6th edition

Mathematics for Elementary and Middle School Teachers with Activities*

Sybilla Beckmann

*Available in print or for download. See Preface for details.

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Mathematics

for Elementary and Middle School Teachers

with Activities*

6th edition

Sybilla Beckmann
University of Georgia

Pearson

* Available in print or for download.
See Preface for details.

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To Will, Joey, and Arianna

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* shows thumbnails of downloads; actual downloads online at **bit.ly/2SWWFUX**

FOREWORD

by Roger Howe, Ph.D., Yale University

We owe a debt of gratitude to Sybilla Beckmann for this book.

Mathematics educators commonly hear that teachers need a "deep understanding" of the mathematics they teach. In this text, this pronouncement is not mere piety, it is the guiding spirit.

With the 1989 publication of its *Curriculum and Evaluation Standards for School Mathematics*, the National Council of Teachers of Mathematics (NCTM) initiated a new era of ferment and debate about mathematics education. The NCTM *Standards* achieved widespread acceptance in the mathematics education community. Many states created or rewrote their standards for mathematics education to conform to the NCTM *Standards*, and the National Science Foundation funded large-scale curriculum development projects to create mathematics programs consistent with the *Standards*' vision.

But this rush of activity largely ignored a major lesson from the 1960s' "New Math" era of mathematics education reform: in order to enable curricular reform, it is vital to raise the level of teachers' capabilities in the classroom. In 1999, the publication of *Knowing and Teaching Elementary Mathematics* by Liping Ma finally focused attention on teachers' understanding of mathematics principles they were teaching. Ma adapted interview questions (originally developed by Deborah Ball) to compare the basic mathematics understanding of American teachers and Chinese teachers. The differences were dramatic. Where American teachers' understanding was foggy, the Chinese teachers' comprehension was crystal clear. This vivid evidence showed that the difference in Asian and American students' achievement, revealed in many international comparisons, correlated to a difference in the mathematical knowledge of the teaching corps.

The Mathematical Education of Teachers, published by the Conference Board on the Mathematical Sciences (CBMS), was one response to Ma's work. Its first recommendation gave official voice to the dictum: "Prospective teachers need mathematics courses that develop a deep understanding of the mathematics they will teach." This report provided welcome focus on the problem, but the daunting task of creating courses to fulfill this recommendation remained.

Sybilla Beckmann has risen admirably to that challenge. Keeping mathematical principles firmly in mind while listening attentively to her students and addressing the needs of the classroom, she has written a text that links mathematical principles to their day-to-day uses. For example, in Chapter 4, Multiplication, the first section is devoted to the meaning of multiplication. First, it is defined through grouping: $A \times B$ means "the number of objects in A groups of B objects each." Beckmann then analyzes other common situations where multiplication arises to show that the definition applies to each. The section problems, then, do not simply provide practice in multiplication; they require students to show how the definition applies.

Subsequent sections continue to connect applications of multiplication (e.g., finding areas, finding volumes) to the definition. This both extends students' understanding of the definition and unites varied applications under a common roof. Reciprocally, the applications are used to develop the key properties of multiplication, strengthening the link between principle and practice. In the next chapter, the definition of multiplication is revisited and adapted to include fraction multiplication as well as whole numbers. Rather than emphasizing the procedure for multiplying fractions, this text focuses on how the procedure follows from the definitions.

Here and throughout the book, students are taught not merely specific mathematics, but the coherence of mathematics and the need for careful definitions as a basis for reasoning. By inculcating the point of view that mathematics makes sense and is based on precise language and careful reasoning, this book conveys far more than knowledge of specific mathematical topics: it can transmit some of the spirit of doing mathematics and create teachers who can share that spirit with their students. I hope the book will be used widely, with that goal in mind.

ABOUT THE AUTHOR

Sybilla Beckmann is a Josiah Meigs Distinguished Professor of Mathematics, Emeritus, at the University of Georgia. She received her PhD in mathematics from the University of Pennsylvania and taught at Yale University as a J. W. Gibbs Instructor of Mathematics. Her early research was on Arithmetic Geometry, but her current research is in mathematical cognition, the mathematical education of teachers, and mathematics content for students at all levels, but especially for PreK through the middle grades. She has developed mathematics courses for prospective elementary and middle grades teachers at the University of Georgia and wrote this book for such courses. A member of the writing teams for the *Common Core State Standards for Mathematics* and for NCTM's *Curriculum Focal Points for Prekindergarten Through Grade 8 Mathematics,* she has worked on the development of several state mathematics standards. She was a member of the Committee on Early Childhood Mathematics of the National Research Council and coauthor of its report *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity.* She has also been a member on numerous other national panels and committees working to improve mathematics education. Several years ago she taught an average sixth grade mathematics class every day at a local public school in order to better understand school mathematics teaching. Sybilla has won a number of awards, including the Louise Hay Award for Contributions to Mathematics Education from the Association for Women in Mathematics and the Mary P. Dolciani Award from the Mathematical Association of America.

Sybilla enjoys playing piano, running, swimming, dancing, weaving, and traveling with her family. She and her husband, Will Kazez, live in Athens, Georgia. They look forward to visits from their two children who are now away working and in graduate school.

PREFACE

I wrote *Mathematics for Elementary and Middle School Teachers* to help future elementary and middle school teachers develop a deep understanding of the mathematics they will teach. People commonly think that elementary and middle school mathematics is simple and that it shouldn't require college-level study to teach it. But to teach mathematics well, teachers must know more than just how to carry out basic mathematical procedures; they must be able to explain why mathematics works the way it does. Knowing *why* requires a much deeper understanding than knowing *how*. By learning to explain why mathematics works the way it does, teachers will learn to make sense of mathematics. I hope they will carry this "sense of making sense" into their own future classrooms.

Because I believe in deep understanding, this book focuses on explaining why. Prospective elementary and middle school teachers will learn to explain why the standard procedures and formulas of elementary and middle school mathematics are valid, why nonstandard methods can also be valid, and why other seemingly plausible ways of reasoning are not correct. The book emphasizes key concepts and principles, and it guides prospective teachers in giving explanations that draw on these key concepts and principles. In this way, teachers will come to organize their knowledge around the key concepts and principles of mathematics so they will be able to help their students do likewise.

State and National Standards The Common Core State Standards for Mathematics (CCSS) have made inroads into state-level standards in varying degrees. In some cases, CCSS is the backbone of the state standards; in other cases, CCSS took a backseat to state-generated standards. Across the board, however, "teaching for understanding" is a strong theme, making this text a good choice no matter where you happen to teach. This book acknowledges the importance of CCSS, as seen in the references to the standards, but the team at Pearson and I have made a renewed effort to connect to specific state standards, as shown on the new state-standards website at **bit.ly/2xmJViL**.

To develop this deeper understanding, teachers must study the mathematics they will teach in an especially active way, by engaging in mathematical practices. The eight CCSS Standards for Mathematical Practice outline and summarize some of the processes that are important in mathematics. These Standards for Mathematical Practice ask students to reason, construct, and critique arguments and to make sense of mathematics. They ask students to look for structure and to apply mathematics. They ask students to monitor and evaluate their progress and to persevere. These standards apply not only to K–12 students, but to all of us who study and practice mathematics, including all of us who teach mathematics.

Throughout this book, the Classroom Activities, Problems, and the text itself have been designed to foster ongoing active engagement in mathematical practices while studying mathematics content. At the beginning of each chapter, I briefly describe ways to engage in a few of the practices that are especially suited to the material in that chapter.

I believe *Mathematics for Elementary and Middle School Teachers* is an excellent fit for the recommendations of the Conference Board of the Mathematical Sciences regarding the mathematical preparation of teachers. I also believe that the book helps prepare teachers to teach in accordance with the principles and standards of the National Council of Teachers of Mathematics (NCTM).

Scope of Coverage This book is centered on the mathematical content of prekindergarten through Grade 8. (You may have noted that we added "Middle School" to the title of the text to emphasize the scope of coverage beyond "Elementary.") The text addresses the K–8 CCSS Standards for Mathematical Content from a teacher's perspective, with a focus on how ideas develop and connect and on powerful ways of representing and reasoning about the ideas. Each section is labeled with the grade levels at which the content is typically introduced. The development of the material goes beyond what is expected of elementary and middle school students. For example, in third grade, students learn that shapes can have the same perimeter but different area. Section 12.8 explores that idea beyond the third grade level, by considering all possible shapes that have a fixed perimeter, including circles, and by asking what the full range of possible areas is for all those shapes.

The chapters are designed to help prospective teachers study how mathematical ideas develop across grade levels. For example, in third grade, students learn about areas of rectangles (Section 12.1), and they find areas of shapes by using the additivity principle (Section 12.2). In sixth grade, students use the additivity principle to explain area formulas, such as those for triangles and parallelograms (Sections 12.3, 12.4); in seventh grade they can apply the principle to see where the area formula for circles comes from (Section 12.6); and in eighth grade they can use the principle to explain the Pythagorean theorem (Section 12.9). Thus teachers can see how a simple but powerful idea introduced in third grade leads to important mathematics across grade levels.

New to the Sixth Edition

We changed the title of the text from *Mathematics for Elementary Teachers* to *Mathematics for Elementary and Middle School Teachers* to emphasize what has always been true of this text—it covers the content that teachers will teach in kindergarten through Grade 8.

Changes to Structure and Format

Classroom Activities The activities are now available to students in a consumable workbook format (ISBN: 978-0-13-693756-2) and downloadable within MyLab Math and via the QR codes in the text. The fact that the Activities have, in this edition, been moved out of the Student Edition should not be taken as a sign of decreased importance—on the contrary, we have tried to make them even more accessible and useful.

- For this edition I added quite a few more activities; please see the "Changes to Math Content" section below for details on some of these additions.
- Classroom Activities now contain a "Materials" list at the beginning that calls attention to any necessary resources outside of the worksheet itself. Many of these resources are available as downloadable pages (**bit.ly/2SWWFUX**), ready for printing or photocopying.

Answers to Problems I was never really pleased with the "short answers" that appeared in blue type in the Annotated Instructor Edition, as they had to be extremely succinct to fit in the space available. They gave the impression that overly brief explanations or (worse yet) answers without explanations were acceptable—both of which go against the philosophy of the text. So we now provide QR codes to take you directly to the Instructor Solutions Manual, which provides the full solutions. *Note that as an added measure of security, instructors need to log into their instructor's MyLab Math account to access the solutions after (or prior to) scanning the QR code*.

Access to MyLab Resources Common feedback that we've received from reviewers is that the resources and supplements need to be easier to locate and access within MyLab Math. For this edition we've added QR codes and short URLs to make it easy for you and your students to find resources in MyLab.

- We included ***QR code + short URL*** for resources that would be useful on a smartphone *or* a computer.
- We used ***short URL only*** for resources that are better left to a computer.
- Some QR codes and short URLs are in the Instructor Edition only—these appear in the **blue "annotation" color**.
- The QR codes and short URLs in **black type** are in the Student Edition itself.

Chapter Review Problems To provide more unique assistance to students, we decided to replace the Chapter Summary, which in previous editions simply repeated information from each section, with a set of Review Problems that pull together ideas from the chapter as a whole and better prepare students for chapter tests.

From the Field This feature has been expanded upon in this edition with additional entries that link material in the text to research on the mathematical thinking and learning of elementary and middle school students.

Correlations to State Standards We have created a web page (**bit.ly/2xmJViL**) that contains correlations of the content of this book to major K–8 state standards. (If your state does not appear online initially, please accept my apologies, as we plan to add to the correlations over time.)

Indexes and Bibliographies In this edition we combined the separate indexes and bibliographies for the book itself and the Classroom Activities into one index and one bibliography. (The page references for all Classroom Activity pages are differentiated with the prefix "CA.")

Downloads For this edition we combined three different sets of downloads (aka blackline masters) for the text and its accompanying activities into one comprehensive set and housed them all in one place: **bit.ly/2SWWFUX**. Specific downloads are referenced by number when they are needed.

Changes to Math Content

This edition has been enhanced in significant ways. Throughout, I revised and simplified wording to clarify mathematical ideas and make them more accessible. I also revised the introductions to many sections to provide a better rationale for the material and to engage the ideas that future teachers may already have about the material. I wrote new problems and revised some others to use gender-neutral pronouns. I added a number of From the Field Research entries, many of which connect the material to research with students in elementary and middle school.

The section on problem solving (formerly Section 2.1) was moved to become an independent section, prior to Chapter 1. Additional references to recommendations have been included.

Chapter 1 A new Class Activity and new problems have been added on representing and counting with numbers in bases other than base ten. There are more detailed discussions on what the digits in a number stand for and on how decimals expand the base-ten system. A short section on the approximately equal sign was added.

Chapter 2 The introduction to fractions has been clarified and enhanced. New problems and a new Class Activity on problem-solving with common partitioning have been added based on research. The outdated term "improper fraction" is no longer emphasized.

Chapter 3 There is an enhanced discussion about the distinction between modeling a situation with a "situation equation" and using a possibly different "solution equation" to solve a problem. Additional problems explore viewing subtraction as taking away versus as the difference in a comparison. There is an additional discussion about using the interpretation of subtraction as a difference in a comparison to interpret subtraction with negative numbers.

Chapter 4 An enhanced discussion and a new Class Activity introduce the definition of multiplication. The wording of the definition of multiplication has been simplified, and more guidance on how to use the definition is provided. An additional Class Activity explains and uses the associative property with groups of groups of things.

Chapter 5 The introduction, discussion in the text, and examples in the Class Activities were revised to use the simplified wording for the definition of multiplication from Chapter 4. The simplified wording allows for a closer and more natural connection between multiplication equations and phrases that describe a real-world situation. There is an enhanced discussion about using estimation with decimal multiplication. There is a new discussion, Class Activity, and problems on decimal multiplication word problems. Additional problems relate decimal multiplication to area. The introduction to multiplication with negative numbers was revised to explain why word problems are not emphasized in that section. There is an additional Class Activity on multiplication with powers.

Chapter 6 Explanations and examples were revised to use the simplified wording for the definition of multiplication given in Chapters 4 and 5. There are additional problems on interpreting whole-number-with-remainder answers and mixed-number answers to whole number division problems. An additional Class Activity develops reasoning about base-ten bundles as a foundation for the standard whole number division algorithm. The Class Activities now offer a more gradual development for understanding why dividing by a fraction is equivalent to multiplying by the reciprocal.

Chapter 7 Revisions and additional problems in Class Activities clarify how ideas of ratio and proportion fit coherently with and extend ideas of multiplication and division. Explanations and examples were revised to use the simplified wording for the definition of multiplication given in Chapters 4 and 5. Based on research, additional guidance in the text and Class Activity problems address common errors in formulating equations. New interactive figures have been developed to illustrate how quantities can vary together in a fixed multiplicative relationship.

Chapter 8 Additional opportunities for problem-solving with factors have been included. A QR code and short URL (**bit.ly/3dldGQb**) have been added that link to an online section about how various number systems are related.

Chapter 9 The discussion on the logic of solving equations, and how the imagery of a pan-balance is helpful, has been elaborated and clarified, including additional Class Activity problems. Discussions about the role of covariation have been added, and students have additional opportunities to create and interpret graphs of covarying quantities. Based on research, additional guidance addresses common errors in formulating equations. A QR code and short URL (**bit.ly/3dldGQb**) have been added that link to an online section about series.

Chapter 10 An additional activity introduces the need for angles. Revisions help to clarify the link between the rotation and static view of angles. Additional problems offer opportunities to make and reason about simple protractors by folding paper, to clarify how protractors measure angles. Additional problems explore how locations in a plane are related to intersections of circles. A QR code and short URL (**bit.ly/3dldGQb**) have been added that link to an online section about visualization.

Chapter 11 Additional problems and revisions have been made concerning making and interpreting rulers and common errors that occur from misinterpreting how rulers measure length.

Chapter 12 Additional emphasis was placed on counting units as the simplest and most direct way to determine area. Additional problems and revisions allow the connection between the area formula for rectangles and the “equal groups” definition of multiplication to be explored more thoroughly.

Chapter 13 Additional emphasis was placed on counting units as the simplest and most direct way to determine volume.

Chapter 14 The descriptions of the different types of symmetry have been simplified to make them easier to understand. Material on creating symmetrical designs has been re-inserted. A new interactive figure was developed to illustrate how to view geometric similarity as involving quantities that vary but are in a fixed relationship.

Chapter 15 The discussion on the distinction between statistical questions and other (mathematical) questions has been elaborated and problems exploring the distinction have been added. A discussion on bias and the role of randomness in avoiding bias has been added. A new problem gives students the opportunity to classify questions about graphs. A Class Activity has been extended to explore the connection between the "fair share" view of the mean with the "balance point" view. Some Class Activities have been revised so they can be used either with electronic simulations or with hands-on materials.

Chapter 16 Based on research, a new Class Activity was added to help students make a connection between long-run relative frequency and probability. A new problem has been added to explore probabilities related to infectious disease. Revisions to the section on using fraction arithmetic to calculate probabilities clarify how the calculations are justified based on the meaning of multiplication, addition, and subtraction.

I am enthusiastic about these changes and am sure students and professors will be as well.

Changes to MyLab Math

- **Exercise review and fine-tuning**—Burak Ölmez (University of Southern California) reviewed all of the MyLab exercises for appropriateness and suggested additions/deletions to better align the content of MyLab to support the text.
- **Redesign for ease of navigation**—We streamlined the organization of the content in MyLab Math and made it more visually appealing. Hopefully this will help you and your students make better use of what's there!
- **Integrated Mathematics and Pedagogy (IMAP) video exercises**—In reviewing the MyLab usage statistics, we discovered that that the IMAP video exercises are very popular! So we added more of them. Now all IMAP videos have at least one assignable exercise so you have a way to check that your students actually watched the video.
- **Interactive Figures**—We added more interactive figures (in editable GeoGebra format) to serve as both teaching and learning tools. You can assign the interactive figures as part of homework via MyLab Math.
- **Video labeling**—We moved what used to be labeled "Common Core Videos" into the new category "Conceptual Understanding Videos." For each Conceptual Understanding Video and Demonstrations Video, we note the relevant Common Core State Standard(s).
- **Mindset and Study Skills**—We've added material on Growth Mindset and common Study Skills (e.g., time management) because research indicates that success in mathematics has a strong affective component and because some students need help with study skills in order to succeed in the course.

Content Features

Organization of Chapters The book is organized around the operations instead of around the different types of numbers. In my view, focusing on the operations has two key advantages. The first is a more advanced, unified perspective, which emphasizes that a given operation (addition, subtraction, multiplication, or division) retains its meaning across all the different types of numbers. Prospective teachers who have already studied numbers and operations in the traditional way for years will find that method enables them to take a broader view and to consider a different perspective. A second advantage is that fractions, decimals, and percents—traditional weak spots—can be studied repeatedly throughout a course, rather than only at the end. The repeated coverage of fractions, decimals, and percents allows students to gradually become used to reasoning with these numbers, so they aren't overwhelmed when they get to multiplication and division of fractions and decimals.

A special section on solving problems and explaining solutions appears prior to Chapter 1. (I moved it from its previous location in Chapter 2 based on reviewer feedback.) The section helps students think about how explanations in math can be different from explanations in other fields of knowledge. This section can be covered at any time but might be especially helpful prior to Chapter 2.

Below, based on a suggestion from users of the text, is a chart that shows the dependencies of the sections. Note that the chart shows only immediate dependencies, not ones further down (i.e., dependents of dependents).

Section	Depends On	Section	Depends On	Section	Depends On
1.1	independent	7.1	5.1	12.1	4.3
1.2	1.1	7.2	7.1, 6.2	12.2	12.1
1.3	1.2	7.3	7.2	12.3	12.2
1.4	1.3	7.4	7.3	12.4	somewhat on 12.3
2.1	1.1, somewhat on 1.2	7.5	7.3	12.5	12.3
2.2	2.1	7.6	6.2, 5.1, 4.4	12.6	12.2
2.3	2.2	8.1	somewhat on 4.1	12.7	12.1
2.4	2.2	8.2	8.1	12.8	12.6
3.1	independent	8.3	8.1	12.9	12.3, 10.1
3.2	3.1, 1.1	8.4	8.1	13.1	somewhat on 10.4
3.3	1.2	8.5	8.1	13.2	13.1, 10.3, somewhat on 11.2
3.4	2.2, somewhat on 1.2	8.6	standard division algorithm, in 6.3	13.3	4.3, somewhat on 11.2
3.5	3.2, 1.2	8.7 (online)	8.6, 1.2	13.4	13.3
4.1	independent	9.1	order of operations, in 4.4	14.1	10.1
4.2	4.1, 1.1, 1.2	9.2	9.1	14.2	14.1
4.3	4.1	9.3	9.2	14.3	14.1, 10.3
4.4	4.1	9.4	9.3	14.4	10.3, somewhat on 14.3, 10.4
4.5	4.4, 4.3	9.5	9.2, 4.1	14.5	somewhat on 7.2
4.6	4.4, 4.3, 4.2	9.6	9.5	14.6	14.5, 14.1
5.1	4.1, 2.2	9.7	9.6	14.7	14.5, somewhat on 11.2, 4.3
5.2	4.2	9.8 (online)	9.5	15.1	somewhat on 7.2
5.3	4.3, 4.4	10.0 (online)	independent	15.2	15.1
5.4	5.2, 4.2	10.1	independent	15.3	15.2, somewhat on 6.2
6.1	4.1, somewhat on 5.3	10.2	10.1	15.4	15.3
6.2	6.1, 2.1	10.3	independent	16.1	somewhat on 2.1
6.3	6.2, 4.4, 1.1	10.4	10.3, 10.1	16.2	4.1
6.4	6.2	11.1	independent	16.3	16.2, 16.1
6.5	6.2, somewhat on 6.4	11.2	11.1	16.4	16.1, 5.1
6.6	6.1, 4.2	11.3	11.1, 1.4		
		11.4	11.2, 6.2, 4.3		

Notes on Topic Coverage A new Class Activity (Class Activity 1-E Counting in Other Bases) and new problems (Practice Exercise 8 and Problems 13, 14, and 15 in Section 1.1) guide students to represent and count numbers in bases other than base ten. Arithmetic in bases other than ten is implied in several activities and problems. For example, Class Activities 3-M Regrouping in Base 12, and 3-N Regrouping in Base 60, and as well as Practice exercise 6, and Problems 7–11 in Section 3.3 also involve the idea of other bases. These activities and problems allow students to grapple with the significance of the base in place value without getting bogged down in the mechanics of arithmetic in other bases.

Visual representations, including number lines, double number lines, strip diagrams (also known as tape diagrams), and base-ten drawings are used repeatedly throughout the book and help prospective teachers learn to explain and make sense of mathematical ideas, solution methods, and standard notation. Chapter 9 introduces U.S. teachers to the impressive strip-diagram method for solving algebra word problems, which is used in Grades 3–6 in Singapore, where children get the top math scores in the world. The text shows how reasoning about strip diagrams leads to standard algebraic techniques.

Class Activities Class Activities were written as part of the text and are central and integral to full comprehension. The activities are now available to students in a consumable workbook format (ISBN: 978-0-13-693756-2) and downloadable within MyLab Math and via the QR codes in the text. The fact that the Activities have, in this edition, been moved out of the Student Edition should not be taken as a sign of decreased importance—on the contrary, we have tried to make them even more accessible and useful.

All good teachers of mathematics know mathematics is not a spectator sport. We must actively think through mathematical ideas to make sense of them for ourselves. When students work on problems in the class activities—first on their own, then in a pair or a small group, and then within a whole-class discussion—they have a chance to think through the mathematical ideas several times. By discussing mathematical ideas and explaining their solution methods to each other, students can deepen and extend their thinking. As every mathematics teacher knows, students really learn mathematics when they have to explain it to someone else.

A number of activities and problems offer opportunities to critique reasoning. For example, in Class Activities 2-S, 3-O, 7-A, 7-O, 12-R, 14-T, 15-E, and 16-B students investigate common errors in comparing fractions, adding fractions, distinguishing proportional relationships from those that are not, determining perimeter, working with similar shapes, displaying data, and probability. Since most misconceptions have a certain plausibility about them, it is important to understand what makes them mathematically incorrect. By examining what makes misconceptions incorrect, teachers deepen their understanding of key concepts and principles, and they develop their sense of valid mathematical reasoning. I also hope that, by studying and analyzing these misconceptions, teachers will be able to explain to their students why an erroneous method is wrong, instead of just saying, "You can't do it that way."

Standard and Nonstandard Methods Other Class Activities and problems examine calculation methods that are nonstandard but nevertheless correct. For example, in Class Activities 2-R, 2-U, 3-L, 3-P, 4-N, 6-G, and 6-O, teachers explore ways to compare fractions, calculate with percents, subtract whole numbers, add and subtract mixed numbers, multiply mentally, divide whole numbers, and divide fractions in nonstandard but logically valid ways. When explaining why nonstandard methods are correct, teachers have further opportunities to draw on key concepts and principles and to see how these concepts and principles underlie calculation methods. By examining nonstandard methods, teachers also learn there can be more than one correct way to solve a problem. They see how valid logical reasoning, not convention or authority, determines whether a method is correct. I hope that, having studied and analyzed a variety of valid solution methods, teachers will be prepared to value their students' creative mathematical activity. Surely a student who has found an unusual but correct solution method would be discouraged if told the method is incorrect. Such a judgment also conveys to the student entirely the wrong message about what mathematics is.

Studying nonstandard methods of calculation provides valuable opportunities, but the common methods deserve to be studied and appreciated. These methods are remarkably clever and make highly efficient use of underlying principles. Because of these methods, we know that a wide range of problems can always be solved straightforwardly. The common methods are major human achievements and part of the world's heritage; like all mathematics, they are especially wonderful because they cross boundaries of culture and language.

Textbook Features

Practice Exercises Practice Exercises in each section of the text give students the opportunity to try out problems. Solutions appear in the text immediately after the Practice Exercises, providing students with many examples of the kinds of good explanations they should learn to write. By attempting the Practice Exercises themselves and checking their solutions against the solutions provided, students will be better prepared to provide good explanations in their homework.

Problems Following the Practice Exercises, the Problems are opportunities for students to explain the mathematics they have learned, without being given an answer at the end of the text. Problems are typically assigned as homework. Solutions appear in the Instructor's Solutions Manual.

From the Field boxes have been expanded to include additional research related to the mathematical content of the section.

Section Summary and Study Items are provided at the end of each section to help students organize their thinking and focus on key ideas as they study.

Chapter Review Problems Each chapter concludes with Review Problems that pull together the ideas from across the chapter. Hints are provided for the Review Problems.

Special Labeling

The **core** icon denotes central material. These problems and activities are highly recommended for mastery of the material.

* Problems with an asterisk are more challenging, involve an extended investigation, or are designed to extend students' thinking beyond the central areas of study.

CCSS The **Common Core** icon indicates that the section addresses the Common Core State Standards for Mathematics at the given grade level. The treatment of the topic goes beyond what students at that grade level are expected to do because teachers need to know how mathematical ideas develop and progress. CCSS have also been noted on almost every activity.

Skills Review MyLab Math Course for *Mathematics for Elementary and Middle School Teachers*

MyLab Math is available to accompany Pearson's market leading text offerings. To complement the inquiry-based approach of this text, a special ***Skills Review*** MyLab Math course contains practice exercises, a complete eText, and many other resources. The Skills Review course differs from traditional MyLab courses in that the algorithmic homework exercises contained in it are not designed to *match* those in the textbook, but rather to *complement* them. Specifically, the online exercises are designed to help students develop fluency with procedures for which they lack confidence, complementing the primary means of assessment in the textbook, activities, and Instructor's Testing Manual. The online exercises have been revised for this edition to better align with the content of the textbook.

What follows are descriptions of the many other resources designed to support faculty and students within MyLab Math.

Activities

- **Activities Manual** (ISBN: 978-0-13-693756-2 and downloadable in MyLab Math)—Contains classroom activities, written by the author in conjunction with the text, that are integral to full comprehension.
- **Additional Activities** (downloadable by chapter in MyLab Math) consist of activities not included in the manual.
- **Downloads** (homepage: **bit.ly/2SWWFUX**) include all blackline masters necessary to support the activities. The downloads are referenced by number within the activities.

Videos

Video Type / Intended Audience	Contents	Easy Access	Assignable?
Demonstration Videos for future teachers	Author or other expert demonstrating math processes/procedures for future teachers	QR codes in the Student Edition	Videos assignable in MyLab Math, and many have corresponding assessment questions.
Conceptual Understanding Videos for future teachers	Author or other expert builds conceptual understanding of key concepts for future teachers.	QR codes in the Student Edition	Videos assignable in MyLab Math, and many have corresponding assessment questions.
IMAP Videos (Integrated Mathematics and Pedagogy) for future teachers	Videos show elementary and middle school students working through problems, providing great insight into student thinking.	QR codes in the Student Edition IMAP Video homepage: **bit.ly/3eUNjlR**	• At least one assignable MyLab exercise for every IMAP video. • Many IMAP videos supported by worksheets. • IMAP Implementation Guide in MyLab provides specifics for integrating IMAP into your course.

Video Type / Intended Audience	Contents	Easy Access	Assignable?
Active Teacher/Active Learner Videos for faculty members	• Classroom Videos show the author or other faculty member in class working with future teachers to learn a specific concept using inquiry-based methods. • Teaching Tip Videos by the author share best practices for faculty teaching this course, focusing on use of inquiry-based methods.	Short URLs in Instructor's Edition Homepage for these videos: **bit.ly/3d2At37**	Not applicable

Interactive Figures and eManipulatives (homepage: **bit.ly/3aQ9571**) Interactive Figures allow students to explore and manipulate the mathematical concepts in a tangible way, leading to a more durable understanding. Interactive Figures are programmed in GeoGebra and are fully editable. eManipulatives correspond to the physical manipulatives designed for K–8 students. Corresponding exercises in MyLab Math make these tools truly assignable. Additional Interactive Figures (in editable GeoGebra format) have been added for this revision.

Core Instructor Resources

Annotated Instructor's Edition (ISBN: 978-0-13-693776-0)—Contains QR codes and short URLs that provide direct access to resources within MyLab Math to support instruction. *Note that as an added measure of security, instructors need to log into their instructor's MyLab Math account to access the following resources after scanning a QR code.*

- **Instructor's Resource Manual**—Written by the author, this manual includes solutions and guidance for each Class Activity, general advice on teaching the course, advice for struggling students, and sample syllabi, as well as support and ideas for each chapter.
- **Instructor's Solutions Manual**—Written by Michael Matthews of University of Nebraska at Omaha, the solutions manual contains worked-out solutions to all Problems in the text.
- **Instructor's Testing Manual**—Written by the author, this manual includes guidance on assessment, including sample test problems.

Supplemental Instructional Resources

- **From the Field: Research** (homepage: **bit.ly/3aX00ti**)—Summaries of research on children's mathematical thinking and learning by I. Burak Ölmez, Eric Siy, and the author expand on the content in the From the Field boxes in the text.
- **From the Field: Children's Lit** (homepage: **bit.ly/2Smc5BQ**)—Summaries of children's literature related to mathematics by Kirsten Keels and the author expand on the content in the From the Field boxes in the text.
- **When Will I Ever Teach This?** (homepage: **bit.ly/35ji3sd**) activities manual uses pages from real elementary school textbooks to demonstrate where topics occur in the curriculum and how they are presented to students.
- **Image Resource Library** (homepage: **bit.ly/2WbQE7r**) that contains all art from the text and is available for instructors to use in their own presentations and handouts.

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Sybilla Beckmann
Athens, Georgia

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Solving Problems and Explaining Solutions

Typical Grade Levels All grades

Fractions are fertile ground for problem solving and reasoning. Before we study fractions, let's think about how to solve problems and explain solutions. We will examine some simple but sensible guidelines for solving problems and think about what qualifies as a good explanation *in mathematics*. A main theme of this book is explaining why: Why are the familiar procedures and formulas of elementary mathematics valid? Why is a student's response incorrect? Why is a different way of carrying out a calculation often perfectly correct? We will be seeking mathematical answers to these questions.

What Is the Role of Problem Solving?

Mathematics exists to solve problems. With mathematics, we can solve a vast variety of problems in technology, science, business and finance, medicine, daily life, and mathematics itself. The potential uses of mathematics are limited only by human ingenuity. Solving problems is not only the most important *end* of mathematics, it is also a *means* for learning mathematics. Mathematicians have long known that good problems can deepen our thinking about mathematics, guide us to new ways of using mathematical techniques, help us recognize connections between topics in mathematics, and force us to confront mathematical misconceptions we may hold. By working on good problems, we learn mathematics better.

Several organizations and groups concerned with teaching and learning mathematics in prekindergarten through high school recommend that all students engage in problem solving, both in mathematics and in other contexts. Some standards on problem solving appear in the next box.

STANDARDS ON PROBLEM SOLVING

- The *Principles and Standards for School Mathematics* of the National Council of Teachers of Mathematics (NCTM) [NCTM00] includes this Process Standard:
- **Problem Solving** Instructional programs from prekindergarten through grade 12 should enable all students to
  - build new mathematical knowledge through problem solving;
  - solve problems that arise in mathematics and in other contexts;
  - apply and adapt a variety of appropriate strategies to solve problems;
  - monitor and reflect on the process of mathematical problem solving.

  (From Principles and Standards for School Mathematics. Copyright © by National Council of Teachers of Mathematics. Used by permission of National Council of Teachers of Mathematics.)
- The *Common Core State Standards for Mathematics (CCSS)* [CCSS10] includes this Standard for Mathematical Practice:
  - **SMP1 Make sense of problems and persevere in solving them.** Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. . . . [They] plan a solution pathway rather than simply jumping into a solution attempt. . . . They monitor and evaluate their progress and change course if necessary. . . . Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" (From Common Core Standards for Mathematical Practice. Published by Common Core Standards Initiative.)

How Do We Become Good Problem Solvers?

In 1945, the mathematician George Polya presented a four-step guideline for solving problems in his book *How to Solve It* [Pol88]. These four steps are simple and sensible, and they have become a framework for thinking about problem solving.

Polya's Steps

1. Understand the problem.
2. Devise a plan.
3. Carry out the plan.
4. Look back.

(From *How to Solve it: A New Aspect of Mathematical Method* by George Polya. Published by Princeton University Press, © 1945.)

Teachers and researchers have used Polya's steps to develop more detailed lists of prompts or questions to help students monitor their thinking during problem solving.

Polya's first step, *understand the problem*, is the most important. It may seem obvious that if you don't understand a problem, you won't be able to solve it, but it is easy to rush into a problem and try to do "something like we did in class" before you think about what the problem is asking. So *slow down* and read problems carefully. In some cases, drawing a diagram or a simple math picture can help you understand the problem.

In the midst of solving a problem (Polya's steps 2 and 3), it's important to monitor and reflect on the process of problem solving. Think about what information you know, what information you are looking for, and how to relate those pieces of information. If you get stuck, think about whether you have seen similar problems. Consider whether you could adapt or modify the reasoning you used for another problem to the problem at hand. Be willing to try another approach to solving the problem.

It's important to persevere when attempting to solve a problem. Students sometimes think they can solve a problem only if they've seen one just like it before, but this is not true. Your common sense and natural thinking abilities are powerful tools that will serve you well if you use them. By persevering, you will develop these thinking abilities.

Polya's fourth step, *look back*, gives you an opportunity to catch mistakes. Check to see if your answer is plausible. For example, if the problem was to find the height of a telephone pole, then answers such as 2.3 feet or 513 yards are unlikely—look for a mistake somewhere. Looking back also gives you an opportunity to make connections: Have you seen this type of answer before? What did you learn from this problem? Could you use these ideas in some other way? Is there another way to solve the problem? When you look back, you have an opportunity to learn from your own work.

IMAP
Topic: Teaching through problem-solving
Videos:
- Elise
- Tonya's Class
- Whole Class

bit.ly/3eUNjlR

Solving Problems for Yourself Students sometimes wonder why they need to solve problems themselves: Why can't the teacher just *show* us how to solve the problem? Of course, teachers do show solutions to many problems. However, sometimes teachers should step back and *guide* their students, helping them to use fundamental concepts and principles to *figure out* how to solve a problem. Why? Because the process of grappling with a problem can help students understand the underlying concepts and principles. Teachers who are too quick to tell students how to solve problems may actually rob them of valuable learning experiences. The process of making sense of things for yourself is the essence of education—use it in mathematics and do not underestimate its power.

Recommendations for Teaching Problem Solving Research has uncovered aspects of problem solving that have implications for teaching and learning. The next box briefly summarizes recommendations from the Institute of Education Sciences (see [Wool2]).

IMPROVING MATHEMATICAL PROBLEM SOLVING IN GRADES 4 THROUGH 8

Recommendation 1. Prepare problems and use them in whole-class instruction.

Recommendation 2. Assist students in monitoring and reflecting on the problem-solving process.

Recommendation 3. Teach students how to use visual representations.

Recommendation 4. Expose students to multiple problem-solving strategies.

Recommendation 5. Help students recognize and articulate mathematical concepts and notation.

(From *Improving Mathematical Problem Solving in Grades 4 Through 8,* published by the Institute of Education Sciences, U.S. Department of Education. See [Wool2].)

One important recommendation is for students to monitor and reflect on their thinking as they solve problems. To help themselves monitor and reflect, students can ask questions such as, "What is this problem asking about?" and "What are some ways I might approach this problem?" and "Is my approach working? Is there another way I might approach this problem?" Teachers can engage in dialogues that build on students' ideas so that students can clarify and refine how they are thinking and verbalize ways to approach a problem.

To improve at anything, including problem solving, it is important to know that ability is not fixed, but can be improved by working at it. According to pioneering work of Dr. Carol Dweck (see her book, Mindset [Dwe0G]), success can be dramatically influenced by how we think about our talents and abilities.

Another recommendation to improve problem solving is to learn how to use visual representations, including strip diagrams and other math drawings.

How Can We Use Strip Diagrams and Other Math Drawings?

Visual representations can often help us make sense of a problem, formulate a solution strategy, and explain a line of reasoning. Simple drawings that show relationships between quantities and are quick and easy to make can be especially helpful. We call such drawings **math drawings**. Math drawings should be as simple as possible and include only those details that are relevant to solving the problem.

One type of math drawing is the **strip diagram**, also called a **tape diagram**. Strip diagrams use lengths of rectangular strips to represent quantities, as shown on the left in the figure on the next page. Because strip diagrams use lengths, they connect readily with number lines. Strip diagrams can help students formulate equations, including algebraic equations. Strip diagrams are used throughout this book.

What Does It Mean to Model with Mathematics?

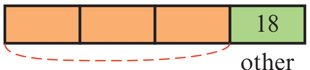
Closely related to mathematical problem solving is the practice of modeling with mathematics, in which students apply mathematics to solve problems that arise in context, including in everyday life. The box on the next page shows a standard on mathematical modeling for students in all grades from Kindergarten through high school.

Video: Using a math drawing to solve a word problem

bit.ly/2W2ngR9

Problem: In an aquarium, $\frac{3}{4}$ of the fish are goldfish. There are 18 other fish. How many goldfish are there?

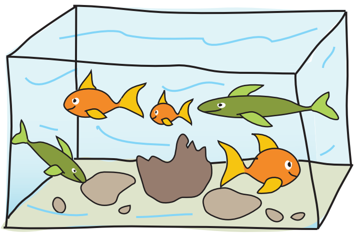
Math drawing (a strip diagram):



3 parts,
18 goldfish in each part

$3 \times 18 = 54$ goldfish

Not a math drawing:



A STANDARD ON MATHEMATICAL MODELING

The *Common Core State Standards for Mathematics (CCSS)* [CCSS10] includes this Standard for Mathematical Practice:

SMP4 Model with mathematics.
Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community.

Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

FROM THE FIELD Research

Wickstrom, M. H., & Aytes, T. (2018). Elementary modeling: Connecting counting with sharing. *Teaching Children Mathematics, 24*(5), 300–307.

Mathematical modeling is the fourth of eight Standards for Mathematical Practice in the Common Core State Standards for Mathematics. This article discusses mathematical modeling at the elementary school level. A class of second graders were engaged in mathematical modeling when the students were presented with a container of goldfish crackers and asked what they wondered about the container. The students were especially interested in how many goldfish each would receive. After students discussed constraints and assumptions, such as, "We should all get the same," groups of students found various strategies for sharing the goldfish fairly. The article concludes by highlighting several important features of mathematical modeling at the elementary school level.

What Is the Role of Explanations in Mathematics?

In mathematics, we want more than just answers to problems. We want to know why a method works. Why does it give the correct answer to the problem? As a teacher, you will need to explain why the mathematics you are teaching works the way it does. However, there is an even more compelling reason for providing explanations. When you try to explain something to someone else, you clarify your own thinking, and you learn more yourself. When you try to explain a solution, you may find that you don't understand it as well as you thought. The exercise of explaining is valuable because it provides an opportunity to learn more, to uncover an error, or to clear up a misconception. Even if you understood the solution well, you will understand it better after explaining it.

At its core, mathematics is about ideas. So explanations and lines of reasoning are just as important in mathematics as skills, procedures, and formulas. Evidence is strong that the practice of asking and answering deep explanatory questions is important for learning, according to the Practice Guide, *Organizing Instruction and Study to Improve Student Learning,* published by the Institute of Education Sciences (see [Pas07]), which states:

> Recommendation 7: Help students build explanations by asking and answering deep questions.

Communicating about mathematics gives both children and adults an opportunity to make sense of mathematics. According to the NCTM [NCTM00, p. 56]

> *From children's earliest experiences with mathematics*, it is important to help them understand that assertions should always have reasons. Questions such as "Why do you think it is true?" and "Does anyone think the answer is different, and why do you think so?" help students see that statements need to be supported or refuted by evidence.
>
>

When we communicate about mathematics in order to explain and convince, we must use *reasoning*. Logical reasoning is the essence of mathematics. In mathematics, everything but the fundamental starting assumptions has a reason, and the whole structure of mathematics is built up by reasoning.

The next box shows standards on reasoning, proof, and communication for students at all grades from prekindergarten or kindergarten through high school.

STANDARDS ON REASONING, PROOF, AND COMMUNICATION

The *Principles and Standards for School Mathematics* of the National Council of Teachers of Mathematics (NCTM) [NCTM00] includes these Process Standards:

- **Communication** Instructional programs from prekindergarten through grade 12 should enable all students to
  - organize and consolidate their mathematical thinking through communication;
  - communicate their mathematical thinking coherently and clearly to peers, teachers, and others;
  - analyze and evaluate the mathematical thinking and strategies of others;
  - use the language of mathematics to express mathematical ideas precisely.

- **Reasoning and Proof** Instructional programs from prekindergarten through grade 12 should enable all students to
  - recognize reasoning and proof as fundamental aspects of mathematics;
  - make and investigate mathematical conjectures;
  - develop and evaluate mathematical arguments and proofs;
  - select and use various types of reasoning and methods of proof.

  (From Principles and Standards for School Mathematics. Copyright © by National Council of Teachers of Mathematics. Used by permission of National Council of Teachers of Mathematics.)
- The *Common Core State Standards for Mathematics (CCSS)* [CCSS10] includes this Standard for Mathematical Practice:
  - **SMP3 Construct viable arguments and critique the reasoning of others.** Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments . . . . They justify their conclusions, communicate them to others, and respond to the arguments of others . . . . Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in the argument—explain what it is . . . . (From Common Core Standards for Mathematical Practice. Published by Common Core Standards Initiative.)

How Are Explanations in Mathematics Different from Other Explanations?

In mathematics, we seek particular kinds of explanations: those using logical reasoning and based on initial assumptions that are either explicitly stated or assumed to be understood by the reader or listener. Explanations can vary according to different areas of knowledge. There are many different kinds of explanations—even of the same phenomenon. For example, consider this question: Why are there seasons?

The simplest answer is "because that's just the way it is." Every year, we observe the passing of the seasons, and we expect to see the cycle of spring, summer, fall, and winter continue indefinitely. The cycle of seasons is an observed fact that has been documented since humans began to keep records. We could stop here, but when we ask why there are seasons, we are searching for a deeper explanation.

A poetic explanation for the seasons might refer to the cycles of birth, death, and rebirth around us. In our experiences, nothing remains unchanged forever, and many things are parts of a cycle. The cycle of the seasons is one of the many cycles that we observe.

Most cultures have stories that explain why we have seasons. The ancient Greeks, for example, explained the seasons with the story of Persephone and her mother, Demeter, who tends the earth. When Pluto, god of the underworld, stole Persephone, forcing her to become his bride, Demeter was heartbroken. Pluto and Demeter arranged a compromise, and Persephone could stay with her mother for half a year and return to Pluto in the underworld for the remaining half of every year. When Persephone is in the underworld, Demeter is sad and does not tend the earth. Leaves fall from the trees, flowers die, and it is fall and winter. When Persephone returns, Demeter is happy again and tends the earth. Leaves grow on the trees, flowers bloom, and it is spring and summer. This is a beautiful story, but we can still ask for another kind of explanation.


Spring

Summer



Fall


Winter

Modern scientists explain the reason for the seasons by the tilt of the earth's axis relative to the plane in which the earth travels around the sun. When the northern hemisphere is tilted toward the sun, it is summer there; when it is tilted away from the sun, it is winter. Perhaps this settles the matter, but a seeker could still ask for more. Why are the earth and sun positioned the way they are? Why does the earth revolve around the sun and not fly off into space? These questions can lead again to poetry or to the spiritual or to further physical theories. Maybe they lead to an endless cycle of questions.

How Do We Write Good Mathematical Explanations?

While an oral explanation helps you develop your solution to a problem, written explanations push you to polish, refine, and clarify your ideas. This is as true in mathematical writing as in any other kind of writing, and it is true at all levels. You should write explanations of your solutions to problems, and your students should write explanations of their solutions, too. Some elementary school teachers have successfully integrated mathematics and writing in their classrooms and use writing to help their students develop their understanding of mathematics.

Like any kind of writing, it takes work and practice to write good mathematical explanations. When you solve a problem, do not attempt to write the final draft of your solution right from the start. Use scratch paper to work on the problem and collect your ideas. Then, write your solution as part of the *looking back* stage of problem solving. Think of your explanations as an essay. As with any essay that aims to convince, what counts is not only factual correctness but also persuasiveness, explanatory power, and clarity of expression. In mathematics, we persuade by giving a thorough, logical argument, in which chains of logical deductions are strung together connecting the starting assumptions to the desired conclusion.

CHARACTERISTICS OF GOOD EXPLANATIONS IN MATHEMATICS

1. The explanation is factually correct, or nearly so, with only minor, inconsequential flaws.
2. The explanation addresses the specific question or problem that was posed. It is focused, detailed, and precise. Key points are emphasized. There are no irrelevant or distracting points.
3. The explanation is clear, convincing, and logical. A clear and convincing explanation is characterized by the following:
   - **a.** The explanation could be used to teach another (college) student, possibly even one who is not in the class.
   - **b.** The explanation could be used to convince a skeptic.
   - **c.** The explanation does not require the reader to make a leap of faith.
   - **d.** If applicable, supporting math drawings, diagrams, and equations are used appropriately and as needed.
   - **e.** The explanation is coherent.
   - **f.** Clear, complete sentences are used.

Good mathematical explanations are thorough. They should not have gaps that require leaps of faith. On the other hand, a good explanation should not belabor points that are well known to the audience or not central to the explanation. For example, if your solution contains the calculation $356 \div 7$, a college-level explanation need not describe how the calculation is carried out, except when it is necessary for the solution. Unless your instructor tells you otherwise, assume that you are writing your explanations for your classmates.

The box above lists characteristics of good mathematical explanations. When you write an explanation, check whether it has these characteristics. The more you work at writing explanations, and the more you ponder and analyze what makes good explanations, the better you will write explanations and the better you will understand the mathematics involved. Note that the solutions to the practice exercises in each section provide you with many examples of the kinds of explanations you should learn to write.

Numbers and the Base-Ten System

1

What are numbers and where do they come from? The concept of numbers has evolved over the course of history, and the way children learn about numbers parallels this development. When did humans first become aware of numbers? The answer is uncertain, but it is at least many tens of thousands of years ago. Some scholars believe that numbers date back to the beginning of human existence, citing as a basis for their views the primitive understanding of numbers observed in some animals. (See [Deh11] for a fascinating account of this and also of the human mind's capacity to comprehend numbers.)

In this chapter, we discuss elementary ideas about numbers, which reveal surprising intricacies that we are scarcely aware of as adults. We will study the base-ten system—a remarkably powerful and efficient system for writing numbers and a major achievement in human history. Not only does the base-ten system allow us to express arbitrarily large numbers and arbitrarily small numbers—as well as everything between—but it also enables us to quickly compare numbers and assess the ballpark size of a number. The base-ten system is familiar to adults, but its slick compactness hides its inner workings. We will examine with care those inner workings of the base-ten system that children must grasp to make sense of numbers.

In this chapter, we focus on the following topics and practices within the *Common Core State Standards for Mathematics (CCSS)*. For other standards, including some state standards, see **bit.ly/2xmJViL**.

Standards for Mathematical Content in the CCSS

In the domain of *Counting and Cardinality* (Kindergarten) young children learn to say and write small counting numbers and to count collections of things. In the domain of *Numbers and Operations in Base Ten* (Kindergarten through Grade 5), students learn to use the powerful base-ten system. This system starts with the idea of making groups of ten and gradually extends this idea to the greater and to the smaller place values of decimals.

Standards for Mathematical Practice in the CCSS

Opportunities to engage in all eight of the Standards for Mathematical Practice described in the CCSS occur throughout the study of counting and the base-ten system. The following standards are especially appropriate for emphasis while studying counting and the base-ten system:

- **SMP2 Reason abstractly and quantitatively.** Students engage in this practice when they make sense of number words and symbols by viewing numbers as representing quantities and when they use numbers to describe quantities.
- **SMP5 Use appropriate tools strategically.** The base-ten system represents numbers in a very compact, abstract way. By reasoning with appropriate tools, such as drawings of tens and ones or number lines that show decimals, students learn to make sense of the powerful base-ten system.
- **SMP7 Look for and make use of structure.** The base-ten system has a uniform structure, which creates symmetry and patterns. Students engage in this practice when they seek to understand how increasingly greater base-ten units can always be created and how the structure of the base-ten system allows us to compare numbers and find numbers between numbers.

From Common Core Standards for Mathematical Practice. Published by Common Core Standards Initiative.

1.1 The Counting Numbers

Typical Grade Levels: Pre–K, Grades K–4

What are numbers, and why do we have them? We use numbers to tell us "how many" or "how much," in order to communicate specific, detailed information about collections of things and about quantities of stuff. Although there are many different kinds of numbers (e.g., fractions, decimals, and negative numbers), the most basic numbers, and the starting point for young children, are the **counting numbers**—the numbers 1, 2, 3, 4, 5, 6, ... (The dots indicate that the list continues without end.)

How Is It Different to View the Counting Numbers as a List and for Describing Set Size?

Counting numbers can be thought about in two distinctly different ways. Connecting these two views is a major mathematical idea that very young children must grasp before they can do school arithmetic.

CLASS ACTIVITY bit.ly/3eVPwxk

1-A The Counting Numbers as a List, p. CA-1

One way to think about the counting numbers is as a list. The list of counting numbers starts with 1 and continues, with every number having a unique successor. Except for the number 1, every number in the list has a unique predecessor. So the list of counting numbers is an *ordered* list. Every counting number appears exactly once in this ordered list. The ordering of the list of counting numbers is important because of the second way of thinking about the counting numbers.

The second way to think about the counting numbers is as "telling how many." In other words, a counting number describes how many things are in a set[1]. The number of things in a set is called the **cardinality of a set**. Think for a moment about how surprisingly abstract the notion of cardinality is. We use the number 3 to quantify a limitless variety of collections—3 cats, 3 toy dinosaurs, 3 jumps, 3 claps, and so on. The number 3 is the abstract, common aspect that all examples of sets of 3 things share.

For sets of up to about 3, 4, or 5 objects, we can usually recognize the number of objects in the set immediately, without counting the objects one by one. Even very young children, who can't yet count, can distinguish between 1 and 2 crackers and between 2 and 3 crackers. The process of immediate recognition of the exact number of objects in a set is called *subitizing* and is discussed further in [NRC09]. But in general, we must count the objects in a set to determine how many there are. The process of counting the objects in a set connects the "list view" of the counting numbers with the "cardinality view." As adults, this connection is so familiar that we are usually not even aware of it. But for young children, this connection is not obvious, and grasping it is an important milestone (see [NRC09]).

How Do Children Connect Counting to Cardinality?

CLASS ACTIVITY bit.ly/3eVPwxk

1-B Connecting Counting Numbers as a List with Cardinality, p. CA-2

When we count a set of objects one by one, we make a **one-to-one correspondence** between an initial portion of the list of counting numbers and the set. For example, when a child counts a set of 5 blocks, the child makes a one-to-one correspondence between the list 1, 2, 3, 4, 5 and the set of blocks. This means that each block is paired with exactly one number and each number is paired with exactly one block, as indicated in Figure 1.1. Such a one-to-one correspondence connects the "list" view of the counting numbers with cardinality. However, another critical piece to understanding this connection relies on something adults typically take for granted but is not obvious to young children: *The last number we say when we count a set of objects tells us the total number of objects in the set*, as indicated in Figure 1.2. It is for this reason that the order of the counting numbers is so important, unlike the order of the letters of the alphabet, for example.

CCSS K.CC.4

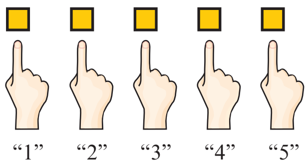


Figure 1.1 Counting 5 blocks makes a one-to-one correspondence between the list 1, 2, 3, 4, 5 and the blocks.

[1]A **set** is a collection of distinct "things." These things can include concepts and ideas, such as the concept of an infinitely long straight line, and imaginary things, such as heffalumps.

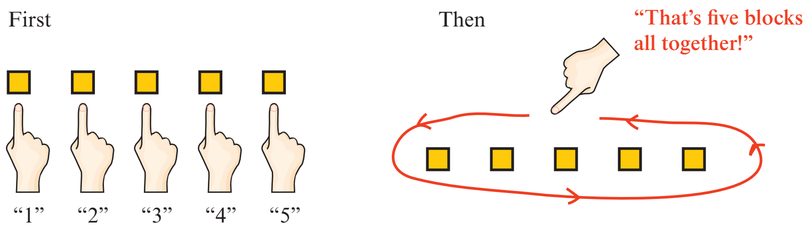


Figure 1.2 When we count objects, the last number we say tells us the total number of objects.

The connection between the "list" and "cardinality" views of the counting numbers is especially important in understanding that numbers *later* in the list correspond with *larger* quantities and that numbers *earlier* in the list correspond with *smaller* quantities. In particular, starting at any counting number, the next number in the list describes the size of a set that has one more object in it, and the previous number in the list describes the size of a set that has one less object in it.

What Are the Origins of the Base-Ten System for Representing Counting Numbers?

Let's think about the list of counting numbers and the symbols we use to represent these numbers. The symbols for the first nine counting numbers (1, 2, 3, 4, 5, 6, 7, 8, 9) have been passed along to us by tradition and could have been different. Instead of the symbol 4, we could be using a completely different symbol. In fact, one way to represent 4 is simply with 4 tally marks. So why don't we just use tally marks to represent counting numbers?

Suppose a shepherd living thousands of years ago used tally marks to keep track of his sheep. If his tally marks were not organized, he likely had a hard time comparing the number of sheep he had on different days, as shown in Figure 1.3. But if he grouped his tally marks, comparing the number of sheep was easier.

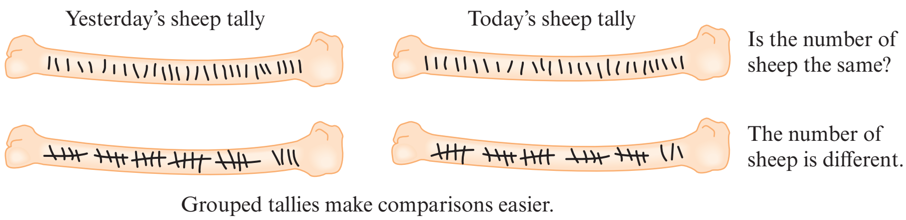


Figure 1.3 A shepherd's tally of sheep.

As people began to live in cities and engage in trade, they needed to work with larger numbers. But tally marks are cumbersome to write in large quantities. Instead of writing tally marks, it's more efficient to write a single symbol that represents a group of tally marks, such as the Roman numeral V, which represents 5. Recording 50 sheep might have been written as:

VVVVVVVVVV

However, it's hard to read all those Vs, so the Romans devised new symbols: X for 10, L for 50, C for 100, D for 500, and M for 1000. These symbols are fine for representing numbers up to a few thousand, but what about representing 10,000? Once again,

MMMMMMMMMM

is difficult to read. What about 100,000 or 1,000,000? To represent these, one might want to create yet more new symbols.

Because the list of counting numbers is infinitely long, creating more new symbols is a problem. How can each counting number in this infinitely long list be represented uniquely?

Starting with our Hindu–Arabic symbols—1, 2, 3, 4, 5, 6, 7, 8, 9—how can one continue the list without resorting to creating an endless string of new symbols? The solution to this problem was not obvious and was a significant achievement in the history of human thought. The **base-ten system**, or **decimal system**, is the ingenious system we use today to write (and say) counting numbers without resorting to creating more and more new symbols. The base-ten system requires using only ten distinct symbols—the **digits** 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. The key innovation of the base-ten system is that rather than using new symbols to represent larger and larger numbers, it uses *place value*.

What Is Place Value in the Base-Ten System?

Place value means that the quantity that a digit in a number represents depends—in a very specific way—on the position of the digit in the number.

Do Class Activity 1-C before you read on.

CLASS ACTIVITY bit.ly/3eVPwxk

1-C How Many Are There? p. CA-4

IMAP
Topic: Place value
Video: Zenaida
bit.ly/3eUNjlR

How did you and your classmates organize the toothpicks in the Class Activity 1-C? If you made bundles of 10 toothpicks and then made 10 bundles of 10 to make a bundle of 100, then you began to reinvent place value and the base-ten system. Place value for the base-ten system works by creating larger and larger units by *repeatedly bundling them in groups of ten.*

Ten plays a special role in the base-ten system, but its importance is not obvious to children. Teachers must repeatedly draw children's attention to the role that ten plays. For example, a young child might be able to count that there are 14 beads in a collection, such as the collection shown in Figure 1.4, but the child may not realize that 14 actually stands for 1 ten and 4 ones. With the perspective presented in Figure 1.4, the numbers 10, 11, 12, 13, 14 are just the counting numbers that follow 9. Young children begin to learn about place value and the base-ten system when they (1) learn to organize collections of between 10 and 19 small objects into one group of ten and some ones, as in Figure 1.5, and (2) understand that the digit 1 in 14 does not stand for "one"

Figure 1.4 The important role that ten plays is not obvious just from counting.

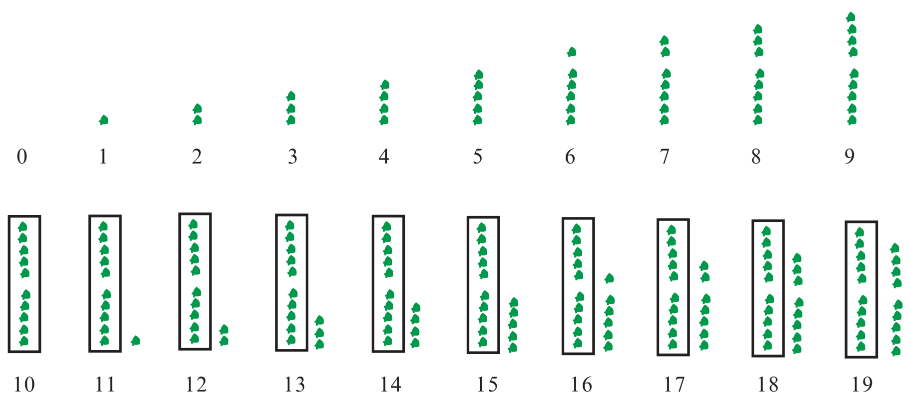


Figure 1.5 Organizing collections of objects to show base-ten structure.

but instead stands for *1 group of ten*. Some teachers like to help young students learn the meaning of the digit 1 in the numbers 10 to 19 with the aid of cards, such as the ones shown in Figure 1.6. These cards show how a number such as 17 is made up of 1 ten and 7 ones.

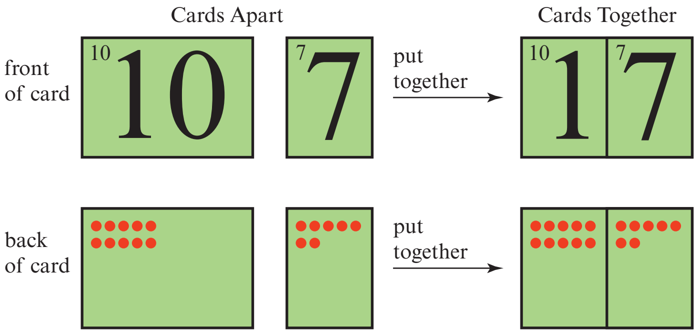


Figure 1.6

Children extend their understanding of the base-ten system in two ways: (1) viewing a group of ten as a unit in its own right and (2) understanding that a two-digit number such as 37 stands for 3 tens and 7 ones and can be represented with bundled objects and simple math drawings like those in Figure 1.7.

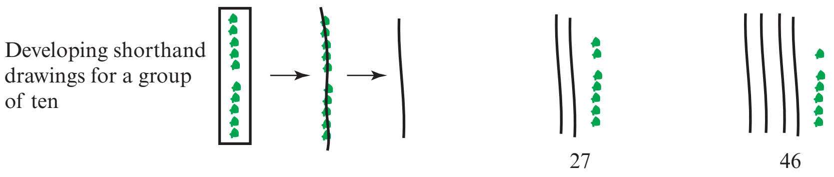


Figure 1.7

CCSS
1.NBT.2a
2.NBT.1a
4.NBT.1

Just as 10 ones are grouped to make a new unit of ten, 10 tens are grouped to make a new unit of one hundred, and 10 hundreds are grouped to make a new unit of one thousand, as indicated in Figure 1.8. Continuing in this way, 10 thousands are grouped to make a new unit of ten thousand, and so on. Thus, arbitrarily large **base-ten units** can be made by grouping 10 of the previously made units. These increasingly large units are represented in successive places to the left in a number. *The value of each place in a number is ten times the value of the place to its immediate right*, as indicated in Figure 1.9, which shows the standard names (used in the United States) of the place values up to the billions place.

In general, a string of digits, such as 1234, stands for the total amount that all its places taken together represent. Within the string, each digit stands for that many of that place's value. In the number 1234, the 1 stands for 1 thousand, the 2 stands for 2 hundreds, the 3 stands for 3 tens, and the 4 stands for 4 ones, giving

1 thousand and 2 hundreds and 3 tens and 4 ones,

which is the total number of toothpicks pictured in Figure 1.10.

A string of digits that represents a number, such as 1234, is called the **base-ten representation** or **base-ten expansion** of the number. To clarify the meaning of the base-ten representation of a number, we sometimes write it in one of the following forms, called **expanded form**:

$$1000 + 200 + 30 + 4$$

$$1 \text{ thousand} + 2 \text{ hundreds} + 3 \text{ tens} + 4 \text{ ones}$$

$$1 \cdot 1000 + 2 \cdot 100 + 3 \cdot 10 + 4$$

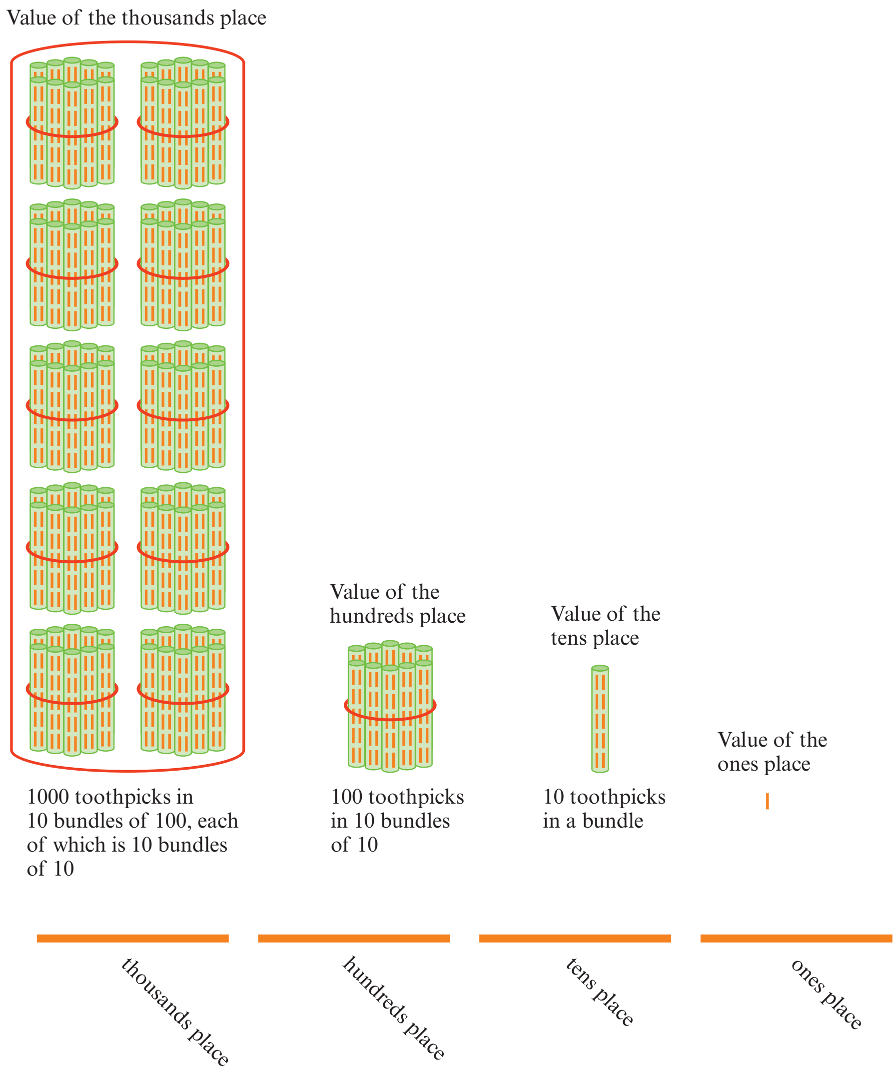


Figure 1.8 Base-ten units and values of places in the base-ten system.

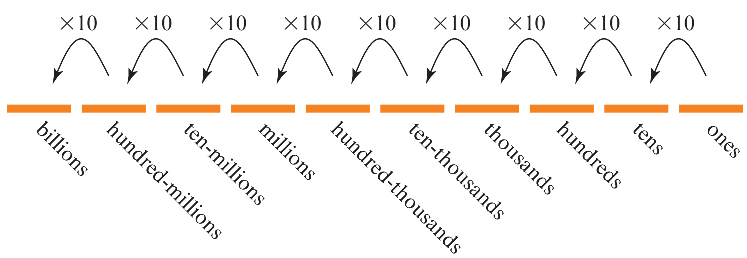


Figure 1.9 Each place's value is ten times the value of the place to its right.

Video: Understanding place value as repeated bundling in groups of ten

bit.ly/3cWP7sv

The expanded form shows how the number 1234 is composed of its **place value parts**, 1 thousand, 2 hundreds, 3 tens, and 4 ones.

Notice that the way the toothpicks in **Figure 1.10** are organized corresponds with the base-ten representation for the total number of toothpicks being depicted. Notice also how compactly this rather large number of toothpicks is represented by the short string 1234. Now imagine showing ten times as many toothpicks as are in the bundle of 1000. This would be a lot of toothpicks, yet writing this number as 10,000 only takes 5 digits!

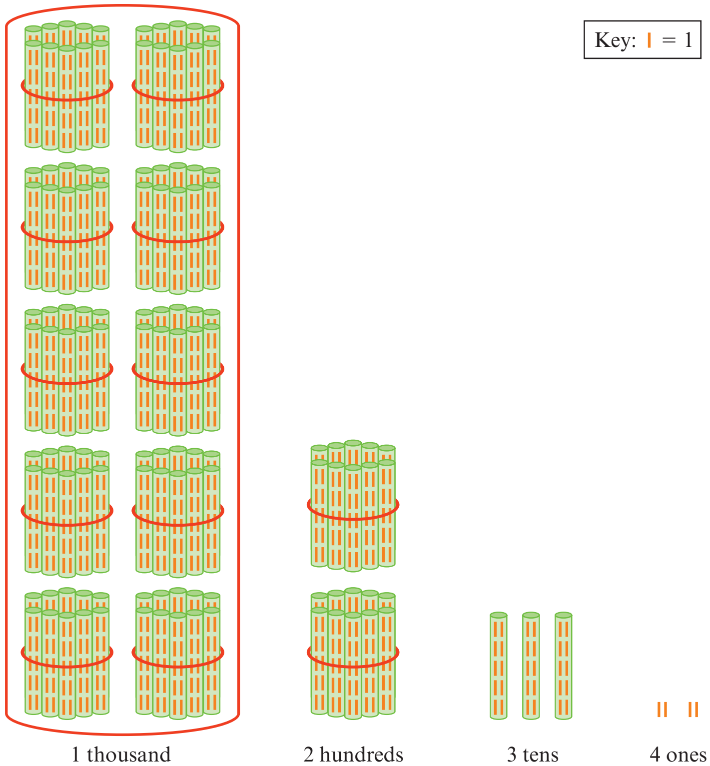


Figure 1.10 Showing how 1234 is composed of its place value parts.

Although the base-ten system is highly efficient and practical, children have difficulty learning what written numbers represent because they must keep the place values in mind. Because these values are not shown explicitly, even interpreting written numbers requires a certain level of abstract thinking.

For example, in the number 247, what do the digits 2, 4, and 7 mean? The 7 means 7 *ones*, but the 4 means 4 *tens* and the 2 means 2 *hundreds*. If we unbundle the 4 tens, as in Figure 1.11, we see that the 4 *tens* also stand for 40 *ones*. If we unbundle the 2 hundreds, as in Figure 1.12, we see that the 2 *hundreds* also stand for 20 *tens* and for 200 *ones*.

In summary, by using place value, the base-ten system allows us to write any counting number, no matter how large, using only the ten digits 0 through 9. The key idea of place value is to create larger and larger units by making the value of each new place ten times the value of the place to its right. By using place value, every counting number can be expressed in a unique way as a string of digits.

In the number 247, what does the 4 stand for?

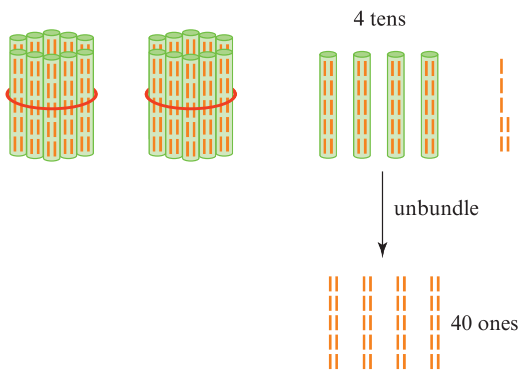


Figure 1.11

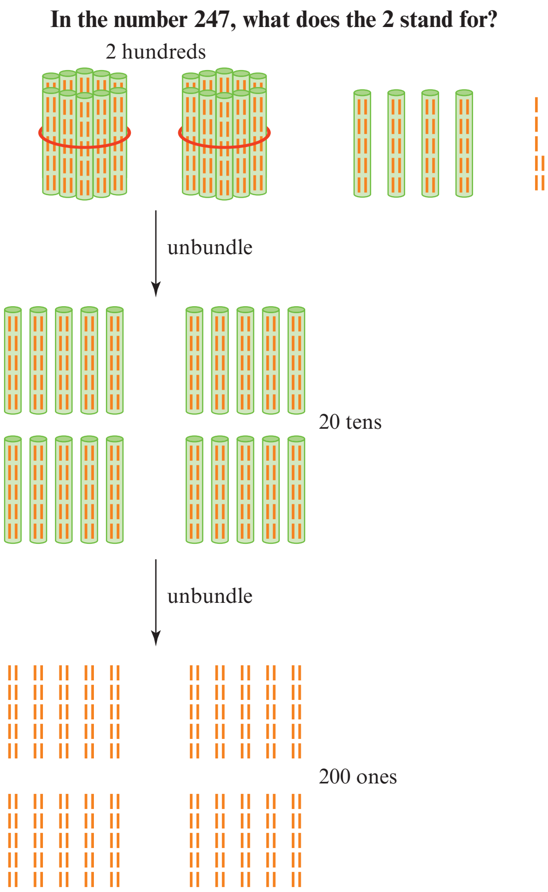


Figure 1.12

CLASS ACTIVITIES bit.ly/3eVPwxk

1-D What Do the Digits in a Counting Number Mean? p. CA-5

1-E Counting in Other Bases, p. CA-6

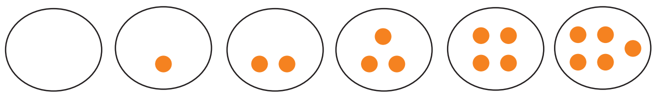
What Is Difficult about Number Words?

Before young children learn to write the symbols for the counting numbers, they learn to say the number words. Unfortunately, some of the words used to say the counting numbers in English do not correspond well with the base-ten representations of these numbers. This may make the early learning of numbers more difficult for children who speak English than for children who speak some other languages.

The English words we use for the first ten counting numbers are arbitrary and could have been different. For example, instead of the word *four* we could be using a completely different word. Numbers greater than 10 are more difficult for English speakers than for speakers of some other languages. The difficulty arises because the way we say the counting numbers from 11 to 19 in English does not correspond to their base-ten representations. Notice that "eleven" does not sound like "one ten and one," which is what 11 stands for, nor does "twelve" sound very much like "one ten and two," which is what 12 stands for. To add to the confusion, "thirteen, fourteen, . . . , nineteen" sound like the reverses of "one ten and three, one ten and four, . . . , one ten and nine," which is what 13, 14, . . . , 19 represent. From 20 onward, most of the English words for counting numbers correspond fairly well to their base-ten representations. For example, "twenty" sounds roughly like "two tens," "sixty-three" sounds very much like "six tens and three," and "two-hundred eighty-four" sounds very much like "2 hundreds and eight tens and four," which is what 284 represents. Note, however, that it's easy for children to confuse decade numbers with teen numbers because their pronunciation is so similar. For example, "sixty" and "sixteen" sound similar.

Why Do We Need Zero?

The notion of zero may seem natural to us today, but our early ancestors struggled to discover and make sense of zero. Although humans have always been acquainted with the notion of "having none," as in having no sheep or having no food to eat, the concept of 0 as a number was introduced far later than the counting numbers—not until sometime just before 800 A.D. (See [Bel92].) Even today, the notion of zero is difficult for many children to grasp. This difficulty is not surprising: Although the counting numbers can be represented nicely by sets of objects, you have to show *no* objects in order to represent the number 0 in a similar fashion. But how does one *show* no objects? We might use a math drawing like the one below.



Representing whole numbers

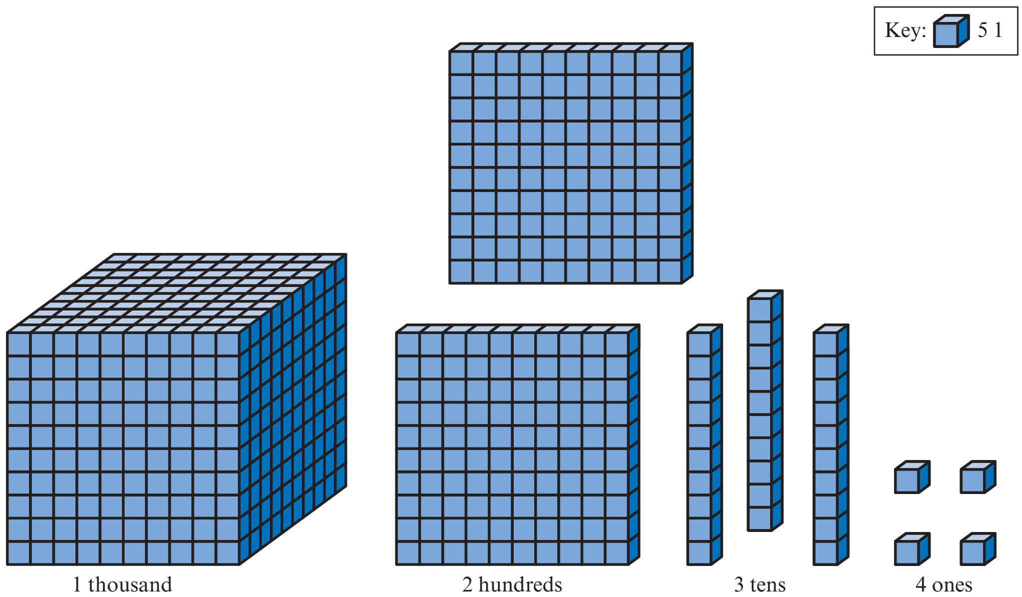
So why do we need 0? To make place value work! To write three hundred, for example, we must show that the 3 is in the hundreds place and then show that there are no tens and no ones. We do this by writing 300.

Whole numbers are counting numbers together with zero:

$$0, 1, 2, 3, 4, 5, \ldots$$

What Are Other Ways to Represent Counting Numbers and Whole Numbers?

So far, we have seen how to represent counting numbers with bundles of toothpicks so as to highlight the base-ten structure of numbers. Instead of bundles, some teachers use base-ten blocks, such as the ones shown below. If the small cube represents one, then a stack of 10 small cubes make a "long," which represents a ten. Ten longs placed side-by-side make a "flat," which represents a hundred. A stack of ten flats make a "large cube," which represents a thousand. All together, the blocks below show the number 1234 as 1 thousand and 2 hundreds and 3 tens and 4 ones.



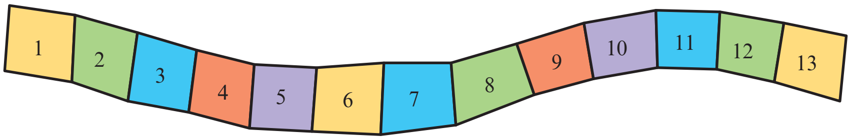
Representing 1234 with base-ten blocks

One disadvantage of base-ten blocks is that for some students, it may be difficult to see each unit as made from 10 of the previous unit. For example, some students might focus only on the outer surface of the large cube and not view it as made from 10 flats.

Another way to represent counting numbers and whole numbers is on a *number line*.

What Ideas Lead to Number Lines?

We can use counting numbers to count events as well as the number of physical things in a collection. For example, children might count how many times they have hopped or jumped or how many steps they have taken. Many children's games involve moving a game piece along a path. If the path is labeled with successive counting numbers, like the one below, we can call the path a **number path**. Number paths are informal precursors to the concepts of distance and length and to the mathematical concept of a number line.



A **number line** is a line on which we have chosen one location as 0 and another location, typically to the right of 0, as 1. Number lines stretch infinitely far in both directions, although, in practice we can only show a small portion of a number line (and that portion may or may not include 0 and 1). The segment (or interval) from 0 to 1 stands for *one*; its length, namely the distance from 0 to 1, is called a **unit**; and the choice of a unit is called the **scale** of the number line. Once choices for the locations of 0 and 1 have been made, each counting number is represented by the point on the number line located that many units to the right of 0.

For example, the number 13 is 13 units to the right of 0, and the distance between 0 and 13 is 13 units. We can think about plotting the number 13 as placing 13 one-unit-long segments end to end, starting from 0, and going to the right, as in Figure 1.13. The number 13 is located at the right end of the 13th segment.

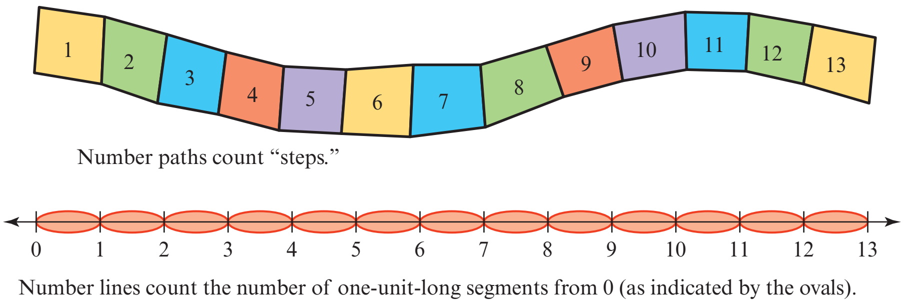


Figure 1.13

Notice that because we plot counting numbers on a number line by repeating the same one-unit-long segment, the distance between a counting number and its successor is always the same, namely 1 unit. For example, the distance between 5 and 6 is 1 unit, and the distance between 1000 and 1001 is also 1 unit. To understand number lines, we must think about numbers in terms of measurement and distance.

Number lines are an important way to represent numbers because they allow the concept of a number to be expanded to decimals, fractions, and negative numbers, and they unify different kinds of numbers and present them as a coherent whole.

CCSS 2.MD.6

Although number paths and number lines are similar, there is a critical distinction between them. This distinction makes using number paths with the youngest children better than using number lines. Number paths clearly show distinct "steps" along the path that children can

IMAP
Topic: Place Value
Video: Maryann
bit.ly/3eUNjlR

count, just as they might count their own steps, hops, or jumps. In contrast, to interpret a number line correctly, we must rely on the ideas of length and distance from 0, as indicated in Figure 1.13. Instead of focusing on length, young children tend to count "tick marks" along a number line. The habit of counting tick marks instead of attending to length can lead to omitting 0 and to misinterpretations about locations of fractions on number lines. We will examine some of these errors when we study fractions in Chapter 2.

FROM THE FIELD Research

Burris, J. T. (2013). Virtual place value. *Teaching Children Mathematics, 20*(4), 228–236.

This study investigated how four third-grade classes engaged with base-ten blocks to build and identify quantities and to write corresponding numbers during a unit on place value. Two classes used virtual base-ten blocks; the other two used concrete base-ten blocks. Both the virtual and the concrete base-ten blocks could be grouped, ungrouped, and regrouped into units, tens, hundreds, and thousands. The students' reasoning was analyzed in terms of a conceptual framework of counting stages. At the most basic stage, students are able to count by ones but are unable to view a number such as 32 as 3 tens and 2 ones. At the most advanced stage, students are able to count by tens and ones and can move fluidly between ways of thinking about a number. For example, they can view the 3 in 32 both as thirty and as 3 tens. The study found that in both groups, most students were at the most advanced stage. A difference between the two groups was in how efficiently students could create equivalent representations of a number, a skill that is directly useful for reasoning about multi-digit algorithms. Students using the virtual base-ten blocks could compose and decompose numbers more readily because they reused quantities on screen to create equivalent representations. In contrast, students using concrete base-ten blocks had to trade blocks to construct equivalent representations.

Clements, D. H., Fuson, K. C., & Sarama, J. (2017). What is developmentally appropriate teaching? *Teaching Children Mathematics, 24*(3), 179–188.

This article debunks criticisms of the Common Core State Standards for Mathematics. It explains why the standards are developmentally appropriate and important for children in the early grades.

SECTION SUMMARY AND STUDY ITEMS

Section 1.1 The Counting Numbers

The counting numbers are the numbers 1, 2, 3, 4, ... There are two distinct ways to think about the counting numbers: (1) the counting numbers form an ordered list, and (2) counting numbers tell how many objects are in a set (i.e., the cardinality of a set). When we count the number of objects in a set, we connect the two views of counting numbers by making a one-to-one correspondence between an initial portion of the list of counting numbers and the objects in the set. The last number word that we say when counting a number of objects tells us how many objects there are.

In the base-ten system, every counting number can be written using only the ten symbols (or digits) 0, 1, 2, . . . , 9. The base-ten system uses place value, which means that the value a digit in a number represents depends on the location of the digit in the number. Each place has the value of a base-ten unit, and base-ten units are created by repeated bundling by ten. Each base-ten unit is the value of a place. The value of a place is ten times the value of the place to its immediate right.

Unfortunately, the way we say the English names of the counting numbers 11 through 19 does not correspond to the way we write these numbers.

The counting numbers can be displayed on number paths, which are informal precursors to number lines. The whole numbers, 0, 1, 2, 3, . . . , can be displayed on number lines.

Key Skills and Understandings

1. Describe the two views of the counting numbers—as a list and as used for cardinality. Discuss the connections between the list and cardinality views of the counting numbers.
2. Explain what it means for the base-ten system to use place value. Discuss what problem the development of the base-ten system solved.
3. Describe base-ten units and explain how adjacent place values are related in the base-ten system.
4. Describe and make math drawings to represent a given counting number in terms of bundled objects in a way that fits with the base-ten representation for that number of objects.
5. Interpret digits in the base-ten representation of a number in multiple ways (as appropriate). For example, interpret the 2 in 247 as 2 *hundreds*, as 20 *tens*, and as 200 *ones*.
6. Describe how to represent whole numbers on a number line and discuss the difference between a number path (as described in the text) and a number line.

PRACTICE EXERCISES FOR SECTION 1.1

1. If a young child can correctly say the number word list “one, two, three, four, five,” will the child necessarily be able to determine how many bears are in a collection of 5 toy bears that are lined up in a row? Discuss why or why not.

2. If a young child can correctly say the number word list “one, two, three, four, five” and point one by one to each bear in a collection of 5 toy bears while saying the number words, does the child necessarily understand that there are 5 bears in the collection? Discuss why or why not.

3. What problem in the history of mathematics did the development of the base-ten system solve?

4. Make a math drawing showing how to organize 19 objects in a way that fits with the structure of the base-ten system.

5. Describe how to organize 100 toothpicks in a way that fits with the structure of the base-ten system. Explain how your organization reflects the structure of the base-ten system and how it fits with the way we write the number 100.

6. In the number 2,789, what does the 8 mean? Explain how to interpret the 8 in two different mathematically valid ways.

7. In the number 2,789, what does the 7 mean? Explain how to interpret the 7 in three different mathematically valid ways.

8. We have seen how to express numbers in base ten by using place value and the idea of repeatedly bundling in groups of ten. We can use the same ideas to express numbers in other bases. In base ten, the units we use for the place values are ones, tens, hundreds, thousands, and so on, where each unit is 10 times the previous unit. In base two, the units we use for the place values are ones, twos, fours, and so on, where each unit is 2 times the previous unit. In general, in base *N*, each unit is *N* times the previous unit.

a. After four, what are the next two base-two units?

b. Make math drawings showing how to represent the counting numbers from one to ten in base two using bundled toothpicks.

c. Write the counting numbers from one to ten in base two, using a subscript 2 to indicate base two. For example, here are the first three: 1_2, 10_2, 11_2.

d. Suppose you had twenty-five toothpicks. Make a math drawing showing how to bundle them into base-two bundles. Use your drawing to write twenty-five in base two.

Answers to Practice Exercises for Section 1.1

1. No, the child might not be able to determine that there are 5 bears in the collection because the child might not be able to make a one-to-one correspondence between the number words 1, 2, 3, 4, 5 and the bears. For example, the child might point twice to one of the bears and count two numbers for that bear, or the child might skip over a bear while counting. See also the next practice exercise and its answer.

2. No, the child might not understand that the last number word that is said while counting the bears tells how many bears there are in the collection.

3. The base-ten system solved the problem of having to invent more and more new symbols to stand for larger and larger numbers. By using the base-ten system and place value, every counting number can be written using only the ten digits 0, 1, . . . , 9.

4. See Figure 1.5.

5. First, bundle all the toothpicks into bundles of 10. Then gather those 10 bundles of 10 into a single bundle. This repeated bundling in groups of 10 is the basis of the base-ten system. The 1 in 100 stands for this 1 large bundle of 10 bundles of 10.

6. The 8 means 8 *tens*, which we could show as 8 bundles of ten toothpicks. If we unbundled those 8 bundles, we would have 80 individual toothpicks. So we can also interpret the 8 as 80 *ones*.

7. The 7 means 7 *hundreds*, which we could show as 7 bundles of a hundred toothpicks. If we unbundled those 7 bundles of a hundred into bundles of ten, we would have 70 bundles of ten toothpicks. Therefore we can also interpret the 7 as 70 *tens*. If we unbundled those 70 bundles of ten toothpicks, we would have 700 individual toothpicks. Therefore we can also interpret the 7 as 700 *ones*.

8. **a.** Eight and sixteen because four times 2 is eight and eight times 2 is sixteen.

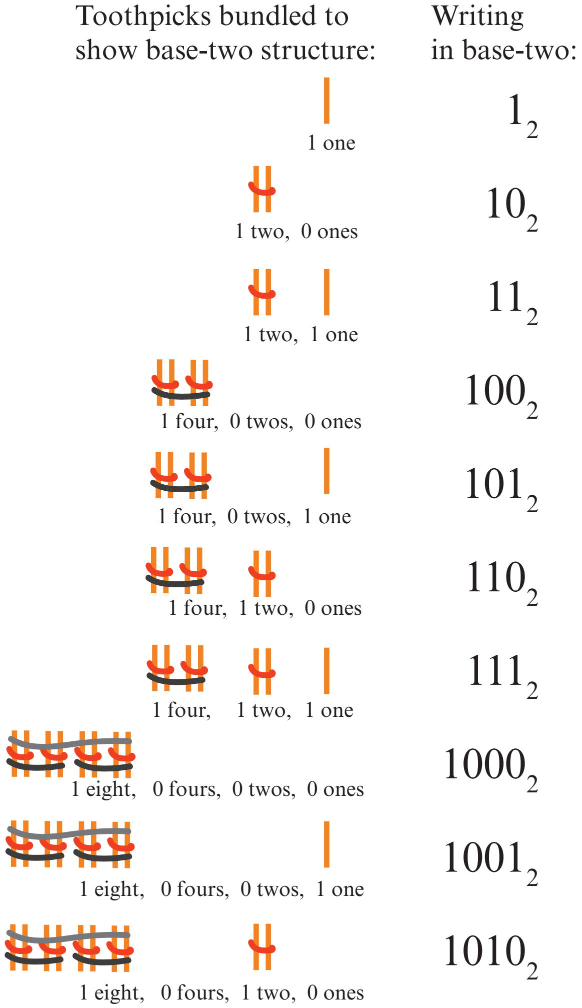


Figure 1.14 The first ten counting numbers in base two.

b. & c. See Figure 1.14.

d. Your math drawing should show 1 bundle of sixteen, 1 bundle of eight, 0 bundles of four, 0 bundles of two, and 1 individual toothpick. Therefore the base-two representation of twenty-five is 11001_2.

PROBLEMS FOR SECTION 1.1

1. In your own words, discuss the connection between the counting numbers as a list and the counting numbers as they are used to describe how many objects are in sets. Include a discussion of what you will need to attend to if you are teaching young children who are learning to count.

2. If you give a child in kindergarten or first grade a bunch of beads or other small objects and ask the child to show you what the 3 in 35 stands for, the child might show you 3 of the beads. You might respond that the 3 really stands for "thirty" and not 3. Of course it's true that the 3 stands for thirty, but is there also another way you could respond, so as to draw attention to base-ten units? How could you organize the beads to make your point?

3. For each of the following collections of small objects, make a simple math drawing and write a brief description for how to organize the objects in a way that corresponds to how we use the base-ten system to write the number for that many objects.

a. 47 beads

b. 328 toothpicks

c. 1000 toothpicks

4. Suppose you have 62 toothpicks and a bunch of rubber bands.

a. Make a simple math drawing and describe how to bundle the toothpicks to represent 62 in terms of base-ten units.

b. Using your answer in part a, explain how to interpret the 6 in 62 in *two* different mathematically valid ways.

5. Suppose you have 358 toothpicks and some rubber bands.

a. Make a simple math drawing and describe how to (repeatedly) bundle the toothpicks to represent 358 in terms of base-ten units.

b. Using your answer in part a, explain how to interpret the 5 in 358 in *two* different mathematically valid ways.

c. Using your answer in part a, explain how to interpret the 3 in 358 in *three* different mathematically valid ways.

6. In your own words, describe how you can use collections of objects (such as toothpicks or Popsicle sticks) to show the values of some base-ten units. Discuss also how the values of adjacent places in base-ten representations of numbers are related.

7. In your own words, discuss the beginning ideas of place value and the base-ten system that young children who can count beyond ten must begin to learn. Include a discussion of some of the hurdles faced by English speakers.

8. Children sometimes mistakenly read the number 1001 as "one hundred one." Why do you think a child would make such a mistake? Make a math drawing showing how to represent 1001 with bundled objects.

9. Explain why the bagged and loose toothpicks pictured in Figure 1.15 are not organized in a way

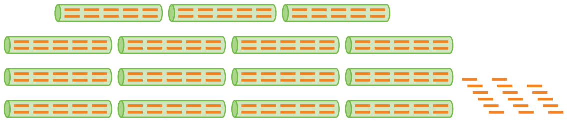


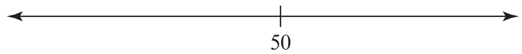
Figure 1.15

that fits with the structure of the base-ten system. Describe how to group these bagged and loose toothpicks so that the same total number of toothpicks is organized in a way that is compatible with the base-ten system.

10. Describe key features of the base-ten system. Compared to more primitive ways of writing numbers, what is one advantage and one disadvantage of the base-ten system?

***11. a.** On the number line below, can you tell where to plot 5? Explain.

b. On the number line below, can you tell where to plot 100? Explain.



c. On the number line below, can you tell where to plot $N + 1$? Explain.

***12.** Draw number lines like the ones in Figure 1.16.

a. Plot 900 on the first number line and explain your reasoning.

b. Plot 250 on the second number line and explain your reasoning.

c. Plot 6200 on the third number line and explain your reasoning.

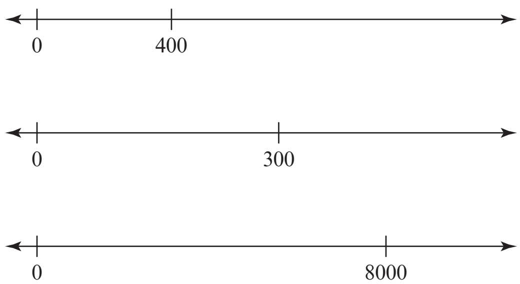


Figure 1.16

13. See Practice Exercise 8 for information about counting in base two.

a. After sixteen, what are the next two base-two units?

b. Make math drawings showing how to represent the counting numbers from eleven to twenty in base two. (Simplified drawings that show less detail than in Practice Exercise 8 are fine.)

c. Write the counting numbers from eleven to twenty in base two.

d. Suppose you had thirty-five toothpicks. Make a (simplified) math drawing showing how to bundle them into base-two bundles. Use that to write thirty-five in base two.

e. Suppose you had forty-five toothpicks. Make a (simplified) math drawing showing how to bundle them into base-two bundles. Use that to write forty-five in base two.

14. See Practice Exercise 8 and Class Activity 1-E for information about counting in bases other than base ten.

a. After ones and threes, what are the next two base-three units?

b. Make math drawings showing how to represent the counting numbers from one to thirty in base three. (Simplified drawings that show less detail than in Practice Exercise 8 are fine.)

c. Write the counting numbers from one to thirty in base three.

d. Suppose you had forty-five toothpicks. Make a (simplified) math drawing showing how to bundle them into base-three bundles. Use that to write forty-five in base three.

15. See Practice Exercise 8 and Class Activity 1-E for information about counting in bases other than base ten.

a. After ones and eights, what are the next two base-eight units?

b. Make math drawings showing how to represent the counting numbers from one to twenty in base eight. (Simplified drawings that show less detail than in Practice Exercise 8 are fine.)

c. Write the counting numbers from one to twenty in base eight.

d. Suppose you had one hundred toothpicks. Make a (simplified) math drawing showing how to bundle them into base-eight bundles. Use that to write one hundred in base eight.

***16.** The students in Ms. Caven's class have a large poster showing a million dots. Now, the students would really like to see a billion of something. Think of at least two different ways that you might attempt to show a billion of something and discuss whether your methods would be feasible. Be specific and back up your explanations with calculations.

1.2 Decimals and Negative Numbers

Typical Grade Levels: Grades 4, 5, 6

The counting numbers are the most basic kinds of numbers, followed by the whole numbers. However, many situations, both practical and theoretical, require other numbers, such as fractions, decimals, and negative numbers. Although fractions, decimals, and negative numbers may appear to be different, they become unified when they are represented on a number line.

What Are the Origins of Decimals and Negative Numbers?

Why do we have decimals (and fractions) and negative numbers? How did these numbers arise?

In ancient times, a farmer filling bags with grain might have had only enough grain to fill the last bag halfway. When trading goods, the farmer needed a way to express partial quantities. In modern times, we buy gasoline by the gallon (or liter), but we don't always buy a whole number of gallons. So we need a precise way to describe numbers that are between whole numbers. Both fractions and decimals arise by creating new units that are less than one, but decimals are created by extending the base-ten system.

The introduction of negative numbers came relatively late in human development. Although the ancient Babylonians may have had the concept of negative numbers around 2000 B.C., negative numbers were not always accepted by mathematicians, even as late as the sixteenth century A.D. ([Bel92]). The difficulty lies in interpreting the meaning of negative numbers. How can negative numbers be represented? This problem may seem perplexing at first, but in fact there are understandable interpretations of negative numbers such as temperature below zero or elevations below sea level or ground level.

How Do Decimals Extend the Base-Ten System?

The essential structure of the base-ten system is that the value of each place is ten times the value of the place to its right. So, moving to the *left* across the places in the base-ten system, the value, of the places are successively *multiplied* by 10. Likewise, moving to the *right*, the values of the places are successively *divided* by 10, as indicated in Figure 1.17.

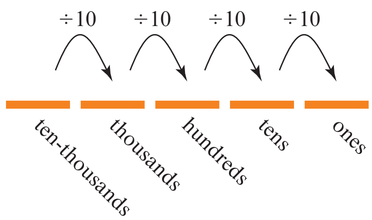


Figure 1.17 Moving to the right, the values of places in the base-ten system are divided by 10.

Using the base-ten system, decimals are created by establishing base-ten units that are smaller than one and placed to the right of the ones place. We indicate the location of the ones place by placing a **decimal point** (.) to its right. Starting at the ones place, we partition (divide) the unit *one* into 10 equal pieces to create a new base-ten unit, a *tenth*. Tenths are recorded in the place to the right of the ones place. Then we partition (divide) a tenth into 10 equal pieces to create a new base-ten unit, a *hundredth*. Hundredths are recorded in the place to the right of the tenths place. Then we partition (divide) a hundredth into 10 equal pieces to create a new base-ten units, a *thousandth*. Thousandths are recorded in the place to the right of the hundredths place. Figure 1.18 depicts the process of dividing place values by 10 to create smaller and smaller place values to the right of the ones place. This process continues without end.

When we use the base-ten system to represent a number as a string of digits, possibly including a decimal point, and possibly having infinitely many nonzero digits to the right of the decimal point, we can say the number is in **decimal notation** and call it a **decimal** or a **decimal number**. We may also refer to the string of digits representing the number as a **decimal representation** or **decimal expansion** of the number.

Just as 2345 stands for the combined amount of 2 thousands, 3 hundreds, 4 tens, and 5 ones, the decimal 2.345 stands for the combined amount of 2 ones, 3 tenths, 4 hundredths, and 5 thousandths and the decimal 23.45 stands for the combined amount of 2 tens, 3 ones, 4 tenths, and 5 hundredths.

Using the base-ten structure, we can represent (some) decimals with bundled objects in the same way that we represent whole numbers with bundled objects. The only difference is that a single object must be allowed to represent a base-ten unit that has value less than one. Although this may seem surprising at first, it is a common idea. After all, a penny represents \$0.01, which is a hundredth of a dollar. Figure 1.19 shows a way to represent decimals with bundled beads. Instead of letting 1 bead stand for *one*, we can let 1 bead stand for a *tenth*, a *hundredth*, a *thousandth* (or any other base-ten unit), and our choice will determine which decimal the collection of bundled beads stands for.

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5.NBT.1

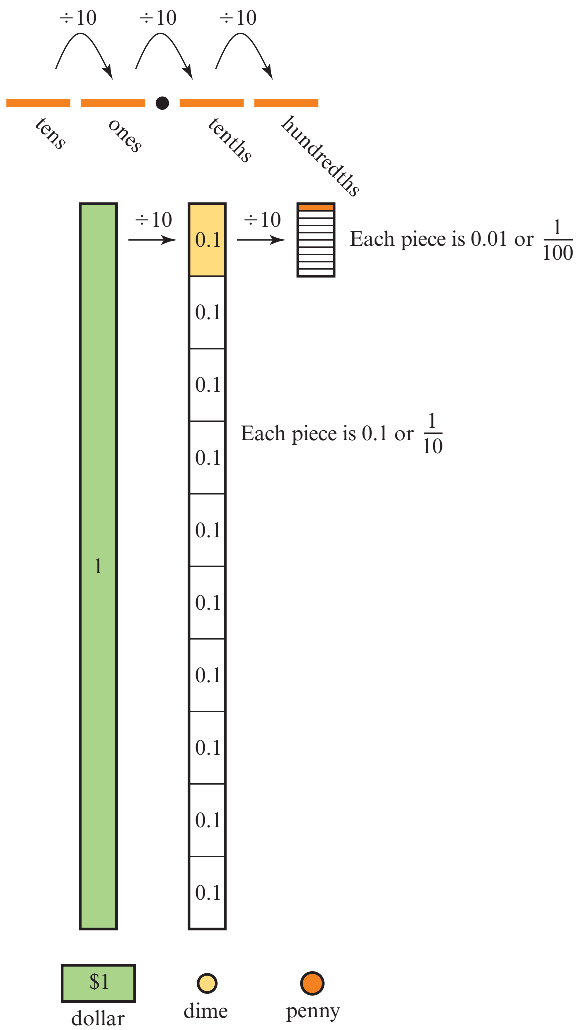
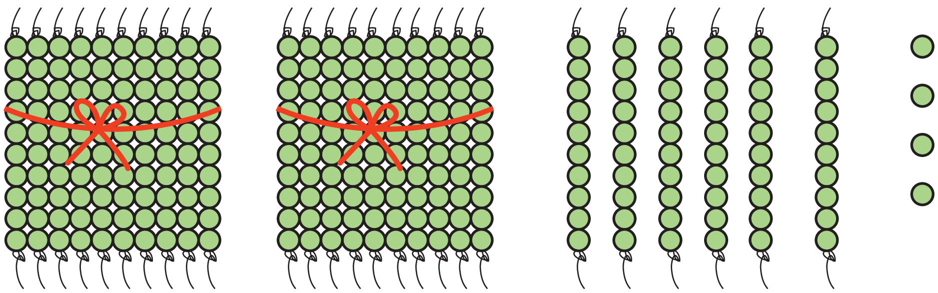


Figure 1.18



The bundled beads represent:

264	26.4	2.64	0.264
if	if	if	if
● = one	● = a tenth	● = a hundredth	● = a thousandth
2 hundreds 6 tens 4 ones	2 tens 6 ones 4 tenths	2 ones 6 tenths 4 hundredths	2 tenths 6 hundredths 4 thousandths

Figure 1.19

IMAP
Topic: Representing decimals
Video: Megan and Donna
bit.ly/3eUNjlR

CLASS ACTIVITY bit.ly/3eVPwxk

1-F Representing Decimals with Bundled Objects, p. CA-8

How Do Decimals as Lengths Develop into Decimals on Number Lines?

A good way to represent positive decimals is as lengths; this way of representing decimals leads naturally to placing decimals on number lines. The meter, which is the main unit of length used in the metric system, is a natural unit to use when representing decimals as lengths because the metric system was designed to be compatible with the base-ten system. Figure 1.20 shows 1 meter partitioned into 10 decimeters. When each decimeter is partitioned into 10 centimeters, we also see 1 meter as 100 centimeters. When each centimeter is partitioned into 10 millimeters, we also see 1 meter as 1000 millimeters.

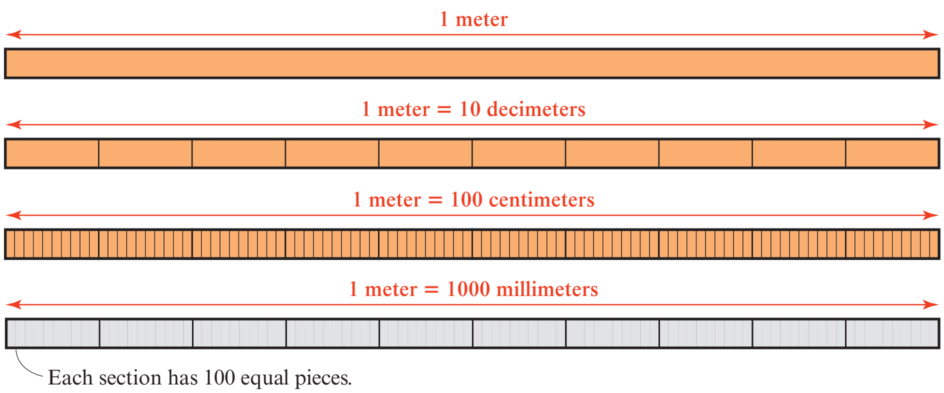


Figure 1.20

If we let 1 meter represent the base-ten unit *one*, then 1 decimeter represents a *tenth*, 1 centimeter represents a *hundredth*, and 1 millimeter represents a *thousandth* (see Figure 1.21). We can join these base-ten units end to end to represent decimals as lengths, such as in Figure 1.22, which shows the decimals 1.2, 1.23, and 1.234.

Although 1 meter is a natural unit of length to use for representing decimals, we can choose *any* length to stand for the base-ten unit *one*. When that length is partitioned into 10 equal lengths, each of those lengths stands for a *tenth*. By continuing to partition lengths into 10 equal lengths, we create *hundredths, thousandths* and so on, as before.

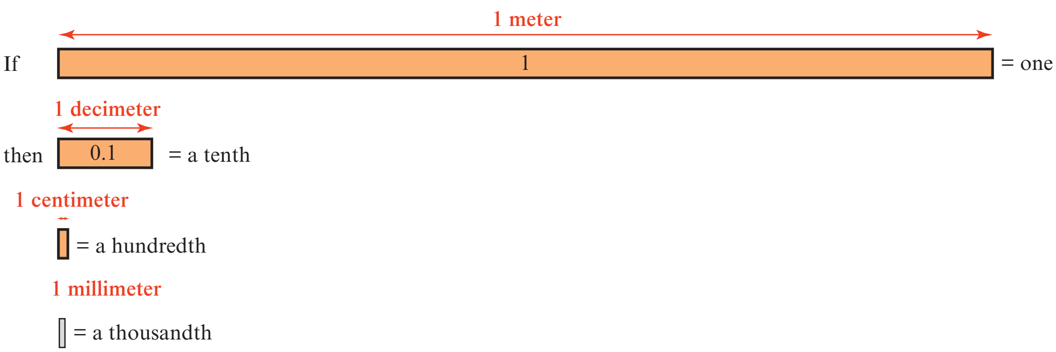


Figure 1.21 Representing base-ten units as lengths.

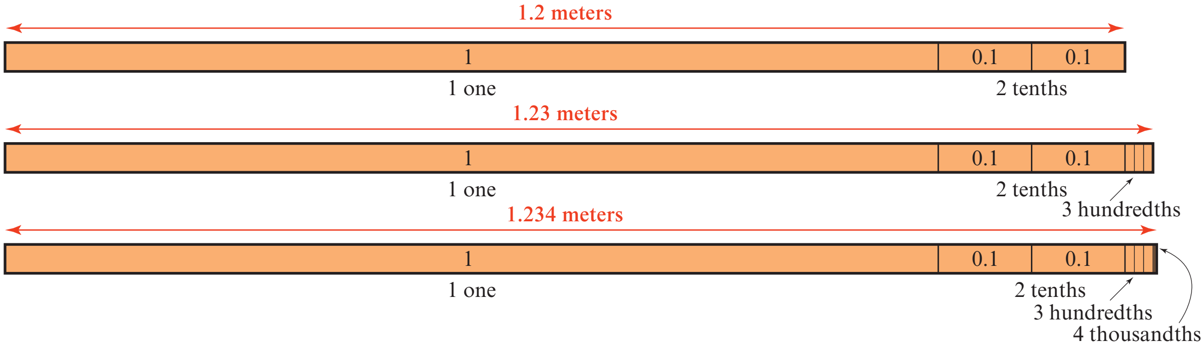


Figure 1.22 Representing decimals as lengths.

In the next Class Activity, you will use strips of paper to represent decimals as lengths. Since strips of paper are not very durable, another option is to use a more durable material such as lengths of plastic tubing (see [Ste02]).

CLASS ACTIVITY bit.ly/3eVPwxk

1-G Representing Decimals as Lengths, p. CA-11

To connect lengths with number lines, we use the same idea for decimals as we did for whole numbers in Section 1.1: Once we have chosen a location for 0 and 1 on a number line, the segment between 0 and 1 is 1 unit long and stands for the base-ten unit *one*. To plot a (positive) decimal, such as 1.234 on the number line, we imagine a strip that is 1.234 units long, so it is 1 one and 2 tenths and 3 hundredths and 4 thousandths long, as in Figure 1.23. We imagine placing this strip so its left endpoint is at 0. Then its right endpoint is where we plot the decimal 1.234. In other words, we plot 1.234, a decimal, 1.234 units to the right of 0 in the same way that we plot 13, a whole number, 13 units to the right of 0. In general, a positive number N is located to the right of 0 at a distance of N units away from 0.

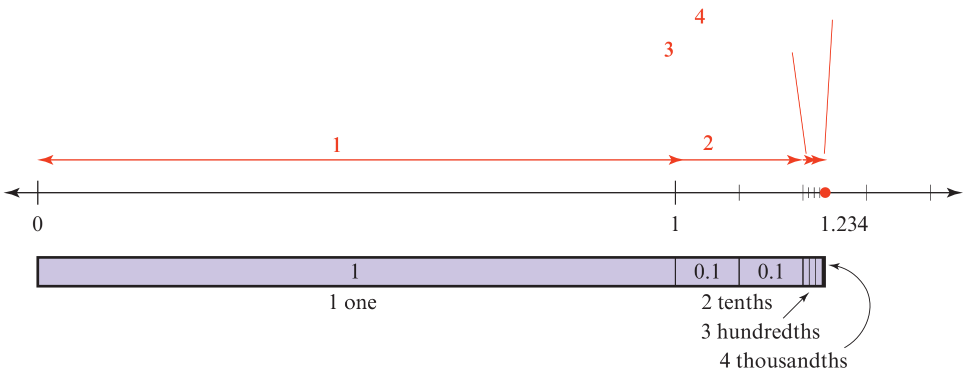


Figure 1.23 Using length and distance from 0 to plot 1.234.

How Do Decimals Fill in a Number Line?

One way to think about decimals is as "filling in" the locations on the number line between the whole numbers. You can think of plotting decimals as points on the number line in successive stages according to the structure of the base-ten system. At the first stage, the whole numbers are placed on a number line so that consecutive whole numbers are one unit apart. (See Figure 1.24).

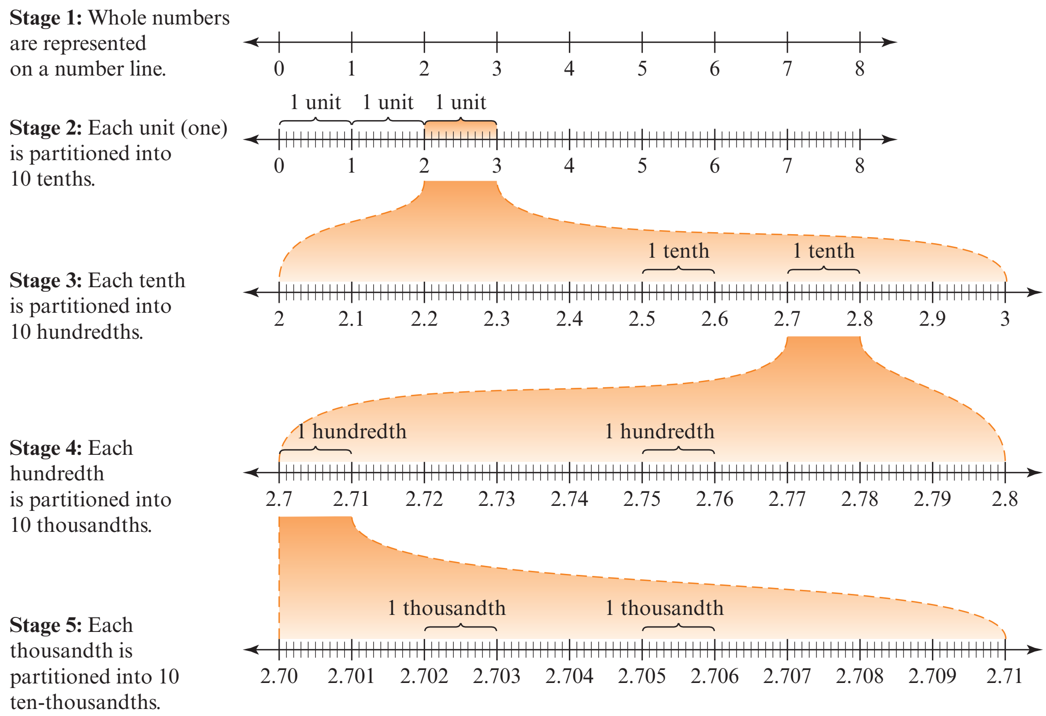


Figure 1.24 Decimals "fill in" number lines.

At the second stage, the decimals that have entries in the tenths place, but no smaller place, are spaced equally between the whole numbers, breaking each interval between consecutive whole numbers into 10 smaller intervals each one-tenth unit long. See the Stage 2 number line in Figure 1.24. Notice that, although the interval between consecutive whole numbers is broken into 10 intervals, there are only 9 tick marks for decimal numbers in the interval, one for each of the 9 nonzero entries, 1 through 9, that go in the tenths places.

We can think of the stages as continuing indefinitely. At each stage in the process of filling in the number line, we plot new decimals. The tick marks for these new decimals should be shorter than the tick marks of the decimal numbers plotted at the previous stage. We use shorter tick marks to distinguish among the stages and to show the structure of the base-ten system.

The digits in a decimal are like an address. When we read a decimal from left to right, we get more and more detailed information about where the decimal is located on a number line. The left-most digit specifies a "big neighborhood" in which the number is located. The next digit to the right narrows the location of the decimal to a smaller neighborhood of the number line. Subsequent digits to the right specify ever smaller neighborhoods in which the decimal is located, as indicated in Figure 1.25. When we read a decimal from left to right, it's almost like specifying a geographic location by giving the country, state, county, zip code, street, and street number, except that decimals can have infinitely more detailed locations.

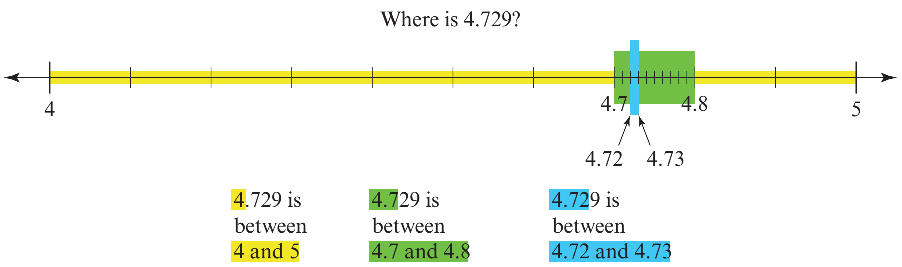


Figure 1.25 Digits to the right in a decimal describe the decimal's location on a number line with ever greater specificity.

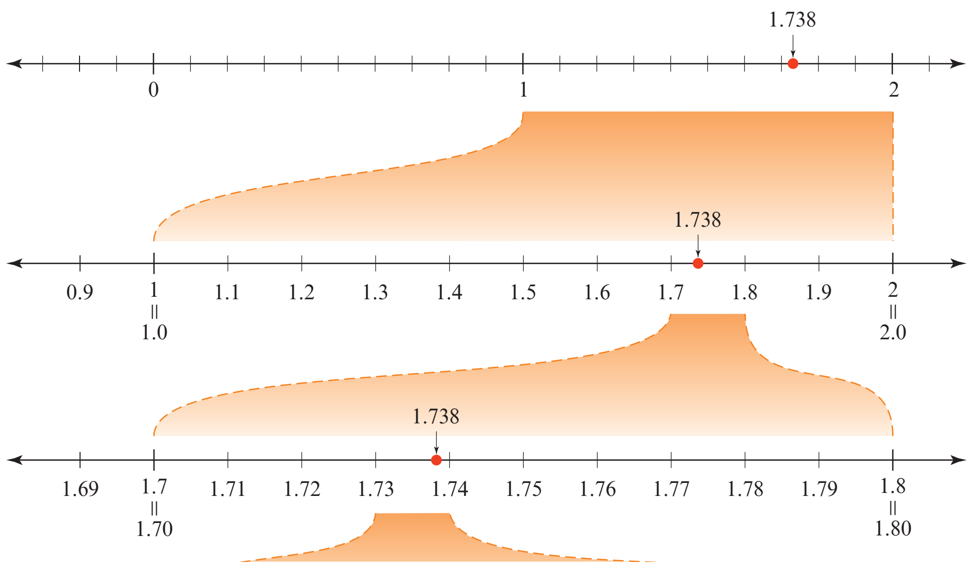


Figure 1.26 Zooming in on the location of 1.738.

By "zooming in" on narrower and narrower portions of the number line, as in **Figure 1.26**, we can see in greater detail where a decimal is located.

When plotting decimals on number lines (or when comparing, adding, or subtracting decimals), it is often useful to append zeros to the rightmost nonzero digit to express explicitly that the values in these smaller places are zero. For example, 1.78, 1.780, 1.7800, 1.78000, and so on, all stand for the same number. These representations show explicitly that the number 1.78 has 0 thousandths, 0 ten-thousandths, and 0 hundred-thousandths. Similarly, we may append zeros to the left of the leftmost nonzero digit in a number to express explicitly that the values in these larger places are zero. For example, instead of writing .58, we may write 0.58, which perhaps makes the decimal point more clearly visible.

CLASS ACTIVITY bit.ly/3eVPwxk

1-H Zooming In on Number Lines, p. CA-12

1-I Numbers Plotted on Number Lines, p. CA-14

What Is Difficult about Decimal Words?

The names for the values of the places to the right of the ones place are symmetrically related to the names of the values of the places to the left of the ones place, as shown in **Figure 1.27**.

Several common errors are associated with the place value names for decimals. One error is not distinguishing clearly between the values of places to the left and right of the decimal point. For example, students sometimes confuse tens with ten**ths** or hundreds with hundred**ths** or thousands with thousand**ths**. The pronunciation is similar, so it's easy to see how this confusion can occur! Teachers must take special care to clearly pronounce the place value names and to make sure students understand the difference. Another error occurs because students expect the symmetry in the place value names to be around the decimal point, not around the

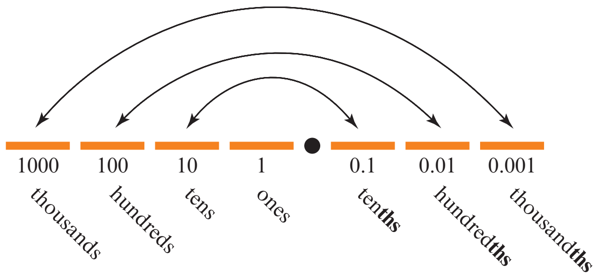


Figure 1.27 Symmetry in the place value names is around the ones place.

ones place. Some students expect there to be a "oneths place" immediately to the right of the decimal point, and they may mistakenly call the hundredths place the tenths place because of this misunderstanding.

A cultural convention is to (1) say decimals according to the value of the rightmost nonzero decimal place and (2) say "and" for the decimal point. For example, we usually say 3.84 as "three and eighty-four hundredths" because the rightmost digit is in the hundredths place. Similarly, we say 1.592 as "one and five-hundred ninety-two thousandths" because the rightmost digit is in the thousandths place. From a mathematical perspective, however, it is perfectly acceptable to say 3.84 as "3 and 8 tenths and 4 hundredths" or "three point eight four." In fact, we can't use the usual cultural conventions when saying decimals that have infinitely many digits to the right of the decimal point. For example, the number pi, which is 3.1415 . . . must be read as "three point one four one five . . ." because there is no rightmost nonzero digit in this number! Furthermore, the conventional way of saying decimals is logical, but the reason for this will not be immediately obvious to students who are just learning about decimals and place value. We will explain why when we discuss adding and subtracting fractions in Chapter 3.

What Are Negative Numbers and Where Are They on Number Lines?

For any number N, its **negative** is also a number and is denoted $-N$. The symbol $(-)$ is called a **minus sign**. For example, the negative of 4 is -4, which can be read *negative four* or *minus four*. The set of numbers consisting of 0, the counting numbers, and the negatives of the counting numbers, is called the **integers**.

$$\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots$$

CCSS 6.NS.5

We can think of a negative number, $-N$, as the "opposite" of N. Negative numbers are commonly used to denote amounts owed, temperatures below zero, and even for locations below ground or below sea level. For example, we could use -100 to represent owing 100 dollars. The temperature $-4°$ Celsius stands for 4 degrees below 0° Celsius. An altitude of -50 feet means 50 feet below sea level. In some places, negative numbers are even used to indicate floor levels. The photo on the next page shows a floor directory in a French department store. Floor 0 is ground level and Floor -1 is one flight below ground level (the basement).

On a number line, we display the negative numbers in the same way as we display 0 and the numbers greater than 0. To the right of 0 on the number line are the **positive numbers**. To the left of 0 on the number line are the **negative numbers**. The number 0 is considered neither positive nor negative.