see how the plurality-with-elimination method works when applied to the Math Club election. For the reader's convenience Table 1-9 shows the preference schedule again.

Number of voters	14	10	8	4	1
1st	*A*	*C*	*D*	*B*	*C*
2nd	*B*	*B*	*C*	*D*	*D*
3rd	*C*	*D*	*B*	*C*	*B*
4th	*D*	*A*	*A*	*A*	*A*

Table 1-9 ■ Preference schedule for the Math Club election

Round 1.

Candidate	*A*	~~*B*~~	*C*	*D*
First-place votes	14	④	11	8

+4

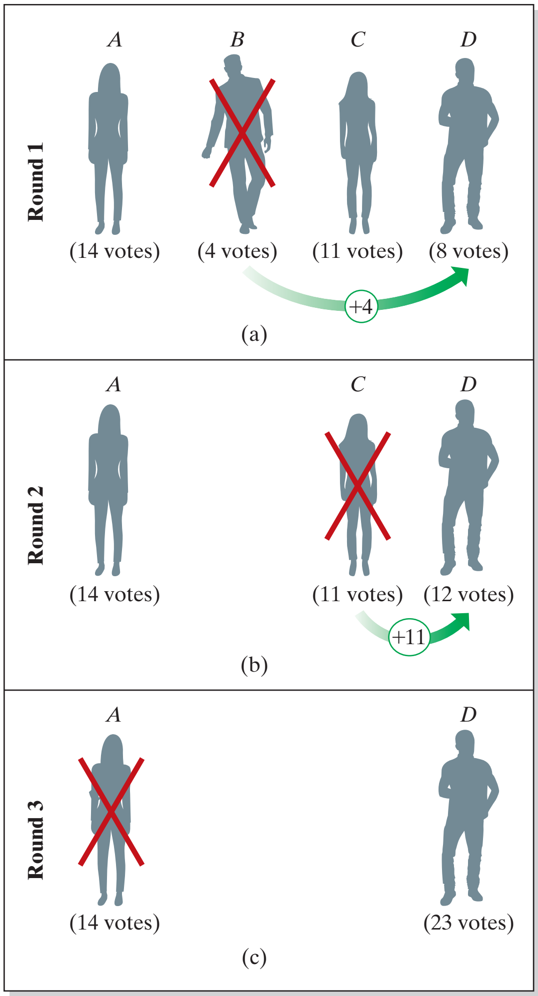


Figure 1-6 Boris is eliminated first, then Carmen, and then Alisha. The last one standing is Dave.

B has the fewest first-place votes and is eliminated first [Fig. 1-6(a)]. After *B* is eliminated, the four votes that originally went to *B* are transferred to *D* (per column 4 of Table 1-9).

Round 2. We now recount the first-place votes. The new tally is

Candidate	*A*	~~*C*~~	*D*
First-place votes	14	11	12

+11

In this round *C* has the fewest first-place votes and is eliminated [Fig. 1-6(b)]. The 11 votes that went to *C* in round 2 are all transferred to *D* (per columns 2 and 5 of Table 1-9).

Round 3. Once again we recount the first-place votes and end up with

Candidate	*A*	*D*
First-place votes	14	23

Now *D* has a majority of the first-place votes and is declared the winner [Fig. 1-6(c)].

In the case of a ranked election we have *D* first, *A* second (eliminated in round 3), *C* third (eliminated in round 2), and *B* last (eliminated in round 1).

Our next example illustrates a few subtleties that can come up when applying the plurality-with-elimination method.

EXAMPLE 1.14 The City of Kingsburg Mayoral Election

Table 1-10 shows the preference schedule for the Kingsburg mayoral election first introduced in Example 1.6. To save money Kingsburg has done away with runoff elections and now uses plurality-with-elimination for all local elections. (Notice that since there are 300 voters voting in this election, a candidate needs 151 or more votes to win.)

Number of voters	93	44	10	30	42	81
1st	*A*	*B*	*C*	*C*	*D*	*E*
2nd	*B*	*D*	*A*	*E*	*C*	*D*
3rd	*C*	*E*	*E*	*B*	*E*	*C*
4th	*D*	*C*	*B*	*A*	*A*	*B*
5th	*E*	*A*	*D*	*D*	*B*	*A*

Table 1-10 ■ Preference schedule for the Kingsburg mayoral election

Round 1. Here C has the fewest number of first-place votes and is eliminated first. Of the 40 votes originally cast for C, 10 are transferred to A (per column 3 of Table 1-10) and 30 are transferred to E (per column 4 of Table 1-10).

Candidate	A	B	~~C~~	D	E
Votes	93	44	40	42	81

+10 **+30**

Round 2. After a recount of the first-place votes, D has the fewest and is eliminated. The 42 votes originally cast for D are now transferred to E, the next eligible candidate since C has already been eliminated (see column 5 of Table 1-10).

Candidate	A	B	~~D~~	E
Number of first-place votes	103	44	42	111

+42

Round 3. After a recount of the first-place votes, E has a majority and is declared the winner.

Candidate	A	B	E
Number of first-place votes	103	44	153

If this were a ranked election, we would continue on to Round 4, only to determine second place between A and B, but this is an election for mayor, and second place, third place, etc., are meaningless.

Several variations of the plurality-with-elimination method are used in real-life elections. One of the most popular goes by the name **instant-runoff voting** (also called **ranked-choice voting** in some places). Instant-runoff voting uses a truncated preference ballot (typically asking for a ranking of the top 3, 4, or 5 candidates). Once the ballots are cast the process works very much like plurality-with-elimination: the candidate(s) with the fewest first-place votes are eliminated and those votes are transferred to the second-place candidates in those ballots; in the next round the candidate(s) with the fewest votes are eliminated and those votes are transferred to the next eligible candidate, and so on. There is one important difference: unlike regular plurality-with-elimination, there is a point at which some votes can no longer be transferred (say your vote was for candidates X, Y, and Z—if and when all three of them are eliminated, there is no one to transfer your vote to). Such votes are called *exhausted votes* and although perfectly legal, they don't count in the final analysis.

Ranked-choice voting is used in several U.S. cities in elections for mayor and city council, including New York City (starting in 2021), San Francisco, Minneapolis, St. Paul, and Oakland, California, as well as in elections for political office in Maine, Australia, Canada, Ireland, and New Zealand. We will illustrate how ranked-choice voting works with the 2014 election for mayor of Oakland.

EXAMPLE 1.15 The 2014 Oakland Mayoral Election

In 2014 a total of 16 candidates were running for mayor of Oakland, an inordinately large number for an election for political office, and it took 15 rounds of elimination before a winner emerged. But Oakland has been using ranked-choice voting since 2010, so the entire elimination process took place inside a computer, and the final results were known without delay.

Of the 16 candidates, only four had a realistic chance of winning, so to keep things simple we will show how the vote count evolved once the other candidates were eliminated (rounds 1 through 12) and just the main four candidates were left.

- **Round 13.**

Dan Siegel	Jean Quan	Rebecca Kaplan	Elizabeth Schaaf
17,402 votes	18,049 votes	18,662 votes	39,941 votes

Dan Siegel has the fewest number of first-place votes and is eliminated. His 17,402 are transferred as follows: 2476 to Quan, 4679 to Kaplan, and 3877 to Schaaf. The remaining 6370 ballots are "exhausted" because all the candidates in those ballots are gone. At this point, the exhausted ballots are discarded [Fig. 1-7(a)].

- **Round 14.**

Jean Quan	Rebecca Kaplan	Elizabeth Schaaf
20,525 votes	23,341 votes	43,818 votes

Now Jean Quan has the fewest number of first-place votes and is eliminated. Her 20,525 are transferred as follows: 5080 go to Kaplan, 4988 go to Schaaf, and 10,457 are exhausted [Fig. 1-7(b)].


Libby Schaaf in front of Oakland City Hall after her election as mayor.

- **Round 15.**

Rebecca Kaplan	Elizabeth Schaaf
28,421 votes	48,806 votes

We are now down to Kaplan and Schaaf, and Schaaf has the majority of the votes.

So this is how Elizabeth "Libby" Schaaf got to be elected the mayor of Oakland in 2014. It took 15 elimination rounds but only one set of ballots—the heavy lifting was done by ranked-choice voting. (Schaaf was reelected mayor in 2018 and will serve as mayor of Oakland at least until 2022.)

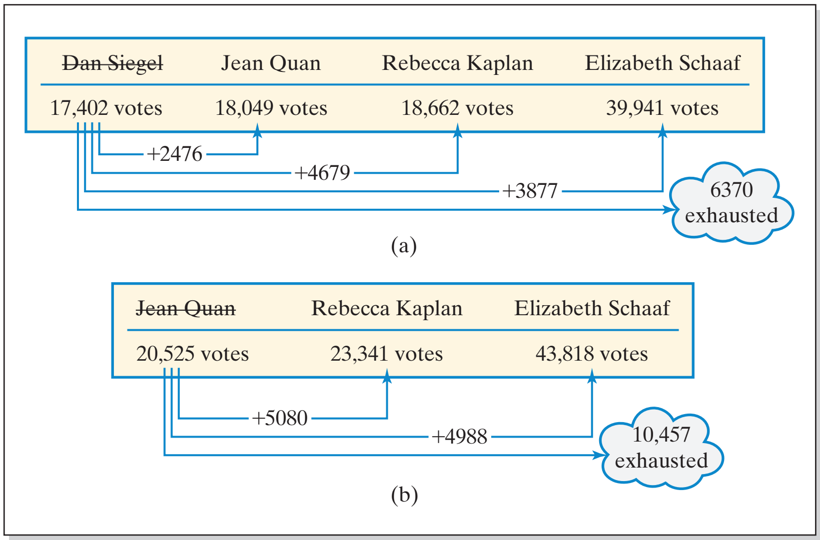
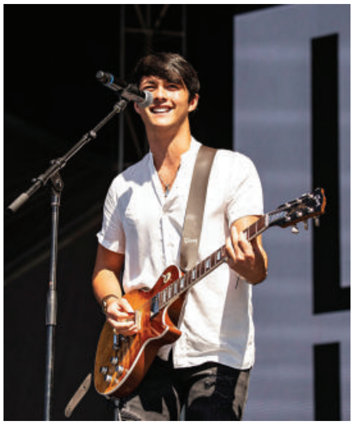


Figure 1-7 Results of 2014 Oakland mayoral election. (*Source:* Alameda County Registrar of Voters, ACGov.org)

As mentioned earlier in this section, the practical advantage of plurality-with-elimination is that it does away with expensive and time-consuming runoff elections. There is one situation, however, where expense is not an issue and delaying the process is part of the game: televised competitions such as *Dancing with the Stars*, *The Voice*, and *American Idol*. The longer the competition goes, the higher the ratings—having many runoffs accomplishes this goal. All of these "elections" work under the same variation of the plurality-with-elimination method: have a round of competition, vote, eliminate the candidate (or candidates) with the fewest votes. The following week have another round of competition and repeat the process. This builds up to the last round of competition, when there are two finalists left. Millions of us get caught up in the hoopla. We will illustrate one such election using the 2019 *American Idol* competition.

EXAMPLE 1.16 The 2019 American Idol Competition



2019 *American Idol* winner Laine Hardy.

We discussed *American Idol* as an election in Example 1.5. Table 1-11 shows the evolution of the 2019 *American Idol* finals. As noted in Example 1.5, the winner is the big deal, but how the candidates place in the competition is also of some relevance for their future musical careers, so we consider *American Idol* a ranked election. Working our way down Table 1-11, we see how the process of elimination played out: Uche' and Dimitrius Graham were eliminated in the first round and tied for 9th-10th place; Alyssa Raghu and Walker Burroughs were eliminated in the second round and tied for 7th-8th place; Jeremiah Lloyd Harmon was eliminated in the third round and placed in 6th place . . . and so it went for a total of five rounds. In the final round of competition it came down to three contestants: Madison VanDenburg, Alejandro Aranda, and Laine Hardy. After a round of singing and voting, Madison was eliminated leaving Alejandro and Laine for the final showdown, where Laine was declared the winner.

Contestant	**Status**	**Place**
Uche'	Eliminated in Round 1	9th–10th place (tie)
Dimitrius Graham	Eliminated in Round 1	9th–10th place (tie)
Alyssa Raghu	Eliminated in Round 2	7th–8th place (tie)
Walker Burroughs	Eliminated in Round 2	7th–8th place (tie)
Jeremiah Lloyd Harmon	Eliminated in Round 3	6th place
Wade Cota	Eliminated in Round 4	4th–5th place (tie)
Laci Kaye Booth	Eliminated in Round 4	4th–5th place (tie)
Madison VanDenburg	Eliminated in final round	3rd place
Alejandro Aranda	Final round runner-up	2nd place
Laine Hardy	Final round winner!	1st place

Table 1-11 ■ 2019 American Idol Voting

1.5 The Method of Pairwise Comparisons

One of the most useful features of a preference schedule is that it allows us to find the winner of any **pairwise comparison** between candidates. Specifically, given any two candidates—call them X and Y—we can count how many voters rank X above Y and how many rank Y above X. The one with the most votes wins the pairwise comparison. This is the basis for a method called the **method of pairwise comparisons** (sometimes also called *Copeland's method*). For each possible pairwise comparison between candidates, give 1 point to the winner, give 0 points to the loser, and when the pairwise comparison ends up in a tie give each candidate $\frac{1}{2}$ point. The candidate with the most points is the winner. (If we are ranking the candidates, the candidate with the second-most points is second, and so on.)

The method of pairwise comparisons is very much like a round-robin tournament: every player plays every other player once; the winner of each "match" gets a point and the loser gets no points (if there is a tie each gets $\frac{1}{2}$ point); and the player with the most points wins the tournament. As usual, we will start with the Math Club election as our first example.

EXAMPLE 1.17 The Math Club Election (Pairwise Comparisons)

Number of voters	14	10	8	4	1
1st	*A*	*C*	*D*	*B*	*C*
2nd	*B*	*B*	*C*	*D*	*D*
3rd	*C*	*D*	*B*	*C*	*B*
4th	*D*	*A*	*A*	*A*	*A*

Table 1-12 ■ Preference schedule for the Math Club election

Table 1-12 shows, once again, the preference schedule for the Math Club election. With four candidates, there are six possible pairwise comparisons to consider (see the first column of Table 1-13). For the sake of brevity, we will go over a couple of these pairwise comparisons in detail and leave the details of the other four to the reader.

- *A* v *B*: The first column of Table 1-12 represents 14 votes for *A* (*A* is ranked higher than *B*); the remaining 23 votes are for *B* (*B* is ranked higher than *A* in the last four columns of the table). The winner of this comparison is *B*.
- *C* v *D*: The first, second, and last columns of Table 1-12 represent votes for *C* (*C* is ranked higher than *D*); the third and fourth columns represent votes for *D* (*D* is ranked higher than *C*). Thus, *C* has 25 votes to *D*'s 12 votes. The winner of this comparison is *C*.

Pairwise comparison	Votes	Winner
(1) *A* v *B*	*A*: 14 votes *B*: 23 votes	*B*
(2) *A* v *C*	*A*: 14 votes *C*: 23 votes	*C*
(3) *A* v *D*	*A*: 14 votes *D*: 23 votes	*D*
(4) *B* v *C*	*B*: 18 votes *C*: 19 votes	*C*
(5) *B* v *D*	*B*: 28 votes *D*: 9 votes	*B*
(6) *C* v *D*	*C*: 25 votes *D*: 12 votes	*C*

Total points: $C = 3$, $B = 2$, $D = 1$, $A = 0$

Table 1-13 ■ Pairwise comparisons for the Math Club election

We continue this way, checking the results of all six possible comparisons (try it now on your own, before you read on!). Once you are done, you should get something along the lines of Table 1-13. A tally of the point totals (shown at the bottom of the table) gives us the outcome of the election: In a winner-only election the winner is *C* (with 3 points); in a ranked election *C* is first (3 points), *B* second (2 points), *D* third (1 point), and *A* fourth (no points).

If you have been paying close attention, you may have noticed that the results of the Math Club election have been different under each of the voting methods we have discussed—both in terms of the winner and in terms of the ranking of the candidates. This can be seen quite clearly in the summary results shown in Table 1-14. It is amazing how much the outcome of an election can depend on the voting method used!

		Ranking			
Method	Winner only	1st	2nd	3rd	4th
Plurality	*A*	*A*	*C*	*D*	*B*
Borda count	*B*	*B*	*C*	*D*	*A*
Plurality-with-elimination	*D*	*D*	*A*	*C*	*B*
Pairwise comparisons	*C*	*C*	*B*	*D*	*A*

Table 1-14 ■ The outcome of the Math Club election under four different voting methods

One more important comment about Example 1.17: Notice that *C* was the *undefeated* champion, as *C* won each of the pairwise comparisons against the other candidates. (We already saw that there is a name for a candidate that beats all the other candidates in pairwise comparisons—we call such a candidate a *Condorcet* candidate.) The method of pairwise comparisons always chooses the Condorcet candidate (when there is one) as the winner of the election, but this is not true with all methods. Under the plurality method, for example, you can have a Condorcet candidate that does not win the election (see Example 1.9).

Although the method of pairwise comparisons is a pretty good method, in real-life elections it is not used as much as the other three methods we discussed. In the next example we will illustrate one interesting and meaningful application of the method—the selection of draft choices in the Women's National Basketball Association (WNBA). Because professional sports teams are extremely secretive about how they make their draft decisions, we will illustrate the general idea with a made-up example.

EXAMPLE 1.18 The WNBA Draft

The Houston Rockettes are the newest expansion team in the Women's National Basketball Association and are awarded the first pick in the upcoming draft. The team's draft committee (made up of coaches, scouts, and team executives) has narrowed down the list to five candidates: Allen, Byers, Castillo, Dixon, and Evans (A, B, C, D, and E for short). After many meetings, the draft committee is ready to vote for the team's first pick in the draft. The election is to be decided using the method of pairwise comparisons.

Table 1-15 shows the preference schedule obtained after each of the 22 members of the draft committee submits a preference ballot ranking the five candidates. There is a total of 10 separate pairwise comparisons to be looked at, and the results are shown in Table 1-16 (we leave it to the reader to check the details).

Number of voters	2	6	4	1	1	4	4
1st	*A*	*B*	*B*	*C*	*C*	*D*	*E*
2nd	*D*	*A*	*A*	*B*	*D*	*A*	*C*
3rd	*C*	*C*	*D*	*A*	*A*	*E*	*D*
4th	*B*	*D*	*E*	*D*	*B*	*C*	*B*
5th	*E*	*E*	*C*	*E*	*E*	*B*	*A*

Table 1-15 ■ Rockettes draft choice election

Pairwise comparison	Votes	Winner (points)
A v B	A: 7 votes B: 15 votes	B (1)
A v C	A: 16 votes C: 6 votes	A (1)
A v D	A: 13 votes D: 9 votes	A (1)
A v E	A: 18 votes E: 4 votes	A (1)
B v C	B: 10 votes C: 12 votes	C (1)
B v D	B: 11 votes D: 11 votes	$B\left(\frac{1}{2}\right); D\left(\frac{1}{2}\right)$
B v E	B: 14 votes E: 8 votes	B (1)
C v D	C: 12 votes D: 10 votes	C (1)
C v E	C: 10 votes E: 12 votes	E (1)
D v E	D: 18 votes E: 4 votes	D (1)
Total points: $A = 3$, $B = 2\frac{1}{2}$, $C = 2$, $D = 1\frac{1}{2}$, $E = 1$		

Table 1-16 ■ Pairwise comparisons for Example 1.18

We can see from Table 1-16 that the winner of the election is A with 3 points. Notice that things are a little trickier here: A is the winner of the election even though the draft committee prefers B to A in a pairwise comparison between the two. We will return to this example in Section 1.6.

You probably noticed in Examples 1-17 and 1-18 that, compared with the other methods, pairwise comparisons takes a lot more work. Each comparison requires a separate calculation, and there seems to be a lot of comparisons that need to be checked. How many? We saw that with 4 candidates there are 6 separate comparisons and with 5 candidates there are 10. As the number of candidates grows, the number of comparisons grows even more. Table 1-17 illustrates the relation between the number of candidates and the number of pairwise comparisons.

Number of candidates	4	5	6	7	8	9	10	. . .	N
Number of pairwise comparisons	6	10	15	21	28	36	45	. . .	$\frac{N(N-1)}{2}$

Table 1-17 ■ The number of pairwise comparisons

A nice formula for counting the number of pairwise comparisons in an election is given below.

In an election with N candidates, the number of pairwise comparisons is

$$N(N-1)/2$$

1.6 Fairness Criteria and Arrow's Impossibility Theorem

So far, this is what we learned: There are many different types of elections and there are different ways to decide their outcome. We examined four different voting methods in some detail, but there are many others that we don't have time to discuss here (see Exercises 68–70 for a small sample). So now comes a different but fundamental question (that may have already crossed your mind): *Of all those voting methods out there, which one is the best?* As simple as it sounds, this question has vexed social scientists and mathematicians for centuries, going back to Condorcet and Borda in the mid 1700s. For multi-candidate elections (three or more candidates) there is no good answer. In fact, we now know that there are limitations to *all* voting methods. This is a very important and famous discovery known as **Arrow's Impossibility Theorem**. In this section we will discuss the basic ideas behind this theorem.

In the late 1940s the American economist Kenneth Arrow turned the question of finding an ideal voting method on its head and asked himself the following: *What would it take for a voting method to at least be a fair voting method*? To answer this question, Arrow set forth a minimum set of requirements that we will call Arrow's **fairness criteria**. (In all fairness, Arrow's original formulation was quite a bit more complicated than the one we present here. The list below is a simplified version.)

- **The majority criterion.** If there is a majority candidate (i.e., a candidate with a majority of the first place votes), then that candidate should be the winner of the election.
- **The Condorcet criterion.** If there is a Condorcet candidate (i.e., a candidate who beats each of the other candidates in a pairwise comparison), then that candidate should be the winner of the election.

- **The monotonicity criterion.** If candidate *X* is the winner of an election, then *X* would still be the winner had a voter ranked *X* higher in his preference ballot. (In other words, a voter should not be able to hurt the winner by moving her up in his ballot.)
- **The independence-of-irrelevant-alternatives (IIA) criterion.** If candidate *X* is the winner of an election, then *X* would still be the winner had one or more of the *irrelevant alternatives* (i.e., losing candidates) not been in the race. (In other words, the winner should not be hurt by the elimination from the election of irrelevant alternatives.)

The above fairness criteria represent some (not necessarily all) of the basic principles we expect a democratic election to satisfy and can be used as a benchmark by which we can measure any voting method. If a method violates any one of these criteria, then there is the potential for unfair results under that method.

The next set of examples illustrates how *violations* of the different fairness criteria might occur.

EXAMPLE 1.19 The Borda Count Violates the Majority Criterion

Number of voters	6	2	3
1st	*A*	*B*	*C*
2nd	*B*	*C*	*D*
3rd	*C*	*D*	*B*
4th	*D*	*A*	*A*

Table 1-18 ■ Preference schedule for Example 1.19

Table 1-18 shows the preference schedule for a small election. The majority candidate in this election is *A* with 6 out of 11 first-place votes. However, when we use the Borda count we get *A*: 29 points, *B*: 32 points, *C*: 30 points, *D*: 19 points, so *B* is the winner!

So here we have a rather messy situation: *A* has a majority of the first-place votes, and yet *A* is *not the winner* under the Borda count method. This is what we mean by "*a violation of the Majority Criterion*"!

Essentially what happened in Example 1.19 is that, on a very small scale, *A* was a "polarizing" candidate—a majority of the voters had *A* as their first choice, but at the same time, there were many voters who had *A* as their last choice. Candidate *B*, on the other hand, was more of a compromise candidate—few first-place votes but enough second- and third-place votes to make a difference and beat *A*. Polarizing candidates (voters either love them or hate them) are quite common in elections for public office, and it is not hard to imagine how real world violations of the majority criterion could easily occur.

EXAMPLE 1.20 The Plurality Method Violates the Condorcet Criterion

Number of voters	49	48	3
1st	*R*	*H*	*F*
2nd	*H*	*S*	*H*
3rd	*F*	*O*	*S*
4th	*O*	*F*	*O*
5th	*S*	*R*	*R*

Table 1-19 ■ Preference schedule for Example 1.20

Let's revisit Example 1.9 (The Fabulous TSU Band Goes Bowling). Table 1-19 shows the preference schedule once again. In this election the Hula Bowl (*H*) is the *Condorcet candidate* (see Example 1.9 for the details), and yet the *winner* under the plurality method is the Rose Bowl (*R*).

Example 1.20 illustrates how, by disregarding the voters' preferences other than first choice, the plurality method can end up choosing a clearly inferior candidate (the Rose Bowl) over a clearly superior Condorcet candidate (the Hula Bowl).

EXAMPLE 1.21 Plurality-with-Elimination Violates the Monotonicity Criterion

This example comes in two parts—a before and an after. The "before" part shows how the voters intend to vote just before they cast their ballots. Table 1-20(a) shows the preference schedule for the "before" election. If all voters vote as shown in Table 1-20(a), *C* will be the winner under the plurality-with-elimination method (*B* is eliminated in the first round and the 8 votes for *B* get transferred to *C* in the second round).

Now imagine that just *before* the ballots are cast the two voters represented by the last column of Table 1-20(a) decide that they really like *C* better than *A* and switch *C* from second place to first place on their ballots. Since this is a change favorable to *C*, and *C* was going to win before, we would expect *C* to remain the winner. Surprisingly, this is not the case—just check it out: the preference schedule after the switch is given by Table 1-20(b). Now *A* is eliminated in the first round, the 7 votes for *A* are transferred to *B*, and *B* becomes the winner of the election!

Number of voters	7	8	10	2
1st	*A*	*B*	*C*	*A*
2nd	*B*	*C*	*A*	*C*
3rd	*C*	*A*	*B*	*B*

(a)

Number of voters	7	8	10	2
1st	*A*	*B*	*C*	*C*
2nd	*B*	*C*	*A*	*A*
3rd	*C*	*A*	*B*	*B*

(b)

Table 1-20 ■ Preference schedules for Example 1.21 (a) before the change and (b) after the change

Looking at Example 1.21 in retrospect we can say that *C* lost the election because of *too many* first-place votes! (Had *C* been able to convince the two voters in the last column not to switch their ballots in her favor, she would have won!) The monotonicity criterion essentially says that this kind of perverse reversal of electoral fortunes represents a violation of a key principle of fairness: *A candidate should never be penalized for getting more votes*. (Just imagine how bizarre it would be to see your typical politician campaigning *not* to get too many first-place votes!)

EXAMPLE 1.22 Pairwise Comparisons Violates the IIA

This example is a continuation of Example 1.18 (The WNBA Draft). Table 1-21 is a repeat of Table 1-15. We saw in Example 1.18 that the winner of the election under the method of pairwise comparisons is *A* (you may want to go back and refresh your memory). The Houston Rockettes are prepared to make *A* their number-one draft choice and offer her a big contract. *A* is happy. End of story? Not quite.

Number of voters	2	6	4	1	1	4	4
1st	*A*	*B*	*B*	*C*	*C*	*D*	*E*
2nd	*D*	*A*	*A*	*B*	*D*	*A*	*C*
3rd	*C*	*C*	*D*	*A*	*A*	*E*	*D*
4th	*B*	*D*	*E*	*D*	*B*	*C*	*B*
5th	*E*	*E*	*C*	*E*	*E*	*B*	*A*

Table 1-21 ■ Rockettes draft choice election: Original preference schedule

Just before the announcement is made, it is discovered that one of the irrelevant alternatives (*C*) should not have been included in the list of candidates. (Nobody had bothered to tell the draft committee that *C* had failed the team physical!) So, *C* is removed from the preference schedule and everything is recalculated. The new preference schedule is now shown in Table 1-22 (it is Table 1-21 after *C* is removed). Table 1-23 shows the result of the six pairwise comparisons between *A*, *B*, *D*, and *E*. We can see that now the winner of the election is *B*! Other than *B*, nobody is happy!

Number of voters	2	6	4	1	1	4	4
1st choice	*A*	*B*	*B*	*B*	*D*	*D*	*E*
2nd choice	*D*	*A*	*A*	*A*	*A*	*A*	*D*
3rd choice	*B*	*D*	*D*	*D*	*B*	*E*	*B*
4th choice	*E*	*E*	*E*	*E*	*E*	*B*	*A*

Table 1-22 ■ Preference schedule after *C* is removed

Pairwise comparison	Votes	Winner
A v B	A: 7 votes B: 15 votes	*B*
A v D	A: 13 votes D: 9 votes	*A*
A v E	A: 18 votes E: 4 votes	*A*
B v D	B: 11 votes D: 11 votes	tie
B v E	B: 14 votes E: 8 votes	*B*
D v E	D: 18 votes E: 4 votes	*D*
Total points: $B = 2\frac{1}{2}$, $A = 2$, $D = 1\frac{1}{2}$, $E = 0$		

Table 1-23 ■ Pairwise comparisons for Table 1.22

Example 1.22 illustrates a typical violation of the IIA: The elimination of an irrelevant alternative (*C*) penalized *A* and made her lose an election she would have otherwise won and in turn allowed *B* to win an election she would have otherwise lost. Clearly this is not fair, and the independence of irrelevant alternatives criterion aims to prevent these types of situations.

The point of the four preceding examples is to illustrate the fact that each of the voting methods we studied in this chapter violates one of Arrow's fairness criteria. In fact, some of the voting methods violate more than one criterion. The full story of which fairness criteria are violated by each voting method is summarized in Table 1-24 on the next page.

Criterion	Plurality	Borda count	Plurality-with-elimination	Pairwise comparisons
Majority	✔	*Yes*	✔	✔
Condorcet	*Yes*	*Yes*	*Yes*	✔
Monotonicity	✔	✔	*Yes*	✔
IIA	*Yes*	*Yes*	*Yes*	*Yes*

Table 1-24 ■ Summary of violations of Arrow's fairness criteria: *Yes* indicates that the method *could violate* the criterion; ✔ indicates that the method *satisfies* the criterion.

If you are looking at Table 1-24 and asking yourself "So what's the point? Why did we spend so much time learning about voting methods that are so flawed?" you have a legitimate gripe. The problem is that we don't have better options: *every voting method—whether already known or yet to be invented—is flawed*. This remarkable fact was discovered in 1949 by Kenneth Arrow and is known as **Arrow's Impossibility Theorem**. To be more precise, Arrow demonstrated that for elections involving three or more candidates *it is mathematically impossible for a voting method to satisfy all four of his fairness criteria*.

In one sense, Arrow's Impossibility Theorem is a bit of a downer. It tells us that no matter how hard we try, democracy will never have a perfectly fair voting method and that the potential for some form of unfairness is built into every election. This does not mean that every election is unfair or that every voting method is equally bad, nor does it mean that we should stop trying to improve the quality of our voting experience.

Conclusion

Elections are the mechanism that allows us to make social decisions in a *democracy*. (In contrast to a *dictatorship*, where social decisions are made by one individual and elections are either meaningless or nonexistent.) The purpose of this chapter is to help you see elections in a new light.

In this chapter we discussed many important concepts, including *preference ballots*, *preference schedules*, *winner-only* versus *ranked elections*, *voting methods*, and *fairness criteria*. We saw plenty of examples of elections—some made-up, some real. We learned some specific skills such as interpreting a preference schedule and calculating the outcome of an election under four different voting methods and variations thereof.

Beyond the specific concepts and skills, there were several important general themes that ran through the chapter:

- *Elections are ubiquitous.* The general public tends to think of elections mostly in terms of political decisions (president, governor, mayor, city council, etc.), but elections are behind almost every meaningful social decision made outside the political arena—Academy Awards, *American Idol*, Heisman Trophy, MVP awards, Homecoming Queen, where to go to dinner, etc.
- *There are many different voting methods.* The outcome of an election can be determined in many different ways. In this chapter we discussed in some detail only four voting methods: *plurality, Borda count, plurality-with-elimination*, and *pairwise comparisons*. By no means do these four exhaust the list—there are many other voting methods, some quite elaborate and exotic.

- *Different voting methods can produce different outcomes.* We saw an extreme illustration of this with the Math Club election: each of the four voting methods produced a different winner. Since there were four candidates, we can say that each of them won the election (just pick the "right" voting method). Of course, this doesn't happen all the time and there are many situations where different voting methods produce the same outcome.
- *Fairness in voting is elusive.* For a voting method to be considered fair there are certain basic criteria that it should consistently satisfy. These are called *fairness criteria*. We introduced four in this chapter (*majority, Condorcet, monotonicity*, and *independence of irrelevant alternatives*), but there are others. Each fairness criterion represents a basic principle we expect a democratic election to satisfy. When a voting method violates any one of these criteria then there is the potential for unfair results under that method. All of the voting methods we discussed in this chapter violate at least one (sometimes several) of the criteria, and there is a good reason why: for elections with three or more candidates it is mathematically impossible for any voting method to satisfy all four fairness criteria. This is a simplified version of *Arrow's Impossibility Theorem.*

One concluding thought about this chapter: One should not interpret Arrow's Impossibility Theorem to mean that democracy is bad and that elections are pointless. The lesson to be learned from Arrow's Impossibility Theorem is that no voting system is perfect, because there are some built-in limitations to the process of making decisions in a democracy. This is a good thing to know, and somewhat surprisingly, it is a knowledge made possible through the power of mathematical ideas.

> “The search of the great minds of recorded history for the perfect democracy, it turns out, is the search for a chimera, a logical self-contradiction.”
>
> *– Paul Samuelson*

Key Concepts

1.1 The Basic Elements of an Election

- **single-choice ballot:** a ballot in which a voter only has to choose one candidate, **4**
- **preference ballot:** a ballot in which the voter has to rank all candidates in order of preference, **4**
- **truncated preference ballot:** a ballot in which a voter only has to rank the top k choices rather than all the choices, **4**
- **ranking (full ranking):** in an election, an outcome that lists *all* the candidates in order of preference (first, second, . . . , last), **4**
- **partial ranking:** in an election, an outcome where just the top k candidates are ranked, **4**
- **preference schedule:** a table that summarizes the preference ballots of all the voters, **7**

1.2 The Plurality Method

- **plurality method:** a voting method that ranks candidates based on the number of first-place votes they receive, **9**
- **insincere voting:** voting for candidates in a manner other than the voter's real preference with the purpose of manipulating the outcome of the election, **11**
- **Condorcet candidate:** a candidate that beats all the other candidates in pairwise comparisons, **12**

1.3 The Borda Count Method

- **Borda count method:** a voting method that assigns points to positions on the ballot and ranks candidates according to the number of points, **12**

1.4 The Plurality-with-Elimination Method

- **plurality-with-elimination method:** a voting method that chooses the candidate with a majority of the votes; when there isn't one it eliminates the candidate(s) with the least votes and transfers those votes to the next highest candidate on those ballots, continuing this way until there is a majority candidate, **15**
- **ranked-choice voting (instant-runoff voting):** a variation of the plurality-with-elimination method based on truncated preference ballots, **17**

1.5 The Method of Pairwise Comparisons

- **method of pairwise comparisons:** a voting method based on head-to-head comparisons between candidates that assigns one point to the winner of each comparison, none to the loser, and $\frac{1}{2}$ point to each of the two candidates in case of a tie, **19**

1.6 Fairness Criteria and Arrow's Impossibility Theorem

- **fairness criteria:** basic rules that define formal requirements for fairness—a fair voting method is expected to always satisfy these basic rules, **22**
- **majority criterion:** a fairness criterion that says that if a candidate receives a majority of the first-place votes, then that candidate should be the winner of the election, **22**
- **Condorcet criterion:** a fairness criterion that says that if there is a Condorcet candidate, then that candidate should be the winner of the election, **22**
- **monotonicity criterion:** a fairness criterion that says that a candidate who would otherwise win an election should not lose the election merely because some voters changed their ballots in a manner that favors that candidate, **23**
- **independence-of-irrelevant-alternatives criterion:** a criterion that says that a candidate who would otherwise win an election should not lose the election merely because one of the losing candidates withdraws from the race, **23**
- **Arrow's Impossibility Theorem:** a theorem that proves that it is mathematically impossible for a voting method to satisfy all of the fairness criteria, **22, 26**

Exercises

WALKING

1.1 Ballots and Preference Schedules

1. Figure 1-8 shows the preference ballots for an election with 21 voters and 5 candidates. Write out the preference schedule for this election.

Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot
1st *C*	1st *A*	1st *B*	1st *A*	1st *C*	1st *D*	1st *A*
2nd *E*	2nd *D*	2nd *E*	2nd *B*	2nd *E*	2nd *C*	2nd *B*
3rd *D*	3rd *B*	3rd *A*	3rd *C*	3rd *D*	3rd *B*	3rd *C*
4th *A*	4th *C*	4th *C*	4th *D*	4th *A*	4th *E*	4th *D*
5th *B*	5th *E*	5th *D*	5th *E*	5th *B*	5th *A*	5th *E*

Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot
1st *B*	1st *A*	1st *D*	1st *D*	1st *A*	1st *C*	1st *A*
2nd *E*	2nd *B*	2nd *C*	2nd *C*	2nd *B*	2nd *E*	2nd *D*
3rd *A*	3rd *C*	3rd *B*	3rd *B*	3rd *C*	3rd *D*	3rd *B*
4th *C*	4th *D*	4th *A*	4th *E*	4th *D*	4th *A*	4th *C*
5th *D*	5th *E*	5th *E*	5th *A*	5th *E*	5th *B*	5th *E*

Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot
1st *B*	1st *C*	1st *A*	1st *C*	1st *A*	1st *D*	1st *D*
2nd *E*	2nd *E*	2nd *B*	2nd *E*	2nd *D*	2nd *C*	2nd *C*
3rd *A*	3rd *D*	3rd *C*	3rd *D*	3rd *B*	3rd *B*	3rd *B*
4th *C*	4th *A*	4th *D*	4th *A*	4th *C*	4th *A*	4th *E*
5th *D*	5th *B*	5th *E*	5th *B*	5th *E*	5th *E*	5th *A*

Figure 1-8

2. Figure 1-9 shows the preference ballots for an election with 17 voters and 4 candidates. Write out the preference schedule for this election.

Ballot	Ballot	Ballot	Ballot	Ballot
1st *C*	1st *B*	1st *A*	1st *C*	1st *B*
2nd *A*	2nd *C*	2nd *D*	2nd *A*	2nd *C*
3rd *D*	3rd *D*	3rd *B*	3rd *D*	3rd *D*
4th *B*	4th *A*	4th *C*	4th *B*	4th *A*

Ballot	Ballot	Ballot	Ballot	Ballot	Ballot
1st *A*	1st *A*	1st *B*	1st *B*	1st *C*	1st *C*
2nd *D*	2nd *C*	2nd *C*	2nd *C*	2nd *A*	2nd *A*
3rd *B*	3rd *D*	3rd *D*	3rd *D*	3rd *D*	3rd *D*
4th *C*	4th *B*	4th *A*	4th *A*	4th *B*	4th *B*

Ballot	Ballot	Ballot	Ballot	Ballot	Ballot
1st *A*	1st *A*	1st *C*	1st *B*	1st *A*	1st *C*
2nd *C*	2nd *D*	2nd *A*	2nd *C*	2nd *D*	2nd *A*
3rd *D*	3rd *B*	3rd *D*	3rd *D*	3rd *B*	3rd *D*
4th *B*	4th *C*	4th *B*	4th *A*	4th *C*	4th *B*

Figure 1-9

3. An election is held to choose the Chair of the Mathematics Department at Tasmania State University. The candidates are Professors Argand, Brandt, Chavez, Dietz, and Epstein (*A*, *B*, *C*, *D*, and *E* for short). Table 1-25 shows the preference schedule for the election.

Number of voters	5	5	3	3	3	2
1st	*A*	*C*	*A*	*B*	*D*	*D*
2nd	*B*	*E*	*D*	*E*	*C*	*C*
3rd	*C*	*D*	*B*	*A*	*B*	*B*
4th	*D*	*A*	*C*	*C*	*E*	*A*
5th	*E*	*B*	*E*	*D*	*A*	*E*

Table 1-25

(a) How many people voted in this election?

(b) How many first-place votes are needed for a majority?

(c) Which candidate had the fewest last-place votes?

4. The student body at Eureka High School is having an election for Homecoming Queen. The candidates are Alicia, Brandy, Cleo, and Dionne (*A*, *B*, *C*, and *D* for short). Table 1-26 shows the preference schedule for the election.

Number of voters	202	160	153	145	125	110	108	102	55
1st	*B*	*C*	*A*	*D*	*D*	*C*	*B*	*A*	*A*
2nd	*D*	*B*	*C*	*B*	*A*	*A*	*C*	*B*	*D*
3rd	*A*	*A*	*B*	*A*	*C*	*D*	*A*	*D*	*C*
4th	*C*	*D*	*D*	*C*	*B*	*B*	*D*	*C*	*B*

Table 1-26

(a) How many students voted in this election?

(b) How many first-place votes are needed for a majority?

(c) Which candidate had the fewest last-place votes?

Exercises 5 through 8 refer to the following format for a preference ballot: The names of the candidates are printed on the ballot in some random order, and the voter is simply asked to rank the candidates [for example, see Fig. 1-1(c)]. For ease of reference we call this the "printed-names" format. (This format makes it easier on the voters and is useful when the names are long or when a misspelled name invalidates the ballot. The main disadvantage is that it tends to favor the candidates who are listed first.)

5. An election is held using the "printed-names" format for the preference ballots. Table 1-27 shows the results of the election. Rewrite Table 1-27 in the conventional preference schedule format used in the text. (Use *A, B, C, D,* and *E* as shorthand for the names of the candidates.)

Number of voters	37	36	24	13	5
Alvarez	3rd	1st	2nd	4th	3rd
Brownstein	1st	2nd	1st	2nd	5th
Clarkson	4th	4th	5th	3rd	1st
Dax	5th	3rd	3rd	5th	4th
Easton	2nd	5th	4th	1st	2nd

Table 1-27

6. An election is held using the "printed-names" format for the preference ballots. Table 1-28 shows the results of the election. Rewrite Table 1-28 in the conventional preference schedule format used in the text. (Use *A, B, C, D,* and *E* as shorthand for the names of the candidates.)

Number of voters	14	10	8	7	4
Andersson	2nd	3rd	1st	5th	3rd
Broderick	1st	1st	2nd	3rd	2nd
Clapton	4th	5th	5th	2nd	4th
Dutkiewicz	5th	2nd	4th	1st	5th
Eklundh	3rd	4th	3rd	4th	1st

Table 1-28

7. Table 1-29 shows a conventional preference schedule for an election. Rewrite Table 1-29 using a format like that in Table 1-27 (as if the ballots were "printed-names" ballots).

Number of voters	14	10	8	7	4
1st	Bob	Bob	Ana	Dee	Eli
2nd	Ana	Dee	Bob	Cat	Bob
3rd	Eli	Ana	Eli	Bob	Ana
4th	Dee	Eli	Dee	Eli	Cat
5th	Cat	Cat	Cat	Ana	Dee

Table 1-29

8. Table 1-30 shows a conventional preference schedule for an election. Rewrite Table 1-30 using a format like that in Table 1-28 (as if the ballots were "printed-names" ballots).

Number of voters	37	36	24	13	5
1st	Ada	Bo	Dina	Ceci	Bo
2nd	Ceci	Ada	Bo	Ada	Dina
3rd	Bo	Dina	Ceci	Eva	Eva
4th	Eva	Ceci	Eva	Bo	Ada
5th	Dina	Eva	Ada	Dina	Ceci

Table 1-30

9. The Demublican Party is holding its annual convention. The 1500 voting delegates are choosing among three possible party platforms: *L* (a liberal platform), *C* (a conservative platform), and *M* (a moderate platform). Seventeen percent of the delegates prefer *L* to *M* and *M* to C. Thirty-two percent of the delegates like *C* the most and *L* the least. The rest of the delegates like *M* the most and *C* the least. Write out the preference schedule for this election.

10. The Epicurean Society is holding its annual election for president. The three candidates are *A*, *B*, and *C*. Twenty percent of the voters like *A* the most and *B* the least. Forty percent of the voters like *B* the most and *A* the least. Of the remaining voters 225 prefer *C* to *B* and *B* to *A*, and 675 prefer *C* to *A* and *A* to *B*. Write out the preference schedule for this election.

1.2 Plurality Method

11. Table 1-31 shows the preference schedule for an election with four candidates (*A*, *B*, *C*, and *D*). Use the plurality method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

Number of voters	27	15	11	9	8	1
1st	*C*	*A*	*B*	*D*	*B*	*B*
2nd	*D*	*B*	*D*	*A*	*A*	*A*
3rd	*B*	*D*	*A*	*B*	*C*	*D*
4th	*A*	*C*	*C*	*C*	*D*	*C*

Table 1-31

12. Table 1-32 shows the preference schedule for an election with four candidates (*A*, *B*, *C*, and *D*). Use the plurality method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

Number of voters	29	21	18	10	1
1st	*D*	*A*	*B*	*C*	*C*
2nd	*C*	*C*	*A*	*B*	*B*
3rd	*A*	*B*	*C*	*A*	*D*
4th	*B*	*D*	*D*	*D*	*A*

Table 1-32

13. Table 1-33 shows the preference schedule for an election with four candidates (*A*, *B*, *C*, and *D*). Use the plurality method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

Number of voters	6	5	4	2	2	2	2
1st	C	A	B	B	C	C	C
2nd	D	D	D	A	B	B	D
3rd	A	C	C	C	A	D	B
4th	B	B	A	D	D	A	A

Table 1-33

14. Table 1-34 shows the preference schedule for an election with four candidates (A, B, C, and D). Use the plurality method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

Number of voters	6	6	5	4	3	3
1st	A	B	B	D	A	B
2nd	C	C	C	A	B	A
3rd	D	A	D	C	C	C
4th	B	D	A	B	D	D

Table 1-34

15. Table 1-35 shows the preference schedule for an election with five candidates (A, B, C, D, and E). The number of voters in this election was very large, so the columns of the preference schedule show percentages rather than actual numbers of voters. Use the plurality method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

Percent of voters	24	23	19	14	11	9
1st	C	D	D	B	A	D
2nd	A	A	A	C	C	C
3rd	B	C	E	A	B	A
4th	E	B	C	D	E	E
5th	D	E	B	E	D	B

Table 1-35

16. Table 1-36 shows the preference schedule for an election with five candidates (A, B, C, D, and E). The number of voters in this election was very large, so the columns of the preference schedule show percentages rather than numbers of voters. Use the plurality method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

Percent of voters	25	21	15	12	10	9	8
1st	C	E	B	A	C	C	C
2nd	E	D	D	D	D	B	E
3rd	D	B	E	B	E	A	D
4th	A	A	C	E	A	E	B
5th	B	C	A	C	B	D	A

Table 1-36

17. Table 1-25 (see Exercise 3) shows the preference schedule for an election with five candidates (A, B, C, D, and E). In this election ties are not allowed to stand, and the following tie-breaking rule is used: *Whenever there is a tie between candidates, the tie is broken in favor of the candidate with the fewer last-place votes.* Use the plurality method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

18. Table 1-26 (see Exercise 4) shows the preference schedule for an election with four candidates (A, B, C, and D). In this election ties are not allowed to stand, and the following tie-breaking rule is used: *Whenever there is a tie between candidates, the tie is broken in favor of the candidate with the fewer last-place votes.* Use the plurality method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

19. Table 1-25 (see Exercise 3) shows the preference schedule for an election with five candidates (A, B, C, D, and E). In this election ties are not allowed to stand, and the following tie-breaking rule is used: *Whenever there is a tie between two candidates, the tie is broken in favor of the winner of a head-to-head comparison between the candidates.* Use the plurality method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

20. Table 1-26 (see Exercise 4) shows the preference schedule for an election with four candidates (A, B, C, and D). In this election ties are not allowed to stand, and the following tie-breaking rule is used: *Whenever there is a tie between two candidates, the tie is broken in favor of the winner of a head-to-head comparison between the candidates.* Use the plurality method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

1.3 Borda Count

21. Table 1-31 (see Exercise 11) shows the preference schedule for an election with four candidates (A, B, C, and D). Use the Borda count method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

22. Table 1-32 (see Exercise 12) shows the preference schedule for an election with four candidates (A, B, C, and D). Use the Borda count method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

23. Table 1-33 (see Exercise 13) shows the preference schedule for an election with four candidates (A, B, C, and D). Use the Borda count method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

24. Table 1-34 (see Exercise 14) shows the preference schedule for an election with four candidates (A, B, C, and D). Use the Borda count method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

25. Table 1-35 (see Exercise 15) shows the preference schedule for an election with five candidates (A, B, C, D, and E). The total number of people that voted in this election was very large, so the columns of the preference schedule show percentages rather than actual numbers of voters. Use the Borda count method to find the complete ranking of the candidates. (*Hint*: The ranking is determined by the percentages and does not depend on the number of voters, so you can pick any number to use for the number of voters. Pick a nice round one.)

26. Table 1-36 (see Exercise 16) shows the preference schedule for an election with five candidates (A, B, C, D, and E). The total number of people that voted in this election was very large, so the columns of the preference schedule show percentages rather than actual numbers of voters. Use the Borda count method to find the complete ranking of the candidates. (*Hint*: The ranking is determined by the percentages and does not depend on the number of voters, so you can pick any number to use for the number of voters. Pick a nice round one.)

27. The 2018 Heisman Award. Table 1-37 shows the results of the balloting for the top three finalists for the 2018 Heisman Award. Find the ranking of the top three finalists and the number of points each one received.

Player	School	1st	2nd	3rd
Dwayne Haskins	Ohio State	46	111	423
Kyler Murray	Oklahoma	517	278	60
Tua Tagovailoa	Alabama	299	431	112

Table 1-37 ■ (*Source:* Heisman Award, www.heisman.com)

28. The 2019 AL Cy Young Award. Table 1-38 shows the top 5 finalists for the 2019 American League Cy Young Award. Find the ranking of the top 5 finalists and the number of points each one received (the point values are the same as those used for the National League Cy Young—see Example 1.12).

Pitcher	1st	2nd	3rd	4th	5th
Shane Bieber	0	0	11	13	5
Gerrit Cole	13	17	0	0	0
Lance Lynn	0	0	0	3	12
Charlie Morton	0	0	18	10	1
Justin Verlander	17	13	0	0	0

Table 1-38 ■ (*Source:* Major League Baseball, mlb.com)

29. An election was held using the conventional Borda count method. There were four candidates (A, B, C, and D) and 110 voters. When the points were tallied (using 4 points for first, 3 points for second, 2 points for third, and 1 point for fourth), A had 320 points, B had 290 points, and C had 180 points. Find how many points D had and give the ranking of the candidates. (*Hint*: Each of the 110 ballots hands out a fixed number of points. Figure out how many, and take it from there.)

30. Imagine that in the voting for the American League Cy Young Award (7 points for first place, 4 points for second, 3 points for third, 2 points for fourth, and 1 point for fifth) there were five candidates (A, B, C, D, and E) and 50 voters. When the points were tallied A had 152 points, B had 133 points, C had 191 points, and D had 175 points. Find how many points E had and give the ranking of the candidates. (*Hint*: Each of the 50 ballots hands out a fixed number of points. Figure out how many, and take it from there.)

1.4 Plurality-with-Elimination

31. Table 1-31 (see Exercise 11) shows the preference schedule for an election with four candidates (A, B, C, and D). Use the plurality-with-elimination method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

32. Table 1-32 (see Exercise 12) shows the preference schedule for an election with four candidates (A, B, C, and D). Use the plurality-with-elimination method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

33. Table 1-33 (see Exercise 13) shows the preference schedule for an election with four candidates (A, B, C, and D). Use the plurality-with-elimination method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

34. Table 1-34 (see Exercise 14) shows the preference schedule for an election with four candidates (A, B, C, and D). Use the plurality-with-elimination method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

35. Table 1-39 shows the preference schedule for an election with five candidates (A, B, C, D, and E). Find the complete ranking of the candidates using the plurality-with-elimination method.

Number of voters	8	7	5	4	3	2
1st	*B*	*C*	*A*	*D*	*A*	*D*
2nd	*E*	*E*	*B*	*C*	*D*	*B*
3rd	*A*	*D*	*C*	*B*	*E*	*C*
4th	*C*	*A*	*D*	*E*	*C*	*A*
5th	*D*	*B*	*E*	*A*	*B*	*E*

Table 1-39

36. Table 1-40 shows the preference schedule for an election with five candidates (A, B, C, D, and E). Find the complete ranking of the candidates using the plurality-with-elimination method.

Number of voters	7	6	5	5	5	5	4	2	1
1st	*D*	*C*	*A*	*C*	*D*	*E*	*B*	*A*	*A*
2nd	*B*	*A*	*B*	*A*	*C*	*A*	*E*	*B*	*C*
3rd	*A*	*E*	*E*	*B*	*A*	*D*	*C*	*D*	*E*
4th	*C*	*B*	*C*	*D*	*E*	*B*	*D*	*E*	*B*
5th	*E*	*D*	*D*	*E*	*B*	*C*	*A*	*C*	*D*

Table 1-40

37. Table 1-35 (see Exercise 15) shows the preference schedule for an election with five candidates (A, B, C, D, and E). The number of voters in this election was very large, so the columns of the preference schedule show percentages rather than actual numbers of voters. Use the plurality-with-elimination method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

38. Table 1-36 (see Exercise 16) shows the preference schedule for an election with five candidates (A, B, C, D, and E). The number of voters in this election was very large, so the columns of the preference schedule show percentages rather than actual numbers of voters. Use the plurality-with-elimination method to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

Top-Two Instant-Runoff Voting. *Exercises 39 and 40 refer to a simple variation of the plurality-with-elimination method called top-two IRV. This method works for winner-only elections. In top-two IRV, instead of eliminating candidates one at a time, we eliminate all the candidates except the top two in the first round and transfer their votes to the two remaining candidates.*

39. Find the winner of the election given in Table 1-39 using the *top-two IRV* method.

40. Find the winner of the election given in Table 1-40 using the *top-two IRV* method.

1.5 Pairwise Comparisons

41. Table 1-31 (see Exercise 11) shows the preference schedule for an election with four candidates (A, B, C, and D). Use the method of pairwise comparisons to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

42. Table 1-32 (see Exercise 12) shows the preference schedule for an election with four candidates (A, B, C, and D). Use the method of pairwise comparisons to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

43. Table 1-33 (see Exercise 13) shows the preference schedule for an election with four candidates (A, B, C, and D). Use the method of pairwise comparisons to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

44. Table 1-34 (see Exercise 14) shows the preference schedule for an election with four candidates (A, B, C, and D). Use the method of pairwise comparisons to

(a) find the winner of the election.

(b) find the complete ranking of the candidates.

45. Table 1-35 (see Exercise 15) shows the preference schedule for an election with five candidates (A, B, C, D, and E). The number of voters in this election was very large, so the columns of the preference schedule give the percent of voters instead of the number of voters. Find the winner of the election using the method of pairwise comparisons.

46. Table 1-36 (see Exercise 16) shows the preference schedule for an election with five candidates (A, B, C, D, and E). The number of voters in this election was very large, so the columns of the preference schedule give the percent of voters instead of the number of voters. Find the winner of the election using the method of pairwise comparisons.

47. Table 1-39 (see Exercise 35) shows the preference schedule for an election with 5 candidates. Find the complete ranking of the candidates using the method of pairwise comparisons. (Assume that ties are broken using the results of the pairwise comparisons between the tying candidates.)

48. Table 1-40 (see Exercise 36) shows the preference schedule for an election with 5 candidates. Find the complete ranking of the candidates using the method of pairwise comparisons.

49. An election with five candidates (A, B, C, D, and E) is decided using the method of pairwise comparisons. If B loses two pairwise comparisons, C loses one, D loses one and ties one, and E loses two and ties one,

- **(a)** find how many pairwise comparisons A loses. (*Hint*: First compute the total number of pairwise comparisons for five candidates.)
- **(b)** find the winner of the election.

50. An election with six candidates (A, B, C, D, E, and F) is decided using the method of pairwise comparisons. If A loses four pairwise comparisons, B and C both lose three, D loses one and ties one, and E loses two and ties one,

- **(a)** find how many pairwise comparisons F loses. (*Hint*: First compute the total number of pairwise comparisons for six candidates.)
- **(b)** find the winner of the election.

1.6 Fairness Criteria

51. Use Table 1-41 to illustrate why the Borda count method violates the Condorcet criterion.

Number of voters	6	2	3
1st	*A*	*B*	*C*
2nd	*B*	*C*	*D*
3rd	*C*	*D*	*B*
4th	*D*	*A*	*A*

Table 1-41

52. Use Table 1-32 to illustrate why the plurality-with-elimination method violates the Condorcet criterion.

53. Use Table 1-42 to illustrate why the plurality method violates the IIA criterion. (*Hint*: Find the winner, then eliminate F and see what happens.)

Number of voters	49	48	3
1st	*R*	*H*	*F*
2nd	*H*	*S*	*H*
3rd	*F*	*O*	*S*
4th	*O*	*F*	*O*
5th	*S*	*R*	*R*

Table 1-42

54. Use the Math Club election (Example 1.10) to illustrate why the Borda count method violates the IIA criterion. (*Hint*: Find the winner, then eliminate D and see what happens.)

55. Use Table 1-43 to illustrate why the plurality-with-elimination method violates the IIA criterion. (*Hint*: Find the winner, then eliminate C and see what happens.)

Number of voters	5	5	3	3	3	2
1st	*A*	*C*	*A*	*D*	*B*	*D*
2nd	*B*	*E*	*D*	*C*	*E*	*C*
3rd	*C*	*D*	*B*	*B*	*A*	*B*
4th	*D*	*B*	*C*	*E*	*C*	*A*
5th	*E*	*A*	*E*	*A*	*D*	*E*

Table 1-43

56. Explain why the method of pairwise comparisons satisfies the majority criterion.

57. Explain why the method of pairwise comparisons satisfies the Condorcet criterion.

58. Explain why the plurality method satisfies the monotonicity criterion.

59. Explain why the Borda count method satisfies the monotonicity criterion.

60. Explain why the method of pairwise comparisons satisfies the monotonicity criterion.

JOGGING

61. Two-candidate elections. Explain why when there are only two candidates, the four voting methods we discussed in this chapter give the same winner and the winner is determined by straight majority. (Assume that there are no ties.)

62. Alternative version of the Borda count. The following simple variation of the conventional Borda count method is sometimes used: last place is worth 0 points, second to last is worth 1 point, . . . , first place is worth $N - 1$ points (where N is the number of candidates). Explain why this variation is equivalent to the conventional Borda count described in this chapter (i.e., it produces exactly the same winner and the same ranking of the candidates).

63. Reverse Borda count. Another commonly used variation of the conventional Borda count method is the following: A first place is worth 1 point, second place is worth 2 points, . . . , last place is worth N points (where N is the number of candidates). The candidate with the *fewest* points is the winner, second *fewest* points is second, and so on. Explain why this variation is equivalent to the original Borda count described in this chapter (i.e., it produces exactly the same winner and the same ranking of the candidates).

64. The average ranking. The average ranking of a candidate is obtained by taking the place of the candidate on each of the ballots, adding these numbers, and dividing by the number of ballots. Explain why the candidate with the best (lowest) average ranking is the Borda winner.

65. The 2006 Associated Press college football poll. The AP college football poll is a ranking of the top 25 college football teams in the country. The voters in the AP poll are a group of sportswriters and broadcasters chosen from across the country. The top 25 teams are ranked using a conventional Borda count: a first-place vote is worth 25 points, a second-place vote is worth 24 points, a third-place vote is worth 23 points, and so on. A last-place vote is worth 1 point. Table 1-44 shows the ranking and total points for each of the top three teams at the end of the 2006 regular season. (The remaining 22 teams are not shown here because they are irrelevant to this exercise.)

Team	Points
1. Ohio State	1625
2. Florida	1529
3. Michigan	1526

Table 1-44

(a) Given that Ohio State was the unanimous first-place choice of all the voters, find the number of voters that participated in the poll.

(b) Given that all the voters had Florida in either second or third place, find the number of second-place and the number of third-place votes for Florida.

(c) Given that all the voters had Michigan in either second or third place, find the number of second-place and the number of third-place votes for Michigan.

66. **The Pareto criterion.** The following fairness criterion was proposed by Italian economist Vilfredo Pareto (1848–1923): *If every voter prefers candidate X to candidate Y, then X should be ranked above Y.*

(a) Explain why the Borda count method satisfies the Pareto criterion.

(b) Explain why the pairwise-comparisons method satisfies the Pareto criterion.

67. **The 2018–2019 NBA Rookie of the Year Award.** Each year, a panel of broadcasters and sportswriters selects the NBA rookie of the year using a modified Borda count. Table 1-45 shows the results of the voting for the top three finalists for the 2018–2019 season.

Player	1st place	2nd place	3rd place	Total points
Luka Doncic	98	2	0	496
Trae Young	2	97	0	301
Deandre Ayton	0	1	63	66

Table 1-45 ■ (*Source:* National Baseball Association, nba.com)

Determine how many points are given for each first-, second-, and third-place vote in this election. *(Hint:* Keep in mind that the points are always positive integers and that 1st place is worth more points than 2nd place and 2nd place is worth more points than 3rd place.)

68. Top-two IRV is a variation of the plurality-with-elimination method in which all the candidates except the top two are eliminated in the first round and their votes transferred to the top two (see Exercises 39 and 40).

(a) Use the Math Club election to show that top-two IRV can produce a different outcome than plurality-with-elimination.

(b) Give an example that illustrates why top-two IRV violates the monotonicity criterion.

(c) Give an example that illustrates why top-two IRV violates the Condorcet criterion.

69. **The Coombs method.** This method is just like the plurality-with-elimination method except that in each round we eliminate the candidate with the *largest number of last-place votes* (instead of the one with the fewest first-place votes).

(a) Find the winner of the Math Club election using the Coombs method.

(b) Give an example that illustrates why the Coombs method violates the Condorcet criterion.

(c) Give an example that illustrates why the Coombs method violates the monotonicity criterion.

70. **Bucklin voting.** (This method was used in the early part of the 20th century to determine winners of many elections for political office in the United States.) The method proceeds in rounds. **Round 1**: Count first-place votes only. If a candidate has a majority of the first-place votes, that candidate wins. Otherwise, go to the next round. **Round 2**: Count *first- and second-place* votes only. If there are any candidates with a majority of votes, the candidate with the most votes wins. Otherwise, go to the next round. **Round 3**: Count *first-, second-, and third-place* votes only. If there are any candidates with a majority of votes, the candidate with the most votes wins. Otherwise, go to the next round. Repeat for as many rounds as necessary.

(a) Find the winner of the Math Club election using the Bucklin method.

(b) Give an example that illustrates why the Bucklin method violates the Condorcet criterion.

(c) Explain why the Bucklin method satisfies the monotonicity criterion.

RUNNING

71. **The 2018–2019 NBA MVP Award.** The National Basketball Association Most Valuable Player is chosen using a modified Borda count. Each of the voters (sportswriters from the U.S and Canada plus one aggregate vote from the fans) submits ballots ranking the top five players from 1st through 5th place. Table 1-46 shows the results of the voting for the top five finalists for the 2018–2019 season. Using the results shown in Table 1-46, determine the point value of each place on the ballot, and (this is the most important