fourth edition

Precalculus

A Unit Circle Approach

Ratti

McWaters

Skrzypek

with

Bernards

Fresh

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FOURTH EDITION

Precalculus

A Unit Circle Approach

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Foreword

We're pleased to present the fourth edition of *Precalculus: A Unit Circle Approach*. Our experience in teaching this material has been exceptionally rewarding. Because students are accustomed to information being delivered by electronic media, the introduction of MyLab™ Math into our courses was, and remains, seamless. With the addition of Jessica Bernards and Wendy Fresh to the author team, the MyLab course has been given a fresh redesign that aligns with the Ratti philosophy. You will now find author created videos over every objective as well as author created assignments, quizzes, and exams. Additionally, we have included interactive figures in both the print and electronic version of the text that will allow students to get a hands on exploration of the topics.

Today's precalculus students and instructors face many challenges. Students arrive with various levels of comprehension from their previous courses. Instead of really learning the concepts presented, students often resort to memorization to pass the course. As a result, a course needs to establish a common starting point for students and engage them in becoming active learners, without sacrificing the solid mathematics essential for conceptual understanding. Instructors in this course must take on the task of providing students with an understanding of precalculus, preparing them for the next step, and ensuring that they find mathematics useful and interesting. Our efforts in this direction have been aided considerably by the many suggestions we have received from users of the previous editions of this text.

Mathematics owes it current identity to contributions from diverse cultures across the world and throughout the ages. In this text we provide references to significant improvements and achievements in mathematics and related areas from sources both ancient and modern. We place a strong emphasis on both concept development and real-life applications. Topics such as functions, graphing, the difference quotient, and limiting processes provide thorough preparation for the study of calculus and will improve students' comprehension of algebra. Just-in-time review throughout the text ensures that all students are brought to the same level before being introduced to new concepts. Numerous applications motivate students to apply the concepts and skills they learn in precalculus to other courses, including the physical, health, and biological sciences, engineering, and economics, and to on-the-job and everyday problem solving. Students are given ample opportunities in this course to think about important mathematical ideas and to practice and apply algebraic skills.

Throughout the text, we emphasize why the material being covered is important and how it can be applied. By thoroughly developing mathematical concepts with clearly defined terminology, students see the "why" behind those concepts, paving the way for a deeper understanding, better retention, less reliance on rote memorization, and ultimately more success. The level of exposition was selected so that the material is accessible to students and provides them with an opportunity to grow.

It is our hope that once you have read through our text, you will see that we were able to fulfill the initial goals of writing for today's students and for you, the instructor.

Marcus McWaters   Lesław Skrzypek   J. S. Ratti

Jessica Bernards   Wendy Fresh

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Preface

Students begin precalculus classes with widely varying backgrounds. Some haven't taken a math course in several years and may need to spend time reviewing prerequisite topics, while others are ready to jump right into new and challenging material. We have provided review material in the Appendix and in some of the early sections of other chapters, we have provided review material in such a way that it can be used or omitted as appropriate for your course. In addition, students may follow several paths after completing a precalculus course. Many will continue their study of mathematics in courses such as finite mathematics, statistics, and calculus. For others, precalculus may be their last mathematics course.

Responding to the current and future needs of all these students was essential in creating this text. We introduce each exercise set with several concept and vocabulary exercises, consisting of fill-in-the-blank and true-false exercises. They are not computation-reliant, but rather test whether students have absorbed the basic concepts and vocabulary of the section. Exercises asking students to extrapolate information from a given graph now appear in much greater number and depth throughout the course. We continue to present our content in a systematic way that illustrates how to study and what to review. We believe that if students use this material well, they will succeed in this course. The changes in this edition result from the thoughtful feedback we have received from students and instructors who have used previous editions of the text. This feedback crucially enhances our own experiences, and we are extremely grateful to the many contributors whose insights are reflected in this new edition.

Key Content Changes

New! Authors. We would like to welcome two new co-authors to the author team, Jessica Bernards and Wendy Fresh! Both Jessica and Wendy are instructors at Portland Community College and have provided wonderful additions to the text and the accompanying MyLab Math course. Jessica and Wendy have had national recognition as instructors and have received several awards for excellence in teaching mathematics.

Revised! Getting Ready for the Next Section. This feature combines two features of previous editions, "Getting Ready for the Next Section" exercises and "Before Starting This Section, Review" objectives. The new structure lists Review Concepts and Review Skills for students to brush up on before beginning the next section. Both the Review Concepts and Review Skills contain section and page number references to make looking up these topics easy. The Review Concepts are meant to be broader topics that students should understand, while Review Skills give exercises from objectives that students will need in the upcoming section.

New! Key Ideas at a Glance. A new feature added to the text for this edition is a single page at the end of each chapter designed to highlight some key concepts for the chapter. In some chapters, this will serve as a comparison between two similar or parallel topics. In other chapters, this will sum up many of the ideas presented in the chapter. This page can serve as a reference to students to look back on when studying or doing exercises. There are also exercises to accompany this feature so that students may test their understanding of the ideas summarized there.

New and Revised! Exercises. We continue to improve the balance of exercises, providing a smooth transition from the less challenging to the more challenging exercises.

Overall, approximately 20% of the exercises have been updated, and more than 500 brand-new exercises have been added. These new exercises primarily consist of applications that connect with students' everyday experiences and enhance students' understanding of graphing.

Revised! Application Exercises. Every section opens with discussion of an application that relates to the topics introduced in that section. This edition continues the trend of pairing an example with this application, but we have also made an effort to include problems in the exercise sets that also tie to this application so that students have an opportunity to apply the mathematics to a real world problem. The section opening, example, and these exercises are easily identified by an accompanying icon.

New! Active Learning Exercises. In many sections throughout the text, exercise sets will end with an Active Learning exercise. This exercise is accompanied by an Interactive Figure powered by GeoGebra, which is accessed through a bit.ly link or by scanning the given QR code. Students will manipulate the figure to explore mathematics in a new way and will use the figure to answer the accompanying exercises.

New! Videos. Videos in the MyLab course have been completely re-made by the authors. Videos can be found at the section and objective level. The videos are available in the MyLab course within the Video & Resource Library, but can also be accessed directly from the text. QR codes can be found at the beginning of each section, and users can use their phone to scan the QR codes and watch the videos.

We have also created videos for select exercises in each section. The text has QR codes next to the beginning of each exercise set, and users can scan the code to find the videos for exercises in that section.

Revised! Diversity, Equity, and Inclusion. We conducted an external review of the text's content to determine how it could be improved to address issues related to diversity, equity, and inclusion. The results of that review informed the revision.

Chapter 1.

- Split section 1.1 into two sections to allow for more time to discuss certain topics. Added new examples involving the distance formula and symmetry.
- Split the previous section 1.4 into two sections to allow for more time to discuss certain topics. Added new examples involving relative maximum values, graphing even and odd functions, average rate of change, and piecewise functions.
- Moved average rate of change material from section 1.3 to the new section 1.5.
- Introduced square root and cube root functions in section 1.6.
- Updated several examples and with newer application context.
- Expanded step by step transformation of functions procedure.
- Split a Procedure in Action example into two parts to give individual attention to finding a parallel line vs a perpendicular line to a given line.
- Included new Active Learning Exercises.

Chapter 2.

- Added more visual aid to the examples on graphing a polynomial, giving a more complete picture of the behavior of graphs of polynomials.
- Added more visual aid to the examples on graphing a rational function, giving a more complete picture of the behavior of graphs of rational functions.
- Included new Active Learning Exercises.

Chapter 3.

- Added new graphs and examples of transformations of exponential functions and logarithmic functions.
- Expanded on the basic properties of logarithms to give some derivations of these properties.
- Added color to the table of logarithmic functions to make it easier to see how changes in the function change the graph itself.
- Included new Active Learning Exercises.

Chapter 4.

- Updated graphs for sine and cosine functions to give additional detail around key points.
- Created new graphics to show the symmetries of the sine and cosine functions.
- Updated graphics showing stretch and compression of sine and cosine graphs.
- Updated graphics showing phase shifts and vertical shifts of sine and cosine graphs.
- Added additional graphics for the inverse sine, cosine, and tangent functions.
- Included new Active Learning Exercises.

Chapter 5.

- Added a graphic to demonstrate one of the Pythagorean identities.
- Added graphics to support trigonometric identities using the symmetries of the sine and cosine functions.
- Split section 5.5 into two sections to allow for more time to discuss certain topics. Added new examples involving trigonometric equations, equations involving multiple angles, and solving trigonometric equations.

Chapter 6.

- Updated and added graphics and explanation for the ambiguous case of SSA triangles.
- Updated solution process for SSA triangles.
- Included new Active Learning Exercises.

Chapter 7.

- Wrote all new section opener discussions for sections 7.1 and 7.3.
- Added new application examples of solving systems of linear equations in two variables, nonlinear systems, Cramer's rule.
- Added new examples of graphing inequalities, partial fraction decomposition.
- Added new figures to demonstrate the common forms of transformations using matrices.
- Included new Active Learning Exercises.

Chapter 8.

- Added many examples of conics seen and used in the real world.
- Included new Active Learning Exercises.

Chapter 9.

- Changed terms used in the sum of the first n terms of an arithmetic sequence so that the derivation of the formula is clearer.
- Added graphics to demonstrate the derivation of the sum of the first n terms of arithmetic and geometric sequences.

Chapter 10.

- Added new section opener application in section 10.1.
- Added new examples and exercises throughout which connect to section opener applications.
- Included new Active Learning Exercises

Features

Chapter Opener. Each chapter opener includes a description of applications (one of them illustrated) relevant to the content of the chapter and the list of topics that will be covered. In one page, students see what they are going to learn and why they are learning it.

Getting Ready for the Next Section. Each section is immediately preceded by a set of concepts and skills that serve as a transition from one section to the next. These sets of problems provide a review of concepts and skills that will be used in the upcoming section.

Section Opener With Application. Each section opens with a list of clearly stated and numbered **Objectives** defined for the section. These objectives are then referenced again in the margin of the lesson at the point where the objective's topic is taught. An **Application** containing a motivating anecdote or an interesting problem then follows. An example later in the section relating to this application and identified by the same icon ( ) is then solved using the mathematics covered in the section. These applications utilize material from a variety of fields: the physical and biological sciences (including health sciences), economics, art and architecture, history, and more. In addition, exercises relating to this application and identified by the same icon ( ) are provided in the exercise sets.

Examples and Practice Problems. **Examples** include a wide range of computational, conceptual, and modern applied problems carefully selected to build confidence, competency, and understanding. Every example has a title indicating its purpose and presents a detailed solution containing annotated steps. All examples are followed by a **Practice Problem** for students to try so that they can check their understanding of the concept covered. Answers to the Practice Problems are provided in the back of the book.

Procedure in Action Examples. These types of examples, interspersed throughout the text, present important procedures in numbered steps. Special **Procedure in Action** examples present important multistep procedures, such as the steps for doing synthetic division, in a two-column format. The steps of the procedure are given in the left column, and an example is worked, following these steps, in the right column. This approach provides students with a clear model with which they can compare when encountering difficulty in their work.

Additional Pedagogical Features

Definitions, ***Theorems***, ***Properties***, and ***Rules*** are all boxed and titled for emphasis and ease of reference.

Warnings appear as appropriate throughout the text to apprise students of common errors and pitfalls that can trip them up in their thinking or calculations.

Summary of Main Facts boxes summarize information related to equations and their graphs, such as those of the conic sections.

A Calculus Symbol appears next to information in the text that is essential for the study of calculus.

Margin Notes

Side Notes provide hints for handling newly introduced concepts.

Recall notes remind students of a key idea learned earlier in the text that will help them work through a current problem.

Technology Connections give students tips on using calculators to solve problems, check answers, and reinforce concepts. Note that the use of graphing calculators is optional in this text.

Do You Know? Features provide students with additional interesting information on topics to keep them engaged in the mathematics presented.

Exercises. The heart of any textbook is its exercises, so we have tried to ensure that the quantity, quality, and variety of exercises meet the needs of all students. Exercises are carefully graded to strengthen the skills developed in the section and are organized using the following categories. Exercises relating to each section's opening application are identified by the icon ( ).

Concepts and Vocabulary exercises begin each exercise set with problems that assess the student's grasp of the definitions and ideas introduced in that section. These true-false and fill-in-the-blank exercises help to rapidly identify gaps in comprehension of the material in that section.

Building Skills exercises develop fundamental skills—each odd-numbered exercise is closely paired with its consecutive even-numbered exercise.

Applying the Concepts exercises use the section's material to solve real-world problems—all are titled and relevant to the topics of the section.

Beyond the Basics exercises provide more challenging problems that give students an opportunity to reach beyond the material covered in the section—these are generally more theoretical in nature and are suitable for honors students, special assignments, or extra credit.

Critical Thinking/Discussion/Writing exercises, appearing as appropriate, are designed to develop students' higher-level thinking skills. Calculator problems, identified by , are included where needed.

Active Learning exercises allow students to explore mathematical concepts in new ways. Students have the chance to manipulate Interactive Figures and answer accompanying questions.

Key Ideas at a Glance. This one page feature found at the end of each chapter highlights some key concepts in each chapter. In some chapters, this will serve as a comparison between two similar or parallel topics. In other chapters, this will sum up many of the ideas presented in the chapter. This page can serve as a reference to students to look back on when studying or doing exercises. There are also exercises to accompany this feature so that students may test their understanding of the ideas summarized there.

Chapter Review and Tests. The chapter-ending material begins with an extensive **Review** featuring a two-column, section-by-section summary of the definitions, concepts, and formulas covered in that chapter, with corresponding examples. This review provides a description and examples of key topics indicating where the material occurs in the text, and encourages students to reread sections rather than memorize definitions out of context. **Review Exercises** provide students with an opportunity to practice what they have learned in the chapter. Then students are given two chapter test options. They can take **Practice Test A** in the usual open-ended format and/or **Practice Test B**, covering the same topics, in a multiple-choice format. Practice Test B has been moved online for this edition, and can be found in the eText. All tests are designed to increase student comprehension and verify that students have mastered the skills and concepts in the chapter. Mastery of these materials should indicate a true comprehension of the chapter and the likelihood of success on the associated in-class examination. **Cumulative Review Exercises** appear at the end of every chapter, starting with Chapter 2, to remind students that mathematics is not modular and that what is learned in the first part of the book will be useful in later parts of the book and on the final examination.

MyLab Math Resources for Success

MyLab Math is available to accompany Pearson's market-leading text options, including ***Precalculus: A Unit Circle Approach*, 4th Edition** (access code required).

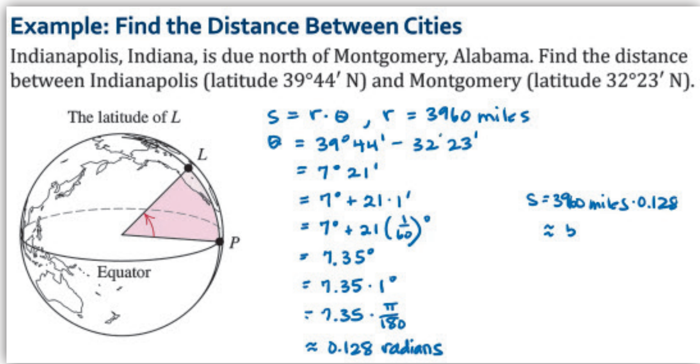
MyLab™ is the teaching and learning platform that empowers you to reach every student. MyLab Math combines trusted author content—including full eText and assessment with immediate feedback—with digital tools and a flexible platform to personalize the learning experience and improve results for each student.

MyLab Math supports all learners, regardless of their ability and background, in order to provide an equal opportunity for success. Accessible resources support learners for a more equitable experience no matter their abilities. And options to personalize learning and address individual gaps help to provide each learner with the specific resources they need in order to achieve success.

Student Resources

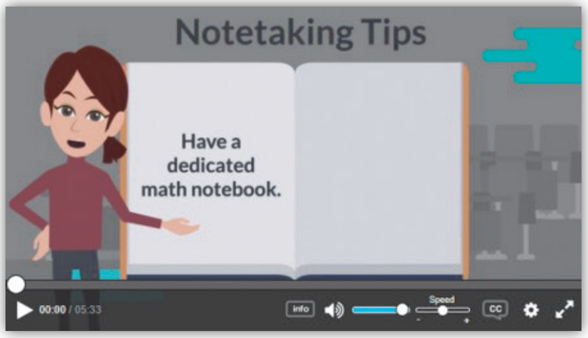
Motivate Your Students—Students are motivated to succeed when they're engaged in the learning experience and understand the relevance and power of math.

▼ **NEW!** **Section Lecture Videos**—Co-authors Jessica Bernards and Wendy Fresh (Portland Community College) have created all new Section videos, segmented and assignable by objective, using their years of teaching experience for online courses and flipped classrooms. Instructors can assign a full objective or only the segment that is needed. The videos allow students an opportunity to learn from experienced master teachers breaking down complex topics in an easy-to-understand manner.



- **NEW!** **Video Notebook** is a note-taking guide that gives students a structured place to take notes and work the example problems as they watch the videos. Definitions and important concepts are highlighted, helpful tips are pointed out along the way. Jessica Bernards and Wendy Fresh author this supplement to make sure students are actively engaged with their learning. The Video Notebook is available as PDFs and customizable Word files in MyLab Math. Instructors can also use the video notebook as a guide when creating their own lecture notes in a traditional lecture class. This way an instructor has pre-built lecture notes ready to go, or an easily adaptable set of lecture notes ready to be modified for the needs of their students.

▼ **NEW!** **Mathematical study skills videos**, created by co-authors Jessica Bernards and Wendy Fresh, motivate students to stick with their math course and offer practical tips to succeed. The animated character, Polly Nomial, guides students through topics such as How Learning Math is Different and Having a Growth Mindset in Math that any math student could benefit from watching. These ten study skills videos have pre-built assignments that include assessment questions that test students' understanding of the content.



- **NEW!** **Personal Inventory Assessments** are a collection of online exercises designed to promote self-reflection and metacognition in students. These 33 assessments include topics such as a Stress Management Assessment, Diagnosing Poor Performance and Enhancing Motivation, and Time Management Assessment.

Address Underpreparedness—Each student learns at a different pace. Personalized learning pinpoints the precise areas where each student needs practice, giving all students the support they need—when and where they need it—to be successful.

NEW! **Integrated Review** can be used in corequisite courses, or simply to help students who enter Precalculus without a full understanding of prerequisite skills and concepts.

- Integrated Review at the chapter level provides a Skills Check assessment to pinpoint which prerequisites topics, if any, students need to review.
- Students who require additional review proceed to a personalized homework assignment to remediate.
- Integrated Review videos and worksheets provide additional instruction.

Instructors who prefer to review at the section level can assign the Enhanced Assignments instead.

Personalized Homework—With Personalized Homework, students take a quiz or test and receive a subsequent homework assignment that is personalized based on their performance. This way, students can focus on just the topics they have not yet mastered.

Other student resources include the following:

- NEW! **Interactive Figures** bring mathematical concepts to life, helping students see the concepts through directed explorations and purposeful manipulation. For this revision, we added many more interactive figures (in editable GeoGebra format) to the Video & Resource Library. The instructional videos that accompany the text now include Interactive Figures to teach key concepts. These figures are assignable in MyLab Math and encourage active learning, critical thinking, and conceptual understanding.
- **Solution Manual**—Written by Beverly Fusfield, the Student's Solution Manual provides detailed worked-out solutions to the odd-numbered end-of-section and Chapter Review exercises as well as solutions to all the Practice Problems, Practice Tests, and Cumulative Review problems. Available in MyLab Math.

Instructor Resources

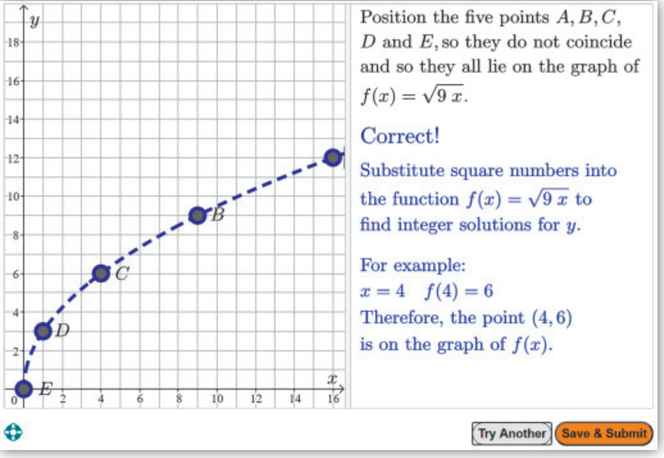
Your course is unique. So whether you'd like to build your own assignments, teach multiple sections, or set prerequisites, MyLab gives you the flexibility to easily create your course to fit your needs.

Pre-Built Assignments are designed to maximize students' performance. All assignments are *fully editable* to make your course your own.

- NEW! **Enhanced Assignments**—These section-level assignments have three unique properties:
  1. They help keep skills fresh with *spaced practice* of previously learned concepts.
  2. Learning aids are strategically turned off for some exercises to ensure students understand how to work the exercises independently.
  3. They contain personalized prerequisite skills exercises for gaps identified in the chapter-level Skills Check Quiz.
- NEW! **Learning Assignments**—Section-level assignments are especially helpful for online classes or flipped classes, where some or all learning takes place independently. These assignments include objective-level videos and interactive figures for student exploration followed by corresponding MyLab questions to ensure engagement and understanding. Instructors can assign the video notebook for students to fill out as they complete these video assignments.

MyLab Math Question Library is correlated to the exercises in the text, reflecting each author's approach and learning style. They regenerate algorithmically to give students unlimited opportunity for practice and mastery. Below are a few exercise types available to assign:

- NEW! **GeoGebra Exercises** are gradable graphing and computational exercises that help students demonstrate their understanding. They enable students to interact directly with the graph in a manner that reflects how students would graph on paper.



- **Setup & Solve Exercises** require students to first describe how they will set up and approach the problem. This reinforces conceptual understanding of the process applied in approaching the problem, promotes long-term retention of the skill, and mirrors what students will be expected to do on a test.
- **Concept and Vocabulary**—Each exercise section begins with exercises that assess the student's grasp of the definitions and ideas introduced in that section. These true-false and fill-in-the-blank exercises help to rapidly identify gaps in comprehension and are assignable in MyLab Math and Learning Catalytics.

Learning Catalytics—With Learning Cataltyics, you'll hear from every student when it matters most. You pose a variety of questions in class (choosing from pre-loaded questions or your own) that help students recall ideas, apply concepts, and develop critical-thinking skills. Your students respond using their own smartphones, tablets, or laptops.

Performance Analytics enable instructors to see and analyze student performance across multiple courses. Based on their current course progress, individuals' performance is identified above, at, or below expectations through a variety of graphs and visualizations.

Now included with Performance Analytics, **Early Alerts** use predictive analytics to identify struggling students—even if their assignment scores are not a cause for concern. In both Performance Analytics and Early Alerts, instructors can email students individually or by group to provide feedback.

Accessibility—Pearson works continuously to ensure our products are as accessible as possible to all students. Currently we work toward achieving WCAG 2.0 AA for our existing products (2.1 AA for future products) and Section 508 standards, as expressed in the Pearson Guidelines for Accessible Educational Web Media (https://wps.pearsoned.com/accessibility/).

Other instructor resources include the following:

- **Instructor Solution Manual**—Written by Bevery Fusfield, the Instructor's Solutions Manual provides complete solutions for all end-of-section exercises, including the Critical Thinking/Discussion/Writing Projects, Practice Problems, Chapter Review exercises, Practice Tests, and Cumulative Review problems.
- **PowerPoint Lecture Slides** feature presentations written and designed specifically for this text, including figures and examples from the text. Accessible versions of the PowerPoints are also available.
- **TestGen** enables instructors to build, edit, print, and administer tests using a computerized bank of questions developed to cover all the objectives of the text. TestGen is algorithmically based, allowing instructors to create multiple but equivalent versions of the same questions or test with the click of a button. Instructors can also modify test bank questions or add new questions. The software and test bank are available for download at pearson.com.
- **Test Bank** features a printable PDF containing all the test exercises available in TestGen. The current version contains 6 forms of tests per chapter in PDF format. Forms A-D are open-ended. Forms E and F are multiple choice.

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We invite all who use this book to send suggestions for improvements to Marcus McWaters at mmm@usf.edu.

Pearson's Commitment to Diversity, Equity, and Inclusion

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DEDICATION

To Our Spouses,

Lata, Debra, Leslie, Kevin, and Jon

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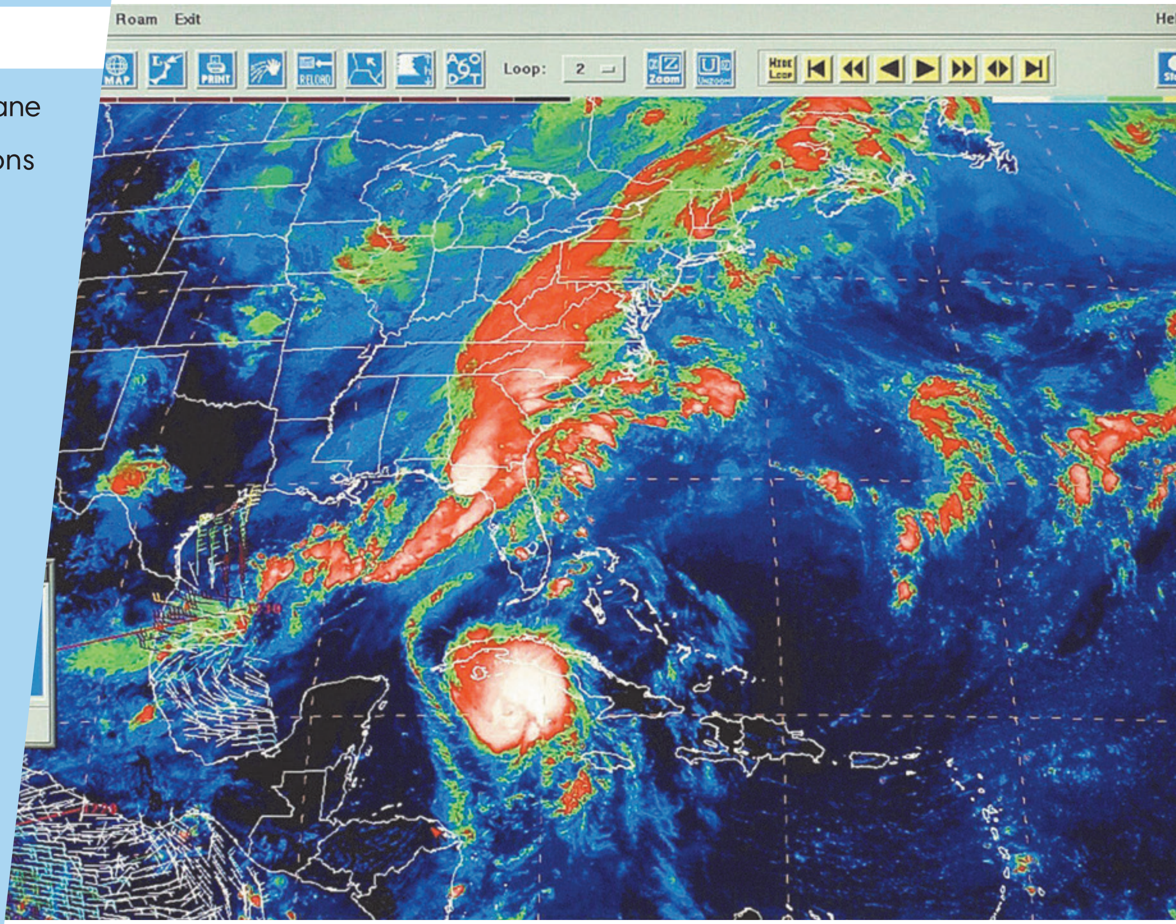
1 Graphs and Functions

TOPICS



Hurricanes are tracked with the use of coordinate grids, and data from fields as diverse as medicine and sports are related and analyzed by means of functions. The material in this chapter introduces you to the versatile concepts that are the everyday tools of all dynamic industries.

GETTING READY ▸ for the Next Section

REVIEW CONCEPTS

The number line (Appendix A.1, page 985)

Equivalent equations (Appendix A.6, page 1019)

Completing squares (Appendix A.6, page 1022)

Interval notation (Appendix A.1, page 987)

REVIEW SKILLS

Evaluating algebraic expressions (Appendix A.1, page 989)

In Exercises GR1– GR8, simplify each expression.

GR1. $\frac{2+5}{2}$ **GR2.** $\frac{-3+7}{2}$ **GR3.** $\frac{-3-7}{2}$

GR4. $\frac{(3-\sqrt{2})-(3+\sqrt{2})}{2}$

GR5. $\sqrt{(5-2)^2+(3-7)^2}$

GR6. $\sqrt{(-8+3)^2+(-5-7)^2}$

GR7. $\sqrt{(2-5)^2+(8-6)^2}$

GR8. $\sqrt{(\sqrt{3}-\sqrt{12})^2+(\sqrt{2}+\sqrt{8})^2}$

Completing squares (Appendix A.6, page 1022)

In Exercises GR9–GR14, add the appropriate term to each binomial so that it becomes a perfect square trinomial. Write and factor the trinomial.

GR9. x^2+4x **GR10.** x^2-6x

GR11. x^2-5x **GR12.** x^2+7x

GR13. $x^2+\frac{3}{2}x$ **GR14.** $x^2-\frac{4}{5}x$

1.1 The Coordinate Plane

Objectives

1 ▸ Plot points in the Cartesian coordinate plane.

2 ▸ Find the distance between two points.

3 ▸ Find the midpoint of a line segment.

Videos for this Section

A Fly on the Ceiling

One day the French mathematician René Descartes noticed a fly buzzing around on a ceiling made of square tiles. He watched the fly and wondered how he could mathematically describe its location. Finally, he realized that he could describe the fly's position by its distance from the walls of the room. Descartes had just discovered the coordinate plane! In fact, the coordinate plane is sometimes called the Cartesian plane in his honor. The discovery led to the development of analytic geometry, the first blending of algebra and geometry.

Although the basic idea of graphing with coordinate axes dates all the way back to Apollonius in the second century B.C., Descartes, who lived in the 1600s, gets the credit for coming up with the two-axis system we use today. In Example 2, we will see how the Cartesian plane helps visualize data on credit card interest rates.

Objective 1 ▸

The Coordinate Plane

A visually powerful device for exploring relationships between numbers is the Cartesian plane. A pair of real numbers in which the order is specified is called an **ordered pair** of real numbers. The ordered pair (a, b) has **first component** a and **second component** b. Two ordered pairs (x, y) and (a, b) are **equal**, and we write $(x, y) = (a, b)$ if and only if $x = a$ and $y = b$.

Just as the real numbers are identified with points on a line, called the ***number line*** or the ***coordinate line***, the sets of ordered pairs of real numbers are identified with points on a plane called the **coordinate plane** or the **Cartesian plane**.

We begin with two coordinate lines, one horizontal and one vertical, that intersect at their zero points. The horizontal line (with positive numbers to the right) is usually called the ***x*-axis**, and the vertical line (with positive numbers up) is usually called the ***y*-axis**. Their point of intersection is called the **origin**. The x-axis and y-axis are called **coordinate axes**, and the plane they form is sometimes called the ***xy*-plane**. The axes divide the plane into four regions called **quadrants**, which are numbered as shown in Figure 1.1. The points on the axes themselves do not belong to any of the quadrants.

The notation $P(a, b)$, or $P = (a, b)$, designates the point P whose *first component* is a and whose *second component* is b. In an xy-plane, the first component, a, is called the ***x*-coordinate** of $P(a, b)$ and the second component, b, is called the ***y*-coordinate** of $P(a, b)$. The signs of the x- and y-coordinates for each quadrant are shown in Figure 1.1. The point corresponding to the ordered pair (a, b) is called the **graph of the ordered pair (a, b)**. However, we frequently ignore the distinction between an ordered pair and its graph.

SIDE NOTE

Although it is common to label the axes as x and y, other letters are also used, especially in applications. In a *uv*-plane or an *st*-plane, the first letter in the name refers to the horizontal axis; the second, to the vertical axis.

René Descartes

(1596–1650) Descartes was born at La Haye, near Tours in southern France. He is often called the father of modern science. Descartes established a new, clear way of thinking about philosophy and science by accepting only those ideas that could be proved by or deduced from first principles. He took as his philosophical starting point the statement **Cogito ergo sum**: "I think; therefore, I am." Descartes made major contributions to modern mathematics, including the Cartesian coordinate system and the theory of equations.

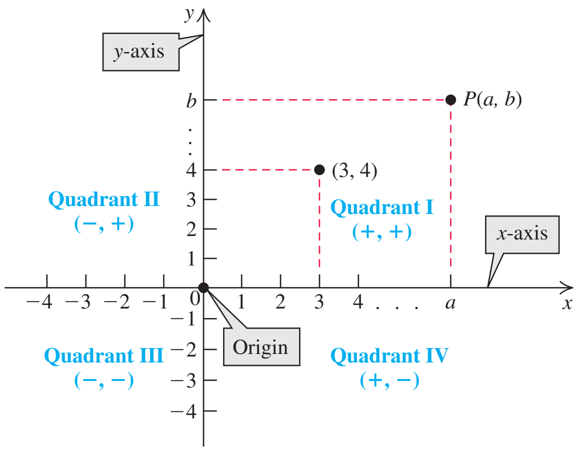


Figure 1.1 ▸ Quadrants in a plane

EXAMPLE 1 Graphing Points

Graph the following points in the xy-plane:

$$A(3, 1),\ B(-2, 4),\ C(-3, -4),\ D(2, -3),\ \text{and } E(-3, 0).$$

Solution

Figure 1.2 shows a coordinate plane along with the graph of the given points. These points are located by moving left, right, up, or down starting from the origin $(0, 0)$.

$A(3, 1)$	3 units right, 1 unit up	$D(2, -3)$	2 units right, 3 units down
$B(-2, 4)$	2 units left, 4 units up	$E(-3, 0)$	3 units left
$C(-3, -4)$	3 units left, 4 units down		

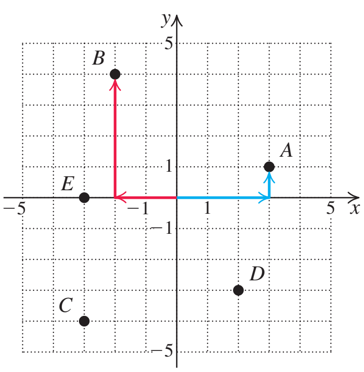


Figure 1.2 ▸ Graphing points

Practice Problem 1. Graph the following points in the xy-plane:

$$P(-2, 2),\ Q(4, 0),\ R(5, -3),\ S(0, -3),\ \text{and } T\left(-2, \frac{1}{2}\right).$$ ■

Pairs of related items can be displayed as ordered pairs. For example, you might write (Jason, 12/1/1982) and (Ashlym, 4/6/1990) for the names and birthdays of two relatives. A more practical example is given next.

EXAMPLE 2 Graphing Data on the Number of Smartphones Sold to Users Worldwide

The data in Table 1.1 show the number of smartphones sold to users worldwide (in millions) over the years 2012–2021.

TABLE 1.1

Year	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021
Number Sold	680.11	969.72	1,244.74	1,423.9	1,495.96	1,536.54	1,556.27	1,540.66	1,378.72	1,535.36

Source: Gartner.com

Graph the ordered pairs (year, smartphones sold), where the first coordinate represents a year and the second coordinate represents the number of smartphones sold in that year.

Solution

We let t represent the years 2012 through 2021 and y represent the number of smartphones sold in each year. Because the data start from the year 2012, we show a break in the t-axis. Alternatively, we could declare a year—say, 2011—as 0. Similar comments apply to the y-axis. The graph of the points (2012, 680.11), (2013, 969.72), (2014, 1,244.74), (2015, 1,423.9), (2016, 1,495.96), (2017, 1,536.54), (2018, 1,556.27), (2019, 1,540.66), (2020, 1,378.72), and (2021, 1,535.36) is shown in Figure 1.3. The figure depicts the number of smartphones sold for every year since 2012.

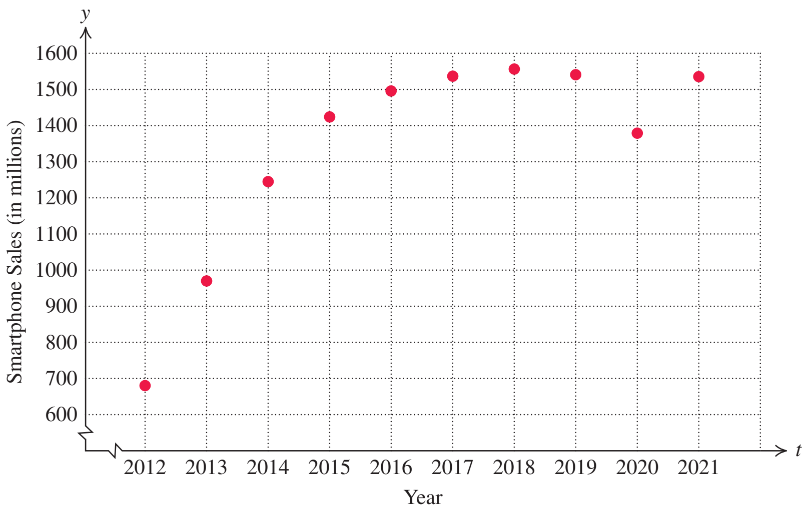


Figure 1.3 ▸ Worldwide smartphones sales

Practice Problem 2. The data in Table 1.2 show the average mortgage interest rates for a 30-year conventional loan in the United States over the years 2011–2020.

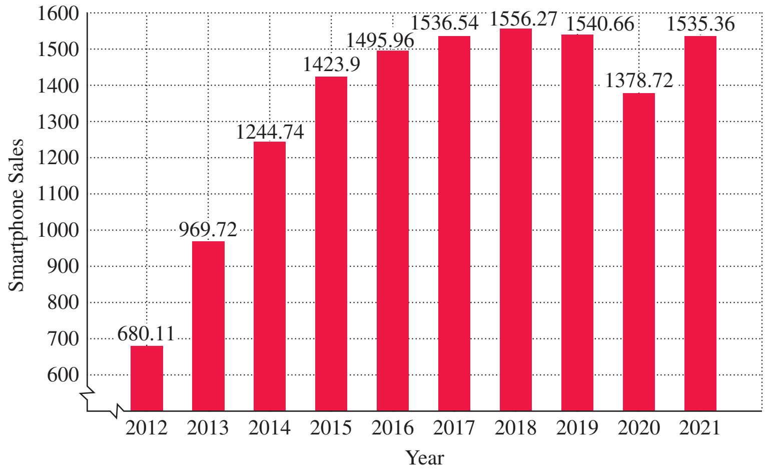
TABLE 1.2

Year	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
Interest Rate	4.45	3.66	3.98	4.17	3.85	3.65	3.99	4.54	3.94	3.11

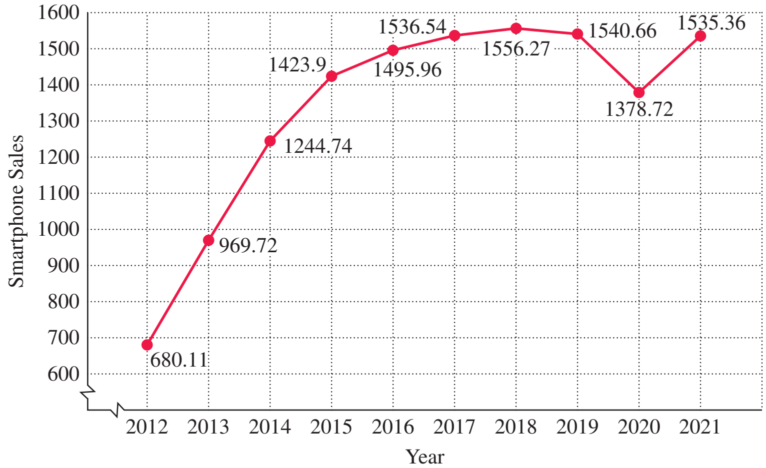
Source: FreddieMac.com

Graph the ordered pairs (year, interest rate), where the first coordinate represents a year and the second coordinate represents the interest rate in that year. ■

The display in Figure 1.3 is called the scatter diagram of the data. There are numerous other ways of visualizing the data. Two such ways are shown in Figure 1.4(a) and Figure 1.4(b).



(a) Bar graph



(b) Line graph

Figure 1.4 ▸ Two methods of visualizing data

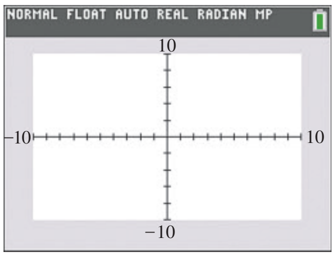
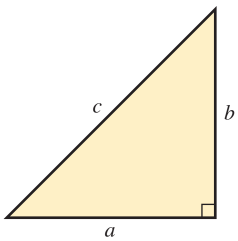


Figure 1.5 ▸ Viewing rectangle

Scales on a Graphing Utility

When drawing a graph, you can use different scales for the x- and y-axes. Similarly, the scale can be set separately for each coordinate axis on a graphing utility. Once scales are set, you get a **viewing rectangle**, where your graphs are displayed. For example, in Figure 1.5, the scale for the x-axis (the distance between each tick mark) is 1, whereas the scale for the y-axis is 2. Although newer calculators, like the TI-84 plus, have two display modes (CLASSIC and MATHPRINT) both display graphs identically. Read more about viewing rectangles in your graphing calculator manual.

Objective 2 ▸



$a^2 + b^2 = c^2$

Figure 1.6 ▸ Pythagorean theorem

Distance Formula

If a Cartesian coordinate system has the same unit of measurement, such as inches or centimeters, on both axes, we can then calculate the distance between any two points in the coordinate plane in the given unit.

Recall that the Pythagorean Theorem states that in a **right triangle** with hypotenuse of length c and the other two sides of lengths a and b,

$$a^2 + b^2 = c^2, \qquad \text{Pythagorean Theorem}$$

as shown in Figure 1.6.

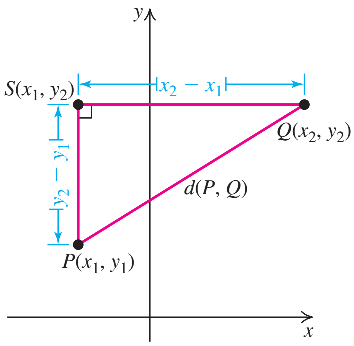


Figure 1.7 ▸ Visualizing the distance formula

Suppose we want to compute the distance $d(P, Q)$ between the two points $P(x_1, y_1)$ and $Q(x_2, y_2)$. We draw a horizontal line through the point Q and a vertical line through the point P to form the right triangle PQS, as shown in Figure 1.7.

The length of the horizontal side of the triangle is $|x_2 - x_1|$, and the length of the vertical side is $|y_2 - y_1|$. By the Pythagorean Theorem, we have

$$[d(P, Q)]^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

$$d(P, Q) = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2} \qquad \text{Take the square root of both sides.}$$

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad |a - b|^2 = (a - b)^2$$

Distance Formula in the Coordinate Plane

Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be any two points in the coordinate plane. Then the distance between P and Q, denoted $d(P, Q)$, is given by the **distance formula**

$$\boldsymbol{d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.}$$

The distance between two points in a coordinate plane is the square root of the sum of the square of the difference between their *x*-coordinates and the square of the difference between their *y*-coordinates.

EXAMPLE 3 Finding the Distance Between Two Points

Find the distance between the points $P(-2, 5)$ and $Q(3, -4)$.

Solution

Let $(x_1, y_1) = (-2, 5)$ and $(x_2, y_2) = (3, -4)$. Then

$$x_1 = -2,\ y_1 = 5,\ x_2 = 3, \text{ and } y_2 = -4.$$

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad \text{Distance formula}$$

$$= \sqrt{[3 - (-2)]^2 + (-4 - 5)^2} \qquad \text{Substitute the values for } x_1, x_2, y_1, y_2.$$

$$= \sqrt{5^2 + (-9)^2} \qquad \text{Simplify.}$$

$$= \sqrt{25 + 81} = \sqrt{106} \approx 10.3 \qquad \text{Simplify; use a calculator.}$$

SIDE NOTE

Remember that in general $\sqrt{a^2 + b^2} \neq a + b$.

Practice Problem 3. Find the distance between the points $(-5, 2)$ and $(-4, 1)$. ■

RECALL

The converse of the Pythagorean Theorem states the following: If in a triangle with side lengths a, b, and c, we have $a^2 + b^2 = c^2$ then the triangle is a right triangle.

In the next example, we use the distance formula and the converse of the Pythagorean Theorem to show that the given triangle is a right triangle.

EXAMPLE 4 Identifying a Right Triangle

Let $A(4, 3)$, $B(1, 4)$, and $C(-2, -5)$ be three points in the plane.

a. Sketch triangle ABC.

b. Find the length of each side of the triangle.

c. Show that ABC is a right triangle.

Solution

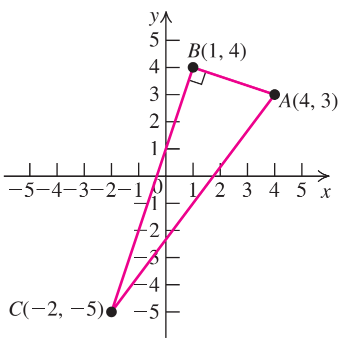


Figure 1.8

a. A sketch of the triangle formed by the three points A, B, and C is shown in Figure 1.8.

b. Using the distance formula, we have

$$d(A, B) = \sqrt{(4 - 1)^2 + (3 - 4)^2} = \sqrt{9 + 1} = \sqrt{10},$$

$$d(B, C) = \sqrt{[1 - (-2)]^2 + [4 - (-5)]^2} = \sqrt{9 + 81} = \sqrt{90} = 3\sqrt{10}, \text{ and}$$

$$d(A, C) = \sqrt{[4 - (-2)]^2 + [3 - (-5)]^2} = \sqrt{36 + 64} = \sqrt{100} = 10.$$

c. We check whether the relationship $a^2 + b^2 = c^2$ holds in this triangle, where a, b, and c denote the lengths of its sides. The longest side, AC, has length 10 units.

$$[d(A, B)]^2 + [d(B, C)]^2 = 10 + 90 \qquad \text{Replace } [d(A, B)]^2 \text{ with 10 and } [d(B, C)]^2 \text{ with 90.}$$
$$= (10)^2$$
$$= [d(A, C)]^2$$

It follows from the converse of the Pythagorean Theorem that the triangle ABC is a right triangle.

Practice Problem 4. Is the triangle with vertices $(6, 2)$, $(-2, 0)$, and $(1, 5)$ an isosceles right triangle—that is, a right triangle with two sides of equal length? ■

EXAMPLE 5 **Applying the Distance Formula to Baseball**

The baseball diamond is in fact a square with a distance of 90 feet between each of the consecutive bases. Use an appropriate coordinate system to calculate the distance the ball travels when the third baseman throws it from third base to first base.

Solution

We can conveniently choose home plate as the origin and place the x-axis along the line from home plate to first base and the y-axis along the line from home plate to third base, as shown in Figure 1.9. The coordinates of home plate (O), first base (A), second base (C), and third base (B) are shown in the figure.

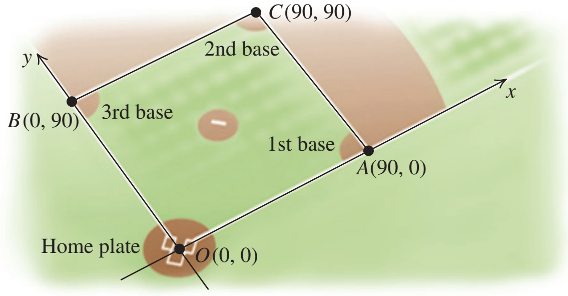


Figure 1.9

We are asked to find the distance between the points $A(90, 0)$ and $B(0, 90)$.

$$d(A, B) = \sqrt{(90 - 0)^2 + (0 - 90)^2} \qquad \text{Distance formula}$$
$$= \sqrt{(90)^2 + (-90)^2}$$
$$= \sqrt{2(90)^2} \qquad \text{Simplify.}$$
$$= 90\sqrt{2} \approx 127.28 \text{ feet} \qquad \text{Simplify; use a calculator.}$$

Practice Problem 5. Young players might play baseball in a square "diamond" with a distance of 60 feet between consecutive bases. Repeat Example 5 for a diamond with these dimensions. ■

Objective 3 ▸

Midpoint Formula

Recall that a point M on the line segment $\overline{PQ}$ is its midpoint if $d(P, M) = d(M, Q)$. See Figure 1.10. We provide the midpoint formula in the xy-plane.

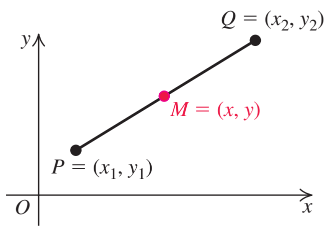


Figure 1.10 ▸ Midpoint of a segment

Midpoint Formula

The coordinates of the midpoint $M = (x, y)$ on the line segment joining $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are given by

$$M = (x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

The x-coordinate of the midpoint of a line segment is the average of the x-coordinates of the segment's endpoints, and the y-coordinate is the average of the y-coordinates of the segment's endpoints. In physics, the midpoint of a line segment is interpreted as the barycenter (center of gravity) of a system of two equal weights placed at the endpoints of a line segment.

We ask you to prove the midpoint formula in Exercise 72.

EXAMPLE 6 Finding the Midpoint of a Line Segment

Find the midpoint of the line segment joining the points $P(-3, 6)$ and $Q(1, 4)$.

Solution

Let $(x_1, y_1) = (-3, 6)$ and $(x_2, y_2) = (1, 4)$. Then

$$x_1 = -3,\ y_1 = 6,\ x_2 = 1, \text{ and } y_2 = 4.$$

$$\begin{aligned}\text{Midpoint} &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) && \text{Midpoint formula}\\ &= \left(\frac{-3 + 1}{2}, \frac{6 + 4}{2}\right) && \text{Substitute values for } x_1, x_2, y_1, y_2.\\ &= (-1, 5) && \text{Simplify.}\end{aligned}$$

The midpoint of the line segment joining the points $P(-3, 6)$ and $Q(1, 4)$ is $M(-1, 5)$.

Practice Problem 6. Find the midpoint of the line segment whose endpoints are $(5, -2)$ and $(6, -1)$. ■

1.1 Exercises

Videos for select exercises

Concepts and Vocabulary

1. A point with a negative first coordinate and a positive second coordinate lies in the ___________ quadrant.
2. Any point on the x-axis has second coordinate ___________.
3. The distance between the points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ is given by the formula $d(P, Q) =$ ___________.
4. The coordinates of the midpoint $M = (x, y)$ of the line segment joining $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are given by $(x, y) =$ ___________.
5. **True or False.** For any points (x_1, y_1) and (x_2, y_2) $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
6. **True or False.** The point $(7, -4)$ is 4 units to the right and 2 units below the point $(3, 2)$.
7. **True or False.** Every point in Quadrant II has a negative y-coordinate.
8. **True or False.** Every point on the y-axis has zero as its x-coordinate.

Building Skills

9. Plot and label each of the given points in a Cartesian coordinate plane and state the quadrant, if any, in which each point is located. $(2, 2), (3, -1), (-1, 0), (-2, -5), (0, 0), (-7, 4)$ $(0, 3), (-4, 2)$
10. **a.** Write the coordinates of any five points on the x-axis. What do these points have in common?
    b. Plot the points $(-2, 1)$, $(0, 1)$, $(0.5, 1)$, $(1, 1)$, and $(2, 1)$. Describe the set of all points of the form $(x, 1)$, where x is a real number.
11. **a.** If the x-coordinate of a point is 0, where does that point lie?
    b. Plot the points $(-1, 1), (-1, 1.5), (-1, 2), (-1, 3)$, and $(-1, 4)$. Describe the set of all points of the form $(-1, y)$, where y is a real number.
12. What figure is formed by the set of all points in a Cartesian coordinate plane that have
    a. x-coordinate equal to -3?
    b. y-coordinate equal to 4?
13. Let $P(x, y)$ be a point in a coordinate plane.
    a. If the point $P(x, y)$ lies above the x-axis, what must be true of y?
    b. If the point $P(x, y)$ lies below the x-axis, what must be true of y?
    c. If the point $P(x, y)$ lies to the left of the y-axis, what must be true of x?
    d. If the point $P(x, y)$ lies to the right of the y-axis, what must be true of x?

14. Let $P(x, y)$ be a point in a coordinate plane. In which quadrant does P lie
a. if x and y are both negative?
b. if x and y are both positive?
c. if x is positive and y is negative?
d. if x is negative and y is positive?

In Exercises 15–24, find (a) the distance between P and Q and (b) the coordinates of the midpoint of the line segment $\overline{PQ}$.

15. $P(2, 1), Q(2, 5)$

16. $P(3, 5),\ Q(-2, 5)$

17. $P(-1, -5),\ Q(2, -3)$

18. $P(-4, 1), Q(-7, -9)$

19. $P(-1,\ 1.5),\ Q(3, -6.5)$

20. $P(0.5, 0.5),\ Q(1, -1)$

21. $P(\sqrt{2}, 4), Q(\sqrt{2}, 5)$

22. $P(v - w, t), Q(v + w, t)$

23. $P(t, k), Q(k, t)$

24. $P(m, n),\ Q(-n, -m)$

In Exercises 25–32, determine whether the given points are collinear. Points are *collinear* if they can be labeled P, Q, and R so that $d(P, Q) + d(Q, R) = d(P, R)$.

25. $(0, 0), (1, 2), (-1, -2)$

26. $(3, 4), (0, 0), (-3, -4)$

27. $(4, -2), (-2, 8), (1, 3)$

28. $(9, 6), (0, -3), (3, 1)$

29. $(-1, 4), (3, 0), (11, -8)$

30. $(-2, 3), (3, 1), (2, -1)$

31. $(4, -4), (15, 1), (1, 2)$

32. $(1, 7), (-7, 8), (-3, 7.5)$

33. Find the coordinates of the points that divide the line segment joining the points $P = (-4, 0)$ and $Q = (0, 8)$ into four equal parts.

34. Repeat Exercise 33 with $P = (-8, 4)$ and $Q = (16, -12)$.

In Exercises 35–42, identify the triangle PQR as an *isosceles* (two sides of equal length), an *equilateral* (three sides of equal length), or a *scalene* (three sides of different lengths) triangle.

35. $P(-5, 5),\ Q(-1, 4),\ R(-4, 1)$

36. $P(3,\ 2), Q(6, 6),\ R(-1, 5)$

37. $P(-4,\ 8),\ Q(0,\ 7),\ R(-3,\ 5)$

38. $P(6,\ 6), Q(-1, -1),\ R(-5,\ 3)$

39. $P(0, -1),\ Q(9, -9),\ R(5,\ 1)$

40. $P(-4,\ 4),\ Q(4,\ 5),\ R(0, -2)$

41. $P(1, -1),\ Q(-1, 1),\ R(-\sqrt{3}, -\sqrt{3})$

42. $P(-0.5, -1),\ Q(-1.5,\ 1),\ R\left(\sqrt{3} - 1, \dfrac{\sqrt{3}}{2}\right)$

43. Show that the points $P(7, -12), Q(-1, 3), R(14, 11)$, and $S(22, -4)$ are the vertices of a square. Find the length of the diagonals.

44. Repeat Exercise 43 for the points $P(8, -10), Q(9, -11), R(8, -12)$, and $S(7, -11)$.

45. Find x such that the point $(x, 2)$ is 5 units from $(2, -1)$.

46. Find y such that the point $(2, y)$ is 13 units from $(-10, -3)$.

47. Find the point on the x-axis that is equidistant from the points $(-5, 2)$ and $(2, 3)$.

48. Find the point on the y-axis that is equidistant from the points $(7, -4)$ and $(8, 3)$.

Applying the Concepts

49. Population. The table shows the total population of the United States in millions. (205 represents 205,000,000.) The population is rounded to the nearest million. Plot the data in a Cartesian coordinate system.

Year	Total Population (millions)
1985	238
1990	250
1995	267
2000	282
2005	296
2010	308
2015	321
2020	333

Source: U.S. Census Bureau.

50. Use the data from Exercise 49 and the midpoint formula for the years 2010 and 2020 to estimate the total population for 2015. Compare your estimate with the table value for 2015.

In Exercises 51–54, use the following vital statistics table.

	Rate per 1000 Population			
Year	Births	Deaths	Marriages	Divorces
1975	14.6	8.6	10.0	4.8
1980	15.9	8.8	10.6	5.2
1985	15.8	8.8	10.1	5.0
1990	16.7	8.6	9.4	4.7
1995	14.8	8.8	8.9	4.4
2000	14.4	8.7	8.5	4.2
2005	14.2	8.1	7.6	3.6
2010	13.0	8.0	6.8	3.6
2015	12.4	7.3	6.5	3.1
2020	11.4	9.7	6.1	2.9

Source: U.S. Census Bureau.

The table shows the rate per 1000 population of live births, deaths, marriages, and divorces from 1975 to 2020. Plot the data in a Cartesian coordinate system and connect the points with line segments.

51. Plot (year, births).

52. Plot (year, deaths).

53. Plot (year, marriages).

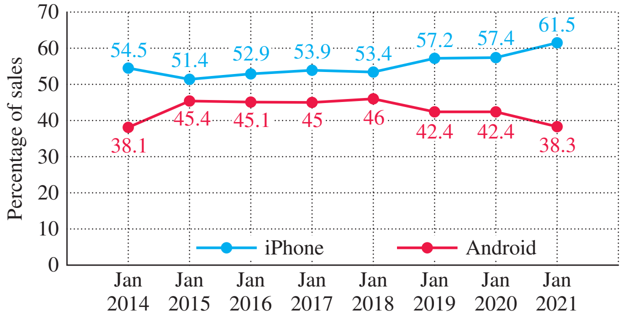
54. Plot (year, divorces).

In Exercises 55 and 56, assume that the graph through all of the given data points is a line segment. Use the midpoint formula to find the estimate.

55. Spending on prescription drugs. Americans spent \$322 billion on prescription drugs in 2016 and \$359 billion in 2020. Estimate the amount Americans spent on prescription drugs in 2018. (*Source:* cms.gov)

56. Internet users. In 2016 there were 3696 million users of the Internet worldwide. By 2020 there were 5053 million such users. Estimate the number of users of the Internet in 2018. (*Source:* http://www.internetworldstats.com)

The following graph shows the percentage of smartphone sales in the United States by two leading platforms. Use this graph for Exercises 57–60. *Source:* https://gs.statcounter.com



57. Use the appropriate line graph to determine the percentage of smartphone sales by Android in January 2014.

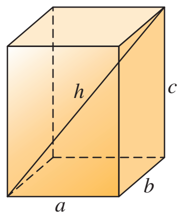
58. Use the appropriate line graph to determine the percentage of smartphone sales by iPhone in January 2017.

59. For the period from January 2014 through January 2021, for which year were the Android sales at a maximum?

60. For the period from January 2014 through January 2021, for which year were the iPhone sales at a maximum?

61. Length of a diagonal. The application of the Pythagorean Theorem in three dimensions involves the relationship between the perpendicular edges of a rectangular block and the solid diagonal of the same block.

In the figure, show that $h^2 = a^2 + b^2 + c^2$.



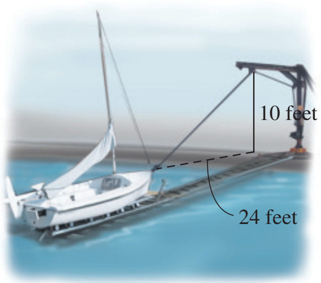
62. Distance. A pilot is flying from Dullsville to Middale to Pleasantville. With reference to an origin, Dullsville is located at $(2, 4)$, Middale at $(8, 12)$, and Pleasantville at $(20, 3)$, all numbers being in 100-mile units.

a. Locate the positions of the three cities on a Cartesian coordinate plane.

b. Compute the distance traveled by the pilot.

c. Compute the direct distance between Dullsville and Pleasantville.

63. Docking distance. A rope is attached to the bow of a sailboat that is 24 feet from the dock. The rope is drawn in over a pulley 10 feet higher than the bow at the rate of 3 feet per second. Find the distance from the boat to the dock after t seconds.



Beyond the Basics

64. Use coordinates to prove that the diagonals of a parallelogram bisect each other. [*Hint:* Choose $(0, 0)\,(a, 0)$, (b, c), and $(a + b, c)$ as the vertices of a parallelogram.]

65. Let $A(2, 3)$, $B(5, 4)$, and $C(3, 8)$ be three points in a coordinate plane. Find the coordinates of the point D such that the points A, B, C, and D form a parallelogram with

a. AB as one of the diagonals. **b.** AC as one of the diagonals.

c. BC as one of the diagonals.

[*Hint:* Use Exercise 64.]

66. Prove that if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram. [*Hint:* Choose $(0, 0)$, $(a, 0)$, (b, c), and (x, y) as the vertices of a quadrilateral and show that $x = b - a$, $y = c$.]

67. a. Show that $(1, 2)$, $(-2, 6)$, $(5, 8)$, and $(8, 4)$ are the vertices of a parallelogram. [*Hint:* Use Exercise 66.]

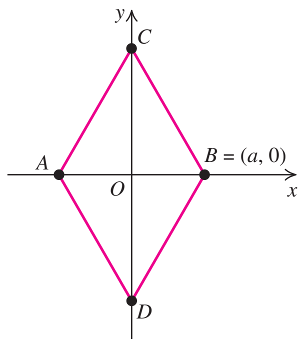
b. Find (x, y) assuming that $(3, 2)$, $(6, 3)$, (x, y), and $(6, 5)$ are the vertices of a parallelogram.

68. Show that the sum of the squares of the lengths of the sides of a parallelogram is equal to the sum of the squares of the lengths of the diagonals. [*Hint:* Choose $(0, 0)$, $(a, 0)$, (b, c), and $(a + b, c)$ as the vertices of the parallelogram.]

69. Prove that in a right triangle, the midpoint of the hypotenuse is the same distance from each of the vertices. [*Hint:* Let the vertices be $(0, 0)$, $(a, 0)$, and $(0, b)$.]

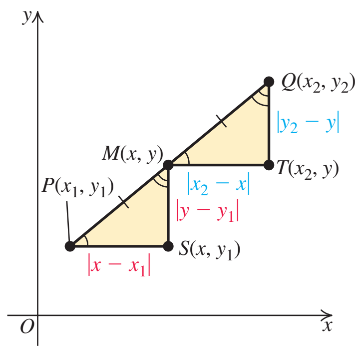
70. Isosceles triangle. An isosceles triangle ABC has right angle at C and hypotenuse of length c. Find the length of each leg of the triangle.

71. Equilateral triangles. Two equilateral triangles ABC and ABD have a common side AB of length $2a$. The side AB lies on the x-axis with $B = (a, 0)$ and midpoint at the origin O. (See the figure.) Find the coordinates of the vertices A, C, and D.



72. Midpoint formula. Use the notation in the accompanying figure to prove the midpoint formula

$$M(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$



Critical Thinking / Discussion / Writing

In Exercises 73–77, describe the set of points $P(x, y)$ in the xy-plane that satisfies the given condition.

73. a. $x = 0$ **b.** $y = 0$

74. a. $xy = 0$ **b.** $xy \neq 0$

75. a. $xy > 0$ **b.** $xy < 0$

76. a. $x^2 + y^2 = 0$ **b.** $x^2 + y^2 \neq 0$

77. a. $x \geq 0$ **b.** $y \geq 0$

78. Describe how to determine the quadrant in which a point lies from the signs of its coordinates.

GETTING READY ▸ for the Next Section

REVIEW CONCEPTS

How to plot points **(Section 1.1, page 3)**

Equivalent equations **(Appendix A.6, page 1019)**

REVIEW SKILLS

Evaluating algebraic expressions **(Appendix A.1, page 989)**

In Exercises GR1–GR4, evaluate each expression for the given values of x and y.

GR1. $x^2 + y^2$

a. $x = \frac{1}{2}, y = \frac{1}{2}$ **b.** $x = \frac{\sqrt{2}}{2}, y = \frac{\sqrt{2}}{2}$

GR2. $(x - 1)^2 + (y + 2)^2$

a. $x = -1, y = 1$ **b.** $x = 4, y = 2$

GR3. $\frac{x}{|x|} + \frac{|y|}{y}$

a. $x = 2, y = -3$ **b.** $x = -4, y = 3$

GR4. $\frac{|x|}{x} + \frac{|y|}{y}$

a. $x = -1, y = -2$ **b.** $x = 3, y = 2$

Completing squares **(Appendix A.6, page 1022)**

In Exercises GR5–GR10, find the appropriate term that should be added so that the binomial becomes a perfect square trinomial.

GR5. $x^2 - 6x$ **GR6.** $x^2 - 8x$

GR7. $y^2 + 3y$ **GR8.** $y^2 + 5y$

GR9. $x^2 - ax$ **GR10.** $x^2 + xy$

1.2 Graphs of Equations

Videos for this Section

Objectives

1 ▸ Sketch a graph by plotting points.

2 ▸ Find the intercepts of a graph.

3 ▸ Find the symmetries in a graph.

4 ▸ Find the equation of a circle.

Leafsmen Discover Deer

A herd of 400 deer was introduced onto a small island called Leafs. The people both liked and admired these beautiful creatures. However, the Leafsmen soon discovered that deer meat is excellent food. Also, a rumor spread throughout Leafs that eating deer meat prolonged one's life. They then began to hunt the deer. The number of deer, y, after t years from the initial introduction of deer into Leafs is described by the equation $y = -t^4 + 96t^2 + 400$. When does the population of deer become extinct in Leafs? (See Example 5.)

Objective 1 ▸

Graph of an Equation

An equation is an algebraic equality relating one or more quantities. An equation involving two unknown quantities describes a relation between these two quantities and specifies how one quantity changes with respect to the other quantity. The two changing (or varying) quantities are often represented by *variables*. The following equations are examples of relationships between two variables:

$$y = 2x + 1; \quad x^2 + y^2 = 4; \quad y = x^2; \quad x = y^2; \quad F = \frac{9}{5}C + 32;$$
$$\text{and} \quad q = -3p^2 + 30.$$

An ordered pair (a, b) is said to **satisfy** an equation with variables x and y if, when a is substituted for x and b is substituted for y in the equation, the resulting statement is true. For example, the ordered pair $(2, 5)$ satisfies the equation $y = 2x + 1$ because replacing x with 2 and y with 5 yields $5 = 2(2) + 1$, and simplifies to $5 = 5$, which is a true statement. The ordered pair $(5, -2)$ does not satisfy this equation because replacing x with 5 and y with -2 yields $-2 = 2(5) + 1$, and simplifies to $-2 = 11$, which is false. An ordered pair that satisfies an equation is called a **solution** of the equation.

In an equation involving x and y, if the value of y can be found given the value of x, then we say that y is the **dependent variable** and x is the **independent variable**. In the equation $y = 2x + 1$, for any real number x, there is a corresponding value of y. Hence, we have infinitely many solutions of the equation $y = 2x + 1$. When these solutions are graphed or plotted as points in the coordinate plane, they constitute the *graph of the equation*.

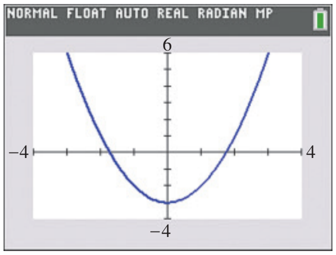
The graph of an equation is a geometric visualization of its solution set. In short, the coordinate plane allows us to investigate algebraic equations geometrically and to read many properties directly from their graphs.

Graph of an Equation

The **graph of an equation** in two variables, such as $\boldsymbol{x}$ and $\boldsymbol{y}$, is the set of all ordered pairs (a, b) in the coordinate plane that satisfy the equation.

TECHNOLOGY CONNECTION

Calculator graph of $y = x^2 - 3$



EXAMPLE 1 Sketching a Graph by Plotting Points

Sketch the graph of $y = x^2 - 3$.

Solution

There are infinitely many solutions of the equation $y = x^2 - 3$. To find a few, we choose integer values of x between -3 and 3. Then we find the corresponding values of y as shown in Table 1.3.

TABLE 1.3

x	$y = x^2 - 3$	(x, y)
-3	$y = (-3)^2 - 3 = 9 - 3 = 6$	$(-3, 6)$
-2	$y = (-2)^2 - 3 = 4 - 3 = 1$	$(-2, 1)$
-1	$y = (-1)^2 - 3 = 1 - 3 = -2$	$(-1, -2)$
0	$y = 0^2 - 3 = 0 - 3 = -3$	$(0, -3)$
1	$y = 1^2 - 3 = 1 - 3 = -2$	$(1, -2)$
2	$y = 2^2 - 3 = 4 - 3 = 1$	$(2, 1)$
3	$y = 3^2 - 3 = 9 - 3 = 6$	$(3, 6)$

We plot the seven solutions (x, y) and join them with a smooth curve, as shown in Figure 1.11. This curve is the graph of the equation $y = x^2 - 3$.

Practice Problem 1. Sketch the graph of $y = -x^2 + 1$. ■

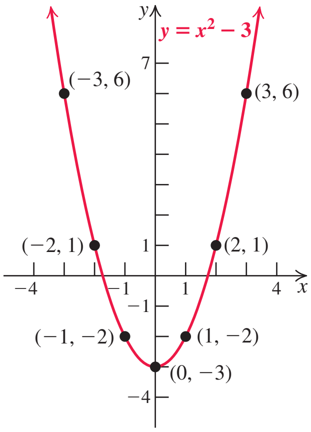


Figure 1.11 ▸ Graph of a parabola

The bowl-shaped curve sketched in Figure 1.11 is called a *parabola.* It is easy to find parabolas in everyday settings. For example, when you throw a ball, the path it travels is a parabola. Also, the reflector behind a car's headlight is parabolic in shape.

Example 1 suggests the following three steps for sketching the graph of an equation by plotting points.

Sketching a Graph by Plotting Points

Step 1 ▸ Make a representative table of solutions of the equation.

Step 2 ▸ Plot the solutions as ordered pairs in the coordinate plane.

Step 3 ▸ Connect the solutions in Step 2 with a smooth curve.

Comment This point-plotting technique has obvious pitfalls. For instance, many different curves pass through the same four points. See Figure 1.12. Assume that these points are solutions of a given equation. There is no way to guarantee that any curve we pass through the plotted points is the actual graph of the equation. However, in general, more plotted solutions result in a more accurate graph of the equation.

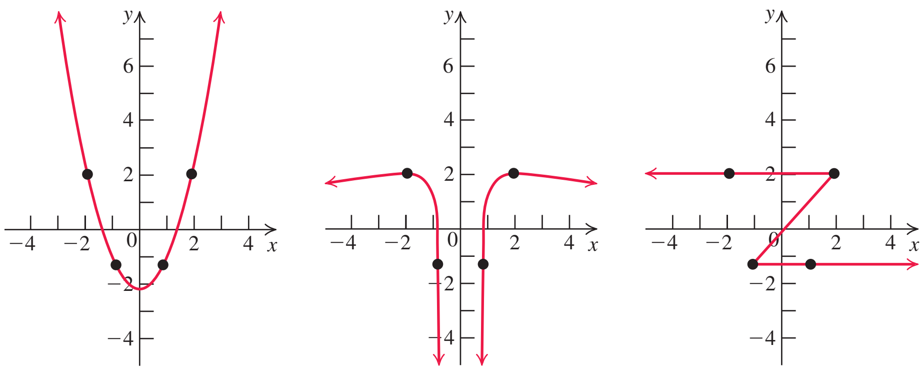


Figure 1.12 ▸ Several graphs through same four points

One way to graph an equation that reduces plotting errors is to plot enough points so that the graph becomes self-evident. Graphing calculators and computer programs are especially good for this purpose. A better way is to identify the shape of the curve the equation represents based on its algebraic features. For instance, we can identify the equation in Example 1 as a quadratic equation. We will soon learn how to graph the parabola that is the graph of the quadratic equation $y = ax^2 + bx + c$. Throughout this book we will investigate various techniques that will allow us to graph an equation that minimizes the amount of point plotting.

Objective 2 ▸

Intercepts

We examine the points where a graph intersects (crosses or touches) the coordinate axes. Because all points on the x-axis have a y-coordinate of 0, any point where a graph intersects the x-axis has the form $(a, 0)$. See Figure 1.13. The number a is called an ***x*-intercept** of the graph. Similarly, any point where a graph intersects the y-axis has the form $(0, b)$, and the number b is called a ***y*-intercept** of the graph.

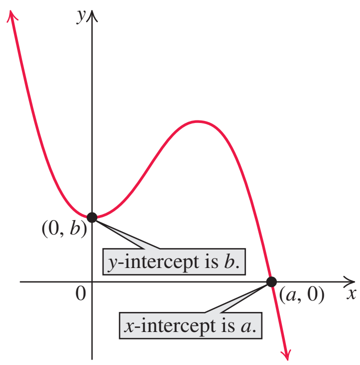


Figure 1.13 ▸ Intercepts of a graph

Finding the Intercepts of the Graph of an Equation

Step 1 ▸ To find the x-intercepts of the graph of an equation, set $y = 0$ in the equation and solve for x.

Step 2 ▸ To find the y-intercepts of the graph of an equation, set $x = 0$ in the equation and solve for y.

Note that only real number solutions in Steps 1 and 2 correspond to the intercepts.

SIDE NOTE

Do not try to calculate the x-intercept by setting $x = 0$. The term x-*intercept* denotes the x-coordinate of the point where the graph touches or crosses the x-axis; so the y-coordinate must be 0.

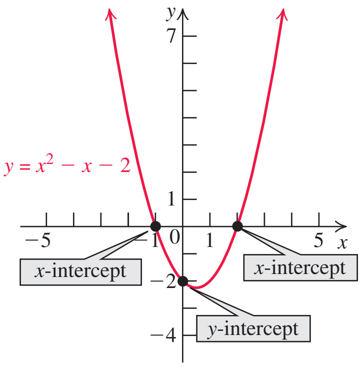


Figure 1.14 ▸ Intercepts of a graph

EXAMPLE 2 Finding Intercepts

Find the x- and y-intercepts of the graph of the equation $y = x^2 - x - 2$.

Solution

Step 1 ▸ Set $y = 0$ in the equation and solve for x.

$y = x^2 - x - 2$	Original equation
$0 = x^2 - x - 2$	Set $y = 0$.
$0 = (x + 1)(x - 2)$	Factor.
$x + 1 = 0 \quad \text{or} \quad x - 2 = 0$	Zero-product property
$x = -1 \quad \text{or} \quad x = 2$	Solve each equation for x.

The x-intercepts are -1 and 2.

Step 2 ▸ Set $x = 0$ in the equation and solve for y.

$y = x^2 - x - 2$	Original equation
$y = 0^2 - 0 - 2$	Set $x = 0$.
$y = -2$	Solve for y.

The y-intercept is -2.

The graph of the equation $y = x^2 - x - 2$ is shown in Figure 1.14.

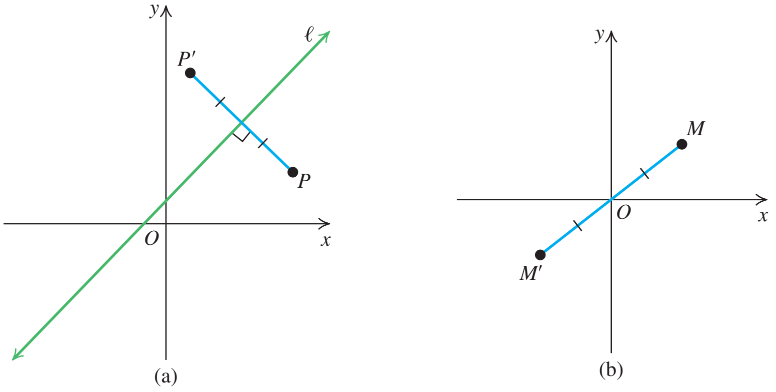
Practice Problem 2. Find the intercepts of the graph of $y = 2x^2 + 3x - 2$. ■

Objective 3 ▸

Symmetry

The concept of **symmetry** helps us sketch graphs of equations. A graph has symmetry if one portion of the graph is a *mirror image* of another portion. As shown in Figure 1.15(a), if a line ℓ is an **axis of symmetry**, or **line of symmetry**, we can construct the mirror image of any point P not on ℓ by first drawing the perpendicular line segment from P to ℓ. Then we extend this segment an equal distance on the other side to a point P' so that the line ℓ perpendicularly bisects the line segment $\overline{PP'}$. In Figure 1.15(a), we say that the point P' is the *symmetric image* of the point P about the line ℓ.

Two points M and M' are **symmetric about a point** Q if Q is the midpoint of the line segment $\overline{MM'}$. Figure 1.15(b) illustrates the symmetry about the origin O. Symmetry lets us use information about part of the graph to draw the remainder of the graph.



(a) Symmetric points about a line

(b) Symmetric points about the origin

Figure 1.15

The following three types of symmetries occur frequently.

Symmetries

1. A graph is **symmetric with respect to (or about) the *y*-axis** if for every point (x, y) on the graph, the point $(-x, y)$ is also on the graph. See Figure 1.16(a) and (d).
2. A graph is **symmetric with respect to (or about) the *x*-axis** if for every point (x, y) on the graph, the point $(x, -y)$ is also on the graph. See Figure 1.16(b) and (e).
3. A graph is **symmetric with respect to (or about) the origin** if for every point (x, y) on the graph, the point $(-x, -y)$ is also on the graph. See Figure 1.16(c) and (f).

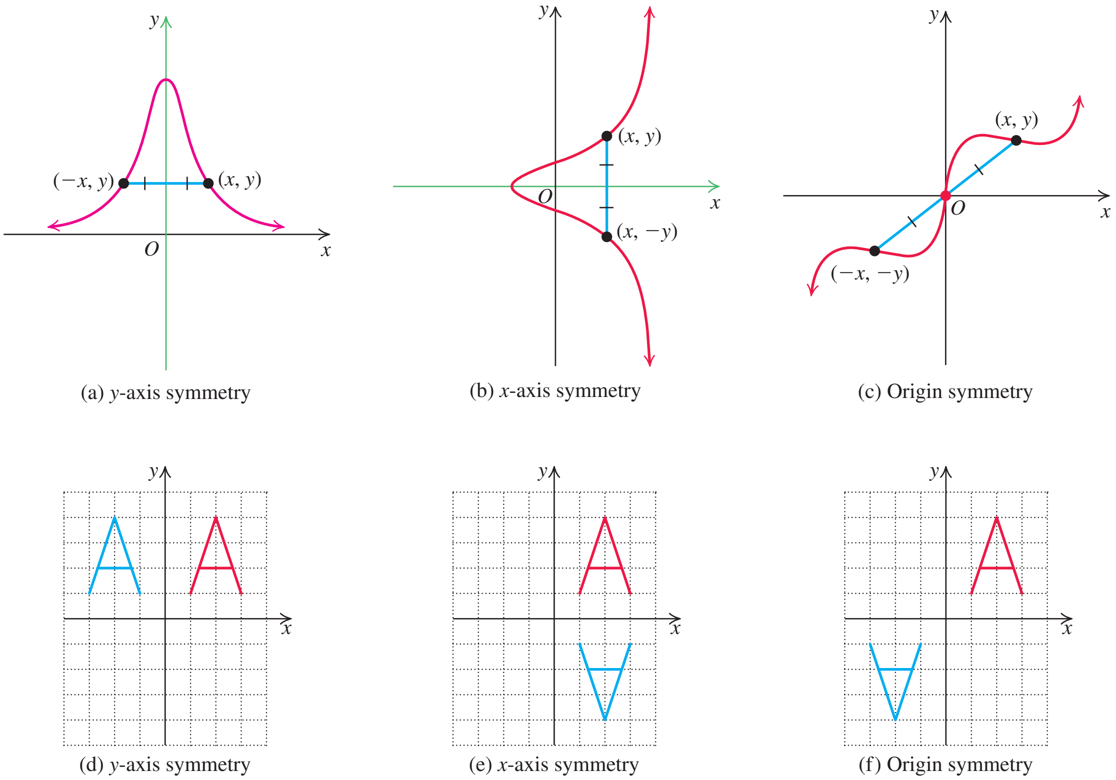


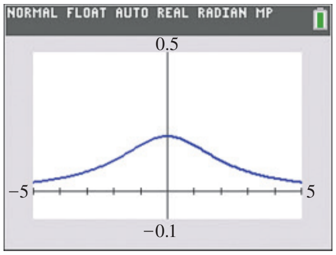
Figure 1.16 ▸ Three types of symmetry

Tests for Symmetry

1. The graph of an equation is symmetric about the y-axis if replacing x with $-x$ results in an equivalent equation.
2. The graph of an equation is symmetric about the x-axis if replacing y with $-y$ results in an equivalent equation.
3. The graph of an equation is symmetric about the origin if replacing x with $-x$ and y with $-y$ results in an equivalent equation.

TECHNOLOGY CONNECTION

The calculator graph of $y = 1/(x^2 + 5)$ reinforces the result of Example 3.



SIDE NOTE

Note that if *only* even powers of x appear in an equation, then the graph is symmetric with respect to the y-axis because for any integer n, $(-x)^{2n} = x^{2n}$.

EXAMPLE 3 Checking for Symmetry

Determine whether the graph of the equation $y = \dfrac{1}{x^2 + 5}$ is symmetric about the y-axis.

Solution

Replace x with $-x$ to see if $(-x, y)$ also satisfies the equation.

$$y = \frac{1}{x^2 + 5} \qquad \text{Original equation}$$

$$y = \frac{1}{(-x)^2 + 5} \qquad \text{Replace } x \text{ with } -x.$$

$$y = \frac{1}{x^2 + 5} \qquad \text{Simplify: } (-x)^2 = x^2.$$

Because replacing x with $-x$ gives us the original equation, the graph of $y = \dfrac{1}{x^2 + 5}$ is symmetric with respect to the y-axis.

Practice Problem 3. Check whether the graph of $x^2 - y^2 = 1$ is symmetric about the y-axis. ■

PROCEDURE IN ACTION

EXAMPLE 4 Sketching a Graph Using Symmetry

Objective

Use symmetry to sketch the graph of an equation.

Step 1 ▸ Test for all three symmetries.

About the ***x*-axis**: Replace y with $-y$.

About the ***y*-axis**: Replace x with $-x$.

About the **origin**: Replace x with $-x$ and y with $-y$.

Example

Use symmetry to sketch the graph of $y = 4x - x^3$.

1.

x-axis	y-axis	origin
Replace y with $-y$.	Replace x with $-x$.	Replace x with $-x$ and y with $-y$.
$-y = 4x - x^3$ $y = -4x + x^3$	$y = 4(-x) - (-x)^3$ $y = -4x + x^3$	$-y = 4(-x) - (-x)^3$ $-y = -4x + x^3$ $y = 4x - x^3$
No	**No**	**Yes**

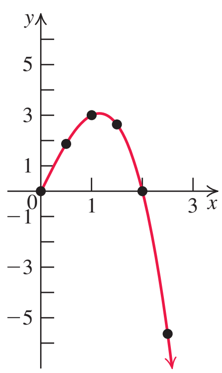
Step 2 ▸ Make a table of values using any symmetries found in Step 1.

2. Origin symmetry: If (x, y) is on the graph, so is $(-x, -y)$. Use only positive x-values in the table.

x	0	0.5	1	1.5	2	2.5
$y = 4x - x^3$	0	1.875	3	2.625	0	−5.625

Step 3 ▸ Plot the points from the table and draw a smooth curve through them.

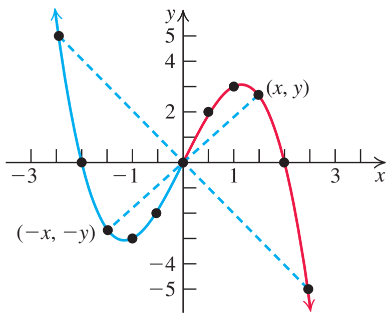
3.



Graph of $y = 4x - x^3$, $x \geq 0$

Step 4 ▸ Extend the portion of the graph found in Step 3, using symmetries.

4.



Graph of $y = 4x - x^3$

Practice Problem 4. Check whether the graph of $x^2 = y^3$ is symmetric about the x-axis, the y-axis, and the origin. ■

EXAMPLE 5 Sketching a Graph by Using Intercepts and Symmetry

Initially, 400 deer are on Leafs island. The number y of deer on the island after t years is described by the equation

$$y = -t^4 + 96t^2 + 400.$$

a. Sketch the graph of the equation $y = -t^4 + 96t^2 + 400$.

b. Adjust the graph in part **a** to account for only the physical aspects of the problem.

c. When do deer become extinct on Leafs?

Solution

a. (i) We find all intercepts. If we set $t = 0$ in the equation $y = -t^4 + 96t^2 + 400$, we obtain $y = 400$. Thus, the y-intercept is 400.

To find the t-intercepts, we set $y = 0$ in the given equation.

$$y = -t^4 + 96t^2 + 400 \quad \text{Original equation}$$
$$0 = -t^4 + 96t^2 + 400 \quad \text{Set } y = 0.$$
$$t^4 - 96t^2 - 400 = 0 \quad \text{Multiply both sides by } -1 \text{ and interchange sides.}$$
$$(t^2 + 4)(t^2 - 100) = 0 \quad \text{Factor.}$$
$$(t^2 + 4)(t + 10)(t - 10) = 0 \quad \text{Factor } t^2 - 100.$$
$$t^2 + 4 = 0 \quad \text{or} \quad t + 10 = 0 \quad \text{or} \quad t - 10 = 0 \quad \text{Zero-product property}$$
$$t = -10 \quad \text{or} \quad t = 10 \quad \text{Solve for } t\text{; there is no real solution of } t^2 + 4 = 0.$$

The t-intercepts are -10 and 10.

(ii) We check for symmetry. Note that t replaces x as the independent variable.

Symmetry about the *t*-axis:
Replacing y with $-y$ gives $-y = -t^4 + 96t^2 + 400$. The pair $(0, 400)$ is on the graph of $y = -t^4 + 96t^2 + 400$ but not on $-y = -t^4 + 96t^2 + 400$. Consequently, the graph is not symmetric about the t-axis.

Symmetry about the *y*-axis:
Replacing t with $-t$ in the equation $y = -t^4 + 96t^2 + 400$, we obtain $y = -(-t)^4 + 96(-t)^2 + 400 = -t^4 + 96t^2 + 400$, which is the original equation. Thus, $(-t, y)$ also satisfies the equation and the graph is symmetric about the y-axis.

Symmetry about the origin:
Replacing t with $-t$ and y with $-y$ in the equation $y = -t^4 + 96t^2 + 400$, we obtain $-y = -(-t)^4 + 96(-t)^2 + 400$, or $-y = -t^4 + 96t^2 + 400$. As we saw when we discussed symmetry in the t-axis, $(0, 400)$ is a solution but $(0, -400)$ is not a solution of this equation; so the graph is not symmetric with respect to the origin.

(iii) We sketch the graph by plotting points for $t \geq 0$ (see Table 1.4) and then using symmetry about the y-axis (see Figure 1.17).

TABLE 1.4

t	$y = -t^4 + 96t^2 + 400$	(t, y)
0	400	$(0, 400)$
1	495	$(1, 495)$
5	2175	$(5, 2175)$
7	2703	$(7, 2703)$
9	1615	$(9, 1615)$
10	0	$(10, 0)$
11	-2625	$(11, -2625)$

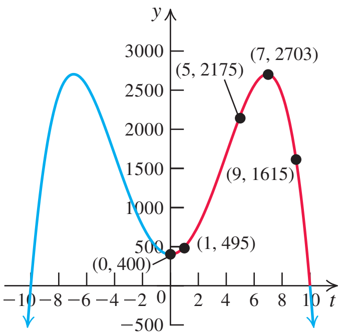


Figure 1.17

b. The graph pertaining to the physical aspects of the problem is the red portion of the graph in Figure 1.17.

c. The positive t-intercept, 10, gives the time in years when the deer population of Leafs is 0; so deer are extinct after 10 years.

Practice Problem 5. Repeat Example 5, assuming that the initial deer population is 324 and the number of deer on the island after t years is given by the equation $y = -t^4 + 77t^2 + 324$. ■

Objective 4 ▸

Circles

The Cartesian coordinate plane allows us to describe various geometric curves using algebraic equations. We illustrate this situation in the case of a circle. We begin with the geometric definition.

Circle

A **circle** is a set of points in a Cartesian coordinate plane that are at a fixed distance r from a specified point (h, k). The fixed distance r is called the **radius** of the circle, and the specified point (h, k) is called the **center** of the circle.

Standard Form A point $P(x, y)$ is on the circle if and only if its distance from the center $C(h, k)$ is r. Using the notation for the distance between the points P and C, we have the following:

$$d(P, C) = r$$

$$\sqrt{(x-h)^2 + (y-k)^2} = r \qquad \text{Distance formula}$$

$$(x-h)^2 + (y-k)^2 = r^2. \qquad \text{Square both sides.}$$

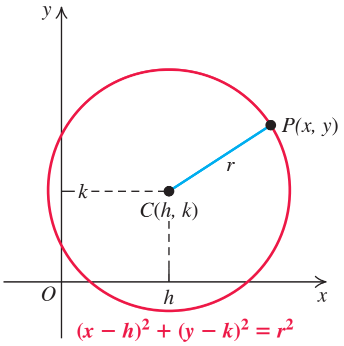


Figure 1.18

The equation $(x - h)^2 + (y - k)^2 = r^2$ is an equation of a circle with radius r and center (h, k). A point (x, y) is on the circle of radius r with center $C(h, k)$ if and only if it satisfies this equation. Figure 1.18 is the graph of a circle with center $C(h, k)$ and radius r.

The Standard Form for the Equation of a Circle

The **standard form** of an equation of a circle with center (h, k) and radius r is

(1) $$(x - h)^2 + (y - k)^2 = r^2$$

Employing the Cartesian coordinate plane as a geometric visualization tool, we see that the graph of any equation in x and y that can be written in the standard form $(x - h)^2 + (y - k)^2 = r^2$ is a circle with center (h, k) and radius r.

EXAMPLE 6 Finding the Equation of a Circle

Find the standard form of the equation of the circle with center $(7, -3)$ and that passes through the point $P = (5, -2)$.

Solution

$(x - h)^2 + (y - k)^2 = r^2$ — Standard form

$(x - 7)^2 + (y - (-3))^2 = r^2$ — Replace h with 7 and k with -3.

(2) $(x - 7)^2 + (y + 3)^2 = r^2$ — $-(-3) = 3$

Because the point $P = (5, -2)$ lies on the circle, its coordinates satisfy equation (2). So

$(5 - 7)^2 + (-2 + 3)^2 = r^2$ — Replace x with 5 and y with -2.

$5 = r^2$ — Simplify.

Replacing r^2 with 5 in equation (2) gives the required standard form

$$(x - 7)^2 + (y + 3)^2 = 5.$$

Practice Problem 6. Find the standard form of the equation of the circle with center $(3, -6)$ and radius 10. ■

If an equation in two variables can be written in standard form (1), then its graph is a circle with center (h, k) and radius r.

EXAMPLE 7 Graphing a Circle

Specify the center and radius and graph each circle.

a. $x^2 + y^2 = 1$

b. $(x + 2)^2 + (y - 3)^2 = 25$

Solution

a. The equation $x^2 + y^2 = 1$ can be rewritten as

$$(x - 0)^2 + (y - 0)^2 = 1^2.$$

Comparing this equation with equation (1), we conclude that the given equation is an equation of the circle with center $(0, 0)$ and radius 1. The graph is shown in Figure 1.19. This circle is called the **unit circle**.

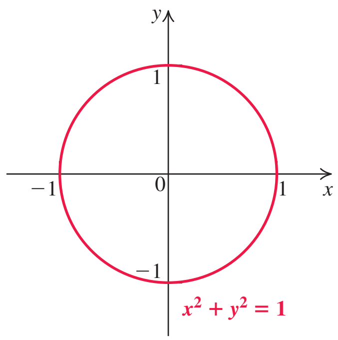


Figure 1.19 ▶ The unit circle

b. Rewriting the equation $(x+2)^2+(y-3)^2=25$ as

$$[x-(-2)]^2+(y-3)^2=5^2, \qquad x+2=x-(-2)$$

we see that the graph of this equation is the circle with center $(-2,\ 3)$ and radius 5. The graph is shown in Figure 1.20 in the margin below.

Practice Problem 7. Graph the equation $(x-2)^2+(y+1)^2=36$. ■

WARNING

It is easy to make sign errors when identifying the center of the circle. The direct method of finding the coordinates of the center is to set the expression inside each of the squares to zero.

$$(x+2)^2+(y-3)^2=25.$$

$x+2=0$ gives $x=-2$ and $y-3=0$ gives $y=3$. We identify the center of this circle as $(-2,\ 3)$.

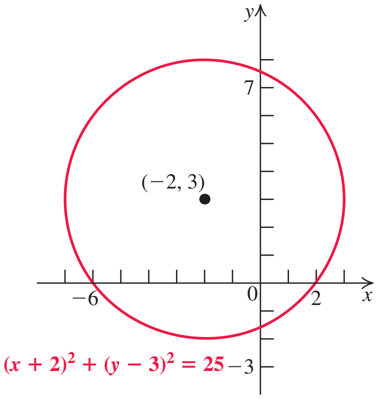


Figure 1.20 ▸ Circle with radius 5 and center at $(-2,\ 3)$

SIDE NOTE

If $r\ge x\ge 0$, then $r^2\ge x^2$, so $r^2-x^2\ge 0$. If $-r\le -x\le 0$, then $r\ge x\ge 0$, and again $r^2-x^2\ge 0$. So if $-r\le x\le r$, we have $r^2-x^2\ge 0$.

Semicircles

Letting $h=0$ and $k=0$ in equation (1), we have

(3) $$x^2+y^2=r^2$$

Equation (3) is the standard form for a circle with center at the origin and radius r.

We solve equation (3) for y:

$$y^2=r^2-x^2 \qquad \text{Subtract } x^2 \text{ from both sides.}$$
$$y=\pm\sqrt{r^2-x^2} \qquad \text{Square root property}$$

or $$y=\sqrt{r^2-x^2} \quad \text{and} \quad y=-\sqrt{r^2-x^2}.$$

Similarly, solving equation (3) for x, we have two equations,

$$x=\sqrt{r^2-y^2} \quad \text{and} \quad x=-\sqrt{r^2-y^2}.$$

The graphs of these four equations are **semicircles** (half circles), shown in Figure 1.21.

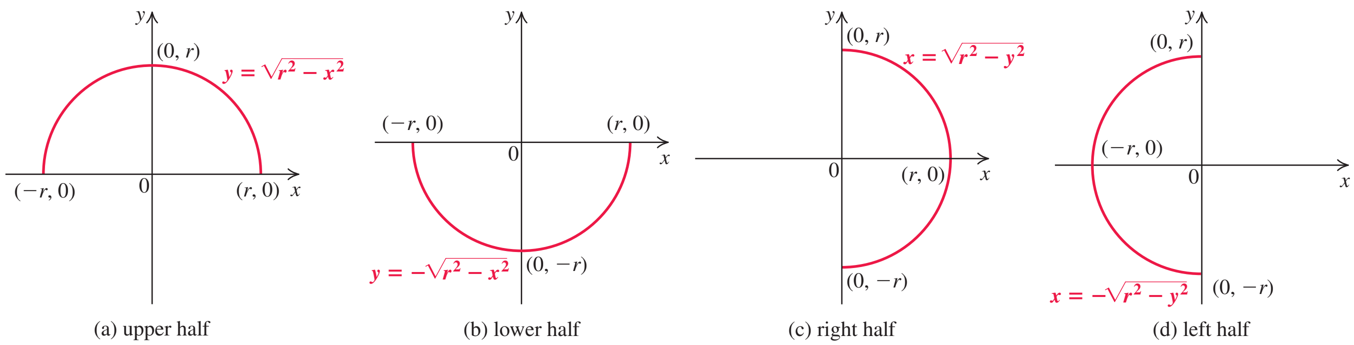


Figure 1.21 ▸ Semicircles

General Form Consider a circle with center $C=(2,-3)$ and radius $r=5$. Its standard form is

$$(x-2)^2+(y+3)^2=5^2.$$

Expanding $(x-2)^2$ and $(y+3)^2$, we have

$$(x^2-4x+4)+(y^2+6y+9)=25 \qquad (a\pm b)^2=a^2\pm 2ab+b^2$$
$$x^2+y^2-4x+6y+13=25 \qquad \text{Rewrite.}$$
$$x^2+y^2-4x+6y-12=0. \qquad \text{Simplify.}$$

This last equation is the **general form** of the equation of the circle with standard form $(x-2)^2+(y+3)^2=5^2$. In general, if we expand the squared expressions in the standard equation of a circle,

$$(x-h)^2+(y-k)^2=r^2,$$

and then simplify, we obtain an equation of the form

$$x^2+y^2+ax+by+c=0. \quad (4)$$

Equation (4) is called the *general form* of the equation of a circle. The graph of the equation $Ax^2+By^2+Cx+Dy+E=0$ is also a circle if $A\neq 0$ and $A=B$. You can convert this equation to the general form by dividing both sides by A. See Exercises 91 and 92.

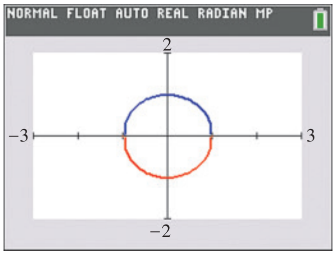
TECHNOLOGY CONNECTION

To graph the equation $x^2+y^2=1$ on a graphing calculator, we graph the two equations of the upper and lower semicircles,

$Y_1=\sqrt{1-x^2}$ and

$Y_2=-\sqrt{1-x^2}$,

in the same window. The calculator graph does not look quite like a circle. We use the ZSquare option to make the display look like a circle.



General Form of the Equation of a Circle

The **general form** of the equation of a circle is

$$x^2+y^2+ax+by+c=0.$$

On the other hand, if we are given an equation in general form, we can convert it to standard form by completing the squares on the x- and y-terms. This gives

$$(x-h)^2+(y-k)^2=d. \quad (5)$$

If $d>0$, the graph of equation (5) is a circle with center (h, k) and radius $\sqrt{d}$. If $d=0$, the graph of equation (5) is the point (h, k). If $d<0$, there is no graph.

RECALL

To convert x^2+bx to a perfect square, add $\left(\frac{1}{2}b\right)^2$ to get

$x^2+bx+\left(\frac{1}{2}b\right)^2=\left(x+\frac{b}{2}\right)^2$.

This completes the square.

EXAMPLE 8 Converting the General Form to Standard Form

Find the center and radius of the circle with equation

$$x^2+y^2-6x+8y+10=0.$$

Solution

Complete the squares on both the x-terms and y-terms to get standard form.

$x^2+y^2-6x+8y+10=0$	Original equation
$(x^2-6x)+(y^2+8y)=-10$	Group the x-terms and y-terms.
$(x^2-6x+9)+(y^2+8y+16)=-10+9+16$	Complete the squares by adding 9 and 16 to both sides.
$(x-3)^2+(y+4)^2=15$	Factor and simplify.
$(x-3)^2+[y-(-4)]^2=\left(\sqrt{15}\right)^2$	

The last equation tells us that we have $h=3$, $k=-4$, and $r=\sqrt{15}$. Therefore, the circle has center $(3, -4)$ and radius $\sqrt{15}\approx 3.9$.

Practice Problem 8. Find the center and radius of the circle with equation $x^2+y^2+4x-6y-12=0$. ■

1.2 Exercises

Videos for select exercises

Concepts and Vocabulary

1. The graph of an equation in two variables such as x and y is the set of all ordered pairs (a, b) ______________.
2. If $(-2, 4)$ is a point on a graph that is symmetric with respect to the y-axis, then the point ______________ is also on the graph.
3. If $(0, -5)$ is a point of a graph, then -5 is a(n) ______________ intercept of the graph.
4. An equation in standard form of a circle with center $(1, 0)$ and radius 2 is ______________.
5. **True or False.** The graph of the equation $3x^2 - 2x + y + 3 = 0$ is a circle.
6. **True or False.** If a graph is symmetric about the x-axis, then it must have at least one x-intercept.
7. **True or False.** The center of the circle with equation $(x + 3)^2 + (y + 4)^2 = 9$ is the point $(3, 4)$.
8. **True or False.** If $(-2, 3)$ is a point on a graph that is symmetric with respect to the origin, then the point $(2, -3)$ is also on the graph.

Building Skills

In Exercises 9–14, determine whether the given points are on the graph of the equation.

	Equation	Points
9.	$y = x - 1$	$(-3, -4), (1, 0), (4, 3), (2, 3)$
10.	$2y = 3x + 5$	$(-1, 1), (0, 2), \left(-\frac{5}{3}, 0\right), (1, 4)$
11.	$y = \sqrt{x + 1}$	$(3, 2), (0, 1), (8, -3), (8, 3)$
12.	$y = \frac{1}{x}$	$\left(-3, \frac{1}{3}\right), (1, 1), (0, 0), \left(2, \frac{1}{2}\right)$
13.	$x^2 - y^2 = 1$	$(1, 0), (0, -1), \left(2, \sqrt{3}\right), \left(2, -\sqrt{3}\right)$
14.	$y^2 = x$	$(1, -1), (1, 1), (0, 0), \left(2, -\sqrt{2}\right)$

In Exercises 15–36, graph each equation by plotting points. Let $x = -3, -2, -1, 0, 1, 2,$ and 3 where applicable.

15. $y = x + 1$ **16.** $y = x - 1$ **17.** $y = 2x$

18. $y = \frac{1}{2}x$ **19.** $y = x^2$ **20.** $y = -x^2$

21. $y = |x|$ **22.** $y = |x + 1|$

23. $y = |x| + 1$ **24.** $y = -|x| + 1$

25. $y = 4 - x^2$ **26.** $y = x^2 - 4$

27. $y = \sqrt{9 - x^2}$ **28.** $y = -\sqrt{9 - x^2}$

29. $y = x^3$ **30.** $y = -x^3$

31. $y^3 = x$ **32.** $y^3 = -x$

33. $x = |y|$ **34.** $|x| = |y|$

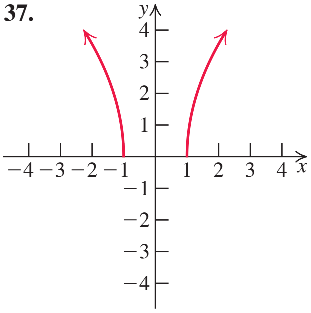
35. $y = |2 - x|$ **36.** $|x| + |y| = 1$

In Exercises 37–46, find

a. x- and y-intercepts.

b. symmetries (if any) about the x-axis, the y-axis, and the origin.

37.



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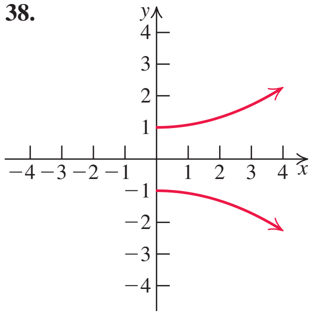

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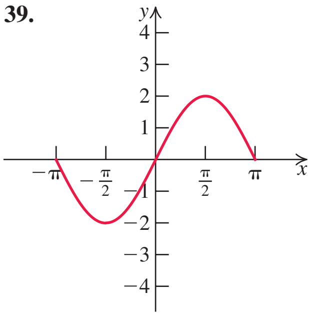

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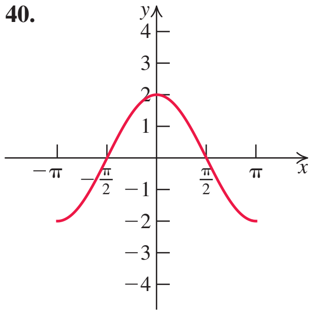

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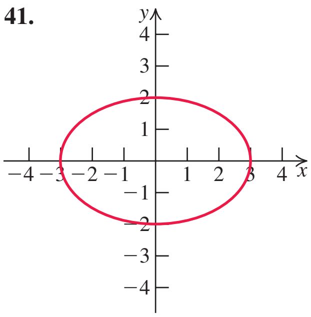

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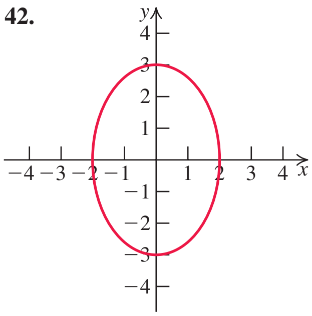

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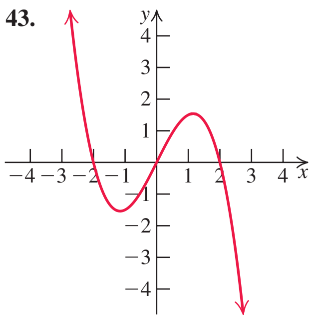

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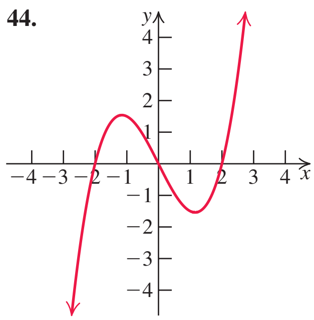

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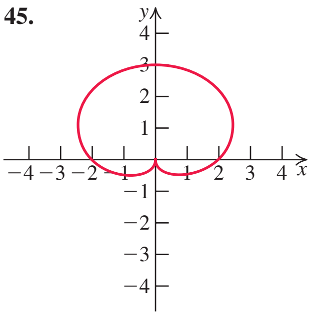

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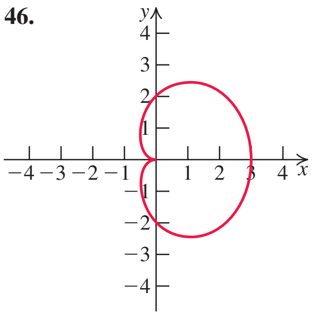
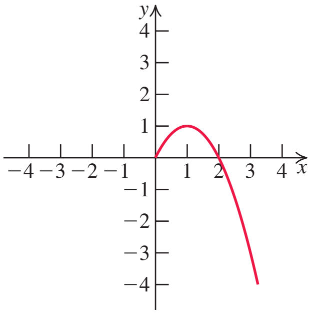


In Exercises 47–50, complete the given graph so that it has the indicated symmetry.



47. Symmetry about the x-axis

48. Symmetry about the y-axis

49. Symmetry about the origin

50. Symmetry about the x-axis and symmetry about the y-axis

In Exercises 51–64, find the x- and y-intercepts of the graph of each equation (if any).

51. $3x + 4y = 12$

52. $2x + 3y = 5$

53. $\frac{x}{5} + \frac{y}{3} = 1$

54. $\frac{x}{2} - \frac{y}{3} = 1$

55. $y = \frac{x+2}{x-1}$

56. $x = \frac{y-2}{y+1}$

57. $y = x^2 - 6x + 8$

58. $x = y^2 - 5y + 6$

59. $x^2 + y^2 = 4$

60. $(x-1)^2 + y^2 = 9$

61. $y = \sqrt{9 - x^2}$

62. $y = \sqrt{x^2 - 1}$

63. $xy = 1$

64. $y = x^2 + 1$

In Exercises 65–74, test each equation for symmetry with respect to the x-axis, the y-axis, and the origin.

65. $y = x^2 + 1$

66. $x = y^2 + 1$

67. $y = x^3 + x$

68. $y = 2x^3 - x$

69. $y = 5x^4 + 2x^2$

70. $y = -3x^6 + 2x^4 + x^2$

71. $y = -3x^5 + 2x^3$

72. $y = 2x^2 - |x|$

73. $x^2y^2 + 2xy = 1$

74. $x^2 + y^2 = 16$

In Exercises 75–78, specify the center and the radius of each circle.

75. $(x-2)^2 + (y-3)^2 = 36$

76. $(x+1)^2 + (y-3)^2 = 16$

77. $(x+2)^2 + (y+3)^2 = 11$

78. $\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{3}{2}\right)^2 = \frac{3}{4}$

In Exercises 79–88, find the standard form of the equation of a circle that satisfies the given conditions. Graph each equation.

79. Center $(0, 1)$; radius 2

80. Center $(1, 0)$; radius 1

81. Center $(-1, 2)$; radius $\sqrt{2}$

82. Center $(-2, -3)$; radius $\sqrt{7}$

83. Center $(3, -4)$; passing through the point $(-1, 5)$

84. Center $(-1, 1)$; passing through the point $(2, 5)$

85. Center $(1, 2)$; touching the x-axis

86. Center $(1, 2)$; touching the y-axis

87. Diameter with endpoints $(7, 4)$ and $(-3, 6)$

88. Diameter with endpoints $(2, -3)$ and $(8, 5)$

In Exercises 89–94, find

a. the center and radius of each circle.

b. the x- and y-intercepts of the graph of each circle.

89. $x^2 + y^2 - 2x - 2y - 4 = 0$

90. $x^2 + y^2 - 4x - 2y - 15 = 0$

91. $2x^2 + 2y^2 + 4y = 0$

92. $3x^2 + 3y^2 + 6x = 0$

93. $x^2 + y^2 - x = 0$

94. $x^2 + y^2 + 1 = 0$

Applying the Concepts

In Exercises 95 and 96, a graph is described *geometrically* as the path of a point $P(x, y)$ on the graph. Find an equation for the graph described.

95. Geometry. $P(x, y)$ is on the graph if and only if the distance from $P(x, y)$ to the x-axis is equal to its distance to the y-axis.

96. Geometry. $P(x, y)$ is the same distance from the two points $(1, 2)$ and $(3, -4)$.

97. Saving and spending. Sketch a graph (years, money) that shows the amount of money available to you if you save \$100 each month until you have \$2400 and then withdraw \$80 each month until the \$2400 is gone.

98. Tracking distance during a workout. Sketch a graph (minutes, miles) that shows the distance you have traveled from your starting point if you jog at 6 mph for 10 minutes, rest for 10 minutes, and then walk at 3 mph back to your starting point.

99. Corporate profits. The equation $P = -0.5t^2 - 3t + 8$ describes the monthly profits (in millions of dollars) of ABCD Corp. for the year 2022, with $t = 0$ representing July 2022.

a. How much profit did the corporation make in March 2022?

b. How much profit did the corporation make in October 2022?

c. Sketch the graph of the equation.

d. Find the t-intercepts. What do they represent?

e. Find the P-intercept. What does it represent?

100. **Female students in college.** The equation

$$P = 18t^2 - 69t + 11{,}264$$

models the approximate number (in thousands) of female college students in the United States for the academic years 2015–2020, with $t = 0$ representing 2015.

a. Sketch the graph of the equation.

b. Find the P-intercept. What does it represent?

(*Source:* https://nces.ed.gov)

101. **Motion.** An object is thrown up from the top of a building that is 320 feet high. The equation $y = -16t^2 + 128t + 320$ gives the object's height (in feet) above the ground at any time t (in seconds) after the object is thrown.

a. What is the height of the object after 0, 1, 2, 3, 4, 5, and 6 seconds?

b. Sketch the graph of the equation $y = -16t^2 + 128t + 320$.

c. What part of the graph represents the physical aspects of the problem?

d. What are the intercepts of this graph, and what do they mean?

102. **Diving for treasure.** A treasure-hunting team of divers is placed in a computer-controlled diving cage. The equation $d = \frac{40}{3}t - \frac{2}{9}t^2$ describes the depth d (in feet) that the cage will descend in t minutes.

a. Sketch the graph of the equation $d = \frac{40}{3}t - \frac{2}{9}t^2$.

b. What part of the graph represents the physical aspects of the problem?

c. What is the total time of the entire dive?

Beyond the Basics

103. In the same coordinate system, sketch the graphs of the two circles with equations $x^2 + y^2 - 4x + 2y - 20 = 0$ and $x^2 + y^2 - 4x + 2y - 31 = 0$ and find the area of the region bounded between the two circles.

104. Find the equation of a circle with radius 5 and x-intercepts -4 and 4. [*Hint:* Center must be on the y-axis; there are two such circles.]

Critical Thinking / Discussion / Writing

105. Sketch the graph of $y^2 = 2x$ and explain how this graph is related to the graphs of $y = \sqrt{2x}$ and $y = -\sqrt{2x}$.

106. Show that a graph that is symmetric with respect to the x-axis and the y-axis must also be symmetric with respect to the origin. Give an example to show that the converse is not true.

107. **a.** Show that a circle with diameter having endpoints $A(0, 1)$ and $B(6, 8)$ intersects the x-axis at the roots of the equation $x^2 - 6x + 8 = 0$.

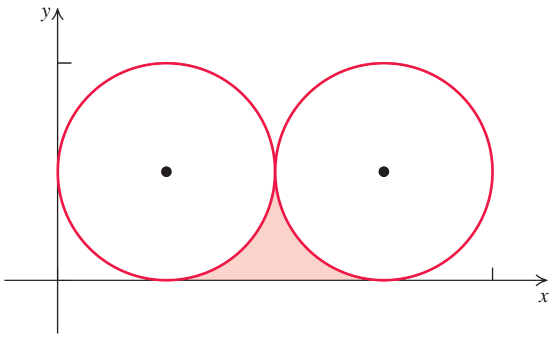
b. Show that a circle with diameter having endpoints $A(0, 1)$ and $B(a, b)$ intersects the x-axis at the roots of the equation $x^2 - ax + b = 0$.

c. Use graph paper, ruler, and compass to approximate the roots of the equation $x^2 - 3x + 1 = 0$.

108. The figure shows two circles each with radius r.

a. Write the coordinates of the center of each circle.

b. Find the area of the shaded region.



GETTING READY ▸ for the Next Section

REVIEW CONCEPTS

Solving linear equations **(Appendix A.6, pages 1020–1021)**

Graphs of equations **(Section 1.2, page 12)**

REVIEW SKILLS

Properties of rational expressions **(Appendix A.3, pages 1000–1005)**

In Exercises GR1–GR6, perform the indicated operations.

GR1. $\dfrac{5-3}{6-2}$ **GR2.** $\dfrac{1-2}{-2-2}$

GR3. $\dfrac{2-(-3)}{3-13}$ **GR4.** $\dfrac{3-1}{-2-(-6)}$

GR5. $\dfrac{\frac{1}{2}-\frac{1}{4}}{\frac{3}{8}-\left(-\frac{1}{4}\right)}$ **GR6.** $\dfrac{\frac{3}{4}-1}{\frac{1}{2}-\frac{1}{6}}$

Solving linear equations **(Appendix A.6, pages 1020–1021)**

In Exercises GR7–GR10, solve each equation for the specified variable.

GR7. $2x + 3y = 6$ for y **GR8.** $\frac{x}{2} - \frac{y}{5} = 3$ for y

GR9. $y - 2 - \frac{2}{3}(x + 1) = 0$ for y

GR10. $0.1x + 0.2y = 0$ for y

1.3 Lines

Objectives

1 ▸ Find the slope of a line.

2 ▸ Write the point-slope form of the equation of a line.

3 ▸ Write the slope-intercept form of the equation of a line.

4 ▸ Recognize the equations of horizontal and vertical lines.

5 ▸ Recognize the general form of the equation of a line.

6 ▸ Find equations of parallel and perpendicular lines.

7 ▸ Model data using linear regression.

Videos for this Section

A Texan's Tall Tale

Gunslinger Wild Bill Longley's *first* burial took place on October 11, 1878, after he was hanged before a crowd of thousands in Giddings, Texas.

Rumors persisted that Longley's hanging had been a hoax and that he had somehow faked his death and escaped execution. In 2001, Longley's descendants had his grave opened to determine whether the remains matched Wild Bill's description: a tall white male, age 27. Both the skeleton and some personal effects suggested that this was indeed Wild Bill. Modern science lent a hand, too: The DNA of Wild Bill's sister's descendant Helen Chapman was a perfect match.

Now the notorious gunman could be buried back in the Giddings cemetery—for the *second* time. How did the scientists conclude from the skeletal remains that Wild Bill was approximately 6 feet tall? (See Example 8.)

Objective 1 ▸

Slope of a Line

In this section, we study various forms of first-degree equations in two variables. Because the graphs of these equations are straight lines, they are called **linear equations**. Just as we measure weight or temperature by a number, we measure the "steepness" of a line by a number called its **slope**.

Consider two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on a line, as shown in Figure 1.22. We say that the ***rise*** is the change in y-coordinates between the points (x_1, y_1) and (x_2, y_2) and that the ***run*** is the corresponding change in the x-coordinates. A positive run indicates change to the right, as shown in Figure 1.22; a negative run indicates change to the left. Similarly, a positive rise indicates upward change; a negative rise indicates downward change. A zero run or rise means no change in the corresponding coordinate.

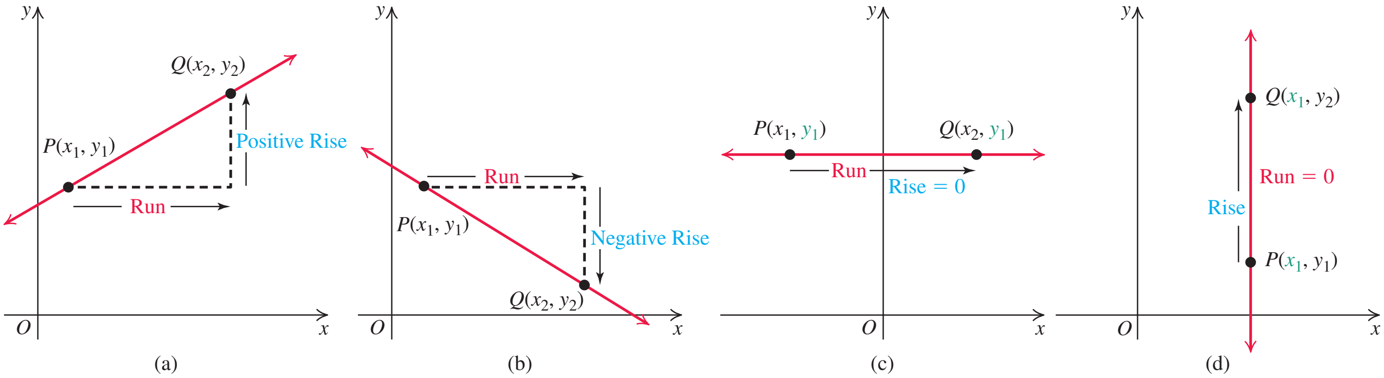


Figure 1.22 ▸ Slope of a line

SIDE NOTE

The symbols Δy (read "delta y") and Δx (read delta x") are used to indicate a "change in y" and a "change in x," respectively. So the slope is sometimes denoted by $m = \dfrac{\Delta y}{\Delta x}$.

Slope of a Line

The **slope** of a nonvertical line that passes through the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is denoted by m and is defined by

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2.$$

The slope of a vertical line is undefined.

Because $m = \dfrac{\text{rise}}{\text{run}} = \dfrac{\text{change in } y\text{-coordinates}}{\text{change in } x\text{-coordinates}}$, if the change in x, Δx, is one unit to the right (x increases by 1 unit), then the change in y, Δy, equals the slope of the line. The slope of a line is therefore the **change in y per unit change in x**. In other words, the slope of a line measures the rate of change of y with respect to x. Roofs, staircases, graded landscapes, and mountainous roads all have slopes. For example, if the pitch (slope) of a section of the roof is 0.4, then for every horizontal distance of 10 feet in that section, the roof ascends 4 feet.

EXAMPLE 1 Finding and Interpreting the Slope of a Line

Sketch the graph of the line that passes through $P(1, -1)$ and $Q(3, 3)$. Find and interpret the slope of the line.

Solution

The graph of the line is sketched in Figure 1.23. The slope m of this line is given by

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y\text{-coordinates}}{\text{change in } x\text{-coordinates}}$$

$$= \frac{(y\text{-coordinate of } Q) - (y\text{-coordinate of } P)}{(x\text{-coordinate of } Q) - (x\text{-coordinate of } P)}$$

$$= \frac{(3) - (-1)}{(3) - (1)} = \frac{3 + 1}{3 - 1} = \frac{4}{2} = 2.$$

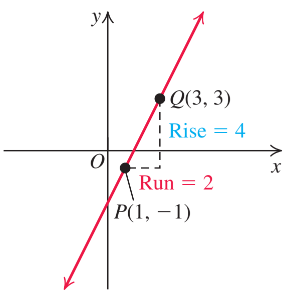


Figure 1.23 ▸ Interpreting slope

Interpretation A slope of 2 means that the value of y increases two units for every one unit increase in the value of x.

Practice Problem 1. Find and interpret the slope of the line containing the points $(-7, 5)$ and $(6, -3)$. ■

Figure 1.24 shows several lines passing through the origin. The slope of each line is labeled.

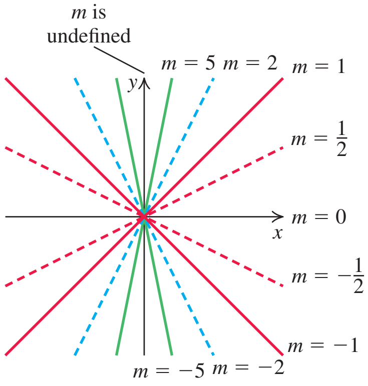


Figure 1.24 ▸ Slopes of lines

Main Facts about Slopes of Lines

1. Scanning graphs from left to right, lines with positive slopes rise and lines with negative slopes fall.
2. The greater the absolute value of the slope, the steeper the line.
3. The slope of a vertical line is undefined.
4. The slope of a horizontal line is zero.

WARNING

Be careful when using the formula for finding the slope of a line joining the points (x_1, y_1) and (x_2, y_2). Be sure to subtract coordinates in the same order.

You can use either

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{or} \quad m = \frac{y_1 - y_2}{x_1 - x_2},$$

but it is incorrect to use

$$m = \frac{y_1 - y_2}{x_2 - x_1} \quad \text{or} \quad m = \frac{y_2 - y_1}{x_1 - x_2}.$$

Figure 1.25 shows how to construct a line with given slope, $\pm\frac{2}{3}$, with rise $= \pm 2$ and run $= 3$.

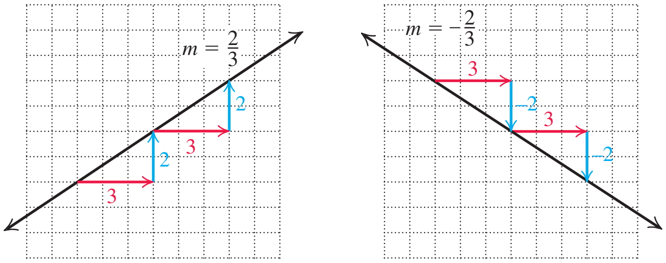


Figure 1.25 ▸ Constructing a line

Objective 2 ▸

Point–Slope Form

We now find the equation of a line ℓ that passes through the point $A(x_1, y_1)$ and has slope m. Let $P(x, y)$, with $x \neq x_1$, be any point in the plane. Then $P(x, y)$ is on the line ℓ if and only if the slope of the line passing through $P(x, y)$ and $A(x_1, y_1)$ is m. This is true if and only if

$$\frac{y - y_1}{x - x_1} = m.$$

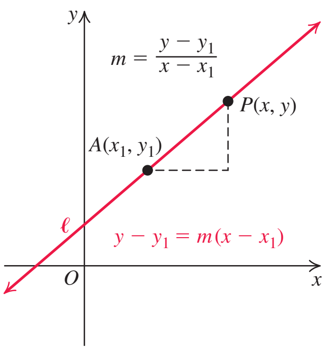


Figure 1.26 ▸ Point–slope form

See Figure 1.26.

Multiplying both sides by $x - x_1$ gives $y - y_1 = m(x - x_1)$, which is also satisfied when $x = x_1$ and $y = y_1$.

The Point–Slope Form of the Equation of a Line

If a line has slope m and passes through the point (x_1, y_1), then the **point–slope form** of an equation of the line is

$$\boldsymbol{y - y_1 = m(x - x_1).}$$

EXAMPLE 2 Finding an Equation of a Line with Given Point and Slope

Find the point–slope form of the equation of the line passing through the point $(1, -2)$ with slope $m = 3$. Then solve for y.

Solution

$$
\begin{aligned}
y - y_1 &= m(x - x_1) && \text{Point–slope form}\\
y - (-2) &= 3(x - 1) && \text{Substitute } x_1 = 1,\ y_1 = -2, \text{ and } m = 3.\\
y + 2 &= 3x - 3 && \text{Simplify.}\\
y &= 3x - 5 && \text{Solve for } y.
\end{aligned}
$$

Slope (3); The y-intercept is -5.

Practice Problem 2. Find the point–slope form of the equation of the line passing through the point $(-2, -3)$ and with slope $-\frac{2}{3}$. Then solve for y. ■

EXAMPLE 3 Finding an Equation of a Line Passing Through Two Given Points

Find the point–slope form of the equation of the line ℓ passing through the points $(-2, 1)$ and $(3, 7)$. Then solve for y.

Solution

We first find the slope m of the line ℓ.

$$m = \frac{7 - 1}{3 - (-2)} = \frac{6}{3 + 2} = \frac{6}{5} \qquad m = \frac{y_2 - y_1}{x_2 - x_1}$$

We use $m = \frac{6}{5}$ and either of the two given points when substituting into the point–slope form:

$$y - y_1 = m(x - x_1).$$

With $(x_1, y_1) = (3, 7)$

$$y - 7 = \frac{6}{5}(x - 3)$$

$$y = \frac{6}{5}x + \frac{17}{5}. \quad \text{Simplify}$$

With $(x_1, y_1) = (-2, 1)$

$$y - 1 = \frac{6}{5}(x - (-2))$$

$$y = \frac{6}{5}x + \frac{17}{5}. \quad \text{Simplify}$$

Same result

SIDE NOTE

You can avoid operations on fractions

$$y - 7 = \frac{6}{5}(x - 3) \quad \text{Multiply by 5.}$$

$$
\begin{aligned}
5(y - 7) &= 6(x - 3)\\
5y - 35 &= 6x - 18\\
5y &= 6x + 17
\end{aligned}
$$

and at the end divide by 5

$$y = \frac{6}{5}x + \frac{17}{5}.$$

Practice Problem 3. Find the point–slope form of the equation of a line passing through the points $(-3, -4)$ and $(-1, 6)$. Then solve for y. ■

Objective 3 ▸

Slope–Intercept Form

One of the most convenient forms of an equation describing a straight line is called the slope–intercept form, where we know the slope of the line and the given point is the y-intercept.

EXAMPLE 4 Finding an Equation of a Line with a Given Slope and y-intercept

Find the point–slope form of the equation of the line with slope m and y-intercept b. Then solve for y.

Solution

Because the line has y-intercept b, the line passes through the point $(0, b)$. See Figure 1.27.

$$
\begin{aligned}
y - y_1 &= m(x - x_1) && \text{Point–slope form}\\
y - b &= m(x - 0) && \text{Substitute } x_1 = 0 \text{ and } y_1 = b.\\
y - b &= mx && \text{Simplify.}\\
y &= mx + b && \text{Solve for } y.
\end{aligned}
$$

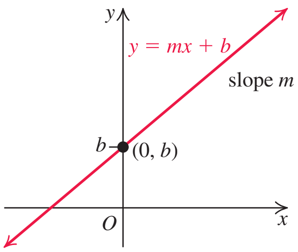


Figure 1.27 ▸ Slope–intercept form

Practice Problem 4. Find the point–slope form of the equation of the line with slope 2 and y-intercept -3. Then solve for y. ■

DO YOU KNOW?

No one is sure why the letter m is used for slope. Many people suggest that m comes from the French **monter** (to climb), but Descartes, who was French, did not use m. Professor John Conway of Princeton University has suggested that m could stand for "modulus of slope."

Slope–Intercept Form of the Equation of a Line

The **slope–intercept form** of the equation of the line with slope m and y-intercept b is

$$y = mx + b.$$

The linear equation in the form $y = mx + b$ displays the slope m (the coefficient of x) and the y-intercept b (the constant term). The number m tells which way and how much the line is tilted; the number b tells where the line intersects the y-axis.

EXAMPLE 5 Graph Using the Slope and y-intercept

Graph the line whose equation is $y = \frac{2}{3}x + 2$.

Solution

The equation

$$y = \frac{2}{3}x + 2$$

is in the slope–intercept form with slope $\frac{2}{3}$ and y-intercept 2. To sketch the graph, find two points on the line and draw a line through the two points. Use the y-intercept as one of the points, and then use the slope to locate a second point. Because $m = \frac{2}{3}$, let 2 be the rise and 3 be the run. From the point $(0, 2)$, move three units to the right (run) and two units up (rise). This gives $(0 + \text{run}, 2 + \text{rise}) = (0 + 3, 2 + 2) = (3, 4)$ as the second point.

The line we want joins the points $(0, 2)$ and $(3, 4)$ and is shown in Figure 1.28.

As an *alternative solution*, we can locate the second point on the line by choosing some value of x and finding the corresponding value of y directly from the given equation. Choosing $x = 3$ will give us the corresponding value of $y = \frac{2}{3} \cdot 3 + 2 = 4$. This gives $(3, 4)$ as a second point. Note that some choices of x are better than others. Choosing $x = 1$ would give us the point $\left(1, \frac{8}{3}\right)$, which would not be easy to plot accurately on a Cartesian grid.

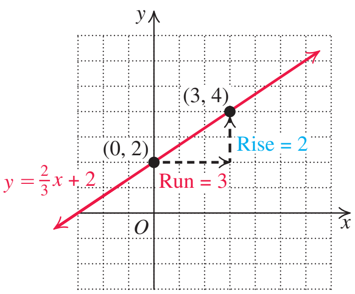


Figure 1.28 ▸ Locating a second point on a line

Practice Problem 5. Graph the line with slope $-\frac{2}{3}$ that contains the point $(0, 4)$. ■

SIDE NOTE

To graph a line with integer slope, such as 3, write $3 = \frac{3}{1}$ where the rise $= 3$ and run $= 1$.

Objective 4 ▸

Equations of Horizontal and Vertical Lines

Consider the horizontal line through the point (h, k). The y-coordinate of every point on this line is k. So we can write its equation as $y = k$. This line has slope $m = 0$ and y-intercept k. Similarly, an equation of the vertical line through the point (h, k) is $x = h$. This line has undefined slope and x-intercept h. See Figure 1.29.

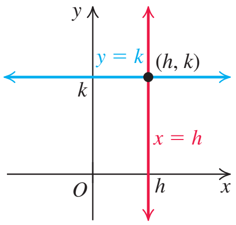


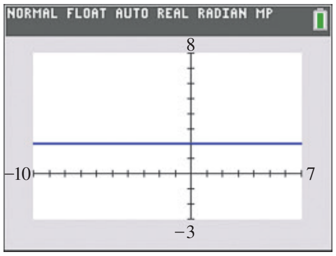
Figure 1.29 ▸ Horizontal and vertical lines

Horizontal and Vertical Lines

An equation of a horizontal line through (h, k) is $\boldsymbol{y = k}$.
An equation of a vertical line through (h, k) is $\boldsymbol{x = h}$.

TECHNOLOGY CONNECTION

Calculator graph of $y = 2$



EXAMPLE 6 Recognizing Horizontal and Vertical Lines

Discuss the graph of each equation in the xy-plane.

a. $y = 2$ **b.** $x = 4$

Solution

a. The equation $y = 2$ may be considered as an equation in two variables x and y by writing

$$0 \cdot x + y = 2.$$

Any ordered pair of the form $(x, 2)$ is a solution of this equation. Some solutions are $(-1, 2)$, $(0, 2)$, $(2, 2)$, and $(7, 2)$. It follows that the graph is a line parallel to the x-axis and two units above it, as shown in Figure 1.30. Its slope is 0.

SIDE NOTE

Sometimes it is easier to remember that

(1) given the equation $y = 2$, there is no change in y, because y is constant. The line has to be horizontal $(\Delta y = 0)$.

(2) given the equation $x = 4$, there is no change in x, because x is constant. The line has to be vertical $(\Delta x = 0)$.

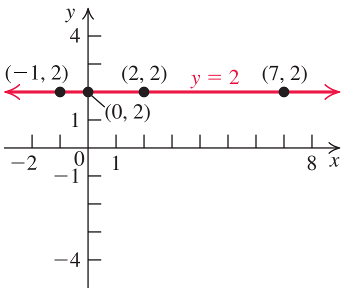


Figure 1.30 ▸ Horizontal line

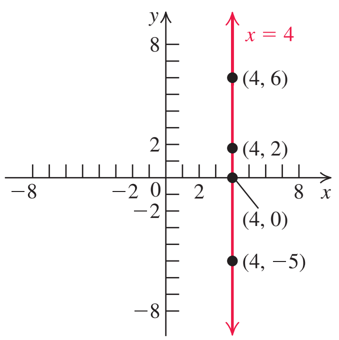


Figure 1.31 ▸ Vertical line

b. The equation $x = 4$ may be written as

$$x + 0 \cdot y = 4.$$

Any ordered pair of the form $(4, y)$ is a solution of this equation. Some solutions are $(4, -5)$, $(4, 0)$, $(4, 2)$, and $(4, 6)$. The graph is a line parallel to the y-axis and four units to the right of it, as shown in Figure 1.31. Its slope is undefined.

TECHNOLOGY CONNECTION

You cannot graph $x = 4$ by using the $Y =$ key on your calculator.

Practice Problem 6. Sketch the graphs of the lines $x = -3$ and $y = 7$. ■

Objective 5 ▸

General Form of the Equation of a Line

An equation of the form

$$ax + by + c = 0,$$

where a, b, and c are constants and a and b are not both zero, is called the *general form* of a linear equation. Consider two possible cases: $b \neq 0$ and $b = 0$.

Suppose $b \neq 0$: We can isolate y on one side and rewrite the equation in slope–intercept form.

$$\begin{aligned} ax + by + c &= 0 && \text{Original equation} \\ by &= -ax - c && \text{Subtract } ax + c \text{ from both sides.} \\ y &= -\frac{a}{b}x - \frac{c}{b} && \text{Divide by } b. \end{aligned}$$

The result is an equation of a line in slope–intercept form with slope $m = -\frac{a}{b}$ and y-intercept equal to $-\frac{c}{b}$.

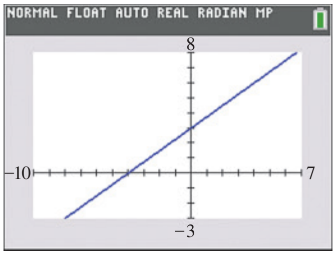
Suppose $b = 0$: We know that $a \neq 0$, since not both a and b are zero, and we can solve for x.

$$\begin{aligned} ax + 0 \cdot y + c &= 0 && \text{Replace } b \text{ with } 0. \\ ax + c &= 0 && \text{Simplify.} \\ x &= -\frac{c}{a} && \text{Solve for } x. \end{aligned}$$

The graph of the equation $x = -\frac{c}{a}$ is a vertical line.

TECHNOLOGY CONNECTION

Calculator graph of $y = \frac{3}{4}x + 3$



General Form of the Equation of a Line

The graph of every linear equation

$$\boldsymbol{ax + by + c = 0,}$$

where a, b, and c are constants and a and b are not both zero, is a line. The equation $ax + by + c = 0$ is called the **general form** of the equation of a line.

EXAMPLE 7 Graphing a Linear Equation Using Intercepts

Find the slope, y-intercept, and x-intercept of the line with equation

$$3x - 4y + 12 = 0.$$

Then sketch the graph.

Solution

First, solve for y to write the equation in slope–intercept form:

$$\begin{aligned} 3x - 4y + 12 &= 0 && \text{Original equation} \\ 4y &= 3x + 12 && \text{Subtract } 3x + 12; \text{ multiply by } -1. \\ y &= \frac{3}{4}x + 3. && \text{Divide by } 4. \end{aligned}$$

This equation tells us that the slope m is $\frac{3}{4}$ and the y-intercept is 3. To find the x-intercept, set $y = 0$ in the original equation and obtain $3x + 12 = 0$, or $x = -4$. So the x-intercept is -4.

We can sketch the graph of the equation if we can find two points on the graph. So we use the intercepts and sketch the line joining the points $(-4, 0)$ and $(0, 3)$. The graph is shown in Figure 1.32.

Alternatively, we can verify that the **two-intercept form** of the equation of the line with x-intercept a and y-intercept b is

$$\frac{x}{a} + \frac{y}{b} = 1.$$

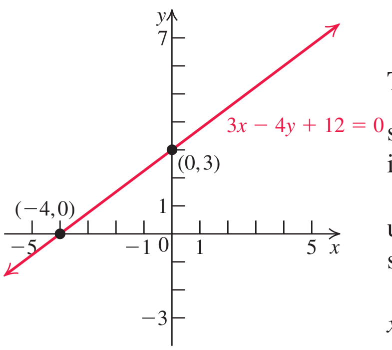


Figure 1.32 ▶ Graphing a line using intercepts