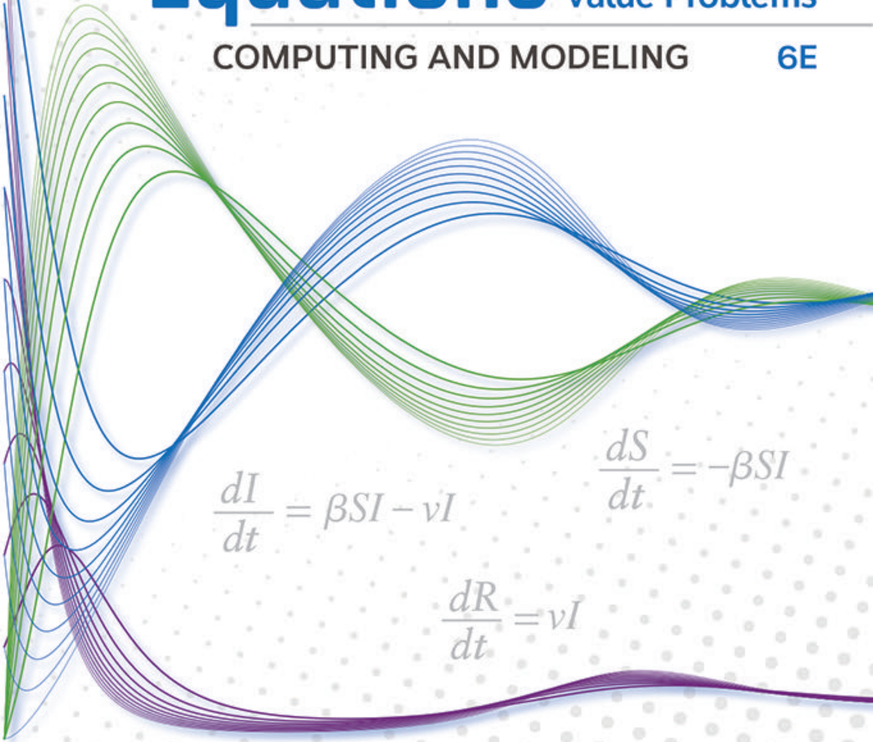
Differential Equations and Boundary Value Problems

COMPUTING AND MODELING

6E



C. Henry **Edwards** David E. **Penney** David T. **Calvis**

DIFFERENTIAL EQUATIONS AND BOUNDARY VALUE PROBLEMS

Computing and Modeling

Sixth Edition

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About the Cover

New to this edition of the text is an introduction to the *SIR* model for the spread of infectious disease. "Compartmental models" like the *SIR* model have long been used to predict the spread of various illnesses, and are currently playing an important role in the worldwide effort to combat COVID-19. Such models divide a given population into three "compartments," namely those who are Susceptible, Infected, and Recovered, while capturing the dynamics among these categories through a system of non-linear differential equations. We discuss in detail a variation of the *SIR* model called an *SIRS* model; not only is this topic timely, but the analysis of this model provides a rich showcase for the theory of autonomous systems presented in Chapter 6.

Our cover design brings together the key elements of this *SIRS* model, combining its three differential equations with a graphic portrayal (adapted from Fig. 6.4.27 in the text) of the evolution over time of the categories S, I, and R.

Library of Congress Cataloging-in-Publication Data

Names: Edwards, C. Henry (Charles Henry), author.| Penney, David E., author. | Calvis, David, author.
Title: Differential equations and boundary value problems : computing and modeling / C. Henry Edwards, David E. Penney, The University of Georgia, David Calvis, Baldwin Wallace University.
Description: Sixth edition. | Hoboken, NJ : Pearson, [2023] | Includes bibliographical references and index.
Identifiers: LCCN 2021051094 | ISBN 9780137540129 | ISBN 9780137540167
Subjects: LCSH: Differential equations.| Boundary value problems.
Classification: LCC QA371 .E28 2023
LC record available at https://lccn.loc.gov/2021051094

ScoutAutomatedPrintCode

Rental Edition
ISBN-10: 0-13-754012-4
ISBN-13: 978-0-13-754012-9

Pearson

Print Offer
ISBN-10: 0-13-754016-7
ISBN-13: 978-0-13-754016-7

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APPLICATION MODULES

The modules listed here follow the indicated sections in the text. Most provide computing projects that illustrate the content of the corresponding text sections. These projects typically provide brief segments of appropriate computer syntax at the point of student need; over time, the student develops the ability to use technology to address a wide range of problems in differential equations.

Maple, *Mathematica*, MATLAB®, and/or Python versions of these investigations are included in the website that accompanies this text as well as in MyLab Math. Within the sections of this textbook, students are provided with short URLs that link directly to the relevant online resources. Go to `bit.ly/3E5bU2W` for a page containing links to all of these online materials.

- **1.3** Computer-Generated Slope Fields and Solution Curves
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- **1.5** Indoor Temperature Oscillations
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- **7.2** Transforms of Initial Value Problems
- **7.3** Damping and Resonance Investigations
- **7.5** Engineering Functions

- **8.2** Automatic Computation of Series Coefficients
- **8.3** Automating the Frobenius Series Method
- **8.4** The Exceptional Case by Reduction of Order
- **8.6** Riccati Equations and Modified Bessel Functions

- **9.2** Computer Algebra Calculation of Fourier Coefficients
- **9.3** Fourier Series of Piecewise Smooth Functions
- **9.5** Heated-Rod Investigations
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- **10.1** Numerical Eigenfunction Expansions
- **10.2** Numerical Heat Flow Investigations
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- **10.4** Bessel Functions and Heated Cylinders

PREFACE

This is a textbook for the standard introductory differential equations course taken by science and engineering students. Its updated content reflects the wide availability of technical computing environments like *Maple*, *Mathematica*, and MATLAB that now are used extensively by practicing engineers and scientists. The traditional manual and symbolic methods are augmented with coverage also of qualitative and computer-based methods that employ numerical computation and graphical visualization to develop greater conceptual understanding. A bonus of this more comprehensive approach is accessibility to a wider range of more realistic applications of differential equations.

Principal Features of This Revision

This 6th edition is a comprehensive and wide-ranging revision.

In addition to fine-tuning the exposition (both text and graphics) in numerous sections throughout the book, new applications have been inserted (including biological), and we have exploited throughout the new interactive computer technology that is now available to students on devices ranging from desktop and laptop computers to smart phones and graphing calculators. It also utilizes computer algebra systems such as *Mathematica*, *Maple*, and MATLAB as well as online platforms such as Wolfram|Alpha and GeoGebra.

There have been additions to content throughout the text, including an expanded Application Module for Section 6.4 to discuss COVID-19. However, the class tested table of contents of the book remains unchanged. Therefore, instructors' notes and syllabi will not require revision to continue teaching with this new edition.

A conspicuous feature of this edition is the insertion of about 16 new Interactive Figures, which illustrate how interactive computer applications with slider bars or touchpad controls can be used to change initial values or parameters in a differential equation, allowing the user to immediately see in real time the resulting changes in the structure of its solutions.

Some illustrations of the various types of revision and updating exhibited in this edition:

New Interactive Technology and Graphics New figures inserted throughout illustrate the facility offered by modern computing technology platforms for the user to interactively vary initial conditions and other parameters in real time. Thus, using a mouse or touchpad, the initial point for an initial value problem can be dragged to a new location, and the corresponding solution curve is automatically redrawn and dragged along with its initial point. For instance, see Figure 1.3.5 (page 20) and Figure 3.2.4 (page 154). Using slider bars in an interactive graphic, the coefficients or other parameters in a linear system can be varied, and the corresponding changes in its direction field and phase plane portrait are automatically shown; for instance see Figure 5.3.21 in the Application for Section 5.3 (page 324). The number of terms used in the partial sum of the Fourier series of a square-wave function can be varied, and the resulting graphical change demonstrating Gibbs's phenomenon is shown immediately; see Figure 9.1.3 (page 587).

New Exposition In a number of sections, new text and graphics have been inserted to enhance student understanding of the subject matter. For instance, see the revised approach to Problems 1 through 4 in Section 1.3 (page 25), the enhanced discussion in the Summary that concludes Chapter 1 (page 73), the historical note pointing to the accomplishments of Katherine Johnson following Application 4.1 (page 242), the many improvements in the presentation of the Lorenz attractor and supporting graphics (page 446), and the improved presentation of the

Application Modules throughout the text. Examples and accompanying graphics have been updated throughout to illustrate current technology; this includes the incorporation of Python as a standard programming platform.

New Content Continuing our recent trend, this edition features a new application of differential equations to the life sciences. In addition to the FitzHugh-Nagumo equations of neuroscience, the Application Module following Section 6.4 now also includes an introduction to the use of differential equations in epidemiology, specifically the *SIR* model widely used in forecasting models today. We explain the early history of the use of differential equations to model the spread of disease and then present the *SIR* model itself, together with its underlying rationale. As a case study, we analyze in detail a related model that takes account of possible reinfection. The phase analysis of this variant of the *SIR* model can be done in the plane, rather than three dimensions, allowing the use of tools developed for nonlinear autonomous systems earlier in the chapter. The modeling of the spread of disease is of course a subject of great contemporary interest in itself. However, this new treatment also reinforces the utility of differential equations throughout the sciences, not just in traditional fields like physics and engineering. Characterized by the same careful and thorough exposition found throughout the text as a whole, this new unit will provide the student with yet another lens through which to view the subject of differential equations.

New Look This text may look notably different than previous editions because of the introduction of color! The addition of multiple colors to the text has allowed us to enhance graphs and figures so that students can more easily discern different solutions in the figures. Marginal notes have been added to give additional help in understanding the mathematics done in the text. Application topics can now be identified in the exercise set with new run-in problem titles. Finally, new headers in the Application Modules now make it clear where the author's exposition ends and the student's investigation begins; look for the Your Turn headers.

Computing Features

The following features highlight the computing technology that distinguishes much of our exposition.

- Over 750 *computer-generated figures* show students vivid pictures of direction fields, solution curves, and phase plane portraits that bring symbolic solutions of differential equations to life.
- About 44 *application modules* follow key sections throughout the text. Most of these applications outline "technology neutral" investigations illustrating the use of technical computing systems and seek to actively engage students in the application of new technology.
- A fresh *numerical emphasis* that is afforded by the early introduction of numerical solution techniques in Chapter 2 (on mathematical models and numerical methods). Here and in Chapter 4, where numerical techniques for systems are treated, a concrete and tangible flavor is achieved by the inclusion of numerical algorithms presented in parallel fashion for systems ranging from graphing calculators to MATLAB.

Modeling Features

Mathematical modeling is a goal and constant motivation for the study of differential equations. To sample the range of applications in this text, take a look at the following questions:

- What explains the commonly observed time lag between indoor and outdoor daily temperature oscillations? (Section 1.5)
- What makes the difference between doomsday and extinction in alligator populations? (Section 2.1)

- How do a unicycle and a twoaxle car react differently to road bumps? (Sections 3.7 and 5.4)
- How can you predict the time of next perihelion passage of a newly observed comet? (Section 4.3)
- Why might an earthquake demolish one building and leave standing the one next door? (Section 5.4)
- What determines whether two species will live harmoniously together, or whether competition will result in the extinction of one of them and the survival of the other? (Section 6.3)
- How can differential equations be used to predict the spread of a disease, or to develop a strategy for "flattening the curve" of its infected population? (Application Module 6.4)
- Why and when does non-linearity lead to chaos in biological and mechanical systems? (Section 6.5)
- If a mass on a spring is periodically struck with a hammer, how does the behavior of the mass depend on the frequency of the hammer blows? (Section 7.6)
- Why are flagpoles hollow instead of solid? (Section 8.6)
- What explains the difference in the sounds of a guitar, a xylophone, and drum? (Sections 9.6, 10.2, and 10.4)

Organization and Content

We have reshaped the usual approach and sequence of topics to accommodate new technology and new perspectives. For instance:

- After a precis of first-order equations in Chapter 1 (though with the coverage of certain traditional symbolic methods streamlined a bit), Chapter 2 offers an early introduction to mathematical modeling, stability and qualitative properties of differential equations, and numerical methods—a combination of topics that frequently are dispersed later in an introductory course. Chapter 3 includes the standard methods of solution of linear differential equations of higher order, particularly those with constant coefficients, and provides an especially wide range of applications involving simple mechanical systems and electrical circuits; the chapter ends with an elementary treatment of endpoint problems and eigenvalues.
- Chapters 4 and 5 provide a flexible treatment of linear systems. Motivated by current trends in science and engineering education and practice, Chapter 4 offers an early, intuitive introduction to first-order systems, models, and numerical approximation techniques. Chapter 5 begins with a self-contained treatment of the linear algebra that is needed, and then presents the eigenvalue approach to linear systems. It includes a wide range of applications (ranging from railway cars to earthquakes) of all the various cases of the eigenvalue method. Section 5.5 includes a fairly extensive treatment of matrix exponentials, which are exploited in Section 5.6 on nonhomogeneous linear systems.
- Chapter 6 on nonlinear systems and phenomena ranges from phase plane analysis to ecological and mechanical systems to a concluding section on chaos and bifurcation in dynamical systems. Section 6.5 presents an elementary introduction to such contemporary topics as period-doubling in biological and mechanical systems, the pitchfork diagram, and the Lorenz strange attractor (all illustrated with vivid computer graphics).
- Laplace transform methods (Chapter 7) and power series methods (Chapter 8) follow the material on linear and nonlinear systems, but can be covered at any earlier point (after Chapter 3) the instructor desires.
- Chapters 9 and 10 treat the applications of Fourier series, separation of variables, and Sturm-Liouville theory to partial differential equations and boundary value problems.

After the introduction of Fourier series, the three classical equations—the wave and heat equations and Laplace's equation—are discussed in the last three sections of Chapter 9. The eigenvalue methods of Chapter 10 are developed sufficiently to include some rather significant and realistic applications.

This book includes enough material appropriately arranged for different courses varying in length from one quarter to two semesters. Many courses choose to omit chapters 8, 9, and 10 (the chapters on Boundary Values Problems), but all the content is included in this one version of the text now.

Student and Instructor Resources

The answer section has been expanded considerably to increase its value as a learn-ing aid. It now includes the answers to most odd-numbered problems plus a good many even-numbered ones. The **Instructor's Solutions Manual** (0-13-754027-2), available at www.pearson.com and within MyLab Math, provides worked-out solutions for most of the problems in the book, and the **Student Solutions Manual** (0-13-754031-0) contains solutions for most of the odd-numbered problems.

The effectiveness of the 44 application modules located throughout the text is greatly enhanced by the material at the new Expanded Applications website. Nearly all of the application modules in the text are marked with and a unique short URL—a web address that leads directly to an Expanded Applications page containing a wealth of resources supporting that module. Typical Expanded Applications materials include an enhanced and expanded PDF version of the text with further discussion or additional applications, together with computer files in a variety of platforms, including *Mathematica*, *Maple*, MATLAB, and in some cases Python and/or TI calculator. These files provide all code appearing in the text as well as equivalent versions in other platforms, allowing students to immediately use the material in the Application Module on the computing platform of their choice. In addition to the URLs dispersed throughout the text, the Expanded Applications can be accessed via this homepage: bit.ly/3E5bU2W.

MyLab Math Resources for Success

MyLab Math is available to accompany Pearson's market-leading text options, including **Edwards' *Differential Equations and Boundary Value Problems: Computing and Modeling*, 6th Edition** (access code required).

MyLab™ is the teaching and learning platform that empowers you to reach every student. MyLab Math combines trusted author content—including full eText and assessment with immediate feedback—with digital tools and a flexible platform to personalize the learning experience and improve results for each student.

MyLab Math supports all learners, regardless of their ability and background, to provide an equal opportunity for success. Accessible resources support learners for a more equitable experience no matter their abilities. And options to personalize learning and address individual gaps helps to provide each learner with the specific resources they need to achieve success.

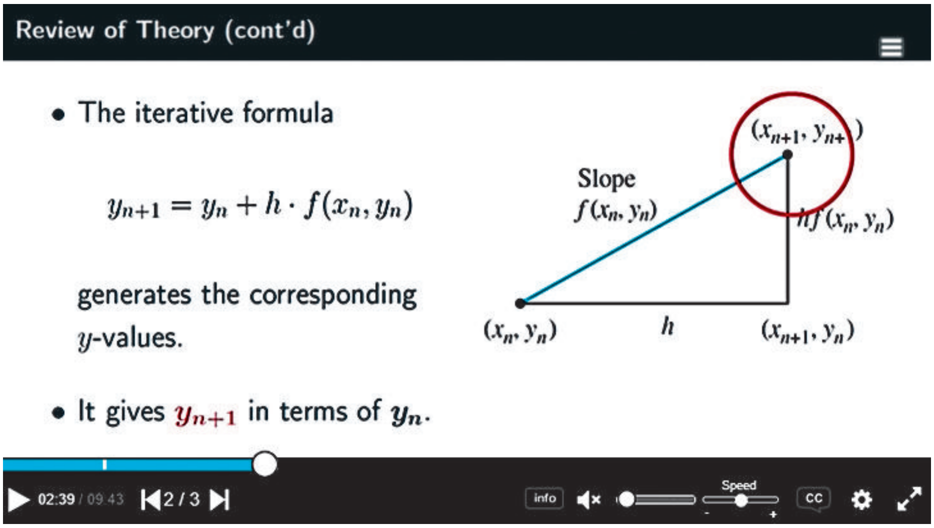
Student Resources

Pearson eText is a simple-to-use, mobile-optimized, personalized reading experience available within MyLab Math. It lest student highlight, take notes and create flashcards all in one place, even when offline. Seamlessly integrated videos bring concepts to life.

More! Exercises with immediate feedback— Over 1000 assignable exercises are based on the textbook exercises, and regenerate algorithmically to give students unlimited opportunity for practice and mastery. MyLab Math provides helpful feedback when students enter incorrect answers and includes optional learning aids including Help Me Solve This, View an Example, videos, and an eText.

- **New! Set-up & Solve Exercises** require students to first describe how they will set up and approach the problem. This reinforces conceptual understanding of the process applied in approaching the problem, promotes long-term retention of the skill, and mirrors what students will be expected to do on a test.

Instructional videos provide meaningful support to students as a learning aid within exercises, alongside key examples in the eText or for self-study within the Video & Resource Library. Instructors can assign videos within MyLab homework; use videos in class or provide as a supplementary resource on specific topics.



- **Solution Manual** – The Student's Solution Manual provides detailed worked-out solutions to most of the odd-numbered exercises in Edwards' Differential Equations and Boundary Value Problems: Computing and Modeling. Available in MyLab Math.

Instructor Resources

Your course is unique. So whether you'd like to build your own assignments, teach multiple sections, or set prerequisites, MyLab gives you the flexibility to easily create your course to fit your needs.

Address gaps in prerequisite skills with the assignable **Additional Review for Differential Equations** chapter, which contains support for students with just-in-time remediation of key calculus and precalculus objective and exercises, to ensure they are adequately prepared with the prerequisite skills needed to successfully complete their course work.

Personalized Homework - With Personalized Homework, students take a quiz or test and receive a subsequent homework assignment that is personalized based on their performance. This way, students can focus on just the topics they have not yet mastered.

Learning Catalytics helps instructors generate class discussion, customize lectures, and promote peer-to-peer learning with real-time analytics. As a student response tool, Learning Catalytics uses students' smartphones, tablets, or laptops to engage them in more interactive tasks and thinking.

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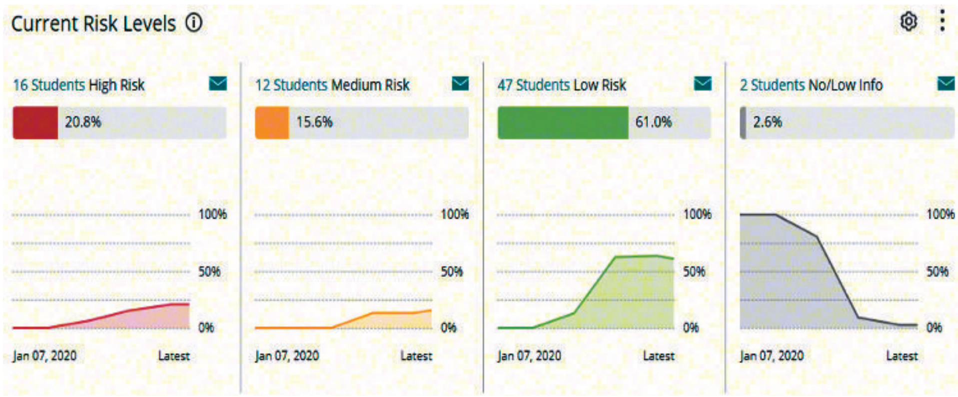
Performance Analytics enable instructors to see and analyze student performance across multiple courses. Based on their current course progress, individuals' performance is identified above, at, or below expectations through a variety of graphs and visualizations.

Now included with Performance Analytics, **Early Alerts** use predictive analytics to identify struggling students — even if their assignment scores are not a cause for concern. In both Performance Analytics and Early Alerts, instructors can email students individually or by group to provide feedback.



Accessibility and achievement go hand in hand. MyLab Math is compatible with the JAWS screen reader, and enables multiple-choice and free-response problem types to be read and interacted with via keyboard controls and math notation input. MyLab Math also works with screen enlargers, including ZoomText, MAGic, and SuperNova. And, all MyLab Math videos have closed-captioning. More information is available at `http://mymathlab.com/accessibility`.

Other instructor resources include:

- **Instructor Solution Manual** – The Instructor's Solutions Manual, available at `www.pearson.com` and within MyLab Math, provides worked-out solutions for most of the problems in the book.
- **Presentation slides** created by author David Calvis available in LaTeX (Beamer) and PDF formats. The slides are ideal for both classroom lecture and student review and combined with Calvis' superlative videos offer a level of support not found in any other Differential Equations course.
- **44 Application Modules –** Follow key sections throughout the text, and actively engage students, most providing computing projects that illustrate the content of the corresponding text sections.
  - Typical materials include an expanded PDF version of the text with further discussion or additional applications, with files in a variety of platforms including *Mathematica, Maple,* and MATLAB.
  - These projects provide brief segments of appropriate computer syntax at the point of student need; over time, the student develops the ability to use technology to address a wide range of problems in differential equations.
  - Students can access the module resources through MyLab Math or directly at `bit.ly/3E5bU2W`.

Learn more at `pearson.com/mylab/math`

Acknowledgments

In preparing this revision, we profited greatly from the advice and assistance of the following very capable and perceptive reviewers. Those who assisted with this sixth edition have an asterisk before their names.

Anthony Aidoo, *Eastern Connecticut State University*
Brent Solie, *Knox College*
Elizabeth Bradley, *University of Louisville*
*Min Chen, *Purdue University*
Gregory Davis, *University of Wisconsin-Green Bay*
Zoran Grujic, *University of Virginia*
Richard Jardine, *Keene State College*
Yang Kuang, *Arizona State University*
Dening Li, *West Virginia University*
*John Lind, *California State University, Chico*
Francisco Sayas-Gonzalez, *University of Delaware*
*Curtis White, *Lee College*
Luther White, *University of Oklahoma*
Hong-Ming Yin, *Washington State University*
Morteza Shafii-Mousavi, *Indiana University-South Bend*

It is a pleasure to (once again) credit Dennis Kletzing and his extraordinary TEXpertise for the attractive presentation of the text and the art in this book. We are grateful to our editor, Brian Fisher, for his support and inspiration of this revision; to Erin Carreiro and Kristina Evans for coordination of the technology tools and content for the text; to Devanshi Tyagi for her supervision of the production of this book; to Straive for their attractive cover design.

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1 First-Order Differential Equations

1.1 Differential Equations and Mathematical Models

The laws of the universe are written in the language of mathematics. Algebra is sufficient to solve many static problems, but the most interesting natural phenomena involve change and are described by equations that relate changing quantities.

Because the derivative $dx/dt = f'(t)$ of the function f is the rate at which the quantity $x = f(t)$ is changing with respect to the independent variable t, it is natural that equations involving derivatives are frequently used to describe the changing universe. An equation relating an unknown function and one or more of its derivatives is called a **differential equation**.

Example 1 The differential equation

$$\frac{dx}{dt} = x^2 + t^2$$

involves both the unknown function $x(t)$ and its first derivative $x'(t) = dx/dt$. The differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 7y = 0$$

involves the unknown function y of the independent variable x and the first two derivatives y' and y'' of y. ■

The study of differential equations has three principal goals:

1. To discover the differential equation that describes a specified physical situation.
2. To find—either exactly or approximately—the appropriate solution of that equation.
3. To interpret the solution that is found.

In algebra, we typically seek the unknown *numbers* that satisfy an equation such as $x^3 + 7x^2 - 11x + 41 = 0$. By contrast, in solving a differential equation, we are challenged to find the unknown *functions* $y = y(x)$ for which an identity such as $y'(x) = 2xy(x)$—that is, the differential equation

$$\frac{dy}{dx} = 2xy$$

—holds on some interval of real numbers. Ordinarily, we will want to find *all* solutions of the differential equation, if possible.

Example 2 If C is a constant and

$$y(x) = Ce^{x^2}, \tag{1}$$

then

$$\frac{dy}{dx} = C\left(2xe^{x^2}\right) = (2x)\left(Ce^{x^2}\right) = 2xy.$$

Thus every function $y(x)$ of the form in Eq. (1) *satisfies*—and thus is a solution of—the differential equation

$$\frac{dy}{dx} = 2xy \tag{2}$$

for all x. In particular, Eq. (1) defines an *infinite* family of different solutions of this differential equation, one for each choice of the arbitrary constant C. By the method of separation of variables (Section 1.4) it can be shown that every solution of the differential equation in (2) is of the form in Eq. (1). ■

Differential Equations and Mathematical Models

The following three examples illustrate the process of translating scientific laws and principles into differential equations. In each of these examples the independent variable is time t, but we will see numerous examples in which some quantity other than time is the independent variable.

Example 3 **Rate of cooling** Newton's law of cooling may be stated in this way: The *time rate of change* (the rate of change with respect to time t) of the temperature $T(t)$ of a body is proportional to the difference between T and the temperature A of the surrounding medium (Fig. 1.1.1). That is,

$$\frac{dT}{dt} = -k(T - A), \tag{3}$$

where k is a positive constant. Observe that if $T > A$, then $dT/dt < 0$, so the temperature is a decreasing function of t and the body is cooling. But if $T < A$, then $dT/dt > 0$, so that T is increasing.

Thus the physical law is translated into a differential equation. If we are given the values of k and A, we should be able to find an explicit formula for $T(t)$, and then—with the aid of this formula—we can predict the future temperature of the body. ■

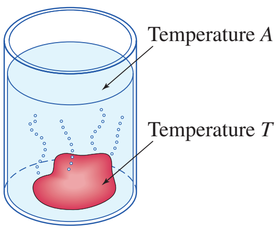

FIGURE 1.1.1. Newton's law of cooling, Eq. (3), describes the cooling of a hot rock in water.

Example 4 **Draining tank** Torricelli's law implies that the *time rate of change* of the volume V of water in a draining tank (Fig. 1.1.2) is proportional to the square root of the depth y of water in the tank:

$$\frac{dV}{dt} = -k\sqrt{y}, \tag{4}$$

where k is a constant. If the tank is a cylinder with vertical sides and cross-sectional area A, then $V = Ay$, so $dV/dt = A \cdot (dy/dt)$. In this case Eq. (4) takes the form

$$\frac{dy}{dt} = -h\sqrt{y}, \tag{5}$$

where $h = k/A$ is a constant. ■

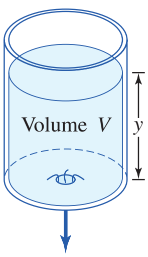

FIGURE 1.1.2. Torricelli's law of draining, Eq. (4), describes the draining of a water tank.

Example 5

Population growth The *time rate of change* of a population $P(t)$ with constant birth and death rates is, in many simple cases, proportional to the size of the population. That is,

$$\frac{dP}{dt} = kP, \tag{6}$$

where k is the constant of proportionality. ■

Let us discuss Example 5 further. Note first that each function of the form

$$P(t) = Ce^{kt} \tag{7}$$

is a solution of the differential equation

$$\frac{dP}{dt} = kP$$

in (6). We verify this assertion as follows:

$$P'(t) = Cke^{kt} = k\left(Ce^{kt}\right) = kP(t)$$

for all real numbers t. Because substitution of each function of the form given in (7) into Eq. (6) produces an identity, all such functions are solutions of Eq. (6).

Thus, even if the value of the constant k is known, the differential equation $dP/dt = kP$ has *infinitely many* different solutions of the form $P(t) = Ce^{kt}$, one for each choice of the "arbitrary" constant C. This is typical of differential equations. It is also fortunate, because it may allow us to use additional information to select from among all these solutions a particular one that fits the situation under study.

Example 6

Population growth Suppose that $P(t) = Ce^{kt}$ is the population of a colony of bacteria at time t, that the population at time $t = 0$ (hours, h) was 1000, and that the population doubled after 1 h. This additional information about $P(t)$ yields the following equations:

$$1000 = P(0) = Ce^0 = C,$$

$$2000 = P(1) = Ce^k.$$

It follows that $C = 1000$ and that $e^k = 2$, so $k = \ln 2 \approx 0.693147$. With this value of k the differential equation in (6) is

$$\frac{dP}{dt} = (\ln 2)P \approx (0.693147)P.$$

Substitution of $k = \ln 2$ and $C = 1000$ in Eq. (7) yields the particular solution

$$P(t) = 1000e^{(\ln 2)t} = 1000(e^{\ln 2})^t = 1000 \cdot 2^t \qquad (\text{because } e^{\ln 2} = 2)$$

that satisfies the given conditions. We can use this particular solution to predict future populations of the bacteria colony. For instance, the predicted number of bacteria in the population after one and a half hours (when $t = 1.5$) is

$$P(1.5) = 1000 \cdot 2^{3/2} \approx 2828.$$ ■

The condition $P(0) = 1000$ in Example 6 is called an **initial condition** because we frequently write differential equations for which $t = 0$ is the "starting time." Figure 1.1.3 shows several different graphs of the form $P(t) = Ce^{kt}$ with $k = \ln 2$. The graphs of all the infinitely many solutions of $dP/dt = kP$ in fact fill the entire two-dimensional plane, and no two intersect. Moreover, the selection of any one point P_0 on the P-axis amounts to a determination of $P(0)$. Because exactly one solution passes through each such point, we see in this case that an initial condition $P(0) = P_0$ determines a unique solution agreeing with the given data.

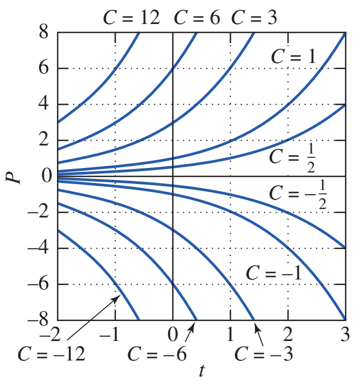

FIGURE 1.1.3. Graphs of $P(t) = Ce^{kt}$ with $k = \ln 2$.

Mathematical Models

Our brief discussion of population growth in Examples 5 and 6 illustrates the crucial process of *mathematical modeling* (Fig. 1.1.4), which involves the following:

1. The formulation of a real-world problem in mathematical terms; that is, the construction of a mathematical model.
2. The analysis or solution of the resulting mathematical problem.
3. The interpretation of the mathematical results in the context of the original real-world situation—for example, answering the question originally posed.

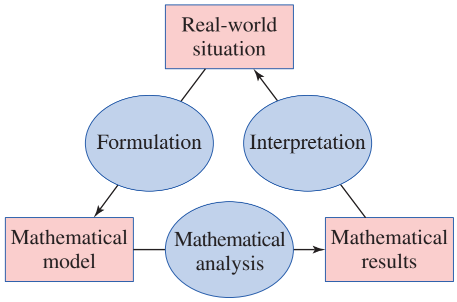

FIGURE 1.1.4. The process of mathematical modeling.

In the population example, the real-world problem is that of determining the population at some future time. A **mathematical model** consists of a list of variables (P and t) that describe the given situation, together with one or more equations relating these variables ($dP/dt = kP$, $P(0) = P_0$) that are known or are assumed to hold. The mathematical analysis consists of solving these equations (here, for P as a function of t). Finally, we apply these mathematical results to attempt to answer the original real-world question.

As an example of this process, think of first formulating the mathematical model consisting of the equations $dP/dt = kP$, $P(0) = 1000$, describing the bacteria population of Example 6. Then our mathematical analysis there consisted of solving for the solution function $P(t) = 1000e^{(\ln 2)t} = 1000 \cdot 2^t$ as our mathematical result. For an interpretation in terms of our real-world situation—the actual bacteria population—we substituted $t = 1.5$ to obtain the predicted population of $P(1.5) \approx 2828$ bacteria after 1.5 hours. If, for instance, the bacteria population is growing under ideal conditions of unlimited space and food supply, our prediction may be quite accurate, in which case we conclude that the mathematical model is adequate for studying this particular population.

On the other hand, it may turn out that no solution of the selected differential equation accurately fits the actual population we're studying. For instance, for *no* choice of the constants C and k does the solution $P(t) = Ce^{kt}$ in Eq. (7) accurately describe the actual growth of the human population of the world over the past few centuries. We must conclude that the differential equation $dP/dt = kP$ is inadequate for modeling the world population—which in recent decades has "leveled off" as compared with the steeply climbing graphs in the upper half ($P > 0$) of Fig. 1.1.3. With sufficient insight, we might formulate a new mathematical model including a perhaps more complicated differential equation, one that takes into account such factors as a limited food supply and the effect of increased population on birth and death rates. With the formulation of this new mathematical model, we may attempt to traverse once again the diagram of Fig. 1.1.4 in a counterclockwise manner. If we can solve the new differential equation, we get new solution functions to compare with the real-world population. Indeed, a successful population analysis may require refining the mathematical model still further as it is repeatedly measured against real-world experience.

But in Example 6 we simply ignored any complicating factors that might affect our bacteria population. This made the mathematical analysis quite simple, perhaps unrealistically so. A satisfactory mathematical model is subject to two contradictory requirements: It must be sufficiently detailed to represent the real-world situation with relative accuracy, yet it must be sufficiently simple to make the mathematical analysis practical. If the model is so detailed that it fully represents the physical situation, then the mathematical analysis may be too difficult to carry out. If the model is too simple, the results may be so inaccurate as to be useless. Thus there is an inevitable tradeoff between what is physically realistic and what is mathematically possible. The construction of a model that adequately bridges this gap between realism and feasibility is therefore the most crucial and delicate step in the process. Ways must be found to simplify the model mathematically without sacrificing essential features of the real-world situation.

Mathematical models are discussed throughout this book. The remainder of this introductory section is devoted to simple examples and to standard terminology used in discussing differential equations and their solutions.

Examples and Terminology

Example 7 If C is a constant and $y(x) = 1/(C - x)$, then

$$\frac{dy}{dx} = \frac{1}{(C-x)^2} = y^2$$

if $x \neq C$. Thus

$$y(x) = \frac{1}{C-x} \tag{8}$$

defines a solution of the differential equation

$$\frac{dy}{dx} = y^2 \tag{9}$$

on any interval of real numbers not containing the point $x = C$. Actually, Eq. (8) defines a *one-parameter family* of solutions of $dy/dx = y^2$, one for each value of the arbitrary constant or "parameter" C. With $C = 1$ we get the particular solution

$$y(x) = \frac{1}{1-x}$$

that satisfies the initial condition $y(0) = 1$. As indicated in Fig. 1.1.5, this solution is continuous on the interval $(-\infty, 1)$ but has a vertical asymptote at $x = 1$. ■

Example 8 Verify that the function $y(x) = 2x^{1/2} - x^{1/2}\ln x$ satisfies the differential equation

$$4x^2y'' + y = 0 \tag{10}$$

for all $x > 0$.

Solution First we compute the derivatives

$$y'(x) = -\tfrac{1}{2}x^{-1/2}\ln x \quad \text{and} \quad y''(x) = \tfrac{1}{4}x^{-3/2}\ln x - \tfrac{1}{2}x^{-3/2}.$$

Then substitution into Eq. (10) yields

$$4x^2y'' + y = 4x^2\left(\tfrac{1}{4}x^{-3/2}\ln x - \tfrac{1}{2}x^{-3/2}\right) + 2x^{1/2} - x^{1/2}\ln x = 0$$

if x is positive, so the differential equation is satisfied for all $x > 0$. ■

The fact that we can write a differential equation is not enough to guarantee that it has a solution. For example, it is clear that the differential equation

$$(y')^2 + y^2 = -1 \tag{11}$$

has *no* (real-valued) solution, because the sum of nonnegative numbers cannot be negative. For a variation on this theme, note that the equation

$$(y')^2 + y^2 = 0 \tag{12}$$

obviously has only the (real-valued) solution $y(x) \equiv 0$. In our previous examples any differential equation having at least one solution indeed had infinitely many.

The **order** of a differential equation is the order of the highest derivative that appears in it. The differential equation of Example 8 is of second order, those in Examples 2 through 7 are first-order equations, and

$$y^{(4)} + x^2 y^{(3)} + x^5 y = \sin x$$

is a fourth-order equation. The most general form of an ***n*th-order** differential equation with independent variable x and unknown function or dependent variable $y = y(x)$ is

$$F\left(x, y, y', y'', \dots, y^{(n)}\right) = 0, \tag{13}$$

where F is a specific real-valued function of $n + 2$ variables.

Our use of the word *solution* has been until now somewhat informal. To be precise, we say that the continuous function $u = u(x)$ is a **solution** of the differential equation in (13) **on the interval** I provided that the derivatives $u', u'', \dots, u^{(n)}$ exist on I and

$$F\left(x, u, u', u'', \dots, u^{(n)}\right) = 0$$

for all x in I. For the sake of brevity, we may say that $u = u(x)$ **satisfies** the differential equation in (13) on I.

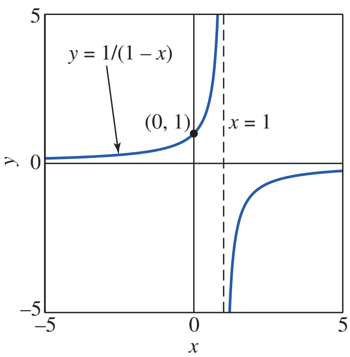

FIGURE 1.1.5. The solution of $y' = y^2$ defined by $y(x) = 1/(1-x)$.

Remark Recall from elementary calculus that a differentiable function on an open interval is necessarily continuous there. This is why only a continuous function can qualify as a (differentiable) solution of a differential equation on an interval. ■

Example 7 *Continued*

Figure 1.1.5 shows the two "connected" branches of the graph $y = 1/(1-x)$. The left-hand branch is the graph of a (continuous) solution of the differential equation $y' = y^2$ that is defined on the interval $(-\infty, 1)$. The right-hand branch is the graph of a *different* solution of the differential equation that is defined (and continuous) on the different interval $(1, \infty)$. So the single formula $y(x) = 1/(1-x)$ actually defines two different solutions (with different domains of definition) of the same differential equation $y' = y^2$. ■

Example 9

If A and B are constants and

$$y(x) = A \cos 3x + B \sin 3x, \tag{14}$$

then two successive differentiations yield

$$y'(x) = -3A \sin 3x + 3B \cos 3x,$$
$$y''(x) = -9A \cos 3x - 9B \sin 3x = -9y(x)$$

for all x. Consequently, Eq. (14) defines what it is natural to call a *two-parameter family* of solutions of the second-order differential equation

$$y'' + 9y = 0 \tag{15}$$

on the whole real number line. Figure 1.1.6 shows the graphs of several such solutions. ■

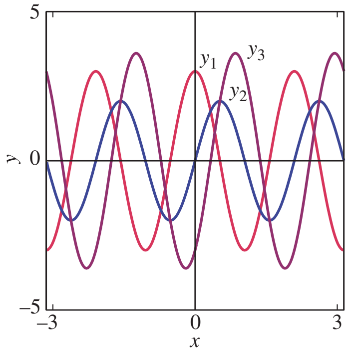

FIGURE 1.1.6. The three solutions $y_1(x) = 3\cos 3x$, $y_2(x) = 2\sin 3x$, and $y_3(x) = -3\cos 3x + 2\sin 3x$ of the differential equation $y'' + 9y = 0$.

Although the differential equations in (11) and (12) are exceptions to the general rule, we will see that an nth-order differential equation ordinarily has an n-parameter family of solutions—one involving n different arbitrary constants or parameters.

In both Eqs. (11) and (12), the appearance of y' as an implicitly defined function causes complications. For this reason, we will ordinarily assume that any differential equation under study can be solved explicitly for the highest derivative that appears; that is, that the equation can be written in the so-called *normal form*

$$y^{(n)} = G\left(x, y, y', y'', \dots, y^{(n-1)}\right), \tag{16}$$

where G is a real-valued function of $n+1$ variables. In addition, we will always seek only real-valued solutions unless we warn the reader otherwise.

All the differential equations we have mentioned so far are **ordinary** differential equations, meaning that the unknown function (dependent variable) depends on only a *single* independent variable. If the dependent variable is a function of two or more independent variables, then partial derivatives are likely to be involved; if they are, the equation is called a **partial** differential equation. For example, the temperature $u = u(x, t)$ of a long thin uniform rod at the point x at time t satisfies (under appropriate simple conditions) the partial differential equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

where k is a constant (called the *thermal diffusivity* of the rod). In Chapters 1 through 8 we will be concerned only with *ordinary* differential equations and will refer to them simply as differential equations.

In this chapter we concentrate on *first-order* differential equations of the form

$$\frac{dy}{dx} = f(x, y). \tag{17}$$

We also will sample the wide range of applications of such equations. A typical mathematical model of an applied situation will be an **initial value problem**, consisting of a differential equation of the form in (17) together with an **initial condition** $y(x_0) = y_0$. Note that we call $y(x_0) = y_0$ an initial condition whether or not $x_0 = 0$. To **solve** the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \tag{18}$$

means to find a differentiable function $y = y(x)$ that satisfies both conditions in Eq. (18) on some interval containing x_0.

Example 10 Given the solution $y(x) = 1/(C - x)$ of the differential equation $dy/dx = y^2$ discussed in Example 7, solve the initial value problem

$$\frac{dy}{dx} = y^2, \quad y(1) = 2.$$

Solution We need only to find a value of C so that the solution $y(x) = 1/(C - x)$ satisfies the initial condition $y(1) = 2$. Substitution of the values $x = 1$ and $y = 2$ in the given solution yields

$$2 = y(1) = \frac{1}{C - 1},$$

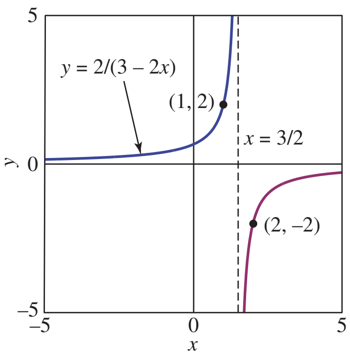

FIGURE 1.1.7. The solutions of $y' = y^2$ defined by $y(x) = 2/(3-2x)$.

so $2C - 2 = 1$, and hence $C = \frac{3}{2}$. With this value of C we obtain the desired solution

$$y(x) = \frac{1}{\frac{3}{2} - x} = \frac{2}{3 - 2x}.$$

Figure 1.1.7 shows the two branches of the graph $y = 2/(3-2x)$. The left-hand branch is the graph on $\left(-\infty, \frac{3}{2}\right)$ of the solution of the given initial value problem $y' = y^2$, $y(1) = 2$. The right-hand branch passes through the point $(2, -2)$ and is therefore the graph on $\left(\frac{3}{2}, \infty\right)$ of the solution of the different initial value problem $y' = y^2$, $y(2) = -2$. ■

The central question of greatest immediate interest to us is this: If we are given a differential equation known to have a solution satisfying a given initial condition, how do we actually *find* or *compute* that solution? And, once found, what can we do with it? We will see that a relatively few simple techniques—separation of variables (Section 1.4), solution of linear equations (Section 1.5), elementary substitution methods (Section 1.6)—are enough to enable us to solve a variety of first-order equations having impressive applications.

1.1 Problems

In Problems 1 through 12, verify by substitution that each given function is a solution of the given differential equation. Throughout these problems, primes denote derivatives with respect to x.

1. $y' = 3x^2$; $y = x^3 + 7$
2. $y' + 2y = 0$; $y = 3e^{-2x}$
3. $y'' + 4y = 0$; $y_1 = \cos 2x$, $y_2 = \sin 2x$
4. $y'' = 9y$; $y_1 = e^{3x}$, $y_2 = e^{-3x}$
5. $y' = y + 2e^{-x}$; $y = e^x - e^{-x}$
6. $y'' + 4y' + 4y = 0$; $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$
7. $y'' - 2y' + 2y = 0$; $y_1 = e^x \cos x$, $y_2 = e^x \sin x$
8. $y'' + y = 3\cos 2x$, $y_1 = \cos x - \cos 2x$, $y_2 = \sin x - \cos 2x$
9. $y' + 2xy^2 = 0$; $y = \dfrac{1}{1+x^2}$
10. $x^2y'' + xy' - y = \ln x$; $y_1 = x - \ln x$, $y_2 = \dfrac{1}{x} - \ln x$
11. $x^2y'' + 5xy' + 4y = 0$; $y_1 = \dfrac{1}{x^2}$, $y_2 = \dfrac{\ln x}{x^2}$
12. $x^2y'' - xy' + 2y = 0$; $y_1 = x\cos(\ln x)$, $y_2 = x\sin(\ln x)$

In Problems 13 through 16, substitute $y = e^{rx}$ into the given differential equation to determine all values of the constant r for which $y = e^{rx}$ is a solution of the equation.

13. $3y' = 2y$
14. $4y'' = y$
15. $y'' + y' - 2y = 0$
16. $3y'' + 3y' - 4y = 0$

In Problems 17 through 26, first verify that $y(x)$ satisfies the given differential equation. Then determine a value of the constant C so that $y(x)$ satisfies the given initial condition. Use a computer or graphing calculator (if desired) to sketch several typical solutions of the given differential equation, and highlight the one that satisfies the given initial condition.

17. $y' + y = 0$; $y(x) = Ce^{-x}$, $y(0) = 2$
18. $y' = 2y$; $y(x) = Ce^{2x}$, $y(0) = 3$
19. $y' = y + 1$; $y(x) = Ce^x - 1$, $y(0) = 5$
20. $y' = x - y$; $y(x) = Ce^{-x} + x - 1$, $y(0) = 10$
21. $y' + 3x^2y = 0$; $y(x) = Ce^{-x^3}$, $y(0) = 7$
22. $e^y y' = 1$; $y(x) = \ln(x + C)$, $y(0) = 0$
23. $x\dfrac{dy}{dx} + 3y = 2x^5$; $y(x) = \frac{1}{4}x^5 + Cx^{-3}$, $y(2) = 1$
24. $xy' - 3y = x^3$; $y(x) = x^3(C + \ln x)$, $y(1) = 17$
25. $y' = 3x^2(y^2 + 1)$; $y(x) = \tan(x^3 + C)$, $y(0) = 1$
26. $y' + y\tan x = \cos x$; $y(x) = (x + C)\cos x$, $y(\pi) = 0$

In Problems 27 through 31, a function $y = g(x)$ is described by some geometric property of its graph. Write a differential equation of the form $dy/dx = f(x, y)$ having the function g as its solution (or as one of its solutions).

27. The slope of the graph of g at the point (x, y) is the sum of x and y.
28. The line tangent to the graph of g at the point (x, y) intersects the x-axis at the point $(x/2, 0)$.
29. Every straight line normal to the graph of g passes through the point $(0, 1)$. Can you *guess* what the graph of such a function g might look like?
30. The graph of g is normal to every curve of the form $y = x^2 + k$ (k is a constant) where they meet.
31. The line tangent to the graph of g at (x, y) passes through the point $(-y, x)$.

Differential Equations as Models

In Problems 32 through 36, write—in the manner of Eqs. (3) through (6) of this section—a differential equation that is a mathematical model of the situation described.

32. The time rate of change of a population P is proportional to the square root of P.
33. The time rate of change of the velocity v of a coasting motorboat is proportional to the square of v.

34. The acceleration dv/dt of a Lamborghini is proportional to the difference between 250 km/h and the velocity of the car.

35. In a city having a fixed population of P persons, the time rate of change of the number N of those persons who have heard a certain rumor is proportional to the number of those who have not yet heard the rumor.

36. In a city with a fixed population of P persons, the time rate of change of the number N of those persons infected with a certain contagious disease is proportional to the product of the number who have the disease and the number who do not.

In Problems 37 through 42, determine by inspection at least one solution of the given differential equation. That is, use your knowledge of derivatives to make an intelligent guess. Then test your hypothesis.

37. $y'' = 0$

38. $y' = y$

39. $xy' + y = 3x^2$

40. $(y')^2 + y^2 = 1$

41. $y' + y = e^x$

42. $y'' + y = 0$

Problems 43 through 46 concern the differential equation

$$\frac{dx}{dt} = kx^2,$$

where k is a constant.

43. **(a)** If k is a constant, show that a general (one-parameter) solution of the differential equation is given by $x(t) = 1/(C - kt)$, where C is an arbitrary constant.

(b) Determine by inspection a solution of the initial value problem $x' = kx^2$, $x(0) = 0$.

44. **(a)** Assume that k is positive, and then sketch graphs of solutions of $x' = kx^2$ with several typical positive values of $x(0)$.

(b) How would these solutions differ if the constant k were negative?

45. Suppose a population P of rodents satisfies the differential equation $dP/dt = kP^2$. Initially, there are $P(0) = 2$ rodents, and their number is increasing at the rate of $dP/dt = 1$ rodent per month when there are $P = 10$ rodents. Based on the result of Problem 43, how long will it take for this population to grow to a hundred rodents? To a thousand? What's happening here?

46. Suppose the velocity v of a motorboat coasting in water satisfies the differential equation $dv/dt = kv^2$. The initial speed of the motorboat is $v(0) = 10$ meters per second (m/s), and v is decreasing at the rate of 1 m/s^2 when $v = 5$ m/s. Based on the result of Problem 43, long does it take for the velocity of the boat to decrease to 1 m/s? To $\frac{1}{10}$ m/s? When does the boat come to a stop?

47. In Example 7 we saw that $y(x) = 1/(C - x)$ defines a one-parameter family of solutions of the differential equation $dy/dx = y^2$. **(a)** Determine a value of C so that $y(10) = 10$. **(b)** Is there a value of C such that $y(0) = 0$? Can you nevertheless find by inspection a solution of $dy/dx = y^2$ such that $y(0) = 0$? **(c)** Figure 1.1.8 shows typical graphs of solutions of the form $y(x) = 1/(C - x)$. Does it appear that these solution curves fill the entire xy-plane? Can you conclude that, given any point (a, b) in the plane, the differential equation $dy/dx = y^2$ has exactly one solution $y(x)$ satisfying the condition $y(a) = b$?

48. **(a)** Show that $y(x) = Cx^4$ defines a one-parameter family of differentiable solutions of the differential equation $xy' = 4y$ (Fig. 1.1.9). **(b)** Show that

$$y(x) = \begin{cases} -x^4 & \text{if } x < 0, \\ x^4 & \text{if } x \geqq 0 \end{cases}$$

defines a differentiable solution of $xy' = 4y$ for all x, but is not of the form $y(x) = Cx^4$. **(c)** Given any two real numbers a and b, explain why—in contrast to the situation in part (c) of Problem 47—there exist infinitely many differentiable solutions of $xy' = 4y$ that all satisfy the condition $y(a) = b$.

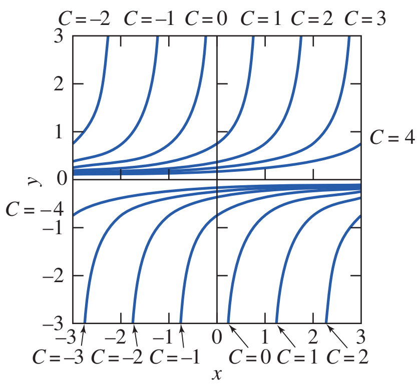

FIGURE 1.1.8. Graphs of solutions of the equation $dy/dx = y^2$.

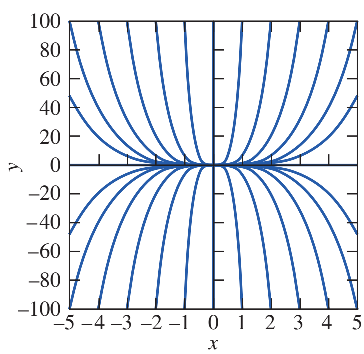

FIGURE 1.1.9. The graph $y = Cx^4$ for various values of C.

1.2 Integrals as General and Particular Solutions

The first-order equation $dy/dx = f(x, y)$ takes an especially simple form if the right-hand-side function f does not actually involve the dependent variable y, so

$$\frac{dy}{dx} = f(x). \tag{1}$$

In this special case we need only integrate both sides of Eq. (1) to obtain

$$y(x) = \int f(x)\,dx + C. \tag{2}$$

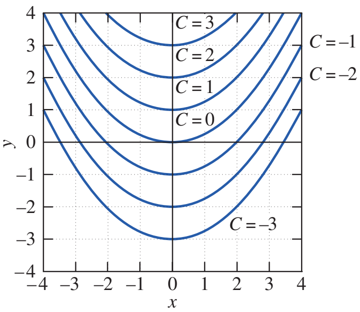

FIGURE 1.2.1. Graphs of $y = \frac{1}{4}x^2 + C$ for various values of C.

This is a **general solution** of Eq. (1), meaning that it involves an arbitrary constant C, and for every choice of C it is a solution of the differential equation in (1). If $G(x)$ is a particular antiderivative of f—that is, if $G'(x) \equiv f(x)$—then

$$y(x) = G(x) + C. \tag{3}$$

The graphs of any two such solutions $y_1(x) = G(x) + C_1$ and $y_2(x) = G(x) + C_2$ on the same interval I are "parallel" in the sense illustrated by Figs. 1.2.1 and 1.2.2. There we see that the constant C is geometrically the vertical distance between the two curves $y(x) = G(x)$ and $y(x) = G(x) + C$.

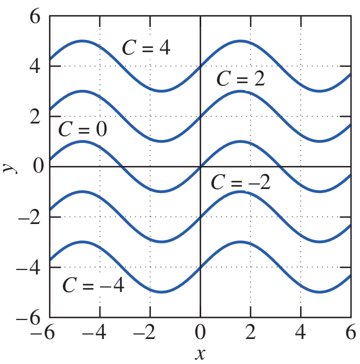

FIGURE 1.2.2. Graphs of $y = \sin x + C$ for various values of C.

To satisfy an initial condition $y(x_0) = y_0$, we need only substitute $x = x_0$ and $y = y_0$ into Eq. (3) to obtain $y_0 = G(x_0) + C$, so that $C = y_0 - G(x_0)$. With this choice of C, we obtain the **particular solution** of Eq. (1) satisfying the initial value problem

$$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0.$$

We will see that this is the typical pattern for solutions of first-order differential equations. Ordinarily, we will first find a *general solution* involving an arbitrary constant C. We can then attempt to obtain, by appropriate choice of C, a *particular solution* satisfying a given initial condition $y(x_0) = y_0$.

Remark As the term is used in the previous paragraph, a *general solution* of a first-order differential equation is simply a one-parameter family of solutions. A natural question is whether a given general solution contains *every* particular solution of the differential equation. When this is known to be true, we call it **the** general solution of the differential equation. For example, because any two antiderivatives of the same function $f(x)$ can differ only by a constant, it follows that every solution of Eq. (1) is of the form in (2). Thus Eq. (2) serves to define **the** general solution of (1). ■

Example 1 **General and particular solution** Solve the initial value problem

$$\frac{dy}{dx} = 2x + 3, \quad y(1) = 2.$$

Solution Integration of both sides of the differential equation as in Eq. (2) immediately yields the general solution

$$y(x) = \int (2x + 3)\,dx = x^2 + 3x + C.$$

Figure 1.2.3 shows the graph $y = x^2 + 3x + C$ for various values of C. The particular solution we seek corresponds to the curve that passes through the point $(1, 2)$, thereby satisfying the initial condition

$$y(1) = (1)^2 + 3 \cdot (1) + C = 2.$$

It follows that $C = -2$, so the desired particular solution is

$$y(x) = x^2 + 3x - 2. \quad ■$$

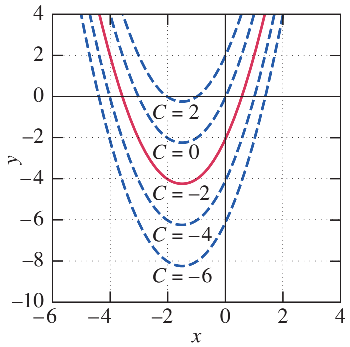

FIGURE 1.2.3. Solution curves for the differential equation in Example 1.

Second-order equations. The observation that the special first-order equation $dy/dx = f(x)$ is readily solvable (provided that an antiderivative of f can be found) extends to second-order differential equations of the special form

$$\frac{d^2y}{dx^2} = g(x), \tag{4}$$

in which the function g on the right-hand side involves neither the dependent variable y nor its derivative dy/dx. We simply integrate once to obtain

$$\frac{dy}{dx} = \int y''(x)\,dx = \int g(x)\,dx = G(x) + C_1,$$

where G is an antiderivative of g and C_1 is an arbitrary constant. Then another integration yields

$$y(x) = \int y'(x)\,dx = \int [G(x) + C_1]\,dx = \int G(x)\,dx + C_1x + C_2,$$

where C_2 is a second arbitrary constant. In effect, the second-order differential equation in (4) is one that can be solved by solving successively the *first-order* equations

$$\frac{dv}{dx} = g(x) \quad \text{and} \quad \frac{dy}{dx} = v(x).$$

Velocity and Acceleration

Direct integration is sufficient to allow us to solve a number of important problems concerning the motion of a particle (or *mass point*) in terms of the forces acting on it. The motion of a particle along a straight line (the x-axis) is described by its **position function**

$$x = f(t) \tag{5}$$

giving its x-coordinate at time t. The **velocity** of the particle is defined to be

$$v(t) = f'(t); \quad \text{that is,} \quad v = \frac{dx}{dt}. \tag{6}$$

Its **acceleration** $a(t)$ is $a(t) = v'(t) = x''(t)$; in Leibniz notation,

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}. \tag{7}$$

Equation (6) is sometimes applied either in the indefinite integral form $x(t) = \int v(t)\,dt$ or in the definite integral form

$$x(t) = x(t_0) + \int_{t_0}^{t} v(s)\,ds,$$

which you should recognize as a statement of the fundamental theorem of calculus (precisely because $dx/dt = v$).

Newton's *second law of motion* says that if a force $F(t)$ acts on the particle and is directed along its line of motion, then

$$ma(t) = F(t); \quad \text{that is,} \quad F = ma, \tag{8}$$

where m is the mass of the particle. If the force F is known, then the equation $x''(t) = F(t)/m$ can be integrated twice to find the position function $x(t)$ in terms of two constants of integration. These two arbitrary constants are frequently determined by the **initial position** $x_0 = x(0)$ and the **initial velocity** $v_0 = v(0)$ of the particle.

Constant acceleration. For instance, suppose that the force F, and therefore the acceleration $a = F/m$, are *constant*. Then we begin with the equation

$$\frac{dv}{dt} = a \quad (a \text{ is a constant}) \tag{9}$$

and integrate both sides to obtain

$$v(t) = \int a\,dt = at + C_1.$$

We know that $v = v_0$ when $t = 0$, and substitution of this information into the preceding equation yields the fact that $C_1 = v_0$. So

$$v(t) = \frac{dx}{dt} = at + v_0. \tag{10}$$

A second integration gives

$$x(t) = \int v(t)\,dt = \int (at + v_0)\,dt = \tfrac{1}{2}at^2 + v_0 t + C_2,$$

and the substitution $t = 0$, $x = x_0$ gives $C_2 = x_0$. Therefore,

$$x(t) = \tfrac{1}{2}at^2 + v_0 t + x_0. \tag{11}$$

Thus, with Eq. (10) we can find the velocity, and with Eq. (11) the position, of the particle at any time t in terms of its *constant* acceleration a, its initial velocity v_0, and its initial position x_0.

Example 2 **Lunar lander** A lunar lander is falling freely toward the surface of the moon at a speed of 450 meters per second (m/s). Its retrorockets, when fired, provide a constant deceleration of 2.5 meters per second per second (m/s^2) (the gravitational acceleration produced by the moon is assumed to be included in the given deceleration). At what height above the lunar surface should the retrorockets be activated to ensure a "soft touchdown" ($v = 0$ at impact)?

Solution We denote by $x(t)$ the height of the lunar lander above the surface, as indicated in Fig. 1.2.4. We let $t = 0$ denote the time at which the retrorockets should be fired. Then $v_0 = -450$ (m/s, negative because the height $x(t)$ is decreasing), and $a = +2.5$, because an upward thrust increases the velocity v (although it decreases the *speed* $|v|$). Then Eqs. (10) and (11) become

$$v(t) = 2.5t - 450 \tag{12}$$

and

$$x(t) = 1.25t^2 - 450t + x_0, \tag{13}$$

where x_0 is the height of the lander above the lunar surface at the time $t = 0$ when the retrorockets should be activated.

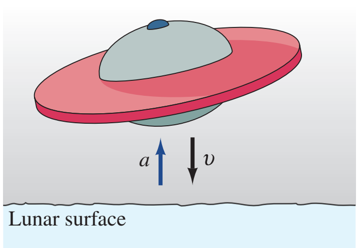

FIGURE 1.2.4. The lunar lander of Example 2.

From Eq. (12) we see that $v = 0$ (soft touchdown) occurs when $t = 450/2.5 = 180$ s (that is, 3 minutes); then substitution of $t = 180$, $x = 0$ into Eq. (13) yields

$$x_0 = 0 - (1.25)(180)^2 + 450(180) = 40{,}500$$

meters—that is, $x_0 = 40.5$ km $\approx 25\frac{1}{6}$ miles. Thus the retrorockets should be activated when the lunar lander is 40.5 kilometers above the surface of the moon, and it will touch down softly on the lunar surface after 3 minutes of decelerating descent. ■

Physical Units

Numerical work requires units for the measurement of physical quantities such as distance and time. We sometimes use ad hoc units—such as distance in miles or kilometers and time in hours—in special situations (such as in a problem involving an auto trip). However, the foot-pound-second (fps) and meter-kilogram-second (mks) unit systems are used more generally in scientific and engineering problems. In fact, fps units are commonly used only in the United States (and a few other countries), while mks units constitute the standard international system of scientific units.

	fps units	**mks units**
Force	pound (lb)	newton (N)
Mass	slug	kilogram (kg)
Distance	foot (ft)	meter (m)
Time	second (s)	second (s)
g	32 ft/s^2	9.8 m/s^2

The last line of this table gives values for the gravitational acceleration g at the surface of the earth. Although these approximate values will suffice for most examples and problems, more precise values are 9.7805 m/s^2 and 32.088 ft/s^2 (at sea level at the equator).

Both systems are compatible with Newton's second law $F = ma$. Thus, 1 N is (by definition) the force required to impart an acceleration of 1 m/s^2 to a mass of 1 kg. Similarly, 1 slug is (by definition) the mass that experiences an acceleration of 1 ft/s^2 under a force of 1 lb. (We will use mks units in all problems requiring mass units and thus will rarely need slugs to measure mass.)

Inches and centimeters (as well as miles and kilometers) also are commonly used in describing distances. For conversions between fps and mks units it helps to remember that

$$1 \text{ in.} = 2.54 \text{ cm (exactly)} \quad \text{and} \quad 1 \text{ lb} \approx 4.448 \text{ N}.$$

For instance,

$$1 \text{ ft} = 12 \,\cancel{\text{in}} \times 2.54 \frac{\text{cm}}{\cancel{\text{in}}} = 30.48 \text{ cm},$$

and it follows that

$$1 \text{ mi} = 5280 \,\cancel{\text{ft}} \times 30.48 \frac{\text{cm}}{\cancel{\text{ft}}} = 160934.4 \text{ cm} \approx 1.609 \text{ km}.$$

Thus a posted U.S. speed limit of 50 mi/h means that—in international terms—the legal speed limit is about $50 \times 1.609 \approx 80.45$ km/h.

Vertical Motion with Gravitational Acceleration

The **weight** W of a body is the force exerted on the body by gravity. Substitution of $a = g$ and $F = W$ in Newton's second law $F = ma$ gives

$$W = mg \tag{14}$$

for the weight W of the mass m at the surface of the earth (where $g \approx 32$ ft/s$^2 \approx 9.8$ m/s^2). For instance, a mass of $m = 20$ kg has a weight of $W = (20 \text{ kg})(9.8 \text{ m/s}^2) = 196$ N. Similarly, a mass m weighing 100 pounds has mks weight

$$W = (100 \text{ lb})(4.448 \text{ N/lb}) = 444.8 \text{ N},$$

so its mass is

$$m = \frac{W}{g} = \frac{444.8 \text{ N}}{9.8 \text{ m/s}^2} \approx 45.4 \text{ kg}.$$

To discuss vertical motion it is natural to choose the y-axis as the coordinate system for position, frequently with $y = 0$ corresponding to "ground level." If we choose the *upward* direction as the positive direction, then the effect of gravity on a vertically moving body is to decrease its height and also to decrease its velocity $v = dy/dt$. Consequently, if we ignore air resistance, then the acceleration $a = dv/dt$ of the body is given by

$$\frac{dv}{dt} = -g. \tag{15}$$

This acceleration equation provides a starting point in many problems involving vertical motion. Successive integrations (as in Eqs. (10) and (11)) yield the velocity and height formulas

$$v(t) = -gt + v_0 \tag{16}$$

and

$$y(t) = -\tfrac{1}{2}gt^2 + v_0 t + y_0. \tag{17}$$

Here, y_0 denotes the initial ($t = 0$) height of the body and v_0 its initial velocity.

Example 3 **Projectile motion** (a) Suppose that a ball is thrown straight upward from the ground ($y_0 = 0$) with initial velocity $v_0 = 96$ (ft/s, so we use $g = 32$ ft/s^2 in fps units). Then it reaches its maximum height when its velocity (Eq. (16)) is zero,

$$v(t) = -32t + 96 = 0,$$

and thus when $t = 3$ s. Hence the maximum height that the ball attains is

$$y(3) = -\tfrac{1}{2} \cdot 32 \cdot 3^2 + 96 \cdot 3 + 0 = 144 \text{ (ft)}$$

(with the aid of Eq. (17)).

(b) If an arrow is shot straight upward from the ground with initial velocity $v_0 = 49$ (m/s, so we use $g = 9.8$ m/s^2 in mks units), then it returns to the ground when

$$y(t) = -\tfrac{1}{2} \cdot (9.8)t^2 + 49t = (4.9)t(-t + 10) = 0,$$

and thus after 10 s in the air. ■

A Swimmer's Problem

Figure 1.2.5 shows a northward-flowing river of width $w = 2a$. The lines $x = \pm a$ represent the banks of the river and the y-axis its center. Suppose that the velocity v_R at which the water flows increases as one approaches the center of the river, and indeed is given in terms of distance x from the center by

$$v_R = v_0 \left(1 - \frac{x^2}{a^2}\right). \tag{18}$$

You can use Eq. (18) to verify that the water does flow the fastest at the center, where $v_R = v_0$, and that $v_R = 0$ at each riverbank.

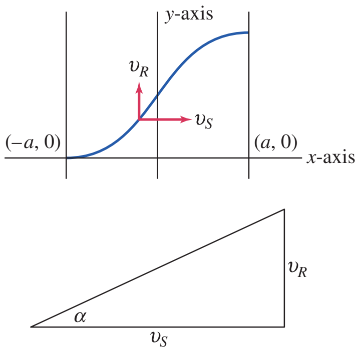

FIGURE 1.2.5. A swimmer's problem (Example 4).

Suppose that a swimmer starts at the point $(-a, 0)$ on the west bank and swims due east (relative to the water) with constant speed v_S. As indicated in Fig. 1.2.5, his velocity vector (relative to the riverbed) has horizontal component v_S and vertical component v_R. Hence the swimmer's direction angle α is given by

$$\tan\alpha = \frac{v_R}{v_S}.$$

Because $\tan\alpha = dy/dx$, substitution using (18) gives the differential equation

$$\frac{dy}{dx} = \frac{v_0}{v_S}\left(1 - \frac{x^2}{a^2}\right) \tag{19}$$

for the swimmer's trajectory $y = y(x)$ as he crosses the river.

Example 4 **River crossing** Suppose that the river is 1 mile wide and that its midstream velocity is $v_0 = 9$ mi/h. If the swimmer's velocity is $v_S = 3$ mi/h, then Eq. (19) takes the form

$$\frac{dy}{dx} = 3(1 - 4x^2).$$

Integration yields

$$y(x) = \int (3 - 12x^2)\,dx = 3x - 4x^3 + C$$

for the swimmer's trajectory. The initial condition $y\left(-\frac{1}{2}\right) = 0$ yields $C = 1$, so

$$y(x) = 3x - 4x^3 + 1.$$

Then

$$y\left(\tfrac{1}{2}\right) = 3\left(\tfrac{1}{2}\right) - 4\left(\tfrac{1}{2}\right)^3 + 1 = 2,$$

so the swimmer drifts 2 miles downstream while he swims 1 mile across the river. ■

1.2 Problems

In Problems 1 through 10, find a function $y = f(x)$ satisfying the given differential equation and the prescribed initial condition.

1. $\dfrac{dy}{dx} = 2x + 1;\ y(0) = 3$
2. $\dfrac{dy}{dx} = (x-2)^2;\ y(2) = 1$
3. $\dfrac{dy}{dx} = \sqrt{x};\ y(4) = 0$
4. $\dfrac{dy}{dx} = \dfrac{1}{x^2};\ y(1) = 5$
5. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x+2}};\ y(2) = -1$
6. $\dfrac{dy}{dx} = x\sqrt{x^2+9};\ y(-4) = 0$
7. $\dfrac{dy}{dx} = \dfrac{10}{x^2+1};\ y(0) = 0$
8. $\dfrac{dy}{dx} = \cos 2x;\ y(0) = 1$
9. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}};\ y(0) = 0$
10. $\dfrac{dy}{dx} = xe^{-x};\ y(0) = 1$

In Problems 11 through 18, find the position function $x(t)$ of a moving particle with the given acceleration $a(t)$, initial position $x_0 = x(0)$, and initial velocity $v_0 = v(0)$.

11. $a(t) = 50,\ v_0 = 10,\ x_0 = 20$
12. $a(t) = -20,\ v_0 = -15,\ x_0 = 5$
13. $a(t) = 3t,\ v_0 = 5,\ x_0 = 0$
14. $a(t) = 2t + 1,\ v_0 = -7,\ x_0 = 4$
15. $a(t) = 4(t+3)^2,\ v_0 = -1,\ x_0 = 1$
16. $a(t) = \dfrac{1}{\sqrt{t+4}},\ v_0 = -1,\ x_0 = 1$
17. $a(t) = \dfrac{1}{(t+1)^3},\ v_0 = 0,\ x_0 = 0$
18. $a(t) = 50\sin 5t,\ v_0 = -10,\ x_0 = 8$

Velocity Given Graphically

In Problems 19 through 22, a particle starts at the origin and travels along the x-axis with the velocity function $v(t)$ whose graph is shown in Figs. 1.2.6 through 1.2.9. Sketch the graph of the resulting position function $x(t)$ for $0 \leqq t \leqq 10$.

19.

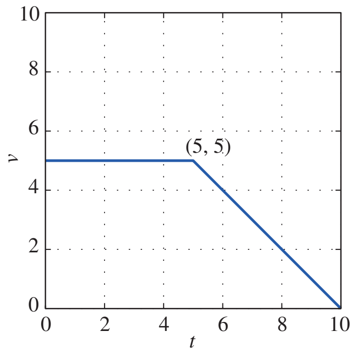

FIGURE 1.2.6. Graph of the velocity function $v(t)$ of Problem 19.

20.

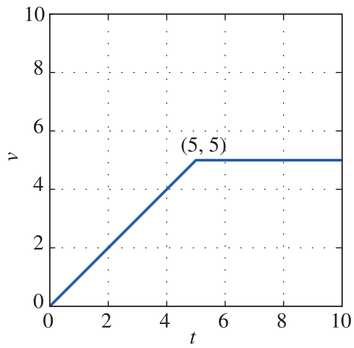

FIGURE 1.2.7. Graph of the velocity function $v(t)$ of Problem 20.

21.

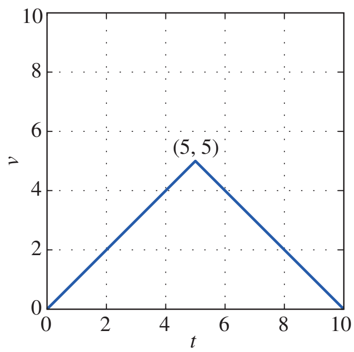

FIGURE 1.2.8. Graph of the velocity function $v(t)$ of Problem 21.

22.

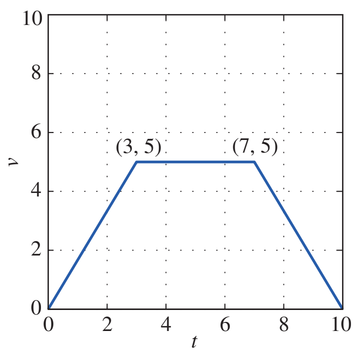

FIGURE 1.2.9. Graph of the velocity function $v(t)$ of Problem 22.

Problems 23 through 28 explore the motion of projectiles under constant acceleration or deceleration. A calculator will be helpful for many of the following problems.

23. What is the maximum height attained by the arrow of part (b) of Example 3?

24. A ball is dropped from the top of a building 400 ft high. How long does it take to reach the ground? With what speed does the ball strike the ground?

25. The brakes of a car are applied when it is moving at 100 km/h and provide a constant deceleration of 10 meters per second per second (m/s^2). How far does the car travel before coming to a stop?

26. A projectile is fired straight upward with an initial velocity of 100 m/s from the top of a building 20 m high and falls to the ground at the base of the building. Find (a) its maximum height above the ground; (b) when it passes the top of the building; (c) its total time in the air.

27. A ball is thrown straight downward from the top of a tall building. The initial speed of the ball is 10 m/s. It strikes the ground with a speed of 60 m/s. How tall is the building?

28. A baseball is thrown straight downward with an initial speed of 40 ft/s from the top of the Washington Monument (555 ft high). How long does it take to reach the ground, and with what speed does the baseball strike the ground?

29. **Variable acceleration** A diesel car gradually speeds up so that for the first 10 s its acceleration is given by

$$\frac{dv}{dt} = (0.12)t^2 + (0.6)t \quad (\text{ft/s}^2).$$

If the car starts from rest ($x_0 = 0$, $v_0 = 0$), find the distance it has traveled at the end of the first 10 s and its velocity at that time.

Problems 30 through 32 explore the relation between the speed of an auto and the distance it skids when the brakes are applied.

30. A car traveling at 60 mi/h (88 ft/s) skids 176 ft after its brakes are suddenly applied. Under the assumption that the braking system provides constant deceleration, what is that deceleration? For how long does the skid continue?

31. The skid marks made by an automobile indicated that its brakes were fully applied for a distance of 75 m before it came to a stop. The car in question is known to have a constant deceleration of 20 m/s^2 under these conditions. How fast—in km/h—was the car traveling when the brakes were first applied?

32. Suppose that a car skids 15 m if it is moving at 50 km/h when the brakes are applied. Assuming that the car has the same constant deceleration, how far will it skid if it is moving at 100 km/h when the brakes are applied?

Problems 33 and 34 explore vertical motion on a planet with gravitational acceleration different than the earth's.

33. On the planet Gzyx, a ball dropped from a height of 20 ft hits the ground in 2 s. If a ball is dropped from the top of a 200-ft-tall building on Gzyx, how long will it take to hit the ground? With what speed will it hit?

34. A person can throw a ball straight upward from the surface of the earth to a maximum height of 144 ft. How high could this person throw the ball on the planet Gzyx of Problem 33?

35. Velocity in terms of height A stone is dropped from rest at an initial height h above the surface of the earth. Show that the speed with which it strikes the ground is $v = \sqrt{2gh}$.

36. Suppose a woman has enough "spring" in her legs to jump (on earth) from the ground to a height of 2.25 feet. If she jumps straight upward with the same initial velocity on the moon—where the surface gravitational acceleration is (approximately) 5.3 ft/s^2—how high above the surface will she rise?

37. At noon a car starts from rest at point A and proceeds at constant acceleration along a straight road toward point B. If the car reaches B at 12:50 P.M. with a velocity of 60 mi/h, what is the distance from A to B?

38. At noon a car starts from rest at point A and proceeds with constant acceleration along a straight road toward point C, 35 miles away. If the constantly accelerated car arrives at C with a velocity of 60 mi/h, at what time does it arrive at C?

39. River crossing If $a = 0.5$ mi and $v_0 = 9$ mi/h as in Example 4, what must the swimmer's speed v_S be in order that he drifts only 1 mile downstream as he crosses the river?

40. River crossing Suppose that $a = 0.5$ mi, $v_0 = 9$ mi/h, and $v_S = 3$ mi/h as in Example 4, but that the velocity of the river is given by the fourth-degree function

$$v_R = v_0\left(1 - \frac{x^4}{a^4}\right)$$

rather than the quadratic function in Eq. (18). Now find how far downstream the swimmer drifts as he crosses the river.

41. Interception of bomb A bomb is dropped from a helicopter hovering at an altitude of 800 feet above the ground. From the ground directly beneath the helicopter, a projectile is fired straight upward toward the bomb, exactly 2 seconds after the bomb is released. With what initial velocity should the projectile be fired in order to hit the bomb at an altitude of exactly 400 feet?

42. Lunar lander A spacecraft is in free fall toward the surface of the moon at a speed of 1000 mph (mi/h). Its retrorockets, when fired, provide a constant deceleration of 20,000 mi/h^2. At what height above the lunar surface should the astronauts fire the retrorockets to insure a soft touchdown? (As in Example 2, ignore the moon's gravitational field.)

43. Solar wind Arthur Clarke's *The Wind from the Sun* (1963) describes Diana, a spacecraft propelled by the solar wind. Its aluminized sail provides it with a constant acceleration of $0.001g = 0.0098$ m/s^2. Suppose this spacecraft starts from rest at time $t = 0$ and simultaneously fires a projectile (straight ahead in the same direction) that travels at one-tenth of the speed $c = 3 \times 10^8$ m/s of light. How long will it take the spacecraft to catch up with the projectile, and how far will it have traveled by then?

44. Length of skid A driver involved in an accident claims he was going only 25 mph. When police tested his car, they found that when its brakes were applied at 25 mph, the car skidded only 45 feet before coming to a stop. But the driver's skid marks at the accident scene measured 210 feet. Assuming the same (constant) deceleration, determine the speed he was actually traveling just prior to the accident.

45. Kinematic formula Use Eqs. (10) and (11) to show that $v(t)^2 - v_0^2 = 2a[x(t) - x_0]$ for all t when the acceleration $a = dv/dt$ is constant. Then use this "kinematic formula"—commonly presented in introductory physics courses—to confirm the result of Example 2.

1.3 Slope Fields and Solution Curves

Consider a differential equation of the form

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

where the right-hand function $f(x, y)$ involves both the independent variable x and the dependent variable y. We might think of integrating both sides in (1) with respect to x, and hence write $y(x) = \int f(x, y(x))\,dx + C$. However, this approach does not lead to a solution of the differential equation, because the indicated integral involves the *unknown* function $y(x)$ itself, and therefore cannot be evaluated explicitly. Actually, there exists *no* straightforward procedure by which a general differential equation can be solved explicitly. Indeed, the solutions of such a simple-looking differential equation as $y' = x^2 + y^2$ cannot be expressed in terms of the ordinary elementary functions studied in calculus textbooks. Nevertheless, the graphical and numerical methods of this and later sections can be used to construct *approximate* solutions of differential equations that suffice for many practical purposes.

Slope Fields and Graphical Solutions

There is a simple geometric way to think about solutions of a given differential equation $y' = f(x, y)$. At each point (x, y) of the xy-plane, the value of $f(x, y)$ determines a slope $m = f(x, y)$. A solution of the differential equation is simply a differentiable function whose graph $y = y(x)$ has this "correct slope" at each point $(x, y(x))$ through which it passes—that is, $y'(x) = f(x, y(x))$. Thus a **solution curve** of the differential equation $y' = f(x, y)$—the graph of a solution of the equation—is simply a curve in the xy-plane whose tangent line at each point (x, y) has slope $m = f(x, y)$. For instance, Fig. 1.3.1 shows a solution curve of the differential equation $y' = x - y$ together with its tangent lines at three typical points.

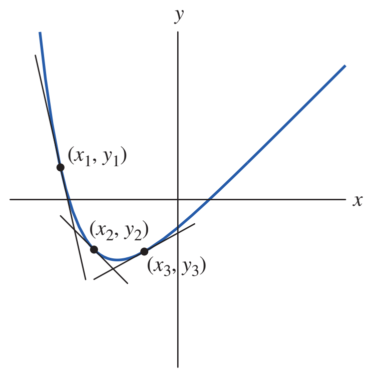

FIGURE 1.3.1. A solution curve for the differential equation $y' = x - y$ together with tangent lines having

- slope $m_1 = x_1 - y_1$ at the point (x_1, y_1);
- slope $m_2 = x_2 - y_2$ at the point (x_2, y_2); and
- slope $m_3 = x_3 - y_3$ at the point (x_3, y_3).

This geometric viewpoint suggests a *graphical method* for constructing *approximate* solutions of the differential equation $y' = f(x, y)$. Through each of a representative collection of points (x, y) in the plane we draw a short line segment having the proper slope $m = f(x, y)$. All these line segments constitute a **slope field** (or a **direction field**) for the equation $y' = f(x, y)$.

Example 1 **Slope fields** Figures 1.3.2 (a)–(d) show slope fields and solution curves for the differential equation

$$\frac{dy}{dx} = ky \tag{2}$$

with the values $k = 2, 0.5, -1$, and -3 of the parameter k in Eq. (2). Note that each slope field yields important qualitative information about the set of all solutions of the differential equation. For instance, Figs. 1.3.2(a) and (b) suggest that each solution $y(x)$ approaches $\pm\infty$ as $x \to +\infty$ if $k > 0$, whereas Figs. 1.3.2(c) and (d) suggest that $y(x) \to 0$ as $x \to +\infty$ if $k < 0$. Moreover, although the sign of k determines the *direction* of increase or decrease of $y(x)$, its absolute value $|k|$ appears to determine the *rate of change* of $y(x)$. All this is apparent from slope fields like those in Fig. 1.3.2, even without knowing that the general solution of Eq. (2) is given explicitly by $y(x) = Ce^{kx}$. ■

A slope field suggests visually the general shapes of solution curves of the differential equation. Through each point a solution curve should proceed in such

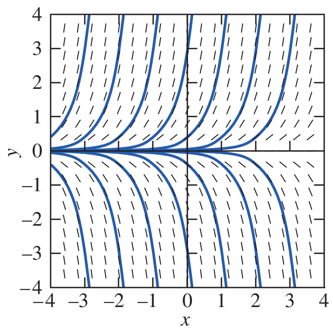

FIGURE 1.3.2(a) Slope field and solution curves for $y' = 2y$.

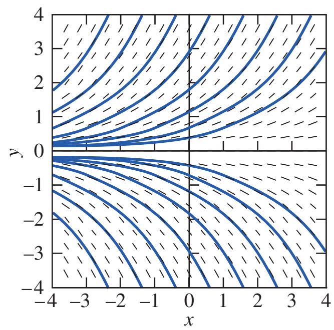

FIGURE 1.3.2(b) Slope field and solution curves for $y' = (0.5)y$.

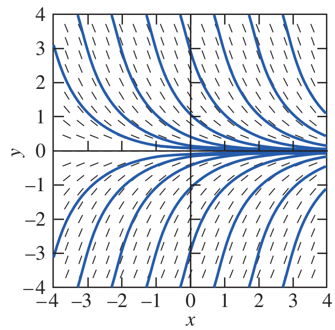

FIGURE 1.3.2(c) Slope field and solution curves for $y' = -y$.

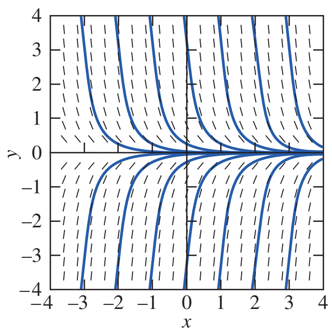

FIGURE 1.3.2(d) Slope field and solution curves for $y' = -3y$.

$x \setminus y$	−4	−3	−2	−1	0	1	2	3	4
−4	0	−1	−2	−3	−4	−5	−6	−7	−8
−3	1	0	−1	−2	−3	−4	−5	−6	−7
−2	2	1	0	−1	−2	−3	−4	−5	−6
−1	3	2	1	0	−1	−2	−3	−4	−5
0	4	3	2	1	0	−1	−2	−3	−4
1	5	4	3	2	1	0	−1	−2	−3
2	6	5	4	3	2	1	0	−1	−2
3	7	6	5	4	3	2	1	0	−1
4	8	7	6	5	4	3	2	1	0

FIGURE 1.3.3. Values of the slope $y' = x - y$ for $-4 \leq x,\ y \leq 4$.

a direction that its tangent line is nearly parallel to the nearby line segments of the slope field. Starting at any initial point (a, b), we can attempt to sketch freehand an approximate solution curve that threads its way through the slope field, following the visible line segments as closely as possible.

Example 2 **Solution curve** Construct a slope field for the differential equation $y' = x - y$ and use it to sketch an approximate solution curve that passes through the point $(-4, 4)$.

Solution Figure 1.3.3 shows a table of slopes for the given equation. The numerical slope $m = x - y$ appears at the intersection of the horizontal x-row and the vertical y-column of the table. If you inspect the pattern of upper-left to lower-right diagonals in this table, you can see that it was easily and quickly constructed. (Of course, a more complicated function $f(x, y)$ on the right-hand side of the differential equation would necessitate more complicated calculations.) Figure 1.3.4 shows the corresponding slope field, and Fig. 1.3.5 shows an approximate solution curve sketched through the point $(-4, 4)$ so as to follow this slope field as closely as possible. At each point it appears to proceed in the direction indicated by the nearby line segments of the slope field. ■

Although a spreadsheet program (for instance) readily constructs a table of slopes as in Fig. 1.3.3, it can be quite tedious to plot by hand a sufficient number of slope segments as in Fig. 1.3.4. However, most computer algebra systems include commands for quick and ready construction of slope fields with as many line segments as desired; such commands are illustrated in the Application material for this section. The more line segments are constructed, the more accurately solution curves can be visualized and sketched. Figure 1.3.6 shows a "finer" slope field

Go To bit.ly/3p4Bp03 to view an interactive version of Fig. 1.3.5.

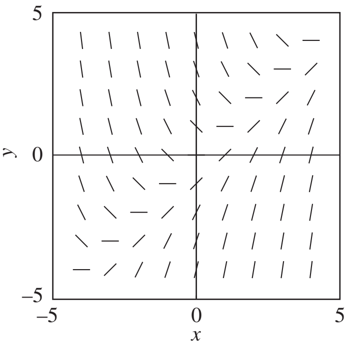

FIGURE 1.3.4. Slope field for $y' = x - y$ corresponding to the table of slopes in Fig. 1.3.3.

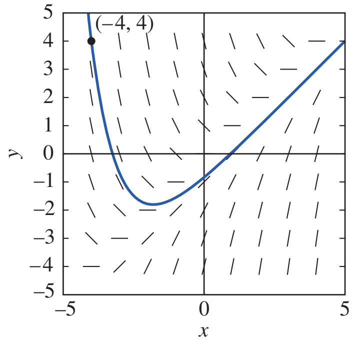

FIGURE 1.3.5. The solution curve through $(-4, 4)$.

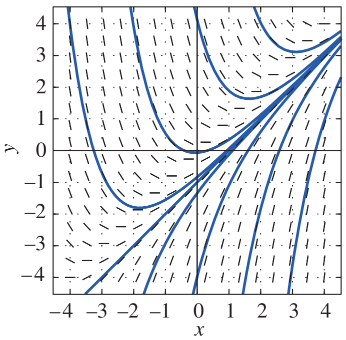

FIGURE 1.3.6. Slope field and typical solution curves for $y' = x - y$.

for the differential equation $y' = x - y$ of Example 2, together with typical solution curves treading through this slope field.

If you look closely at Fig. 1.3.6, you may spot a solution curve that appears to be a straight line! Indeed, you can verify that the linear function $y = x - 1$ is a solution of the equation $y' = x - y$, and it appears likely that the other solution curves approach this straight line as an asymptote as $x \to +\infty$. This inference illustrates the fact that a slope field can suggest tangible information about solutions that is not at all evident from the differential equation itself. Can you, by tracing the appropriate solution curve in this figure, infer that $y(3) \approx 2$ for the solution $y(x)$ of the initial value problem $y' = x - y$, $y(-4) = 4$?

Applications of Slope Fields

The next two examples illustrate the use of slope fields to glean useful information in physical situations that are modeled by differential equations. Example 3 is based on the fact that a baseball moving through the air at a moderate speed v (less than about 300 ft/s) encounters air resistance that is approximately proportional to v. If the baseball is thrown straight downward from the top of a tall building or from a hovering helicopter, then it experiences both the downward acceleration of gravity and an upward acceleration of air resistance. If the y-axis is directed *downward*, then the ball's velocity $v = dy/dt$ and its gravitational acceleration $g = 32$ ft/s^2 are both positive, while its acceleration due to air resistance is negative. Hence its total acceleration is of the form

$$\frac{dv}{dt} = g - kv. \tag{3}$$

A typical value of the air resistance proportionality constant might be $k = 0.16$.

Example 3

Falling baseball Suppose you throw a baseball straight downward from a helicopter hovering at an altitude of 3000 feet. You wonder whether someone standing on the ground below could conceivably catch it. In order to estimate the speed with which the ball will land, you can use your laptop's computer algebra system to construct a slope field for the differential equation

$$\frac{dv}{dt} = 32 - 0.16v. \tag{4}$$

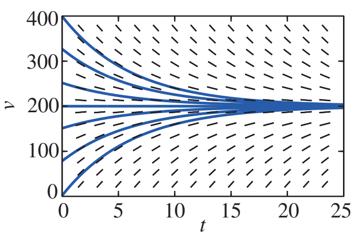

FIGURE 1.3.7. Slope field and typical solution curves for $v' = 32 - 0.16v$.

The result is shown in Fig. 1.3.7, together with a number of solution curves corresponding to different values of the initial velocity $v(0)$ with which you might throw the baseball downward. Note that all these solution curves appear to approach the horizontal line $v = 200$ as an asymptote. This implies that—however you throw it—the baseball should approach the *limiting velocity* $v = 200$ ft/s instead of accelerating indefinitely (as it would in the absence of any air resistance). The handy fact that 60 mi/h = 88 ft/s yields

$$v = 200\,\frac{\cancel{\text{ft}}}{\cancel{\text{s}}} \times \frac{60\ \text{mi/h}}{88\ \cancel{\text{ft/s}}} \approx 136.36\ \frac{\text{mi}}{\text{h}}.$$

Perhaps a catcher accustomed to 100 mi/h fastballs would have some chance of fielding this speeding ball. ■

Comment If the ball's initial velocity is $v(0) = 200$, then Eq. (4) gives $v'(0) = 32 - (0.16)(200) = 0$, so the ball experiences *no* initial acceleration. Its velocity therefore remains unchanged, and hence $v(t) \equiv 200$ is a constant "equilibrium solution" of the differential equation. If the initial velocity is greater than 200, then the initial acceleration given by Eq. (4) is negative, so the ball slows down as it falls. But if the initial velocity is less than 200, then the initial acceleration given by (4) is positive, so the ball speeds up as it falls. It therefore seems quite reasonable that, because of air resistance, the baseball will approach a limiting velocity of 200 ft/s—whatever initial velocity it starts with. You might like to verify that—in the absence of air resistance—this ball would hit the ground at over 300 mi/h. ■

In Section 2.1 we will discuss in detail the logistic differential equation

$$\frac{dP}{dt} = kP(M - P) \tag{5}$$

that often is used to model a population $P(t)$ that inhabits an environment with *carrying capacity* M. This means that M is the maximum population that this environment can sustain on a long-term basis (in terms of the maximum available food, for instance).

Example 4 **Limiting population** If we take $k = 0.0004$ and $M = 150$, then the logistic equation in (5) takes the form

$$\frac{dP}{dt} = 0.0004P(150 - P) = 0.06P - 0.0004P^2. \tag{6}$$

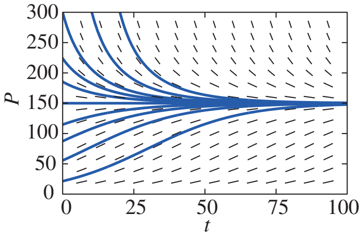

FIGURE 1.3.8. Slope field and typical solution curves for $P' = 0.06P - 0.0004P^2$.

The positive term $0.06P$ on the right in (6) corresponds to natural growth at a 6% annual rate (with time t measured in years). The negative term $-0.0004P^2$ represents the inhibition of growth due to limited resources in the environment.

Figure 1.3.8 shows a slope field for Eq. (6), together with a number of solution curves corresponding to possible different values of the initial population $P(0)$. Note that all these solution curves appear to approach the horizontal line $P = 150$ as an asymptote. This implies that—whatever the initial population—the population $P(t)$ approaches the *limiting population* $P = 150$ as $t \to \infty$. ■

Comment If the initial population is $P(0) = 150$, then Eq. (6) gives

$$P'(0) = 0.0004(150)(150 - 150) = 0,$$

so the population experiences *no* initial (instantaneous) change. It therefore remains unchanged, and hence $P(t) \equiv 150$ is a constant "equilibrium solution" of the differential equation. If the initial population is greater than 150, then the initial rate of change given by (6) is negative, so the population immediately begins to decrease. But if the initial population is less than 150, then the initial rate of change given by (6) is positive, so the population immediately begins to increase. It therefore seems quite reasonable to conclude that the population will approach a limiting value of 150—whatever the (positive) initial population. ■

Existence and Uniqueness of Solutions

Before one spends much time attempting to solve a given differential equation, it is wise to know that solutions actually *exist*. We may also want to know whether there is only one solution of the equation satisfying a given initial condition—that is, whether its solutions are *unique*.

Example 5 Failure of existence **(a)** The initial value problem

$$y' = \frac{1}{x}, \quad y(0) = 0 \tag{7}$$

has *no* solution, because no solution $y(x) = \int (1/x)\,dx = \ln|x| + C$ of the differential equation is defined at $x = 0$. We see this graphically in Fig. 1.3.9, which shows a direction field and some typical solution curves for the equation $y' = 1/x$. It is apparent that the indicated direction field "forces" all solution curves near the y-axis to plunge downward so that none can pass through the point $(0, 0)$.

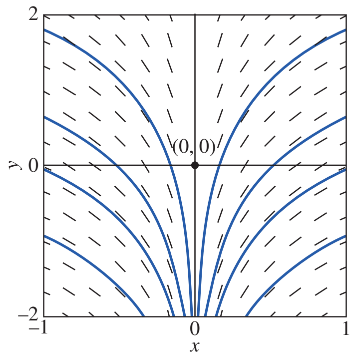

FIGURE 1.3.9. Direction field and typical solution curves for the equation $y' = 1/x$.

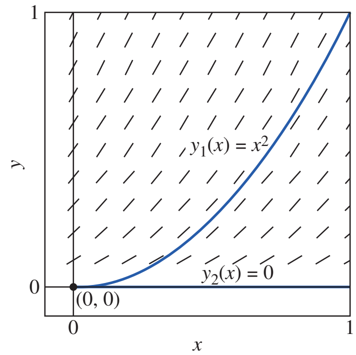

FIGURE 1.3.10. Direction field and two different solution curves for the initial value problem $y' = 2\sqrt{y}$, $y(0) = 0$.

Failure of uniqueness **(b)** On the other hand, you can readily verify that the initial value problem

$$y' = 2\sqrt{y}, \quad y(0) = 0 \tag{8}$$

has the *two* different solutions $y_1(x) = x^2$ and $y_2(x) \equiv 0$ (see Problem 27). Figure 1.3.10 shows a direction field and these two different solution curves for the initial value problem in (8). We see that the curve $y_1(x) = x^2$ threads its way through the indicated direction field, whereas the differential equation $y' = 2\sqrt{y}$ specifies slope $y' = 0$ along the x-axis $y_2(x) = 0$. ■

Example 5 illustrates the fact that, before we can speak of "the" solution of an initial value problem, we need to know that it has *one and only one* solution. Questions of existence and uniqueness of solutions also bear on the process of mathematical modeling. Suppose that we are studying a physical system whose behavior is completely determined by certain initial conditions, but that our proposed mathematical model involves a differential equation *not* having a unique solution satisfying those conditions. This raises an immediate question as to whether the mathematical model adequately represents the physical system.

The theorem stated below implies that the initial value problem $y' = f(x, y)$, $y(a) = b$ has one and only one solution defined near the point $x = a$ on the x-axis, provided that both the function f and its partial derivative $\partial f/\partial y$ are continuous near the point (a, b) in the xy-plane. Methods of proving existence and uniqueness theorems are discussed in the Appendix.

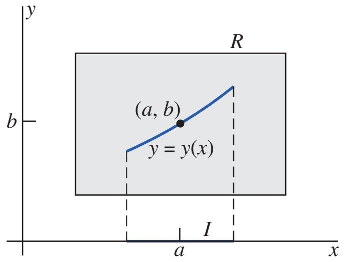

FIGURE 1.3.11. The rectangle R and x-interval I of Theorem 1, and the solution curve $y = y(x)$ through the point (a, b).

THEOREM 1 Existence and Uniqueness of Solutions

Suppose that both the function $f(x, y)$ and its partial derivative $D_y f(x, y)$ are continuous on some rectangle R in the xy-plane that contains the point (a, b) in its interior. Then, for some open interval I containing the point a, the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(a) = b \tag{9}$$

has one and only one solution that is defined on the interval I. (As illustrated in Fig. 1.3.11, the solution interval I may not be as "wide" as the original rectangle R of continuity; see Remark 3 below.)

Remark 1 In the case of the differential equation $dy/dx = -y$ of Example 1 and Fig. 1.3.2(c), both the function $f(x, y) = -y$ and the partial derivative $\partial f/\partial y = -1$ are continuous everywhere, so Theorem 1 implies the existence of a unique solution for any initial data (a, b). Although the theorem ensures existence only on some open interval containing $x = a$, each solution $y(x) = Ce^{-x}$ actually is defined for all x.

Remark 2 In the case of the differential equation $dy/dx = 2\sqrt{y}$ of Example 5(b) and Eq. (8), the function $f(x, y) = 2\sqrt{y}$ is continuous wherever $y > 0$, but the partial derivative $\partial f/\partial y = 1/\sqrt{y}$ is discontinuous when $y = 0$, and hence at the point $(0, 0)$. This is why it is possible for there to exist two different solutions $y_1(x) = x^2$ and $y_2(x) \equiv 0$, each of which satisfies the initial condition $y(0) = 0$.

Remark 3 In Example 7 of Section 1.1 we examined the especially simple differential equation $dy/dx = y^2$. Here we have $f(x, y) = y^2$ and $\partial f/\partial y = 2y$. Both of these functions are continuous everywhere in the xy-plane, and in particular on the rectangle $-2 < x < 2$, $0 < y < 2$. Because the point $(0, 1)$ lies in the interior of this rectangle, Theorem 1 guarantees a unique solution—necessarily a continuous function—of the initial value problem

$$\frac{dy}{dx} = y^2, \quad y(0) = 1 \tag{10}$$

on some open x-interval containing $a = 0$. Indeed this is the solution

$$y(x) = \frac{1}{1 - x}$$

that we discussed in Example 7. But $y(x) = 1/(1 - x)$ is discontinuous at $x = 1$, so our unique continuous solution does not exist on the entire interval $-2 < x < 2$. Thus the solution interval I of Theorem 1 may not be as wide as the rectangle R where f and $\partial f/\partial y$ are continuous. Geometrically, the reason is that the solution curve provided by the theorem may leave the rectangle—wherein solutions of the differential equation are guaranteed to exist—before it reaches the one or both ends of the interval (see Fig. 1.3.12). ■

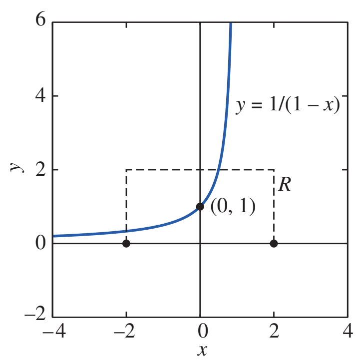

FIGURE 1.3.12. The solution curve through the initial point $(0, 1)$ leaves the rectangle R before it reaches the right side of R.

The following example shows that, if the function $f(x, y)$ and/or its partial derivative $\partial f/\partial y$ fail to satisfy the continuity hypothesis of Theorem 1, then the initial value problem in (9) may have *either* no solution *or* many—even infinitely many—solutions.

Example 6 Consider the first-order differential equation

$$x\frac{dy}{dx} = 2y. \tag{11}$$

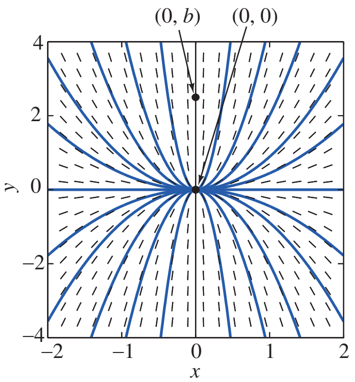

FIGURE 1.3.13. There are infinitely many solution curves through the point $(0, 0)$, but no solution curves through the point $(0, b)$ if $b \neq 0$.

Applying Theorem 1 with $f(x, y) = 2y/x$ and $\partial f/\partial y = 2/x$, we conclude that Eq. (11) must have a unique solution near any point in the xy-plane where $x \neq 0$. Indeed, we see immediately by substitution in (11) that

$$y(x) = Cx^2 \tag{12}$$

satisfies Eq. (11) for any value of the constant C and for all values of the variable x. In particular, the initial value problem

$$x\frac{dy}{dx} = 2y, \quad y(0) = 0 \tag{13}$$

has infinitely many different solutions, whose solution curves are the parabolas $y = Cx^2$ illustrated in Fig. 1.3.13. (In case $C = 0$ the "parabola" is actually the x-axis $y = 0$.)

Observe that all these parabolas pass through the origin $(0, 0)$, but none of them passes through any other point on the y-axis. It follows that the initial value problem in (13) has infinitely many solutions, but the initial value problem

$$x\frac{dy}{dx} = 2y, \quad y(0) = b \tag{14}$$

has no solution if $b \neq 0$.

Finally, note that through any point off the y-axis there passes only one of the parabolas $y = Cx^2$. Hence, if $a \neq 0$, then the initial value problem

$$x\frac{dy}{dx} = 2y, \quad y(a) = b \tag{15}$$

has a unique solution on any interval that contains the point $x = a$ but not the origin $x = 0$. In summary, the initial value problem in (15) has

- a unique solution near (a, b) if $a \neq 0$;
- no solution if $a = 0$ but $b \neq 0$;
- infinitely many solutions if $a = b = 0$. ■

Still more can be said about the initial value problem in (15). Consider a typical initial point off the y-axis—for instance the point $(-1, 1)$ indicated in Fig. 1.3.14. Then for any value of the constant C the function defined by

$$y(x) = \begin{cases} x^2 & \text{if } x \leq 0, \\ Cx^2 & \text{if } x > 0 \end{cases} \tag{16}$$

is continuous and satisfies the initial value problem

$$x\frac{dy}{dx} = 2y, \quad y(-1) = 1. \tag{17}$$

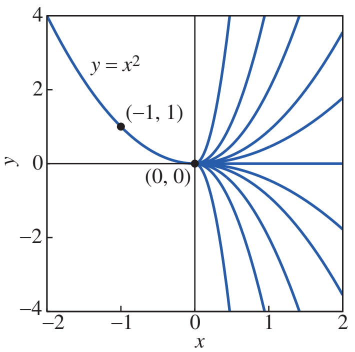

FIGURE 1.3.14. There are infinitely many solution curves through the point $(1, -1)$.

For a particular value of C, the solution curve defined by (16) consists of the left half of the parabola $y = x^2$ and the right half of the parabola $y = Cx^2$. Thus the unique solution curve near $(-1, 1)$ branches at the origin into the infinitely many solution curves illustrated in Fig. 1.3.14.

We therefore see that Theorem 1 (if its hypotheses are satisfied) guarantees uniqueness of the solution near the initial point (a, b), but a solution curve through (a, b) may eventually branch elsewhere so that uniqueness is lost. Thus a solution may exist on a larger interval than one on which the solution is unique. For instance, the solution $y(x) = x^2$ of the initial value problem in (17) exists on the whole x-axis, but this solution is unique only on the negative x-axis $-\infty < x < 0$.

1.3 Problems

In Problems 1 through 4, construct a slope field for the given differential equation by drawing line segments with the appropriate slopes through the points (x, y) with $x, y = -2, -1, 0, 1, 2$.

1. $\dfrac{dy}{dx} = -y - \sin x$

2. $\dfrac{dy}{dx} = x + y$

3. $\dfrac{dy}{dx} = y - \sin x$

4. $\dfrac{dy}{dx} = x - y$

In Problems 5 through 10, we have provided the slope field of the indicated differential equation, together with one or more solution curves (see Figs. 1.3.15 through 1.3.20). Sketch likely solution curves through the additional points marked in each slope field.

5. $\dfrac{dy}{dx} = y - x + 1$

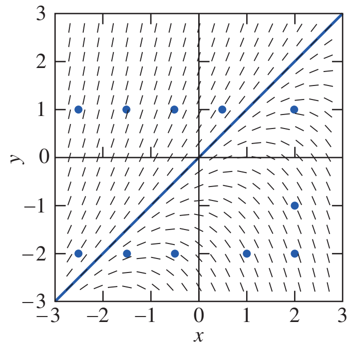

FIGURE 1.3.15.

6. $\dfrac{dy}{dx} = x - y + 1$

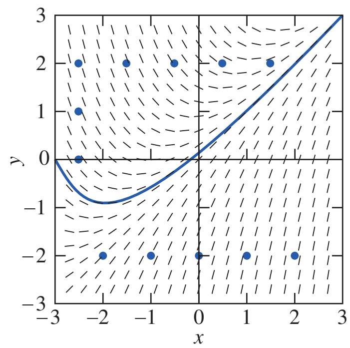

FIGURE 1.3.16.

7. $\dfrac{dy}{dx} = \sin x + \sin y$

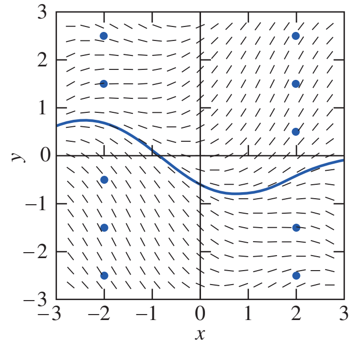

FIGURE 1.3.17.

8. $\dfrac{dy}{dx} = x^2 - y$

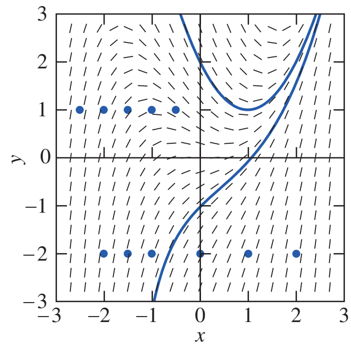

FIGURE 1.3.18.

9. $\dfrac{dy}{dx} = x^2 - y - 2$

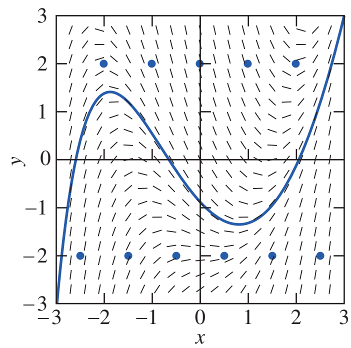

FIGURE 1.3.19.

10. $\dfrac{dy}{dx} = -x^2 + \sin y$

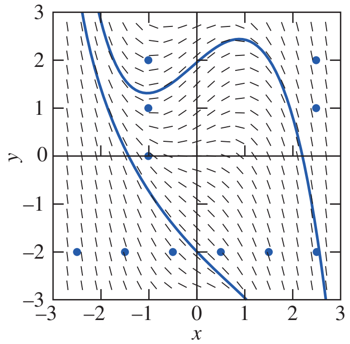

FIGURE 1.3.20.

A more detailed version of Theorem 1 says that, if the function $f(x, y)$ is continuous near the point (a, b), then at least one solution of the differential equation $y' = f(x, y)$ exists on some open interval I containing the point $x = a$ and, moreover, that if in addition the partial derivative $\partial f/\partial y$ is continuous near (a, b), then this solution is unique on some (perhaps smaller) interval J. In Problems 11 through 20, determine whether existence of at least one solution of the given initial value problem is thereby guaranteed and, if so, whether uniqueness of that solution is guaranteed.

11. $\dfrac{dy}{dx} = 2x^2y^2; \quad y(1) = -1$

12. $\dfrac{dy}{dx} = x \ln y; \quad y(1) = 1$

13. $\dfrac{dy}{dx} = \sqrt[3]{y}; \quad y(0) = 1$

14. $\dfrac{dy}{dx} = \sqrt[3]{y}; \quad y(0) = 0$

15. $\dfrac{dy}{dx} = \sqrt{x - y}; \quad y(2) = 2$

16. $\dfrac{dy}{dx} = \sqrt{x - y}; \quad y(2) = 1$

17. $y\dfrac{dy}{dx} = x - 1; \quad y(0) = 1$

18. $y\dfrac{dy}{dx} = x - 1; \quad y(1) = 0$

19. $\dfrac{dy}{dx} = \ln(1 + y^2); \quad y(0) = 0$

20. $\dfrac{dy}{dx} = x^2 - y^2; \quad y(0) = 1$

In Problems 21 and 22, first use the method of Example 2 to construct a slope field for the given differential equation. Then sketch the solution curve corresponding to the given initial condition. Finally, use this solution curve to estimate the desired value of the solution $y(x)$.

21. $y' = x + y, \quad y(0) = 0; \quad y(-4) = ?$

22. $y' = y - x, \quad y(4) = 0; \quad y(-4) = ?$

Problems 23 and 24 are like Problems 21 and 22, but now use a computer algebra system to plot and print out a slope field for the given differential equation. If you wish (and know how), you can check your manually sketched solution curve by plotting it with the computer.

23. $y' = x^2 + y^2 - 1, \quad y(0) = 0; \quad y(2) = ?$

24. $y' = x + \frac{1}{2}y^2, \quad y(-2) = 0; \quad y(2) = ?$

25. Falling parachutist You bail out of the helicopter of Example 3 and pull the ripcord of your parachute. Now $k = 1.6$ in Eq. (3), so your downward velocity satisfies the initial value problem

$$\frac{dv}{dt} = 32 - 1.6v, \quad v(0) = 0.$$

In order to investigate your chances of survival, construct a slope field for this differential equation and sketch the appropriate solution curve. What will your limiting velocity be? Will a strategically located haystack do any good? How long will it take you to reach 95% of your limiting velocity?

26. Deer population Suppose the deer population $P(t)$ in a small forest satisfies the logistic equation

$$\frac{dP}{dt} = 0.0225P - 0.0003P^2.$$

Construct a slope field and appropriate solution curve to answer the following questions: If there are 25 deer at time $t = 0$ and t is measured in months, how long will it take the number of deer to double? What will be the limiting deer population?

The next seven problems illustrate the fact that, if the hypotheses of Theorem 1 are not satisfied, then the initial value problem $y' = f(x, y)$, $y(a) = b$ may have either no solutions, finitely many solutions, or infinitely many solutions.

27. **(a)** Verify that if c is a constant, then the function defined piecewise by

$$y(x) = \begin{cases} 0 & \text{for } x \leqq c, \\ (x - c)^2 & \text{for } x > c \end{cases}$$

satisfies the differential equation $y' = 2\sqrt{y}$ for all x (including the point $x = c$). Construct a figure illustrating the fact that the initial value problem $y' = 2\sqrt{y}$, $y(0) = 0$ has infinitely many different solutions. **(b)** For what values of b does the initial value problem $y' = 2\sqrt{y}$, $y(0) = b$ have (i) no solution, (ii) a unique solution that is defined for all x?

28. Verify that if k is a constant, then the function $y(x) \equiv kx$ satisfies the differential equation $xy' = y$ for all x. Construct a slope field and several of these straight line solution curves. Then determine (in terms of a and b) how many different solutions the initial value problem $xy' = y$, $y(a) = b$ has—one, none, or infinitely many.

29. Verify that if c is a constant, then the function defined piecewise by

$$y(x) = \begin{cases} 0 & \text{for } x \leqq c, \\ (x-c)^3 & \text{for } x > c \end{cases}$$

satisfies the differential equation $y' = 3y^{2/3}$ for all x. Can you also use the "left half" of the cubic $y = (x-c)^3$ in piecing together a solution curve of the differential equation? (See Fig. 1.3.21.) Sketch a variety of such solution curves. Is there a point (a, b) of the xy-plane such that the initial value problem $y' = 3y^{2/3}$, $y(a) = b$ has either no solution or a unique solution that is defined for all x? Reconcile your answer with Theorem 1.

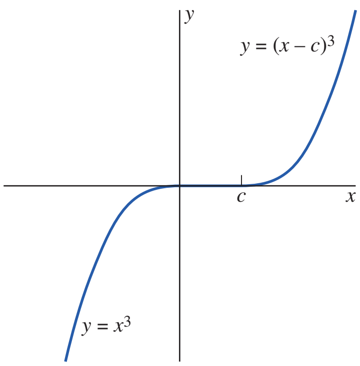

FIGURE 1.3.21. A suggestion for Problem 29.

30. Verify that if c is a constant, then the function defined piecewise by

$$y(x) = \begin{cases} +1 & \text{if } x \leqq c, \\ \cos(x-c) & \text{if } c < x < c+\pi, \\ -1 & \text{if } x \geqq c+\pi \end{cases}$$

satisfies the differential equation $y' = -\sqrt{1-y^2}$ for all x. (Perhaps a preliminary sketch with $c = 0$ will be helpful.) Sketch a variety of such solution curves. Then determine (in terms of a and b) how many different solutions the initial value problem $y' = -\sqrt{1-y^2}$, $y(a) = b$ has.

31. Carry out an investigation similar to that in Problem 30, except with the differential equation $y' = +\sqrt{1-y^2}$. Does it suffice simply to replace $\cos(x-c)$ with $\sin(x-c)$ in piecing together a solution that is defined for all x?

32. Verify that if $c > 0$, then the function defined piecewise by

$$y(x) = \begin{cases} 0 & \text{if } x^2 \leqq c, \\ (x^2-c)^2 & \text{if } x^2 > c \end{cases}$$

satisfies the differential equation $y' = 4x\sqrt{y}$ for all x. Sketch a variety of such solution curves for different values of c. Then determine (in terms of a and b) how many different solutions the initial value problem $y' = 4x\sqrt{y}$, $y(a) = b$ has.

33. If $c \neq 0$, verify that the function defined by $y(x) = x/(cx-1)$ (with the graph illustrated in Fig. 1.3.22) satisfies the differential equation $x^2y' + y^2 = 0$ if $x \neq 1/c$. Sketch a variety of such solution curves for different values of c. Also, note the constant-valued function $y(x) \equiv 0$ that does not result from any choice of the constant c. Finally, determine (in terms of a and b) how many different solutions the initial value problem $x^2y' + y^2 = 0$, $y(a) = b$ has.

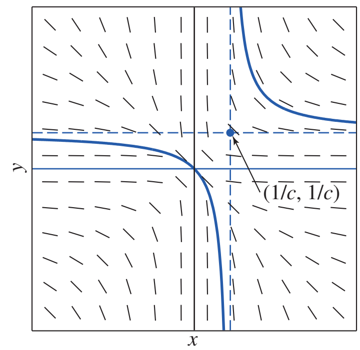

FIGURE 1.3.22. Slope field for $x^2y' + y^2 = 0$ and graph of a solution $y(x) = x/(cx-1)$.

34. **(a)** Use the direction field of Problem 5 to estimate the values at $x = 1$ of the two solutions of the differential equation $y' = y - x + 1$ with initial values $y(-1) = -1.2$ and $y(-1) = -0.8$.

(b) Use a computer algebra system to estimate the values at $x = 3$ of the two solutions of this differential equation with initial values $y(-3) = -3.01$ and $y(-3) = -2.99$.

The lesson of this problem is that small changes in initial conditions can make big differences in results.

35. **(a)** Use the direction field of Problem 6 to estimate the values at $x = 2$ of the two solutions of the differential equation $y' = x - y + 1$ with initial values $y(-3) = -0.2$ and $y(-3) = +0.2$.

(b) Use a computer algebra system to estimate the values at $x = 2$ of the two solutions of this differential equation with initial values $y(-3) = -0.5$ and $y(-3) = +0.5$.

The lesson of this problem is that big changes in initial conditions may make only small differences in results.

1.3 Application Computer-Generated Slope Fields and Solution Curves

Go to **bit.ly/3EmOgiO** for additional discussion and explorations of this topic using computational resources such as *Maple*/*Mathematica*/MATLAB.

Widely available computer algebra systems and technical computing environments include facilities to automate the construction of slope fields and solution curves, as do some graphing calculators (see Figs. 1.3.23–1.3.25).

The additional material available online for this Application includes discussion of *Maple*™, *Mathematica*™, and MATLAB resources for the investigation of differential equations. For instance, the *Maple* command

```
with(DEtools):
DEplot(diff(y(x),x)=sin(x-y(x)), y(x), x=-5..5, y=-5..5);
```

and the *Mathematica* command

```
VectorPlot[{1, Sin[x-y]}, {x, -5, 5}, {y, -5, 5}]
```

FIGURE 1.3.23. TI-84 Plus CE Python graphing calculator and TI Nspire™ CX II CAS graphing calculator. Screenshot from Texas Instruments Incorporated. Courtesy of Texas Instruments Incorporated.

produce slope fields similar to the one shown in Fig. 1.3.25. Figure 1.3.25 itself was generated with the MATLAB program **dfield** [John Polking and David Arnold, *Ordinary Differential Equations Using* MATLAB, 3rd edition, York, NY: Pearson Education, 2003]. The web site **cs.unm.edu/~joel/dfield** provides a freely-available Java version of **dfield**. When a differential equation is entered in the **dfield** setup menu (Fig. 1.3.26), you can (with mouse button clicks) plot both a slope field and the solution curve (or curves) through any desired point (or points). Another freely available and user-friendly MATLAB-based ODE package with impressive graphical capabilities is **Iode**, available at **conf.math.illinois.edu/iode**.

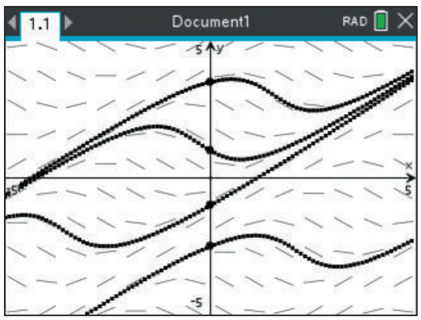

FIGURE 1.3.24. Slope field and solution curves for the differential equation

$$\frac{dy}{dx} = \sin(x - y)$$

with initial points $(0, b)$, $b = -2.5, -1, 1, 3.5$ and window $-5 \leq x, y \leq 5$ on a TI Nspire™ CX II CAS graphing calculator. Courtesy of Texas Instruments Incorporated.

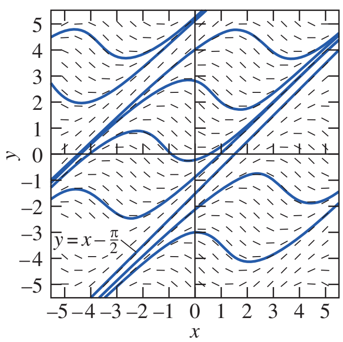

FIGURE 1.3.25. Computer-generated slope field and solution curves for the differential equation $y' = \sin(x - y)$.

Modern technology platforms offer even further interactivity by allowing the user to vary initial conditions and other parameters "in real time." *Mathematica*'s **Manipulate** command was used to generate Fig. 1.3.27, which shows three particular solutions of the differential equation $dy/dx = \sin(x - y)$. The solid curve corresponds to the initial condition $y(1) = 0$. As the "locator point" initially at $(1, 0)$ is dragged—by mouse or touchpad—to the point $(0, 3)$ or $(2, -2)$, the solution curve immediately follows, resulting in the dashed curves shown. The TI Nspire™ CX II CAS similar capability; indeed, as Fig. 1.3.24 appears on the Nspire display, each of the initial points $(0, b)$ can be dragged to different locations using the Nspire's touchpad, with the corresponding solution curves being instantly redrawn.

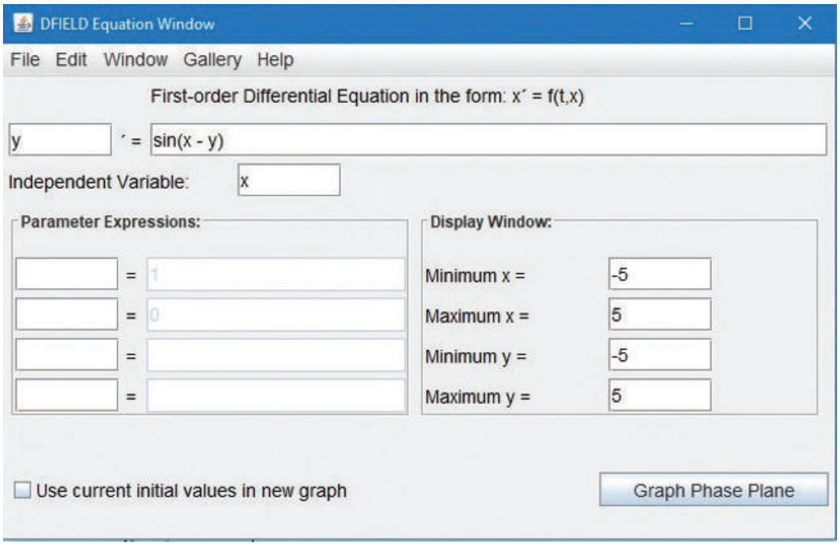

FIGURE 1.3.26. MATLAB **dfield** setup to construct slope field and solution curves for $y' = \sin(x - y)$.

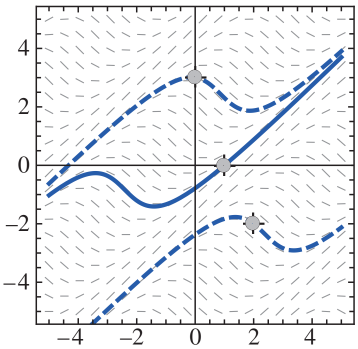

FIGURE 1.3.27. Interactive *Mathematica* solution of the differential equation $y' = \sin(x - y)$. The "locator point" corresponding to the initial condition $y(1) = 0$ can be dragged to any other point in the display, causing the solution curve to be automatically redrawn.

Your Turn

Use a graphing calculator or computer system in the following investigations. You might warm up by generating the slope fields and some solution curves for Problems 1 through 10 in this section.

INVESTIGATION A: Plot a slope field and typical solution curves for the differential equation $dy/dx = \sin(x - y)$, but with a larger window than that of Fig. 1.3.25. With $-10 \leqq x \leqq 10$, $-10 \leqq y \leqq 10$, for instance, a number of apparent straight line solution curves should be visible, especially if your display allows you to drag the initial point interactively from upper left to lower right.

(a) Substitute $y = ax + b$ in the differential equation to determine what the coefficients a and b must be in order to get a solution. Are the results consistent with what you see on the display?

(b) A computer algebra system gives the general solution

$$y(x) = x - 2\tan^{-1}\left(\frac{x - 2 - C}{x - C}\right).$$

Plot this solution with selected values of the constant C to compare the resulting solution curves with those indicated in Fig. 1.3.24. Can you see that *no* value of C yields the linear solution $y = x - \pi/2$ corresponding to the initial condition $y(\pi/2) = 0$? Are there any values of C for which the corresponding solution curves lie close to this straight line solution curve?

INVESTIGATION B: For your own personal investigation, let n be the *smallest* digit in your student ID number that is greater than 1, and consider the differential equation

$$\frac{dy}{dx} = \frac{1}{n}\cos(x - ny).$$

(a) First investigate (as in part (a) of Investigation A) the possibility of straight line solutions.

(b) Then generate a slope field for this differential equation, with the viewing window chosen so that you can picture some of these straight lines, plus a sufficient number of nonlinear solution curves that you can formulate a conjecture about what happens to $y(x)$ as $x \to +\infty$. State your inference as plainly as you can. Given the initial value $y(0) = y_0$, try to predict (perhaps in terms of y_0) how $y(x)$ behaves as $x \to +\infty$.

(c) A computer algebra system gives the general solution

$$y(x) = \frac{1}{n}\left[x + 2\tan^{-1}\left(\frac{1}{x - C}\right)\right].$$

Can you make a connection between this symbolic solution and your graphically generated solution curves (straight lines or otherwise)?

1.4 Separable Equations and Applications

In the preceding sections we saw that if the function $f(x, y)$ does not involve the variable y, then solving the first-order differential equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

is a matter of simply finding an antiderivative. For example, the general solution of

$$\frac{dy}{dx} = -6x \tag{2}$$

is given by

$$y(x) = \int -6x\,dx = -3x^2 + C.$$

If instead $f(x, y)$ *does* involve the dependent variable y, then we can no longer solve the equation merely by integrating both sides: The differential equation

$$\frac{dy}{dx} = -6xy \tag{3a}$$

differs from Eq. (2) only in the factor y appearing on the right-hand side, but this is enough to prevent us from using the same approach to solve Eq. (3a) that was successful with Eq. (2).

And yet, as we will see throughout the remainder of this chapter, differential equations like (3a) often can, in fact, be solved by methods which are based on the idea of "integrating both sides." The idea behind these techniques is to rewrite the given equation in a form that, while equivalent to the given equation, allows both sides to be integrated directly, thus leading to the solution of the original differential equation.

The most basic of these methods, *separation of variables*, can be applied to Eq. (3a). First, we note that the right-hand function $f(x, y) = -6xy$ can be viewed as the *product* of two expressions, one involving only the independent variable x, and the other involving only the dependent variable y:

$$\frac{dy}{dx} = \underbrace{\boxed{(-6x)}}_{\text{depends only on } x} \cdot \underbrace{\boxed{y}}_{\text{depends only on } y} \tag{3b}$$

Next, we informally break up the derivative dy/dx into the "free-floating" differentials dx and dy—a notational convenience that leads to correct results, as we will see below—and then multiply by dx and divide by y in Eq. (3b), leading to

$$\frac{dy}{y} = -6x\,dx. \tag{3c}$$

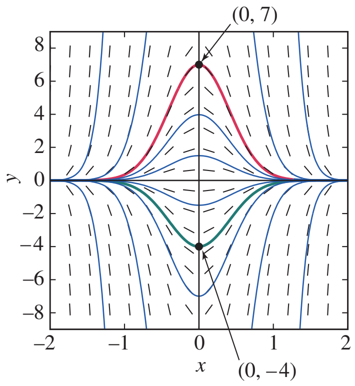

FIGURE 1.4.1. Slope field and solution curves for $y' = -6xy$.

Equation (3c) is an equivalent version of the original differential equation in (3a), but with the variables x and y *separated* (that is, by the equal sign), and this is what allows us to integrate both sides. The left-hand side is integrated with respect to y (with no "interference" from the variable x), and *vice versa* for the right-hand side. This leads to

$$\int \frac{dy}{y} = \int -6x\,dx,$$

or

$$\ln|y| = -3x^2 + C. \tag{4}$$

This gives the general solution of Eq. (3a) *implicitly*, and a family of solution curves is shown in Fig. 1.4.1.

In this particular case we can go on to solve for y to give the *explicit* general solution

$$y(x) = \pm e^{-3x^2+C} = \pm e^{-3x^2}e^C = Ae^{-3x^2}, \tag{5}$$

where A represents the constant $\pm e^C$, which can take on any nonzero value. If we impose an initial condition on Eq. (3a), say $y(0) = 7$, then in Eq. (5) we find that $A = 7$, yielding the particular solution

$$y(x) = 7e^{-3x^2},$$

which is the upper emphasized solution curve shown in Fig. 1.4.1. In the same way, the initial condition $y(0) = -4$ leads to the particular solution

$$y(x) = -4e^{-3x^2},$$

which is the lower emphasized solution curve shown in Fig. 1.4.1.

To complete this example, we note that whereas the constant A in Eq. (5) is nonzero, taking $A = 0$ in (5) leads to $y(x) \equiv 0$, and this is, in fact, a solution of the given differential equation (3a). Thus Eq. (5) actually provides a solution of (3a) for *all* values of the constant A, including $A = 0$. Why did the method of separation of variables fail to capture all solutions of Eq. (3a)? The reason is that in the step in which we actually separated the variables, that is, in passing from Eq. (3b) to (3c), we *divided by* y, thus (tacitly) assuming that $y \neq 0$. As a result, our general solution (5), with its restriction that $A \neq 0$, "missed" the particular solution $y(x) \equiv 0$ corresponding to $A = 0$. Such solutions are known as *singular solutions*, and we say more about them—together with implicit and general solutions—below.

In general, the first-order differential equation (1) is called **separable** provided that $f(x, y)$ can be written as the product of a function of x and a function of y:

$$\frac{dy}{dx} = f(x, y) = g(x)k(y) = \frac{g(x)}{h(y)},$$

where $h(y) = 1/k(y)$. In this case the variables x and y can be *separated*—isolated on opposite sides of an equation—by writing informally the equation

$$h(y)\,dy = g(x)\,dx,$$

which we understand to be concise notation for the differential equation

$$h(y)\frac{dy}{dx} = g(x). \tag{6}$$

(In the preceding example, $h(y) = \frac{1}{y}$ and $g(x) = -6x$.) As illustrated above, we can solve this type of differential equation simply by integrating both sides with respect to x:

$$\int h(y(x))\frac{dy}{dx}\,dx = \int g(x)\,dx + C;$$

equivalently,

$$\int h(y)\,dy = \int g(x)\,dx + C. \tag{7}$$

All that is required is that the antiderivatives

$$H(y) = \int h(y)\,dy \qquad \text{and} \qquad G(x) = \int g(x)\,dx$$

can be found. To see that Eqs. (6) and (7) are equivalent, note the following consequence of the chain rule:

$$D_x[H(y(x))] = H'(y(x))y'(x) = h(y)\frac{dy}{dx} = g(x) = D_x[G(x)],$$

which in turn is equivalent to

$$H(y(x)) = G(x) + C, \tag{8}$$

because two functions have the same derivative on an interval if and only if they differ by a constant on that interval.

Example 1 Solve the differential equation

$$\frac{dy}{dx} = \frac{4-2x}{3y^2-5}. \tag{9}$$

Solution Because

$$\frac{4-2x}{3y^2-5} = (4-2x)\cdot\frac{1}{3y^2-5} = g(x)k(y)$$

is the product of a function that depends only on x, and one that depends only on y, Eq. (9) is separable, and thus we can proceed in much the same way as in Eq. (3a). Before doing so, however, we note an important feature of Eq. (9) not shared by Eq. (3a): The function $k(y) = \dfrac{1}{3y^2-5}$ is not defined for all values of y. Indeed, setting $3y^2 - 5$ equal to zero shows that $k(y)$, and thus $\dfrac{dy}{dx}$ itself, becomes infinite as y approaches either of $\pm\sqrt{\frac{5}{3}}$. Because an infinite slope corresponds to a vertical line segment, we would therefore expect the line segments in the slope field for this differential equation to be "standing straight up" along the two horizontal lines $y = \pm\sqrt{\frac{5}{3}} \approx \pm 1.29$; as Fig. 1.4.2 shows (where these two lines are dashed), this is indeed what we find.

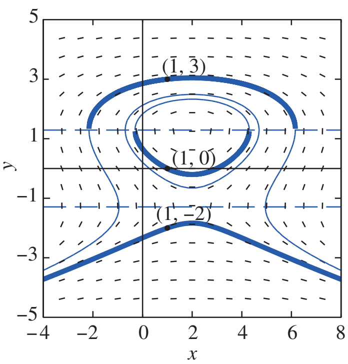

FIGURE 1.4.2. Slope field and solution curves for $y' = (4-2x)/(3y^2-5)$ in Example 1.

What this means for the differential equation (9) is that *no solution curve of this equation can cross either of the horizontal lines* $y = \pm\sqrt{\frac{5}{3}}$, simply because along these lines $\dfrac{dy}{dx}$ is undefined. Effectively, then, these lines divide the plane into three regions—defined by the conditions $y > \sqrt{\frac{5}{3}}$, $-\sqrt{\frac{5}{3}} < y < \sqrt{\frac{5}{3}}$, and $y < -\sqrt{\frac{5}{3}}$—with all solution curves of Eq. (9) remaining confined to one of these regions.

With this in mind, the general solution of the differential equation in Eq. (9) is easy to find, at least in implicit form. Separating variables and integrating both sides leads to

$$\int 3y^2 - 5\,dy = \int 4 - 2x\,dx,$$

and thus

$$y^3 - 5y = 4x - x^2 + C. \tag{10}$$

Note that unlike Eq. (4), the general solution in Eq. (10) cannot readily be solved for y; thus we cannot directly plot the solution curves of Eq. (9) in the form $y = y(x)$, as we would like. However, what we *can* do is rearrange Eq. (10) so that the constant C is isolated on the right-hand side:

$$y^3 - 5y - (4x - x^2) = C. \tag{11}$$

This shows that the solution curves of the differential equation in Eq. (9) are contained in the level curves (also known as contours) of the function

$$F(x, y) = y^3 - 5y - (4x - x^2). \tag{12}$$

Because no particular solution curve of Eq. (9) can cross either of the lines $y = \pm\sqrt{\frac{5}{3}}$—despite the fact that the level curves of $F(x, y)$ freely do so—the particular solution curves of Eq. (9) are those *portions* of the level curves of $F(x, y)$ which avoid the lines $y = \pm\sqrt{\frac{5}{3}}$.

For example, suppose that we wish to solve the initial value problem

$$\frac{dy}{dx} = \frac{4 - 2x}{3y^2 - 5}, \quad y(1) = 3. \tag{13}$$

Substituting $x = 1$ and $y = 3$ into our general solution (10) gives $C = 9$. Therefore our desired solution curve lies on the level curve

$$y^3 - 5y - (4x - x^2) = 9 \tag{14}$$

of $F(x, y)$; Fig. 1.4.2 shows this and other level curves of $F(x, y)$. However, because the solution curve of the initial value problem (13) must pass through the point $(1, 3)$, which lies above the line $y = \sqrt{\frac{5}{3}}$ in the xy-plane, our desired solution curve is restricted to that portion of the level curve (14) which satisfies $y > \sqrt{\frac{5}{3}}$. (In Fig. 1.4.2 the solution curve of the initial value problem (13) is drawn more heavily than the remainder of the level curve (14).) In the same way, Figure 1.4.2 also shows the particular solutions of Eq. (9) subject to the initial conditions $y(1) = 0$ and $y(1) = -2$. In each of these cases, the curve corresponding to the desired particular solution is only a piece of a larger level curve of the function $F(x, y)$. (Note that in fact, some of the level curves of F themselves consist of two pieces.)

Finally, despite the difficulty of solving Eq. (14) for y by algebraic means, we can nonetheless "solve" for y in the sense that, when a specific value of x is substituted in (14), we can attempt to solve numerically for y. For instance, taking $x = 4$ yields the equation

$$f(y) = y^3 - 5y - 9 = 0;$$

Fig. 1.4.3 shows the graph of f. Using technology we can solve for the single real root $y \approx 2.8552$, thus yielding the value $y(4) \approx 2.8552$ for the solution of the initial value problem (13). By repeating this process for other values of x, we can create a table (like the one shown below) of corresponding x- and y-values for the solution of (13); such a table serves effectively as a "numerical solution" of this initial value problem.

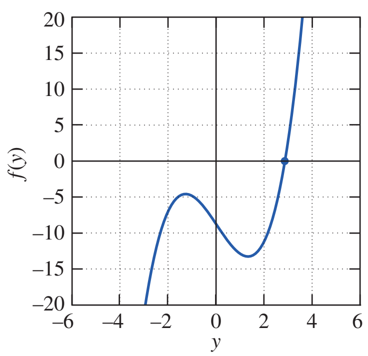

FIGURE 1.4.3. Graph of $f(y) = y^3 - 5y - 9$.

x	−1	0	1	2	3	4	5	6
y	2.5616	2.8552	3	3.0446	3	2.8552	2.5616	1.8342

■

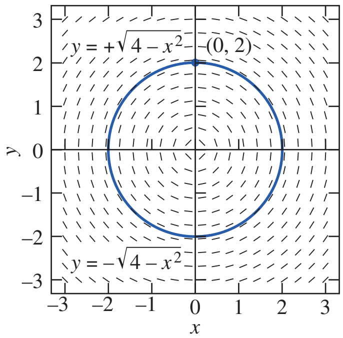

FIGURE 1.4.4. Slope field and solution curves for $y' = -x/y$.

Implicit, General, and Singular Solutions

The equation $K(x, y) = 0$ is commonly called an **implicit solution** of a differential equation if it is satisfied (on some interval) by some solution $y = y(x)$ of the differential equation. But note that a particular solution $y = y(x)$ of $K(x, y) = 0$ may or may not satisfy a given initial condition. For example, differentiation of $x^2 + y^2 = 4$ yields

$$x + y\frac{dy}{dx} = 0,$$

so $x^2 + y^2 = 4$ is an implicit solution of the differential equation $x + yy' = 0$. But only the first of the two explicit solutions

$$y(x) = +\sqrt{4 - x^2} \quad \text{and} \quad y(x) = -\sqrt{4 - x^2}$$

satisfies the initial condition $y(0) = 2$ (Fig. 1.4.4).

Remark 1 You should not assume that every possible algebraic solution $y = y(x)$ of an implicit solution satisfies the same differential equation. For instance, if we multiply the implicit solution $x^2 + y^2 - 4 = 0$ by the factor $(y - 2x)$, then we get the new implicit solution

$$(y - 2x)(x^2 + y^2 - 4) = 0$$

that yields (or "contains") not only the previously noted explicit solutions $y = +\sqrt{4 - x^2}$ and $y = -\sqrt{4 - x^2}$ of the differential equation $x + yy' = 0$, but also the additional function $y = 2x$ that does *not* satisfy this differential equation.

Remark 2 Similarly, solutions of a given differential equation can be either gained or lost when it is multiplied or divided by an algebraic factor. For instance, consider the differential equation

$$(y - 2x)y\frac{dy}{dx} = -x(y - 2x) \tag{15}$$

having the obvious solution $y = 2x$. But if we divide both sides by the common factor $(y - 2x)$, then we get the previously discussed differential equation

$$y\frac{dy}{dx} = -x, \quad \text{or} \quad x + y\frac{dy}{dx} = 0, \tag{16}$$

of which $y = 2x$ is *not* a solution. Thus we "lose" the solution $y = 2x$ of Eq. (15) upon its division by the factor $(y - 2x)$; alternatively, we "gain" this new solution when we multiply Eq. (16) by $(y - 2x)$. Such elementary algebraic operations to simplify a given differential equation before attempting to solve it are common in practice, but the possibility of loss or gain of such "extraneous solutions" should be kept in mind. ■

A solution of a differential equation that contains an "arbitrary constant" (like the constant C appearing in Eqs. (4) and (10)) is commonly called a **general solution** of the differential equation; any particular choice of a specific value for C yields a single particular solution of the equation.

The argument preceding Example 1 actually suffices to show that *every* particular solution of the differential equation $h(y)y' = g(x)$ in (6) satisfies the equation $H(y(x)) = G(x) + C$ in (8). Consequently, it is appropriate to call (8) not merely ***a*** general solution of (6), but ***the*** general solution of (6).

In Section 1.5 we shall see that every particular solution of a *linear* first-order differential equation is contained in its general solution. By contrast, it is common for a nonlinear first-order differential equation to have both a general solution involving an arbitrary constant C and one or several particular solutions that cannot be obtained by selecting a value for C. These exceptional solutions are frequently called **singular solutions**. In Problem 30 we ask you to show that the general solution of the differential equation $(y')^2 = 4y$ yields the family of parabolas $y = (x - C)^2$ illustrated in Fig. 1.4.5, and to observe that the constant-valued function $y(x) \equiv 0$ is a singular solution that cannot be obtained from the general solution by any choice of the arbitrary constant C.

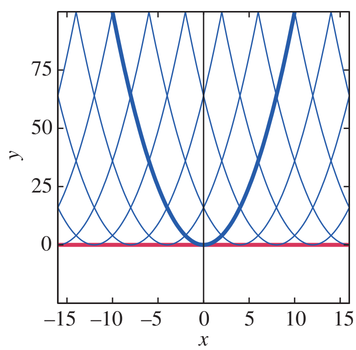

FIGURE 1.4.5. The general solution curves $y = (x - C)^2$ and the singular solution curve $y = 0$ of the differential equation $(y')^2 = 4y$.

Example 2 Find all solutions of the differential equation

$$\frac{dy}{dx} = 6x(y-1)^{2/3}.$$

Solution Separation of variables gives

$$\int \frac{1}{3(y-1)^{2/3}}\,dy = \int 2x\,dx;$$

$$(y-1)^{1/3} = x^2 + C;$$

$$y(x) = 1 + (x^2 + C)^3.$$

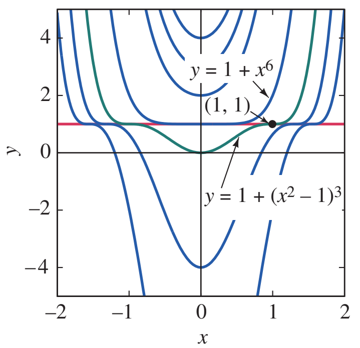

FIGURE 1.4.6. General and singular solution curves for $y' = 6x(y-1)^{2/3}$.

Positive values of the arbitrary constant C give the solution curves in Fig. 1.4.6 that lie above the line $y = 1$, whereas negative values yield those that dip below it. The value $C = 0$ gives the solution $y(x) = 1 + x^6$, but *no* value of C gives the singular solution $y(x) \equiv 1$ that was lost when the variables were separated. Note that the two different solutions $y(x) \equiv 1$ and $y(x) = 1 + (x^2 - 1)^3$ both satisfy the initial condition $y(1) = 1$. Indeed, the whole singular solution curve $y = 1$ consists of points where the solution is not unique and where the function $f(x, y) = 6x(y-1)^{2/3}$ is not differentiable. ■

Natural Growth and Decay

The differential equation

$$\frac{dx}{dt} = kx \quad (k \text{ a constant}) \tag{17}$$

serves as a mathematical model for a remarkably wide range of natural phenomena—any involving a quantity whose time rate of change is proportional to its current size. Here are some examples.

Population Growth: Suppose that $P(t)$ is the number of individuals in a population (of humans, or insects, or bacteria) having *constant* birth and death rates β and δ (in births or deaths per individual per unit of time). Then, during a short time interval Δt, approximately $\beta P(t)\,\Delta t$ births and $\delta P(t)\,\Delta t$ deaths occur, so the change in $P(t)$ is given approximately by

$$\Delta P \approx (\beta - \delta)P(t)\,\Delta t,$$

and therefore

$$\frac{dP}{dt} = \lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t} = kP, \tag{18}$$

where $k = \beta - \delta$.

Compound Interest: Let $A(t)$ be the number of dollars in a savings account at time t (in years), and suppose that the interest is *compounded continuously* at an annual interest rate r. (Note that 10% annual interest means that $r = 0.10$.) Continuous compounding means that during a short time interval Δt, the amount of interest added to the account is approximately $\Delta A = rA(t)\,\Delta t$, so that

$$\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = rA. \tag{19}$$

Radioactive Decay: Consider a sample of material that contains $N(t)$ atoms of a certain radioactive isotope at time t. It has been observed that a constant fraction of those radioactive atoms will spontaneously decay (into atoms of another element or into another isotope of the same element) during each unit of time. Consequently, the sample behaves exactly like a population with a constant death rate and no births. To write a model for $N(t)$, we use Eq. (18) with N in place of P, with $k > 0$ in place of δ, and with $\beta = 0$. We thus get the differential equation

$$\frac{dN}{dt} = -kN. \tag{20}$$

The value of k depends on the particular radioactive isotope.

The key to the method of *radiocarbon dating* is that a constant proportion of the carbon atoms in any living creature is made up of the radioactive isotope ^{14}C of carbon. This proportion remains constant because the fraction of ^{14}C in the atmosphere remains almost constant, and living matter is continuously taking up carbon from the air or is consuming other living matter containing the same constant ratio of ^{14}C atoms to ordinary ^{12}C atoms. This same ratio permeates all life, because organic processes seem to make no distinction between the two isotopes.

The ratio of ^{14}C to normal carbon remains constant in the atmosphere because, although ^{14}C is radioactive and slowly decays, the amount is continuously replenished through the conversion of ^{14}N (ordinary nitrogen) to ^{14}C by cosmic rays bombarding the upper atmosphere. Over the long history of the planet, this decay and replenishment process has come into nearly steady state.

Of course, when a living organism dies, it ceases its metabolism of carbon and the process of radioactive decay begins to deplete its ^{14}C content. There is no replenishment of this ^{14}C, and consequently the ratio of ^{14}C to normal carbon begins to drop. By measuring this ratio, the amount of time elapsed since the death of the organism can be estimated. For such purposes it is necessary to measure the **decay constant** k. For ^{14}C, it is known that $k \approx 0.0001216$ if t is measured in years.

(Matters are not as simple as we have made them appear. In applying the technique of radiocarbon dating, extreme care must be taken to avoid contaminating the sample with organic matter or even with ordinary fresh air. In addition, the cosmic ray levels apparently have not been constant, so the ratio of ^{14}C in the atmosphere has varied over the past centuries. By using independent methods of dating samples, researchers in this area have compiled tables of correction factors to enhance the accuracy of this process.)

Drug Elimination: In many cases the amount $A(t)$ of a certain drug in the bloodstream, measured by the excess over the natural level of the drug, will decline at a rate proportional to the current excess amount. That is,

$$\frac{dA}{dt} = -\lambda A, \tag{21}$$

where $\lambda > 0$. The parameter λ is called the **elimination constant** of the drug.

The Natural Growth Equation

The prototype differential equation $dx/dt = kx$ with $x(t) > 0$ and k a constant (either negative or positive) is readily solved by separating the variables and integrating:

$$\int \frac{1}{x}\,dx = \int k\,dt;$$

$$\ln x = kt + C.$$

Note $e^{\ln x} = x$ and $e^{kt+C} = e^{kt} \cdot e^C$ by properties of the natural exponential function.

Then we solve for x:

$$e^{\ln x} = e^{kt+C}; \quad x = x(t) = e^C e^{kt} = Ae^{kt}.$$

Because C is a constant, so is $A = e^C$. It is also clear that $A = x(0) = x_0$, so the particular solution of Eq. (17) with the initial condition $x(0) = x_0$ is simply

$$x(t) = x_0 e^{kt}. \tag{22}$$

Because of the presence of the natural exponential function in its solution, the differential equation

$$\frac{dx}{dt} = kx \tag{23}$$

is often called the **exponential** or **natural growth equation**. Figure 1.4.7 shows a typical graph of $x(t)$ in the case $k > 0$; the case $k < 0$ is illustrated in Fig. 1.4.8.

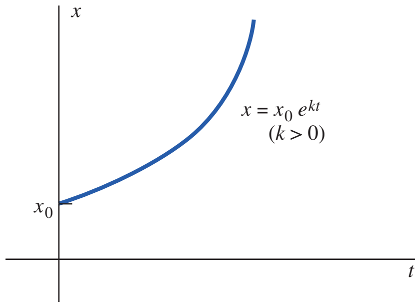

FIGURE 1.4.7. Natural growth.

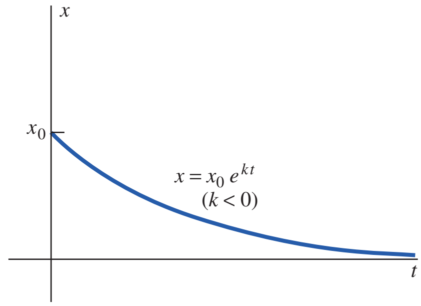

FIGURE 1.4.8. Natural decay.

Example 3 World population According to data listed at **`www.census.gov`**, the world's total population reached 6 billion persons in mid-1999, and was then increasing at the rate of about 212 thousand persons each day. Assuming that natural population growth at this rate continues, we want to answer these questions:

(a) What is the annual growth rate k?

(b) What will be the world population at the middle of the 21st century?

(c) How long will it take the world population to increase tenfold—thereby reaching the 60 billion that some demographers believe to be the maximum for which the planet can provide adequate food supplies?

Solution **(a)** We measure the world population $P(t)$ in billions and measure time in years. We take $t = 0$ to correspond to (mid) 1999, so $P_0 = 6$. The fact that P is increasing by 212,000, or 0.000212 billion, persons per day at time $t = 0$ means that

$$P'(0) = (0.000212)(365.25) \approx 0.07743$$

billion per year. From the natural growth equation $P' = kP$ with $t = 0$ we now obtain

$$k = \frac{P'(0)}{P(0)} \approx \frac{0.07743}{6} \approx 0.0129.$$

Thus the world population was growing at the rate of about 1.29% annually in 1999. This value of k gives the world population function

$$P(t) = 6e^{0.0129t}.$$

(b) With $t = 51$ we obtain the prediction

$$P(51) = 6e^{(0.0129)(51)} \approx 11.58 \text{ (billion)}$$

for the world population in mid-2050 (so the population will almost have doubled in the just over a half-century since 1999).

(c) The world population should reach 60 billion when

$$60 = 6e^{0.0129t}; \quad \text{that is, when } t = \frac{\ln 10}{0.0129} \approx 178;$$

and thus in the year 2177. ■

Note Actually, the rate of growth of the world population is expected to slow somewhat during the next half-century, and the best current prediction for the 2050 population is "only" 9.1 billion. A simple mathematical model cannot be expected to mirror precisely the complexity of the real world.

The decay constant of a radioactive isotope is often specified in terms of another empirical constant, the *half-life* of the isotope, because this parameter is more convenient. The **half-life** τ of a radioactive isotope is the time required for *half* of it to decay. To find the relationship between k and τ, we set $t = \tau$ and $N = \frac{1}{2}N_0$ in the equation $N(t) = N_0 e^{-kt}$, so that $\frac{1}{2}N_0 = N_0 e^{-k\tau}$. When we solve for τ, we find that

$$\tau = \frac{\ln 2}{k}. \tag{24}$$

For example, the half-life of ^{14}C is $\tau \approx (\ln 2)/(0.0001216)$, approximately 5700 years.

Example 4 Radiometric dating A specimen of charcoal found at Stonehenge turns out to contain 63% as much ^{14}C as a sample of present-day charcoal of equal mass. What is the age of the sample?

Solution We take $t = 0$ as the time of the death of the tree from which the Stonehenge charcoal was made and N_0 as the number of ^{14}C atoms that the Stonehenge sample contained then. We are given that $N = (0.63)N_0$ now, so we solve the equation $(0.63)N_0 = N_0 e^{-kt}$ with the value $k = 0.0001216$. Thus we find that

$$t = -\frac{\ln(0.63)}{0.0001216} \approx 3800 \text{ (years)}.$$

Thus the sample is about 3800 years old. If it has any connection with the builders of Stonehenge, our computations suggest that this observatory, monument, or temple—whichever it may be—dates from 1800 B.C. or earlier. ■

Cooling and Heating

According to Newton's law of cooling (Eq. (3) of Section 1.1), the time rate of change of the temperature $T(t)$ of a body immersed in a medium of constant temperature A is proportional to the difference $A - T$. That is,

$$\frac{dT}{dt} = k(A - T), \tag{25}$$

where k is a positive constant. This is an instance of the linear first-order differential equation with constant coefficients:

$$\frac{dx}{dt} = ax + b. \tag{26}$$

It includes the exponential equation as a special case ($b = 0$) and is also easy to solve by separation of variables.

Example 5 Cooling A 4-lb roast, initially at 50°F, is placed in a 375°F oven at 5:00 P.M. After 75 minutes it is found that the temperature $T(t)$ of the roast is 125°F. When will the roast be 150°F (medium rare)?

Solution We take time t in minutes, with $t = 0$ corresponding to 5:00 P.M. We also assume (somewhat unrealistically) that at any instant the temperature $T(t)$ of the roast is uniform throughout. We have $T(t) < A = 375$, $T(0) = 50$, and $T(75) = 125$. Hence

$$\frac{dT}{dt} = k(375 - T);$$

$$\int \frac{1}{375 - T}\,dT = \int k\,dt;$$

$$-\ln(375 - T) = kt + C;$$

$$375 - T = Be^{-kt}.$$

Here B represents e^{-C}.

Now $T(0) = 50$ implies that $B = 325$, so $T(t) = 375 - 325e^{-kt}$. We also know that $T = 125$ when $t = 75$. Substitution of these values in the preceding equation yields

$$k = -\tfrac{1}{75}\ln\left(\tfrac{250}{325}\right) \approx 0.0035.$$

Hence we finally solve the equation

$$150 = 375 - 325e^{(-0.0035)t}$$

for $t = -[\ln(225/325)]/(0.0035) \approx 105$ (min), the total cooking time required. Because the roast was placed in the oven at 5:00 P.M., it should be removed at about 6:45 P.M. ■

Torricelli's Law

Suppose that a water tank has a hole with area a at its bottom, from which water is leaking. Denote by $y(t)$ the depth of water in the tank at time t, and by $V(t)$ the volume of water in the tank then. It is plausible—and true, under ideal conditions—that the velocity of water exiting through the hole is

$$v = \sqrt{2gy}, \tag{27}$$

which is the velocity a drop of water would acquire in falling freely from the surface of the water to the hole (see Problem 35 of Section 1.2). One can derive this formula beginning with the assumption that the sum of the kinetic and potential energy of the system remains constant. Under real conditions, taking into account the constriction of a water jet from an orifice, $v = c\sqrt{2gy}$, where c is an empirical constant between 0 and 1 (usually about 0.6 for a small continuous stream of water). For simplicity we take $c = 1$ in the following discussion.

As a consequence of Eq. (27), we have

$$\frac{dV}{dt} = -av = -a\sqrt{2gy}; \tag{28a}$$

equivalently,

$$\frac{dV}{dt} = -k\sqrt{y} \quad \text{where} \quad k = a\sqrt{2g}. \tag{28b}$$

This is a statement of Torricelli's law for a draining tank. Let $A(y)$ denote the horizontal cross-sectional area of the tank at height y. Then, applied to a thin horizontal slice of water at height $\overline{y}$ with area $A(\overline{y})$ and thickness $d\overline{y}$, the integral calculus method of cross sections gives

$$V(y) = \int_0^y A(\overline{y})\,d\overline{y}.$$

The fundamental theorem of calculus therefore implies that $dV/dy = A(y)$ and hence that

$$\frac{dV}{dt} = \frac{dV}{dy} \cdot \frac{dy}{dt} = A(y)\frac{dy}{dt}. \tag{29}$$

From Eqs. (28) and (29) we finally obtain

$$A(y)\frac{dy}{dt} = -a\sqrt{2gy} = -k\sqrt{y}, \tag{30}$$

an alternative form of Torricelli's law.

Example 6 **Draining bowl** A hemispherical bowl has top radius 4 ft and at time $t = 0$ is full of water. At that moment a circular hole with diameter 1 in. is opened in the bottom of the tank. How long will it take for all the water to drain from the tank?

Solution From the right triangle in Fig. 1.4.9, we see that

$$A(y) = \pi r^2 = \pi\left[16 - (4-y)^2\right] = \pi(8y - y^2).$$

With $g = 32$ ft/s^2, Eq. (30) becomes

$$\pi(8y - y^2)\frac{dy}{dt} = -\pi\left(\tfrac{1}{24}\right)^2\sqrt{2 \cdot 32y};$$

$$\int (8y^{1/2} - y^{3/2})\,dy = -\int \tfrac{1}{72}\,dt;$$

$$\tfrac{16}{3}y^{3/2} - \tfrac{2}{5}y^{5/2} = -\tfrac{1}{72}t + C.$$

Now $y(0) = 4$, so

$$C = \tfrac{16}{3} \cdot 4^{3/2} - \tfrac{2}{5} \cdot 4^{5/2} = \tfrac{448}{15}.$$

The tank is empty when $y = 0$, thus when

$$t = 72 \cdot \tfrac{448}{15} \approx 2150 \text{ (s)};$$

that is, about 35 min 50 s. So it takes slightly less than 36 min for the tank to drain. ■

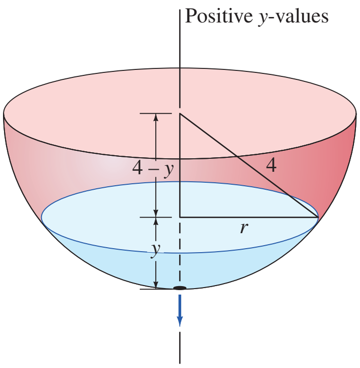

FIGURE 1.4.9. Draining a hemispherical tank.

1.4 Problems

Find general solutions (implicit if necessary, explicit if convenient) of the differential equations in Problems 1 through 18. Primes denote derivatives with respect to x.

1. $\dfrac{dy}{dx} + 2xy = 0$

2. $\dfrac{dy}{dx} + 2xy^2 = 0$

3. $\dfrac{dy}{dx} = y\sin x$

4. $(1+x)\dfrac{dy}{dx} = 4y$

5. $2\sqrt{x}\dfrac{dy}{dx} = \sqrt{1-y^2}$

6. $\dfrac{dy}{dx} = 3\sqrt{xy}$

7. $\dfrac{dy}{dx} = (64xy)^{1/3}$

8. $\dfrac{dy}{dx} = 2x\sec y$

9. $(1-x^2)\dfrac{dy}{dx} = 2y$

10. $(1+x)^2\dfrac{dy}{dx} = (1+y)^2$

11. $y' = xy^3$

12. $yy' = x(y^2+1)$

13. $y^3\dfrac{dy}{dx} = (y^4+1)\cos x$

14. $\dfrac{dy}{dx} = \dfrac{1+\sqrt{x}}{1+\sqrt{y}}$

15. $\dfrac{dy}{dx} = \dfrac{(x-1)y^5}{x^2(2y^3-y)}$

16. $(x^2+1)(\tan y)y' = x$

17. $y' = 1 + x + y + xy$ (*Suggestion:* Factor the right-hand side.)

18. $x^2y' = 1 - x^2 + y^2 - x^2y^2$

Find explicit particular solutions of the initial value problems in Problems 19 through 28.

19. $\dfrac{dy}{dx} = ye^x, \quad y(0) = 2e$

20. $\dfrac{dy}{dx} = 3x^2(y^2+1), \quad y(0) = 1$

21. $2y\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2-16}}, \quad y(5) = 2$

22. $\dfrac{dy}{dx} = 4x^3y - y, \quad y(1) = -3$

23. $\dfrac{dy}{dx} + 1 = 2y, \quad y(1) = 1$

24. $\dfrac{dy}{dx} = y\cot x, \quad y\left(\tfrac{1}{2}\pi\right) = \tfrac{1}{2}\pi$