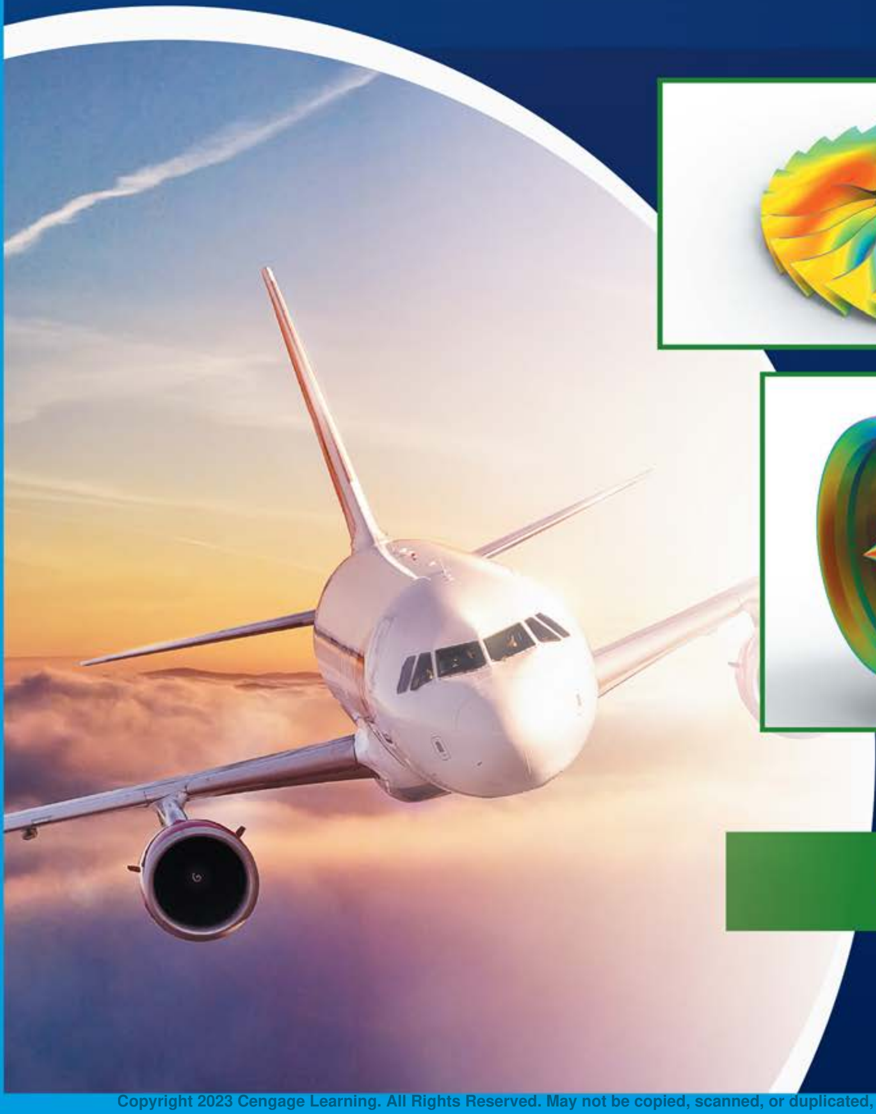
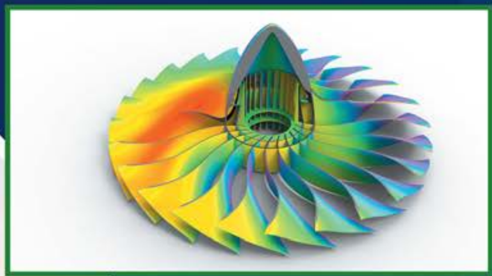
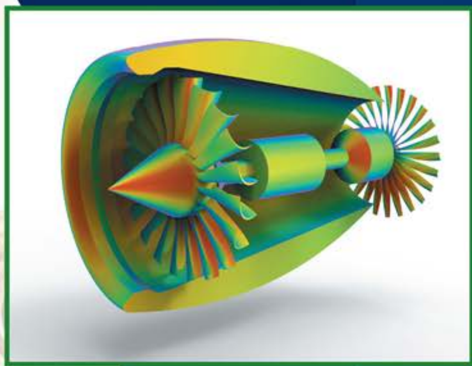
CENGAGE

A First Course in the Finite Element Method

Enhanced Sixth Edition







Daryl L. Logan

CONVERSION FACTORS U.S. Customary Units to SI Units

Quantity Converted from U.S. Customary	To	SI Equivalent
(Acceleration)		
1 foot/second^2 (ft/s^2)	meter/second^2 (m/s^2)	0.3048 m/s^2
1 inch/second^2 (in./s^2)	meter/second^2 (m/s^2)	0.0254 m/s^2
(Area)		
1 foot^2 (ft^2)	meter^2 (m^2)	0.0929 m^2
1 inch^2 (in.^{2})	meter^2 (m^2)	645.2 mm^2
(Density, mass)		
1 pound mass/inch^3 (lbm/in.^{3})	kilogram/meter^3 (kg/m^3)	27.68 Mg/m^3
1 pound mass/foot^3 (lbm/ft^3)	kilogram/meter^3 (kg/m^3)	16.02 kg/m^3
(Energy, Work)		
1 British thermal unit (BTU)	Joule (J)	1055 J
1 foot-pound force (ft-lb)	Joule (J)	1.356 J
1 kilowatt-hour	Joule (J)	3.60×10^6 J
(Force)		
1 kip (1000 lb)	Newton (N)	4.448 kN
1 pound force (lb)	Newton (N)	4.448 N
(Length)		
1 foot (ft)	meter (m)	0.3048 m
1 inch (in.)	meter (m)	25.4 mm
1 mile (mi), (U.S. statute)	meter (m)	1.609 km
1 mile (mi), (international nautical)	meter (m)	1.852 km
(Mass)		
1 pound mass (lbm)	kilogram (kg)	0.4536 kg
1 slug (lb-sec^2/ft)	kilogram (kg)	14.59 kg
1 metric ton (2000 lbm)	kilogram (kg)	907.2 kg
(Moment of force)		
1 pound-foot (lb·ft)	Newton-meter (N·m)	1.356 N·m
1 pound-inch (lb·in.)	Newton-meter (N·m)	0.1130 N·m
(Moment of inertia of an area)		
1 inch^4	meter^4 (m^4)	0.4162×10^{-6} m^4
(Moment of inertia of a mass)		
1 pound-foot-second^2(lb·ft·s^2)	kilogram-meter^2 (kg·m^2)	1.356 kg·m^2
(Momentum, linear)		
1 pound-second (lb·s)	kilogram-meter/second (kg·m/s)	4.448 N·s
(Momentum, angular)		
pound-foot-second (lb·ft·s)	Newton-meter-second (N·m·s)	1.356 N·m·s

CONVERSION FACTORS U.S. Customary Units to SI Units (*Continued*)

Quantity Converted from U.S. Customary	To	SI Equivalent
(Power)		
1 foot-pound/second (ft·lb/s)	Watt (W)	1.356 W
1 horsepower (550 ft·lb/s)	Watt (W)	745.7 W
(Pressure, stress)		
1 atmosphere (std)(14.7.lb/in.^{2})	Newton/meter^2 (N/m^2 or Pa)	101.3 kPa
1 pound/foot^2 (lb/ft^2)	Newton/meter^2 (N/m^2 or Pa)	47.88 Pa
1 pound/inch^2 (lb/in.^{2} or psi)	Newton/meter^2 (N/m^2 or Pa)	6.895 kPa
1 kip/inch^2(ksi)	Newton/meter^2 (N/m^2 or Pa)	6.895 MPa
(Spring constant)		
1 pound/inch (lb/in.)	Newton/meter (N/m)	175.1 N/m
(Temperature)		
$T(°F) = 1.8T(°C) + 32$		
(Velocity)		
1 foot/second (ft/s)	meter/second (m/s)	0.3048 m/s
1 knot (nautical mi/h)	meter/second (m/s)	0.5144 m/s
1 mile/hour (mi/h)	meter/second (m/s)	0.4470 m/s
1 mile/hour (mi/h)	kilometer/hour (km/h)	1.609 km/h
(Volume)		
1 foot^3 (ft^3)	meter^3 (m^3)	0.02832 m^3
1 inch^3 (in.^{3})	meter^3 (m^3)	16.39×10^{-6} m^3

PHYSICAL PROPERTIES IN SI AND USCS UNITS

Property	SI	USCS
Water (fresh)		
specific weight	9.81 kN/m^3	62.4 lb/ft^3
mass density	1000 kg/m^3	1.94 slugs/ft^3
Aluminum		
specific weight	26.6 kN/m^3	169/lb/ft^3
mass density	2710 kg/m^3	5.26 slugs/ft^3
Steel		
specific weight	77.0 kN/m^3	490 lb/ft^3
mass density	7850 kg/m^3	15.2 slugs/ft^3
Reinforced concrete		
specific weight	23.6 kN/m^3	150 lb/ft^3
mass density	2400 kg/m^3	4.66 slugs/ft^3
Acceleration of gravity (on the earth's surface)		
Recommended value	9.81 m/s^2	32.2 ft/s^2
Atmospheric pressure (at sea level)		
Recommended value	101 kPa	14.7 psi

TYPICAL PROPERTIES OF SELECTED ENGINEERING MATERIALS

Material	Ultimate Strength σ_u		0.2% Yield Strength σ_y		Modulus of Elasticity E		Sheer Modulus G	Coefficient of Thermal Expansion, α		Density, ρ	
	ksi	MPa	ksi	MPa	(10^6 psi	GPa)	(10^6 psi)	$10^{-6}/^\circ$F	$10^{-6}/^\circ$C	lb/in.^{3}	kg/m^3
Aluminum											
Alloy 1100-H14 (99 % A1)	14	110(T)	14	95	10.1	70	3.7	13.1	23.6	0.098	2710
Alloy 2024-T3 (sheet and plate)	70	480(T)	50	340	10.6	73	4.0	12.6	22.7	0.100	2763
Alloy 6061-T6 (extruded)	42	260(T)	37	255	10.0	69	3.7	13.1	23.6	0.098	2710
Alloy 7075-T6 (sheet and plate)	80	550(T)	70	480	10.4	72	3.9	12.9	23.2	0.101	2795
Yellow brass (65% Cu, 35% Zn)											
Cold-rolled	78	540(T)	63	435	15	105	5.6	11.3	20.0	0.306	8470
Annealed	48	330(T)	15	105	15	105	5.6	11.3	20.0	0.306	8470
Phosphor bronze											
Cold-rolled (510)	81	560(T)	75	520	15.9	110	5.9	9.9	17.8	0.320	8860
Spring-tempered (524)	122	840(T)	—	—	16	110	5.9	10.2	18.4	0.317	8780
Cast iron											
Gray, 4.5%C, ASTM A-48	25	170(T)	—	—	10	70	4.1	6.7	12.1	0.260	7200
	95	650(C)									
Malleable, ASTM A-47	50	340(T)	33	230	24	165	9.3	6.7	12.1	0.264	7300
	90	620(C)	—	—							

TYPICAL PROPERTIES OF SELECTED ENGINEERING MATERIALS (*Continued*)

Material	Ultimate Strength σ_u ksi	Ultimate Strength σ_u MPa	0.2% Yield Strength σ_y ksi	0.2% Yield Strength σ_y MPa	Modulus of Elasticity E (10^6 psi)	Modulus of Elasticity E (GPa)	Sheer Modulus G (10^6 psi)	Coefficient of Thermal Expansion, α $10^{-6}/°F$	Coefficient of Thermal Expansion, α $10^{-6}/°C$	Density, ρ lb/in.^{3}	Density, ρ kg/m^3
Copper and its alloys											
CDA 145 copper, hard	48	331(T)	44	303	16	110	6.1	9.9	17.8	0.323	8940
CDA 172 beryllium copper, hard	175	1210(T)	240	965	19	131	7.1	9.4	17.0	0.298	8250
CDA 220 bronze, hard	61	421(T)	54	372	17	117	6.4	10.2	18.4	0.318	8800
CDA 260 brass, hard	76	524(T)	63	434	16	110	6.1	11.1	20.0	0.308	8530
Magnesium alloy (8.5% A1)	55	380(T)	40	275	4.5	45	2.4	14.5	26.0	0.065	1800
Monel alloy 400 (Ni-Cu)											
Cold-worked	98	675(T)	85	580	26	180	—	7.7	13.9	0.319	8830
Annealed	80	550(T)	32	220	26	180	—	7.7	13.9	0.319	8830
Steel											
Structural (ASTM-A36)	58	400(T)	36	250	29	200	11.5	6.5	11.7	0.284	7860
High-strength low-alloy ASTM-A242	70	480(T)	50	345	29	200	11.5	6.5	11.7	0.284	7860
Quenched and tempered alloy ASTM-A514	120	825(T)	100	690	29	200	11.5	6.5	11.7	0.284	7860
Stainless, (302)											
Cold-rolled	125	860(T)	75	520	28	190	10.6	9.6	17.3	0.286	7920
Annealed	90	620(T)	40	275	28	190	10.6	9.6	17.3	0.286	7920
Titanium alloy (6% A1, 4% V)	130	900(T)	120	825	16.5	114	6.2	5.3	9.5	0.161	4460
Concrete											
Medium strength	4.0	28(C)	—	—	3.5	25	—	5.5	10.0	0.084	2320
High strength	6.0	40(C)	—	—	4.5	30	—	5.5	10.0	0.084	2320
Granite	35	240(C)	—	—	10	69	—	4.0	7.0	0.100	2770
Glass, 98% silica	7	50(C)	—	—	10	69	—	44.0	80.0	0.079	2190
Melamine	6	41(T)	—	—	2.0	13.4	—	17.0	30.0	0.042	1162
Nylon, molded	8	55(T)	—	—	0.3	2	—	45.0	81.0	0.040	1100
Polystyrene	7	48(T)	—	—	0.45	3	—	40.0	72.0	0.038	1050
Rubbers											
Natural	2	14(T)	—	—	—	—	—	90.0	162.0	0.033	910
Neoprene	3.5	24(T)	—	—	—	—	—			0.045	1250
Timber, air dry, parallel to grain											
Douglas fir, construction grade	7.2	50(C)	—	—	1.5	10.5	—	varies	varies	0.019	525
Eastern spruce	5.4	37(C)	—	—	1.3	9	—	1.7–	3–	0.016	440
Southern pine,construction grade	7.3	50(C)	—	—	1.2	8.3	—	3.0	5.4	0.022	610

The values given in the table are average mechanical properties. Further verification may be necessary for final design or analysis. For ductile materials, the compressive strength is normally assumed to equal the tensile strength. *Abbreviations:* C, compressive strength; T, tensile strength. For an explanation of the numbers associated with the aluminums, cast irons, and steels, see ASM Metals Reference Book, latest ed., American Society for Metals, Metals Park, Ohio 44073

A First Course in the Finite Element Method

ENHANCED SIXTH EDITION

Daryl L. Logan

University of Wisconsin–Platteville

CENGAGE

Australia • Brazil • Canada • Mexico • Singapore • United Kingdom • United States

CENGAGE

***A First Course in the Finite Element Method:* Sixth Edition**
Daryl L. Logan

SVP, Higher Education Product Management: Erin Joyner

VP, Product Management, Learning Experiences: Thais Alencar

Product Director: Mark Santee

Senior Product Manager: Timothy L. Anderson

Product Assistant: Simeon Lloyd-Wingard

Learning Designer: MariCarmen Constable

Senior Content Manager: Sharon Chambliss

Digital Delivery Quality Partner: Nikkita Kendrick

VP, Product Marketing: Jason Sakos

Portfolio Marketing Manager: Taylor Shenberger

Sr. Director, Product Marketing: Jennifer Fink

IP Analyst: Deanna Ettinger

Compositor: Straive

Designer: Gaby Vinales

Cover Image: 3DConcepts/Shutterstock.com, Jag_cz/Shutterstock.com

Library of Congress Control Number: 2021920956

ISBN: 978-0-357-67642-4

Cengage
200 Pier 4 Boulevard
Boston, MA 02210
USA

Cengage is a leading provider of customized learning solutions with employees residing in nearly 40 different countries and sales in more than 125 countries around the world. Find your local representative at **www.cengage.com**.

To learn more about Cengage platforms and services, register or access your online learning solution, or purchase materials for your course, visit **www.cengage.com**.

Printed in the United States of America
Print Number: 01 Print Year: 2021

CONTENTS

PREFACE

Features and Approach

The purpose of this enhanced sixth edition is to provide an introductory approach to the finite element method that can be understood by both undergraduate and graduate students without the usual prerequisites (such as structural analysis and upper level calculus) required by many available texts in this area. The book is written primarily as a basic learning tool for the undergraduate student in civil and mechanical engineering whose main interest is in stress analysis and heat transfer, although material on fluid flow in porous media and through hydraulic networks and electrical networks and electrostatics is also included. The concepts are presented in sufficiently simple form with numerous example problems logically placed throughout the book, so that the book serves as a valuable learning aid for students with other backgrounds, as well as for practicing engineers. The text is geared toward those who want to apply the finite element method to solve practical physical problems.

General principles are presented for each topic, followed by traditional applications of these principles, including longhand solutions, which are in turn followed by computer applications where relevant. The approach is taken to illustrate concepts used for computer analysis of large-scale problems.

The book proceeds from basic to advanced topics and can be suitably used in a two-course sequence. Topics include basic treatments of (1) simple springs and bars, leading to two- and three-dimensional truss analysis; (2) beam bending, leading to plane frame, grid, and space frame analysis; (3) elementary plane stress/strain elements, leading to more advanced plane stress/strain elements and applications to more complex plane stress/strain analysis; (4) axisymmetric stress analysis; (5) isoparametric formulation of the finite element method; (6) three-dimensional stress analysis; (7) plate bending analysis; (8) heat transfer and fluid mass transport; (9) basic fluid flow through porous media and around solid bodies, hydraulic networks, electric networks, and electrostatics analysis; (10) thermal stress analysis; and (11) time-dependent stress and heat transfer.

Additional features include how to handle inclined or skewed boundary conditions, beam element with nodal hinge, the concept of substructure, the patch test, and practical considerations in modeling and interpreting results.

The direct approach, the principle of minimum potential energy, and Galerkin's residual method are introduced at various stages, as required, to develop the equations needed for analysis.

Appendices provide material on the following topics: (A) basic matrix algebra used throughout the text; (B) solution methods for simultaneous equations; (C) basic theory of elasticity; (D) work-equivalent nodal forces; (E) the principle of virtual work; and (F) properties of structural steel shapes.

More than 100 solved examples appear throughout the text. Most of these examples are solved "longhand" to illustrate the concepts. More than 570 end-of-chapter problems are provided to reinforce concepts. The answers to many problems are included in the back of the book to aid those wanting to verify their work. Those end-of-chapter problems to be solved using a computer program are marked with a computer symbol.

Additional Features

Additional features of this edition include updated notation used by most engineering instructors, chapter objectives at the start of each chapter to help students identify what content is most important to focus on and retain summary equations for handy use at the end of each chapter, additional information on modeling, and more comparisons of finite element solutions to analytical solutions.

New Features

Over 140 new problems for solution have been included, and additional design-type problems have been added to chapters 3, 5, 7, 11, and 12. Additional real-world applications from industry have been added to enhance student understanding and reinforce concepts. New space frames, solid-model-type examples, and problems for solution have been added. New examples from other fields now demonstrate how students can use the Finite Element Method to solve problems in a variety of engineering and mathematical physics areas. As in the 5th edition, this edition deliberately leaves out consideration of special purpose computer programs and suggests that instructors choose a program they are familiar with to integrate into their finite element course.

Resources for Instructors

To access instructor resources, including a secure, downloadable Instructor's Solution Manual and Lecture Note PowerPoint Slides, please visit our Instructor Resource Center at http://login.cengage.com.

Digital Resources

CENGAGE | WEBASSIGN

New Digital Solution for Your Engineering Classroom

WebAssign is a powerful digital solution designed by educators to enrich the engineering teaching and learning experience. With a robust computational engine at its core, WebAssign provides extensive content, instant assessment, and superior support. WebAssign's powerful question editor allows engineering instructors to create their own questions or modify existing questions. Each question can use any combination of text, mathematical equations and formulas, sound, pictures, video, and interactive HTML elements. Numbers, words, phrases, graphics, and sound or video files can be randomized so that each student receives a different version of the same question. In addition to common question types such as multiple choice, fill-in-the-blank, essay, and numerical, you can also incorporate robust answer entry palettes (mathPad, chemPad, calcPad, physPad, Graphing Tool) to input and grade symbolic expressions, equations, matrices, and chemical structures using powerful computer algebra systems.

WebAssign Offers Engineering Instructors the Following

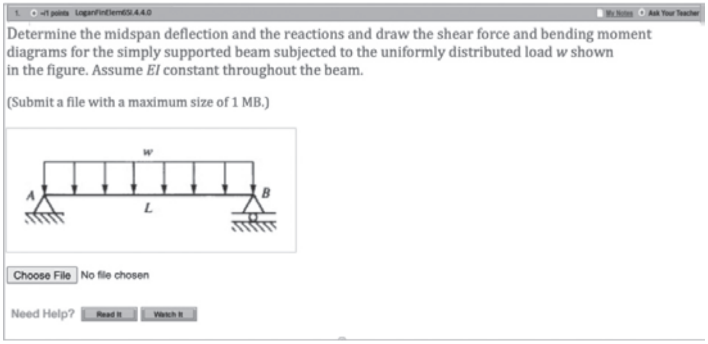
- The ability to create and edit algorithmic and numerical exercises.
- The opportunity to generate randomized iterations of algorithmic and numerical exercises. When instructors assign numerical WebAssign homework exercises (engineering math exercises), the WebAssign program offers them the ability to generate and assign their students differing versions of the same engineering math exercise. The computational engine extends beyond and provides the luxury of solving for correct solutions/answers.
- The ability to create and customize numerical questions, allowing students to enter units, use a specific number of significant digits, use a specific number of decimal places, respond with a computed answer, or answer within a different tolerance value than the default.

Visit www.webassign.com/instructors/features/ to learn more. To create an account, instructors can go directly to the signup page at www.webassign.net/signup.html.

WebAssign Features for Students

Review Concepts at Point of Use

Within WebAssign, a "Read It" button at the bottom of each question links students to corresponding sections of the textbook, enabling access to the MindTap Reader at the precise moment of learning. A "Watch It" button allows a short video to play. These videos help students understand and review the problem they need to complete, enabling support at the precise moment of learning.



My Class Insights

WebAssign's built-in study feature shows performance across course topics so that students can quickly identify which concepts they have mastered and which areas they may need to spend more time on.

Ask Your Teacher

This powerful feature enables students to contact their instructor with questions about a specific assignment or problem they are working on.

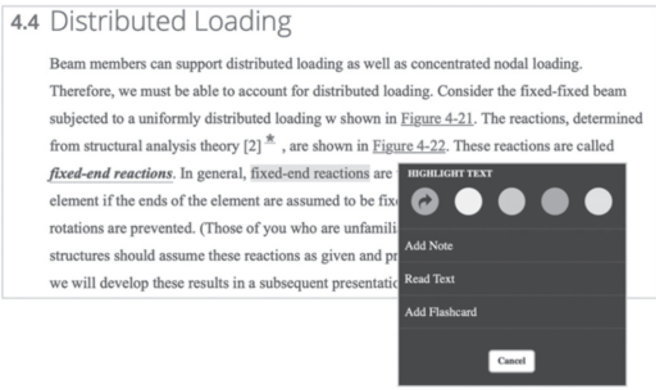
MindTap Reader

Available via WebAssign and our digital subscription service, Cengage Unlimited, MindTap Reader is Cengage's next-generation eBook for engineering students.

The MindTap Reader provides more than just text learning for the student. It offers a variety of tools to help our future engineers learn chapter concepts in a way that resonates with their workflow and learning styles.

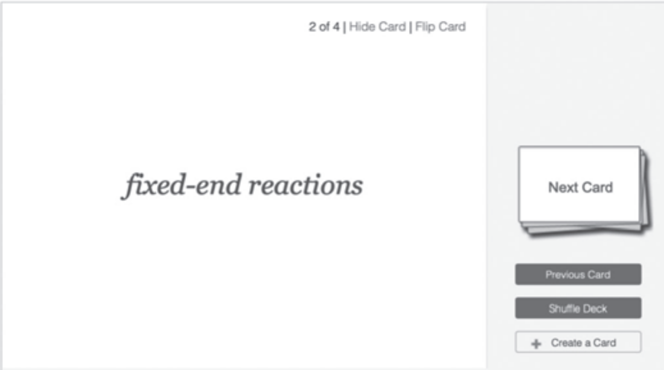
Personalize their experience

Within the MindTap Reader, students can highlight key concepts, add notes, and bookmark pages. These are collected in My Notes, ensuring they will have their own study guide when it comes time to study for exams.



Flexibility at their fingertips

With access to the book's internal glossary, students can personalize their study experience by creating and collating their own custom flashcards. The ReadSpeaker feature reads text aloud to students, so they can learn on the go—wherever they are.



The Cengage Mobile App

Available on iOS and Android smartphones, the Cengage Mobile App provides convenience. Students can access their entire textbook anyplace and anytime. They can take notes, highlight important passages, and have their text read aloud whether they are online or off.

To learn more and download the mobile app, visit www.cengage.com/mobile-app/.

All-You-Can-Learn Access with Cengage Unlimited

Cengage Unlimited is the cost-saving student plan that includes access to our entire library of eTextbooks, online platforms and more—in one place, for one price. For just $119.99 for four months, a student gets online and offline access to Cengage course materials across disciplines, plus hundreds of student success and career readiness skill-building activities. To learn more, visit www.cengage.com/unlimited.

Suggested Topics

Following is an outline of suggested topics for a first course (approximately 44 lectures, 52 minutes each) in which this textbook is used.

Topic	Number of Lectures
Appendix A	1
Appendix B	1
Chapter 1	2
Chapter 2	3
Chapter 3, Sections 3.1–3.11, 3.14 and 3.15	5
Exam 1	1
Chapter 4, Sections 4.1–4.6	4
Chapter 5, Sections 5.1–5.3, 5.5	4
Chapter 6	4
Chapter 7	3
Exam 2	1
Chapter 9	2
Chapter 10	4
Chapter 11	3
Chapter 13, Sections 13.1–13.7	5
Chapter 15	3
Exam 3	1

ACKNOWLEDGMENTS

I express my deepest appreciation to the staff at Cengage Learning especially Tim Anderson, Publisher; Mona Zeftel, Content Developer and Teresa Versaggi, Product Assistant; and to Rose Kernan of RPK Editorial Services for their assistance in producing this new edition.

I am grateful to Dr. Ted Belytschko for his excellent teaching of the finite element method, which aided me in writing this text. Also I want to thank the many students at the University for their suggestions on ways to make topics easier to understand. These suggestions have been incorporated into this edition as well.

I also wish to thank the reviewers for this enhanced sixth edition who include Raghu Agarwal, San Jose State University; Mohammad Alimi, California State University, Fresno; K. Chandrashekhara, Missouri University of Science and Technology; Howie Fang, University of North Carolina, Charlotte; Winfred Foster, Auburn University; Kenneth Miller, St. Cloud State University; Ronald Rorrer, University of Colorado, Denver; and Charles Yang, Wichita State University.

I thank many students at the University of Wisconsin-Platteville (UWP), whose names are credited throughout the book, for contributing various two-and three-dimensional models from the finite element course. Thank you also to UWP graduate student William Gobeli for Table 11-2. Also special thanks to Andrew Heckman, an alum of UWP and Design Engineer at Seagraves Fire Apparatus for permission to use Figure 11-10 and to Mr. Yousif Omer, Structural Engineer at John Deere Dubuque Works for allowing permission to use Figure 1-11. I want to thank the people at Autodesk for their contribution of Figure 9-2b. Finally, I want to thank Ioan Giosan, Senior Design Engineer at Valmont West Coast Engineering for allowing permission to use Figures 1-12 and 1-13.

NOTATION

English Symbols

a_i	generalized coordinates (coefficients used to express displacement in general form)
A	cross-sectional area
$[B]$	matrix relating strains to nodal displacements or relating temperature gradient to nodal temperatures
c	specific heat of a material
$[C']$	matrix relating stresses to nodal displacements
C	direction cosine in two dimensions
C_x, C_y, C_z	direction cosines in three dimensions
$\{d\}$	element and structure nodal displacement matrix, both in global coordinates
$\{d'\}$	local-coordinate element nodal displacement matrix
D	bending rigidity of a plate
$[D]$	matrix relating stresses to strains
$[D']$	operator matrix given by Eq. (10.2.16)
e	exponential function
E	modulus of elasticity
$\{f\}$	global-coordinate nodal force matrix
$\{f'\}$	local-coordinate element nodal force matrix
$\{f_b\}$	body force matrix
$\{f_h\}$	heat transfer force matrix
$\{f_q\}$	heat flux force matrix
$\{f_Q\}$	heat source force matrix
$\{f_s\}$	surface force matrix
$\{F\}$	global-coordinate structure force matrix
$\{F_c\}$	condensed force matrix
$\{F_i\}$	global nodal forces
$\{F_0\}$	equivalent force matrix
$\{g\}$	temperature gradient matrix or hydraulic gradient matrix
G	shear modulus
h	heat-transfer (or convection) coefficient
i, j, m	nodes of a triangular element
I	principal moment of inertia
$[J]$	Jacobian matrix
k	spring stiffness
$[k]$	global-coordinate element stiffness or conduction matrix
$[k_c]$	condensed stiffness matrix, and conduction part of the stiffness matrix in heat-transfer problems
$[k']$	local-coordinate element stiffness matrix
$[k_n]$	convective part of the stiffness matrix in heat-transfer problems
$[K]$	global-coordinate structure stiffness matrix
K_{xx}, K_{yy}	thermal conductivities (or permeabilities, for fluid mechanics) in the x and y directions, respectively

L	length of a bar or beam element
m	maximum difference in node numbers in an element
$m(x)$	general moment expression
m_x, m_y, m_{xy}	moments in a plate
$[m'], [m]$	local element mass matrix
$[m_i']$	local nodal moments
$[M]$	global mass matrix
$[\mathrm{M}^*]$	matrix used to relate displacements to generalized coordinates for a linear-strain triangle formulation
$[M']$	matrix used to relate strains to generalized coordinates for a linear-strain triangle formulation
n_b	bandwidth of a structure
n_d	number of degrees of freedom per node
$[N]$	shape (interpolation or basis) function matrix
N_i	shape functions
p	surface pressure (or nodal heads in fluid mechanics)
p_r, p_z	radial and axial (longitudinal) pressures, respectively
P	concentrated load
$[P']$	concentrated local force matrix
q	heat flow (flux) per unit area or distributed loading on a plate
$\overline{q}$	rate of heat flow
q^*	heat flow per unit area on a boundary surface
Q	heat source generated per unit volume or internal fluid source
Q^*	line or point heat source
Q_x, Q_y	transverse shear line loads on a plate
r, θ, z	radial, circumferential, and axial coordinates, respectively
R	residual in Galerkin's integral
R_b	body force in the radial direction
R_{ix}, R_{iy}	nodal reactions in x and y directions, respectively
s, t, z'	natural coordinates attached to isoparametric element
S	surface area
t	thickness of a plane element or a plate element
t_i, t_j, t_m	nodal temperatures of a triangular element
T	temperature function
T_∞	free-stream temperature
$[T]$	displacement, force, and stiffness transformation matrix
$[T_i]$	surface traction matrix in the i direction
u, v, w	displacement functions in the x, y, and z directions, respectively
u_i, v_i, w_i	x, y, and z displacements at node i, respectively
U	strain energy
ΔU	change in stored energy
v	velocity of fluid flow
V	shear force in a beam
w	distributed loading on a beam or along an edge of a plane element
W	work
x_i, y_i, z_i	nodal coordinates in the x, y, and z directions, respectively
x', y', z'	local element coordinate axes
x, y, z	structure global or reference coordinate axes
$[X]$	body force matrix

X_b, Y_b	body forces in the x and y directions, respectively
Z_b	body force in longitudinal direction (axisymmetric case) or in the z direction (three-dimensional case)

Greek Symbols

α	coefficient of thermal expansion
$\alpha_i, \beta_i, \gamma_i, \delta_i$	used to express the shape functions defined by Eq. (6.2.10) and Eqs. (11.2.5) through (11.2.8)
δ	spring or bar deformation
ϵ	normal strain
$\{\varepsilon_T\}$	thermal strain matrix
k_x, k_y, k_{xy}	curvatures in plate bending
ν	Poisson's ratio
ϕ_i	nodal angle of rotation or slope in a beam element
π_h	functional for heat-transfer problem
π_p	total potential energy
ρ	mass density of a material
ρ_w	weight density of a material
ω	angular velocity and natural circular frequency
Ω	potential energy of forces
ϕ	fluid head or potential, or rotation or slope in a beam
σ	normal stress
$\{\sigma_T\}$	thermal stress matrix
τ	shear stress and period of vibration
θ	angle between the x axis and the local x' axis for two-dimensional problems
θ_p	principal angle
$\theta_x, \theta_y, \theta_z$	angles between the global x, y, and z axes and the local x' axis, respectively, or rotations about the x and y axes in a plate
$[\psi]$	general displacement function matrix

Other Symbols

$\dfrac{d(\ )}{dx}$	derivative of a variable with respect to x
dt	time differential
$(\dot{\ })$	the dot over a variable denotes that the variable is being differentiated with respect to time
[ ]	denotes a rectangular or a square matrix
{ }	denotes a column matrix
(_)	the underline of a variable denotes a matrix
$(\ ')$	the prime next to a variable denotes that the variable is being described in a local coordinate system
$[\ ]^{-1}$	denotes the inverse of a matrix
$[\ ]^T$	denotes the transpose of a matrix
$\dfrac{\partial(\ )}{\partial x}$	partial derivative with respect to x
$\dfrac{\partial(\ )}{\partial\{x\}}$	partial derivative with respect to each variable in $\{d\}$

CHAPTER 1

Introduction

CHAPTER OBJECTIVES

At the conclusion of this chapter, you will be able to:

- Present an introduction to the finite element method.
- Provide a brief history of the finite element method.
- Introduce matrix notation.
- Describe the role of the computer in the development of the finite element method.
- Present the general steps used in the finite element method.
- Illustrate the various types of elements used in the finite element method.
- Show typical applications of the finite element method.
- Summarize some of the advantages of the finite element method.

Prologue

The finite element method is a numerical method for solving problems of engineering and mathematical physics. Typical problem areas of interest in engineering and mathematical physics that are solvable by use of the finite element method include structural analysis, heat transfer, fluid flow, mass transport, and electromagnetic potential.

For physical systems involving complicated geometries, loadings, and material properties, it is generally not possible to obtain analytical mathematical solutions to simulate the response of the physical system. Analytical solutions are those given by a mathematical expression that yields the values of the desired unknown quantities at any location in a body (here total structure or physical system of interest) and are thus valid for an infinite number of locations in the body. These analytical solutions generally require the solution of ordinary or partial differential equations, typically created by engineers, physicists, and mathematicians to eliminate the need for the creation and testing of numerous prototype designs, which may be quite costly. Because of the complicated geometries, loadings, and material properties, the solution to these differential equations is usually not obtainable. Hence, we need to rely on numerical methods, such as the finite element method, that can approximate the solution to these equations.

The finite element formulation of the problem results in a system of simultaneous algebraic equations for solution, rather than requiring the solution of differential equations. These numerical

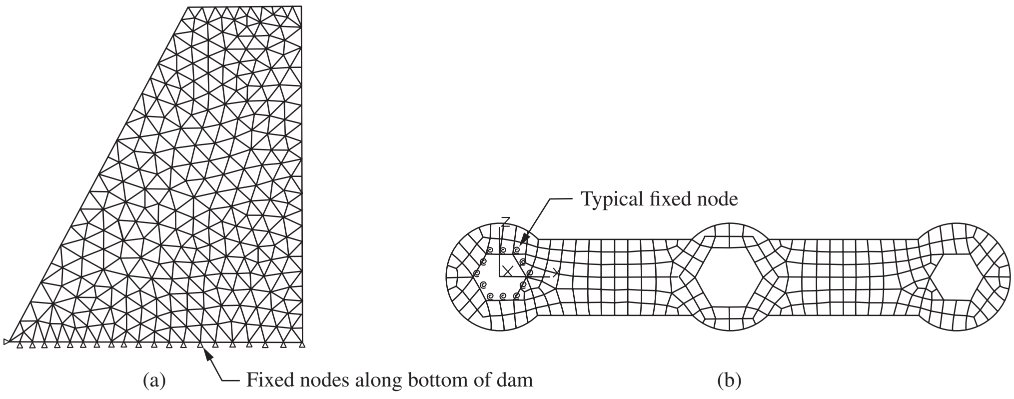


Figure 1–1 Two-dimensional models of (a) discretized dam and (b) discretized bicycle wrench (Applied loads are not shown.) All elements and nodes lie in a plane.

methods yield approximate values of the unknowns at discrete numbers of points in the continuum. Hence, this process of modeling a body by dividing it into an equivalent system of smaller bodies of units (finite elements) interconnected at points common to two or more elements (nodal points or nodes) and/or boundary lines and/or surfaces is called *discretization*. Figure 1–1 shows a cross section of a concrete dam and a bicycle wrench, respectively, that illustrate this process of discretization, where the dam has been divided into 490 plane triangular elements and the wrench has been divided into 254 plane quadrilateral elements. In both models the elements are connected at nodes and along inter element boundary lines. In the finite element method, instead of solving the problem for the entire body in one operation, we formulate the equations for each finite element and then combine them to obtain the solution for the whole body.

Briefly, the solution for structural problems typically refers to determining the displacements at each node and the stresses within each element making up the structure that is subjected to applied loads. In nonstructural problems, the nodal unknowns may, for instance, be temperatures or fluid pressures due to thermal or fluid fluxes.

This chapter first presents a brief history of the development of the finite element method. You will see from this historical account that the method has become a practical one for solving engineering problems only in the past 60 years (paralleling the developments associated with the modern high-speed electronic digital computer). This historical account is followed by an introduction to matrix notation; then we describe the need for matrix methods (as made practical by the development of the modern digital computer) in formulating the equations for solution. This section discusses both the role of the digital computer in solving the large systems of simultaneous algebraic equations associated with complex problems and the development of numerous computer programs based on the finite element method. Next, a general description of the steps involved in obtaining a solution to a problem is provided. This description includes discussion of the types of elements available for a finite element method solution. Various representative applications are then presented to illustrate the capacity of the method to solve problems, such as those involving complicated geometries, several different materials, and irregular loadings. Chapter 1 also lists some of the advantages of the finite element method in solving problems of engineering and mathematical physics. Finally, we present numerous features of computer programs based on the finite element method.

1.1 Brief History

This section presents a brief history of the finite element method as applied to both structural and nonstructural areas of engineering and to mathematical physics. References cited here are intended to augment this short introduction to the historical background.

The modern development of the finite element method began in the 1940s in the field of structural engineering with the work by Hrennikoff [1] in 1941 and McHenry [2] in 1943, who used a lattice of line (one-dimensional) elements (bars and beams) for the solution of stresses in continuous solids. In a paper published in 1943 but not widely recognized for many years, Courant [3] proposed setting up the solution of stresses in a variational form. Then he introduced piecewise interpolation (or shape) functions over triangular subregions making up the whole region as a method to obtain approximate numerical solutions. In 1947 Levy [4] developed the flexibility or force method, and in 1953 his work [5] suggested that another method (the stiffness or displacement method) could be a promising alternative for use in analyzing statically redundant aircraft structures. However, his equations were cumbersome to solve by hand, and thus the method became popular only with the advent of the high-speed digital computer.

In 1954 Argyris and Kelsey [6, 7] developed matrix structural analysis methods using energy principles. This development illustrated the important role that energy principles would play in the finite element method.

The first treatment of two-dimensional elements was by Turner et al. [8] in 1956. They derived stiffness matrices for truss elements, beam elements, and two-dimensional triangular and rectangular elements in plane stress and outlined the procedure commonly known as the *direct stiffness method* for obtaining the total structure stiffness matrix. Along with the development of the high-speed digital computer in the early 1950s, the work of Turner et al. [8] prompted further development of finite element stiffness equations expressed in matrix notation. The phrase *finite element* was introduced by Clough [9] in 1960 when both triangular and rectangular elements were used for plane stress analysis.

A flat, rectangular-plate bending-element stiffness matrix was developed by Melosh [10] in 1961. This was followed by development of the curved-shell bending-element stiffness matrix for axisymmetric shells and pressure vessels by Grafton and Strome [11] in 1963.

Extension of the finite element method to three-dimensional problems with the development of a tetrahedral stiffness matrix was done by Martin [12] in 1961, by Gallagher et al. [13] in 1962, and by Melosh [14] in 1963. Additional three-dimensional elements were studied by Argyris [15] in 1964. The special case of axisymmetric solids was considered by Clough and Rashid [16] and Wilson [17] in 1965.

Most of the finite element work up to the early 1960s dealt with small strains and small displacements, elastic material behavior, and static loadings. However, large deflection and thermal analysis were considered by Turner et al. [18] in 1960 and material nonlinearities by Gallagher et al. [13] in 1962, whereas buckling problems were initially treated by Gallagher and Padlog [19] in 1963. Zienkiewicz et al. [20] extended the method to visco elasticity problems in 1968.

In 1965 Archer [21] considered dynamic analysis in the development of the consistent-mass matrix, which is applicable to analysis of distributed-mass systems such as bars and beams in structural analysis.

With Melosh's [14] realization in 1963 that the finite element method could be set up in terms of a variational formulation, it began to be used to solve nonstructural applications. Field problems, such as determination of the torsion of a shaft, fluid flow, and heat conduction, were solved by Zienkiewicz and Cheung [22] in 1965, Martin [23] in 1968, and Wilson and Nickel [24] in 1966.

Further extension of the method was made possible by the adaptation of weighted residual methods, first by Szabo and Lee [25] in 1969 to derive the previously known elasticity equations used in structural analysis and then by Zienkiewicz and Parekh [26] in 1970 for transient field problems. It was then recognized that when direct formulations and variational formulations are difficult or not possible to use, the method of weighted residuals may at times be appropriate. For example, in 1977 Lyness et al. [27] applied the method of weighted residuals to the determination of magnetic field.

In 1976, Belytschko [28, 29] considered problems associated with large-displacement nonlinear dynamic behavior and improved numerical techniques for solving the resulting systems of equations. For more on these topics, consult the texts by Belytschko, Liu, Moran [58], and Crisfield [61, 62].

A relatively new field of application of the finite element method is that of bioengineering [30, 31]. This field is still troubled by such difficulties as nonlinear materials, geometric nonlinearities, and other complexities still being discovered.

From the early 1950s to the present, enormous advances have been made in the application of the finite element method to solve complicated engineering problems. Engineers, applied mathematicians, and other scientists will undoubtedly continue to develop new applications. For an extensive bibliography on the finite element method, consult the work of Kardestuncer [32], Clough [33], or Noor [57].

1.2 Introduction to Matrix Notation

Matrix methods are a necessary tool used in the finite element method for purposes of simplifying the formulation of the element stiffness equations, for purposes of longhand solutions of various problems, and, most important, for use in programming the methods for high-speed electronic digital computers. Hence matrix notation represents a simple and easy-to-use notation for writing and solving sets of simultaneous algebraic equations.

Appendix A discusses the significant matrix concepts used throughout the text. We will present here only a brief summary of the notation used in this text.

A **matrix** *is a rectangular array of quantities arranged in rows and columns that is often used as an aid in expressing and solving a system of algebraic equations*. As examples of matrices that will be described in subsequent chapters, the force components $(F_{1x}, F_{1y}, F_{1z}, F_{2x}, F_{2y}, F_{2z}, \ldots, F_{nx}, F_{ny}, F_{nz})$ acting at the various nodes or points $(1, 2, \ldots, n)$ on a structure and the corresponding set of nodal displacements $(u_1, v_1, w_1, u_2, v_2, w_2, \ldots, u_n, v_n, w_n)$ can both be expressed as matrices:

$$\{F\} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{1z} \\ F_{2x} \\ F_{2y} \\ F_{2z} \\ \vdots \\ F_{nx} \\ F_{ny} \\ F_{nz} \end{Bmatrix} \qquad \{d\} = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ \vdots \\ u_n \\ v_n \\ w_n \end{Bmatrix} \tag{1.2.1}$$

The subscripts to the right of F identify the node and the direction of force, respectively. For instance, F_{1x} denotes the force at node 1 applied in the x direction. The x, y, and z displacements at a node are denoted by u, v, and w, respectively. The subscript next to u, v, and w denotes the node. For instance, u_1, v_1, and w_1 denote the displacement components in the x, y, and z directions, respectively, at node 1. The matrices in Eqs. (1.2.1) are called *column matrices* and have a size of $n \times 1$. The brace notation { } will be used throughout the text to denote a column matrix. The whole set of force or displacement values in the column matrix is simply represented by $\{F\}$ or $\{d\}$.

The more general case of a known rectangular matrix will be indicated by use of the bracket notation [ ]. For instance, the element and global structure stiffness matrices $[k]$ and $[K]$, respectively, developed throughout the text for various element types (such as those in Figure 1–2 on page 11), are represented by square matrices given as

$$[k] = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & \vdots & & \vdots \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix} \tag{1.2.2}$$

and

$$[K] = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \vdots & \vdots & & \vdots \\ K_{n1} & K_{n2} & \cdots & K_{nn} \end{bmatrix} \tag{1.2.3}$$

where, in structural theory, the elements k_{ij} and K_{ij} are often referred to as *stiffness influence coefficients*.

You will learn that the global nodal forces $\{F\}$ and the global nodal displacements $\{d\}$ are related through use of the global stiffness matrix $[K]$ by

$$\{F\} = [K]\{d\} \tag{1.2.4}$$

Equation (1.2.4) is called the *global stiffness equation* and represents a set of simultaneous equations. It is the basic equation formulated in the stiffness or displacement method of analysis.

To obtain a clearer understanding of elements K_{ij} in Eq. (1.2.3), we use Eq. (1.2.1) and write out the expanded form of Eq. (1.2.4) as

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ \vdots \\ F_{nz} \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \vdots & & & \\ K_{n1} & K_{n2} & \cdots & K_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \vdots \\ w_n \end{Bmatrix} \tag{1.2.5}$$

Now assume a structure to be forced into a displaced configuration defined by $u_1 = 1, v_1 = w_1 = \cdots w_n = 0$. Then from Eq. (1.2.5), we have

$$F_{1x} = K_{11} \qquad F_{1y} = K_{21}, \ldots, F_{nz} = K_{n1} \tag{1.2.6}$$

Equations (1.2.6) contain all elements in the first column of $[K]$. In addition, they show that these elements, $K_{11}, K_{21}, \ldots, K_{n1}$, are the values of the full set of nodal forces required to maintain the imposed displacement state. In a similar manner, the second column in $[K]$ represents the values of forces required to maintain the displaced state $v_1 = 1$ and all other nodal displacement components equal to zero. We should now have a better understanding of the meaning of stiffness influence coefficients.

Subsequent chapters will discuss the element stiffness matrices $[k]$ for various element types, such as bars, beams, plane stress, and three-dimensional stress. They will also cover the procedure for obtaining the global stiffness matrices $[K]$ for various structures and for solving Eq. (1.2.4) for the unknown displacements in matrix $\{d\}$.

Using matrix concepts and operations will become routine with practice; they will be valuable tools for solving small problems longhand. And matrix methods are crucial to the use of the digital computers necessary for solving complicated problems with their associated large number of simultaneous equations.

1.3 Role of the Computer

As we have said, until the early 1950s, matrix methods and the associated finite element method were not readily adaptable for solving complicated problems. Even though the finite element method was being used to describe complicated structures, the resulting large number of algebraic equations associated with the finite element method of structural analysis made the method extremely difficult and impractical to use. However, with the advent of the computer, the solution of thousands of equations in a matter of minutes became possible.

The first modern-day commercial computer appears to have been the UNIVAC, IBM 701, which was developed in the 1950s. This computer was built based on vacuum-tube technology.

Along with the UNIVAC came the punch-card technology whereby programs and data were created on punch cards. In the 1960s, transistor-based technology replaced the vacuum-tube technology due to the transistor's reduced cost, weight, and power consumption and its higher reliability. From 1969 to the late 1970s, integrated circuit-based technology was being developed, which greatly enhanced the processing speed of computers, thus making it possible to solve larger finite element problems with increased degrees of freedom. From the late 1970s into the 1980s, large-scale integration as well as workstations that introduced a windows-type graphical interface appeared along with the computer mouse. The first computer mouse received a patent on November 17, 1970. Personal computers had now become mass-market desktop computers. These developments came during the age of networked computing, which brought the Internet and the World Wide Web. In the 1990s the Windows operating system was released, making IBM and IBM-compatible PCs more user friendly by integrating a graphical user interface into the software.

The development of the computer resulted in the writing of computational programs. Numerous special-purpose and general-purpose programs have been written to handle various complicated structural (and nonstructural) problems. Programs such as [46–56] illustrate the elegance of the finite element method and reinforce understanding of it.

In fact, finite element computer programs now can be solved on single-processor machines, such as a single desktop or laptop personal computer (PC) or on a cluster of computer nodes. The powerful memories of the PC and the advances in solver programs have made it possible to solve problems with over a million unknowns.

To use the computer, the analyst, having defined the finite element model, inputs the information into the computer. This information may include the position of the element nodal coordinates, the manner in which elements are connected, the material properties of the elements, the applied loads, boundary conditions, or constraints, and the kind of analysis to be performed. The computer then uses this information to generate and solve the equations necessary to carry out the analysis.

1.4 General Steps of the Finite Element Method

This section presents the general steps included in a finite element method formulation and solution to an engineering problem. We will use these steps as our guide in developing solutions for structural and nonstructural problems in subsequent chapters.

For simplicity's sake, for the presentation of the steps to follow, we will consider only the structural problem. The nonstructural heat-transfer, fluid mechanics, and electrostatics problems and their analogies to the structural problem are considered in Chapters 13 and 14.

Typically, for the structural stress-analysis problem, the engineer seeks to determine displacements and stresses throughout the structure, which is in equilibrium and is subjected to applied loads. For many structures, it is difficult to determine the distribution of deformation using conventional methods, and thus the finite element method is necessarily used.

There are three primary methods that can be used to derive the finite element equations of a physical system. These are (1) the *direct method* or *direct equilibrium method for structural analysis problems*, (2) the *variational methods consisting of among the subsets energy methods and the principle of virtual work*, and (3) the *weighted residual methods*. We briefly describe these three primary methods as follows, and more details of each will be described later in this section under step 4.

Direct Methods

The direct method, being the simplest and yielding a clear physical insight into the finite element method, is recommended in the initial stages of learning the concepts of the finite element method. However, the direct method is limited in its application to deriving element stiffness matrices for one-dimensional elements involving springs, uniaxial bars, trusses, and beams.

There are two general direct approaches traditionally associated with the finite element method as applied to structural mechanics problems. One approach, called the *force*, or *flexibility*, *method*, uses internal forces as the unknowns of the problem. To obtain the governing equations, first the equilibrium equations are used. Then necessary additional equations are found by introducing compatibility equations. The result is a set of algebraic equations for determining the redundant or unknown forces.

The second approach, called the *displacement*, or *stiffness*, *method*, assumes the displacements of the nodes as the unknowns of the problem. For instance, compatibility conditions requiring that elements connected at a common node, along a common edge, or on a common surface before loading remain connected at that node, edge, or surface after deformation takes place are initially satisfied. Then the governing equations are expressed in terms of nodal displacements using the equations of equilibrium and an applicable law relating forces to displacements.

These two direct approaches result in different unknowns (forces or displacements) in the analysis and different matrices associated with their formulations (flexibilities or stiffnesses). It has been shown [34] that, for computational purposes, the displacement (or stiffness) method is more desirable because its formulation is simpler for most structural analysis problems. Furthermore, a vast majority of general-purpose finite element programs have incorporated the displacement formulation for solving structural problems. Consequently, only the displacement method will be used throughout this text.

Variational Methods

The variational method is much easier to use for deriving the finite element equations for two- and three-dimensional elements than the direct method. However, it requires the existence of a functional, that upon minimizing yields the stiffness matrix and related element equations. For structural/stress analysis problems, we can use the principle of minimum potential energy as the functional, for this principle is a relatively easy physical concept to understand and has likely been introduced to the reader in an undergraduate course in basic applied mechanics [35].

It can be used to develop the governing equations for both structural and nonstructural problems. The variational method includes a number of principles. One of these principles, used extensively throughout this text because it is relatively easy to comprehend and is often introduced in basic mechanics courses, is the theorem of minimum potential energy that applies to materials behaving in a linear-elastic manner. This theorem is explained and used in various sections of the text, such as Section 2.6 for the spring element, Section 3.10 for the bar element, Section 4.7 for the beam element, Section 6.2 for the constant strain triangle plane stress and plane strain element, Section 9.1 for the axisymmetric element, Section 11.2 for the three-dimensional solid tetrahedral element, and Section 12.2 for the plate bending element. A functional analogous to that used in the theorem of minimum potential energy is then employed to develop the finite element equations for the nonstructural problem of heat transfer presented in Chapter 13.

Another variational principle often used to derive the governing equations is the principle of virtual work. This principle applies more generally to materials that behave in a linear-elastic fashion, as well as those that behave in a nonlinear fashion. The principle of virtual work is described in Appendix E for those choosing to use it for developing the general governing finite element equations that can be applied specifically to bars, beams, and two- and three-dimensional solids in either static or dynamic systems.

Weighted Residual Methods

The weighted residual methods [36] allow the finite element method to be applied directly to any differential equation without having the existence of a variational principle. Section 3.12 introduces the Galerkin method (a very well-known residual method) for deriving the bar element stiffness matrix and associated element equations. Section 3.13 introduces other residual methods for solving the governing differential equation for the axial displacement along a bar.

The finite element method involves modeling the structure using small interconnected elements called *finite elements*. A displacement function is associated with each finite element. Every interconnected element is linked, directly or indirectly, to every other element through common (or shared) interfaces, including nodes and/or boundary lines and/or surfaces. By using known stress/strain properties for the material making up the structure, one can determine the behavior of a given node in terms of the properties of every other element in the structure. The total set of equations describing the behavior of each node results in a series of algebraic equations best expressed in matrix notation.

We now present the steps, along with explanations necessary at this time, used in the finite element method formulation and solution of a structural problem. The purpose of setting forth these general steps now is to expose you to the procedure generally followed in a finite element formulation of a problem. You will easily understand these steps when we illustrate them specifically for springs, bars, trusses, beams, plane frames, plane stress, axisymmetric stress, three-dimensional stress, plate bending, heat transfer, fluid flow, and electrostatics in subsequent chapters. We suggest that you review this section periodically as we develop the specific element equations.

Keep in mind that the analyst must make decisions regarding dividing the structure or continuum into finite elements and selecting the element type or types to be used in the analysis (step 1), the kinds of loads to be applied, and the types of boundary conditions or supports to be applied. The other steps, 2 through 7, are carried out automatically by a computer program.

Step 1 Discretize and Select the Element Types

Step 1 involves dividing the body into an equivalent system of finite elements with associated nodes and choosing the most appropriate element type to model most closely the actual physical behavior. The total number of elements used and their variation in size and type within a given body are primarily matters of engineering judgment. The elements must be made small enough to give usable results and yet large enough to reduce computational effort. Small elements (and possibly higher-order elements) are generally desirable where the results are changing rapidly, such as where changes in geometry occur; large elements can be used where results are relatively constant. We will have more to say about discretization guidelines in later chapters, particularly in Chapter 7, where the concept becomes quite significant.

The discretized body or mesh is often created with mesh-generation programs or preprocessor programs available to the user.

The choice of elements used in a finite element analysis depends on the physical makeup of the body under actual loading conditions and on how close to the actual behavior the analyst wants the results to be. Judgment concerning the appropriateness of one-, two-, or three-dimensional idealizations is necessary. Moreover, the choice of the most appropriate element for a particular problem is one of the major tasks that must be carried out by the designer/analyst. Elements that are commonly employed in practice—most of which are considered in this text—are shown in Figure 1–2.

The primary line elements [Figure 1–2(a)] consist of bar (or truss) and beam elements. They have a cross-sectional area but are usually represented by line segments. In general, the cross-sectional area within the element can vary, but throughout this text it will be considered to be constant. These elements are often used to model trusses and frame structures (see Figure 1–3 on page 17, for instance). The simplest line element (called a *linear element*) has two nodes, one at each end, although higher-order elements having three nodes [Figure 1–2(a)] or more (called *quadratic*, *cubic*, etc., *elements*) also exist. Chapter 10 includes discussion of higher-order line elements. The line elements are the simplest of elements to consider and will be discussed in Chapters 2 through 5 to illustrate many of the basic concepts of the finite element method.

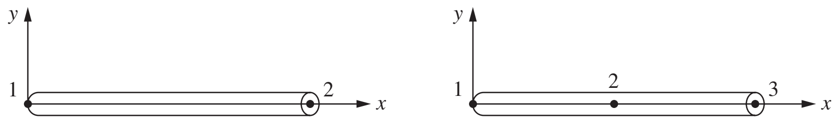
The basic two-dimensional (or plane) elements [Figure 1–2(b)] are loaded by forces in their own plane (plane stress or plane strain conditions). They are triangular or quadrilateral elements. The simplest two-dimensional elements have corner nodes only (linear elements) with straight sides or boundaries (Chapter 6), although there are also higher-order elements, typically with midside nodes [Figure 1–2(b)] (called *quadratic elements*) and curved sides (Chapters 8 and 10). The elements can have variable thicknesses throughout or be constant. They are often used to model a wide range of engineering problems (see Figures 1–4 and 1–5 on pages 17 and 18).

The most common three-dimensional elements [Figure 1–2(c)] are tetrahedral and hexahedral (or brick) elements; they are used when it becomes necessary to perform a three-dimensional stress analysis. The basic three-dimensional elements (Chapter 11) have corner nodes only and straight sides, whereas higher-order elements with midedge nodes (and possible midface nodes) have curved surfaces for their sides [Figure 1–2(c)].

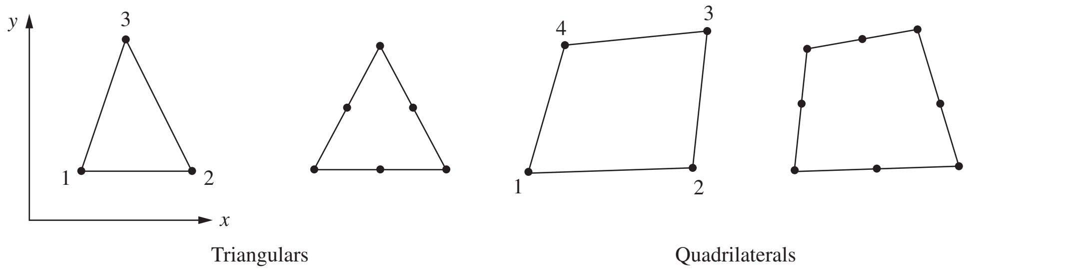
The axisymmetric element [Figure 1–2(d)] is developed by rotating a triangle or quadrilateral about a fixed axis located in the plane of the element through $360°$. This element (described in Chapter 9) can be used when the geometry and loading of the problem are axisymmetric.

Step 2 Select a Displacement Function

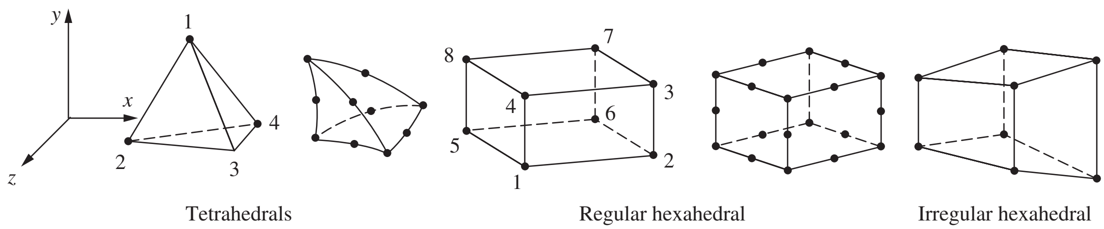
Step 2 involves choosing a displacement function within each element. The function is defined within the element using the nodal values of the element. Linear, quadratic, and cubic polynomials are frequently used functions because they are simple to work with in finite element formulation. However, trigonometric series can also be used. For a two-dimensional element, the displacement function is a function of the coordinates in its plane (say, the x-y plane). The functions are expressed in terms of the nodal unknowns (in the two-dimensional



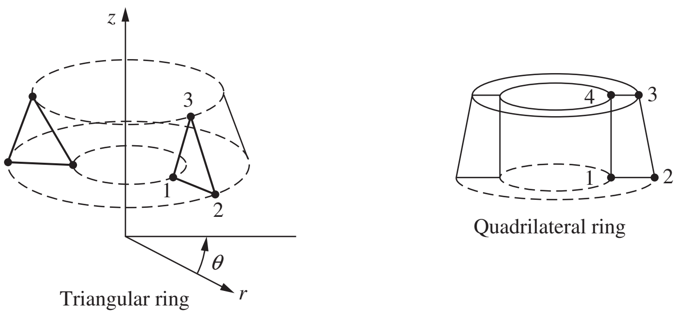
(a) Simple two-noded line element (typically used to represent a bar or beam element) and the higher-order line element



(b) Simple two-dimensional elements with corner nodes (typically used to represent plane stress/strain) and higher-order two-dimensional elements with intermediate nodes along the sides



(c) Simple three-dimensional elements (typically used to represent three-dimensional stress state) and higher-order three-dimensional elements with intermediate nodes along edges



(d) Simple axisymmetric triangular and quadrilateral elements used for axisymmetric problems

Figure 1–2 Various types of simple lowest-order finite elements with corner nodes only and higher-order elements with intermediate nodes problem, in terms of an x and a y component). The same general displacement function can be used repeatedly for each element. Hence the finite element method is one in which a continuous quantity, such as the displacement throughout the body, is approximated by a discrete model composed of a set of piecewise-continuous functions defined within each finite domain or finite element.

For the one-dimensional spring and bar elements, the displacement function is a function of a single coordinate (say x, along the axis of the spring or bar). For the spring and bar elements one can then skip step 2 and go directly to step 3 to derive the element stiffness matrix and equations. This will be explicitly demonstrated in Chapters 2 and 3 for the spring and bar, respectively.

Step 3 Define the Strain/Displacement and Stress/Strain Relationships

Strain/displacement and stress/strain relationships are necessary for deriving the equations for each finite element. In the case of one-dimensional deformation, say, in the x direction, we have strain ε_x related to displacement u by

$$\varepsilon_x = \frac{du}{dx} \tag{1.4.1}$$

for small strains. In addition, the stresses must be related to the strains through the stress/strain law—generally called the *constitutive law*. The ability to define the material behavior accurately is most important in obtaining acceptable results. The simplest of stress/strain laws, Hooke's law, which is often used in stress analysis, is given by

$$\sigma_x = E\varepsilon_x \tag{1.4.2}$$

where σ_x = stress in the x direction and E = modulus of elasticity.

Step 4 Derive the Element Stiffness Matrix and Equations

Initially, the development of element stiffness matrices and element equations was based on the concept of stiffness influence coefficients, which presupposes a background in structural analysis. We now present alternative methods used in this text that do not require this special background.

Direct Equilibrium Method

According to this method, the stiffness matrix and element equations relating nodal forces to nodal displacements are obtained using force equilibrium conditions for a basic element, along with force/deformation relationships. Because this method is most easily adaptable to line or one-dimensional elements, Chapters 2, 3, and 4 illustrate this method for spring, bar, and beam elements, respectively.

Work or Energy Methods

To develop the stiffness matrix and equations for two- and three-dimensional elements, it is much easier to apply a work or energy method [35]. The principle of virtual work (using virtual displacements), the principle of minimum potential energy, and Castigliano's theorem are methods frequently used for the purpose of derivation of element equations.

The principle of virtual work outlined in Appendix E is applicable for any material behavior, whereas the principle of minimum potential energy and Castigliano's theorem are applicable only to elastic materials. Furthermore, the principle of virtual work can be used even when a potential function does not exist. However, all three principles yield identical element equations for linear-elastic materials; thus which method to use for this kind of material in structural analysis is largely a matter of convenience and personal preference. We will present the principle of minimum potential energy—probably the best known of the three energy methods mentioned here—in detail in Chapters 2 and 3, where it will be used to derive the spring and bar element equations. We will further generalize the principle and apply it to the beam element in Chapter 4 and to the plane stress/strain element in Chapter 6. Thereafter, the principle is routinely referred to as the basis for deriving all other stress-analysis stiffness matrices and element equations given in Chapters 8, 9, 11, and 12.

For the purpose of extending the finite element method outside the structural stress analysis field, a **functional**[1] (a function of another function or a function that takes functions as its argument) analogous to the one to be used with the principle of minimum potential energy is quite useful in deriving the element stiffness matrix and equations (see Chapters 13 and 14 on heat transfer and fluid flow, respectively). For instance, letting π denote the functional and $f(x,y)$ denote a function f of two variables x and y, we then have $\pi = \pi(f(x,y))$, where π is a function of the function f. A more general form of a functional depending on two independent variables $u(x,y)$ and $v(x,y)$, where independent variables are x and y in Cartesian coordinates, is given by

$$\pi = \iint F(x, y, u, v, u,_x, u,_y, v,_x, v,_y, u,_{xx}, \ldots v,_{yy})dx\,dy \tag{1.4.3}$$

where the comma preceding the subscripts x and y denotes differentiation with respect to x or y, i.e., $u,_x = \frac{\partial u}{\partial x}$, etc.

Weighted Residuals Methods

The weighted residuals methods are useful for developing the element equations; particularly popular is Galerkin's method. These methods yield the same results as the energy methods wherever the energy methods are applicable. They are especially useful when a functional such as potential energy is not readily available. The weighted residual methods allow the finite element method to be applied directly to any differential equation.

Galerkin's method, along with the collocation, the least squares, and the subdomain *weighted residual methods* are introduced in Chapter 3. To illustrate each method, they will all be used to solve a one-dimensional bar problem for which a known exact solution exists for comparison. As the more easily adapted residual method, Galerkin's method will also be used to derive the bar element equations in Chapter 3 and the beam element equations in Chapter 4

[1] Another definition of a functional is as follows: A functional is an integral expression that implicitly contains differential equations that describe the problem. A typical functional is of the form $I(u) = \int F(x, u, u')dx$ where $u(x)$, x, and F are real so that $I(u)$ is also a real number. Here $u' = \partial u/\partial x$.

and to solve the combined heat-conduction/convection/mass transport problem in Chapter 13. For more information on the use of the methods of weighted residuals, see Reference [36]; for additional applications to the finite element method, consult References [37] and [38].

Using any of the methods just outlined will produce the equations to describe the behavior of an element. These equations are written conveniently in matrix form as

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \cdots & k_{2n} \\ k_{31} & k_{32} & k_{33} & \cdots & k_{3n} \\ \vdots & & & & \vdots \\ k_{n1} & & & \cdots & k_{nn} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{Bmatrix} \tag{1.4.4}$$

or in compact matrix form as

$$\{f\} = [k]\{d\} \tag{1.4.5}$$

where $\{f\}$ is the vector of element nodal forces, $[k]$ is the element stiffness matrix (normally square and symmetric), and $\{d\}$ is the vector of unknown element nodal degrees of freedom or generalized displacements, n. Here generalized displacements may include such quantities as actual displacements, slopes, or even curvatures. The matrices in Eq. (1.4.5) will be developed and described in detail in subsequent chapters for specific element types, such as those in Figure 1–2.

Step 5 Assemble the Element Equations to Obtain the Global or Total Equations and Introduce Boundary Conditions

In this step the individual element nodal equilibrium equations generated in step 4 are assembled into the global nodal equilibrium equations. Section 2.3 illustrates this concept for a two-spring assemblage. Another more direct method of superposition (called the *direct stiffness method*), whose basis is nodal force equilibrium, can be used to obtain the global equations for the whole structure. This direct method is illustrated in Section 2.4 for a spring assemblage. Implicit in the direct stiffness method is the concept of continuity, or compatibility, which requires that the structure remain together and that no tears occur anywhere within the structure.

The final assembled or global equation written in matrix form is

$$\{F\} = [K]\{d\} \tag{1.4.6}$$

where $\{F\}$ is the vector of global nodal forces, $[K]$ is the structure global or total stiffness matrix, (for most problems, the global stiffness matrix is square and symmetric) and $\{d\}$ is now the vector of known and unknown structure nodal degrees of freedom or generalized displacements. It can be shown that at this stage, the global stiffness matrix $[K]$ is a singular matrix because its determinant is equal to zero. To remove this singularity problem, we must invoke certain boundary conditions (or constraints or supports) so that the structure remains in place instead of moving as a rigid body. Further details and methods of invoking boundary conditions are given in subsequent chapters. At this time it is sufficient to note that invoking boundary or support conditions results in a modification of the global Eq. (1.4.6). We also emphasize that the applied known loads have been accounted for in the global force matrix $\{F\}$.

Step 6 Solve for the Unknown Degrees of Freedom (or Generalized Displacements)

Equation (1.4.6), modified to account for the boundary conditions, is a set of simultaneous algebraic equations that can be written in expanded matrix form as

$$\begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \vdots & & & \vdots \\ K_{n1} & K_{n2} & \cdots & K_{nn} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{Bmatrix} \tag{1.4.7}$$

where now n is the structure total number of unknown nodal degrees of freedom. These equations can be solved for the ds by using an elimination method (such as Gauss's method) or an iterative method (such as the Gauss-Seidel method). These two methods are discussed in Appendix B. The ds are called the *primary unknowns*, because they are the first quantities determined using the stiffness (or displacement) finite element method.

Step 7 Solve for the Element Strains and Stresses

For the structural stress-analysis problem, important secondary quantities of strain and stress (or moment and shear force) can be obtained because they can be directly expressed in terms of the displacements determined in step 6. Typical relationships between strain and displacement and between stress and strain—such as Eqs. (1.4.1) and (1.4.2) for one-dimensional stress given in step 3—can be used.

Step 8 Interpret the Results

The final goal is to interpret and analyze the results for use in the design/analysis process. Determination of locations in the structure where large deformations and large stresses occur is generally important in making design/analysis decisions. Postprocessor computer programs help the user to interpret the results by displaying them in graphical form.

1.5 Applications of the Finite Element Method

The finite element method can be used to analyze both structural and nonstructural problems. Typical structural areas include

1. Stress analysis, including truss and frame analysis (such as pedestrian walk bridges, high rise building frames, and windmill towers), and stress concentration problems, typically associated with holes, fillets, or other changes in geometry in a body (such as automotive parts, pressures vessels, medical devices, aircraft, and sports equipment)
2. Buckling, such as in columns, frames, and vessels
3. Vibration analysis, such as in vibratory equipment
4. Impact problems, including crash analysis of vehicles, projectile impact, and bodies falling and impacting objects

Nonstructural problems include

1. Heat transfer, such as in electronic devices emitting heat as in a personal computer microprocessor chip, engines, and cooling fins in radiators
2. Fluid flow, including seepage through porous media (such as water seeping through earthen dams), cooling ponds, and in air ventilation systems as used in sports arenas, etc., air flow around racing cars, yachting boats, and surfboards, etc.
3. Distribution of electric or magnetic potential, such as in antennas and transistors

Finally, some biomechanical engineering problems (which may include stress analysis) typically include analyses of human spine, skull, hip joints, jaw/gum tooth implants, heart, and eye.

We now present some typical applications of the finite element method. These applications will illustrate the variety, size, and complexity of problems that can be solved using the method and the typical discretization process and kinds of elements used.

Figure 1–3 illustrates a control tower for a railroad. The tower is a three-dimensional frame comprising a series of beam-type elements. The 48 elements are labeled by the circled numbers, whereas the 28 nodes are indicated by the uncircled numbers. Each node has three rotation and three displacement components associated with it. The rotations (θs) and displacements (u, v, w) are called the *degrees of freedom*. Because of the loading conditions to which the tower structure is subjected, we have used a three-dimensional model.

The finite element method used for this frame enables the designer/analyst quickly to obtain displacements and stresses in the tower for typical load cases, as required by design codes. Before the development of the finite element method and the computer, even this relatively simple problem took many hours to solve.

The next illustration of the application of the finite element method to problem solving is the determination of displacements and stresses in an underground box culvert subjected to ground shock loading from a bomb explosion. Figure 1–4 shows the discretized model, which included a total of 369 nodes, 40 one-dimensional bar or truss elements used to model the steel reinforcement in the box culvert, and 333 plane strain two-dimensional triangular and rectangular elements used to model the surrounding soil and concrete box culvert. With an assumption of symmetry, only half of the box culvert need be analyzed. This problem requires the solution of nearly 700 unknown nodal displacements. It illustrates that different kinds of elements (here bar and plane strain) can often be used in one finite element model.

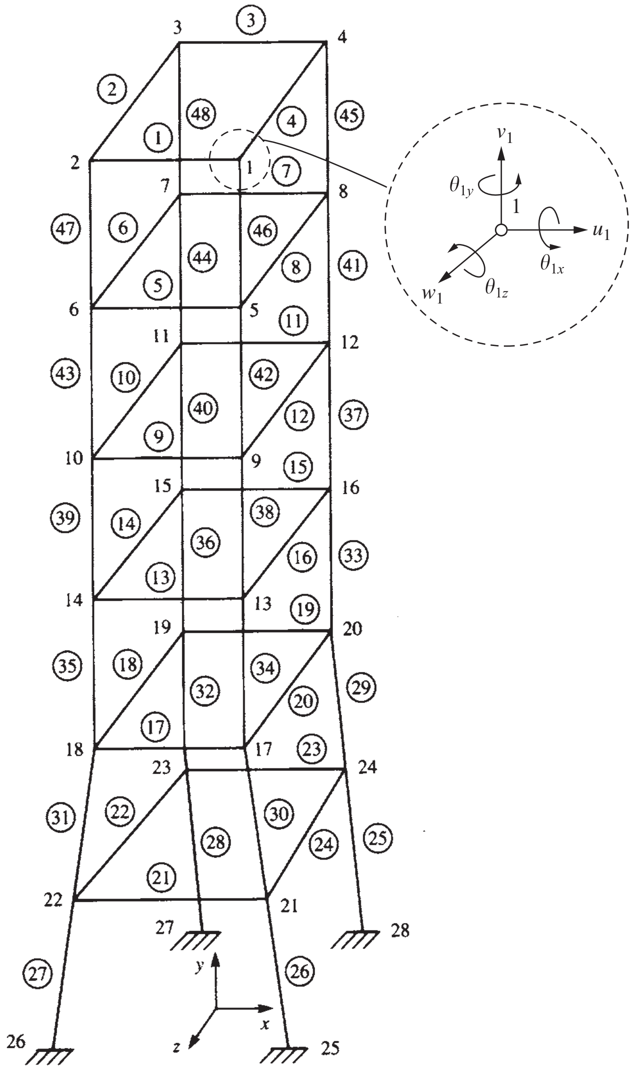
Another problem, that of the hydraulic cylinder rod end shown in Figure 1–5, was modeled by 120 nodes and 297 plane strain triangular elements. Symmetry was also applied to the whole rod end so that only half of the rod end had to be analyzed, as shown. The purpose of this analysis was to locate areas of high stress concentration in the rod end.

Figure 1–6 shows a chimney stack section that is four form heights high (or a total of 32 ft high). In this illustration, 584 beam elements were used to model the vertical and horizontal stiffeners making up the formwork, and 252 flat-plate elements were used to model the inner wooden form and the concrete shell. Because of the irregular loading pattern on the structure, a three-dimensional model was necessary. Displacements and stresses in the concrete were of prime concern in this problem.

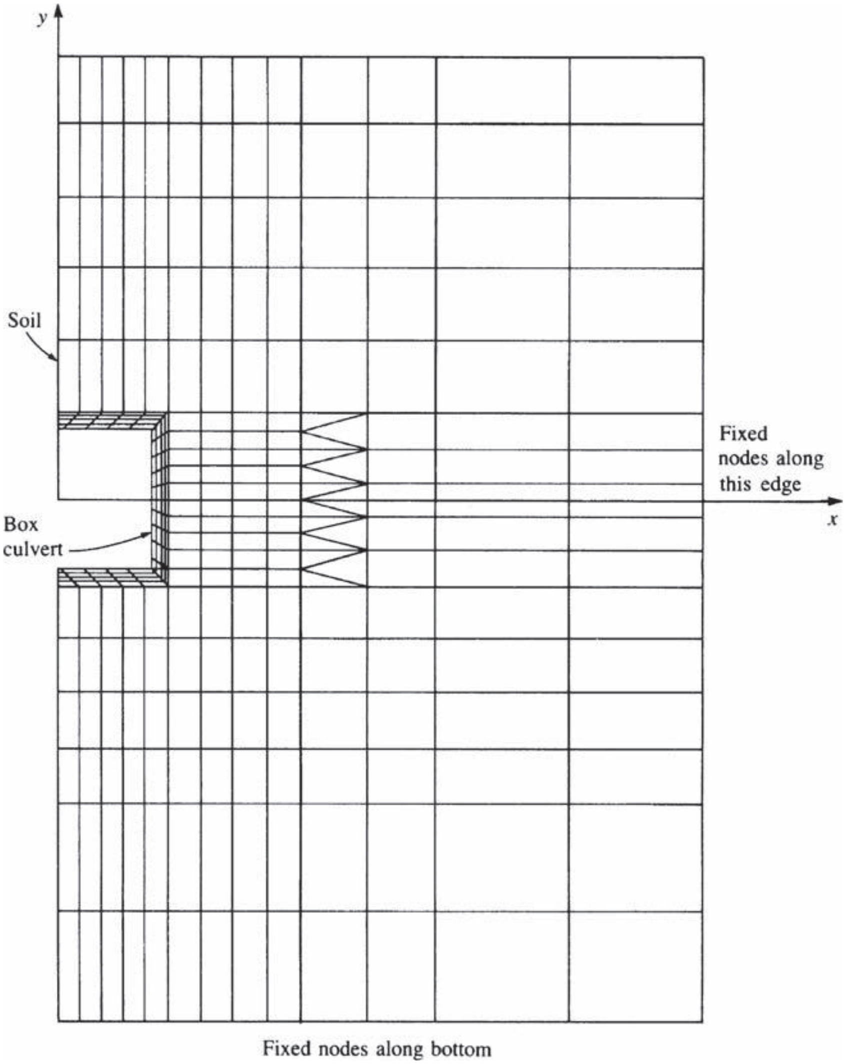
Figure 1–7 shows the finite element discretized model of a proposed steel die used in a plastic film-making process. The irregular geometry and associated potential stress concentrations necessitated use of the finite element method to obtain a reasonable solution. Here 240 axisymmetric elements were used to model the three-dimensional die.

Figure 1–8 illustrates the use of a three-dimensional solid element to model a swing casting for a backhoe frame. The three-dimensional hexahedral elements are necessary to model the irregularly shaped three-dimensional casting. Two-dimensional models certainly would not yield accurate engineering solutions to this problem.

Figure 1–9 illustrates a two-dimensional heat-transfer model used to determine the temperature distribution in earth subjected to a heat source—a buried pipeline transporting a hot gas.



■ **Figure 1–3** Discretized railroad control tower (28 nodes, 48 beam elements) with typical degrees of freedom shown at node 1, for example (By Daryl L. Logan)



■ **Figure 1–4** Discretized model of an underground box culvert (369 nodes, 40 bar elements, and 333 plane strain elements) [39]

Figure 1–10 shows a three-dimensional model of human pelvis which can be used to study stresses in the bone and the cement layer between the bone and the implant.

Figure 1–11 shows a three-dimensional model of a 710G bucket, used to study stress throughout the bucket.

More recently, mechanical event simulation (MES), including nonlinear behavior and contact, such as in roll forming processes, has been studied using finite element analysis [46], as shown in Figure 1–12 and wind mill generator stress analysis under various loading conditions, including wind, ice, and earthquake while the blades are rotating has been performed [46], as shown in Figure 1–13.

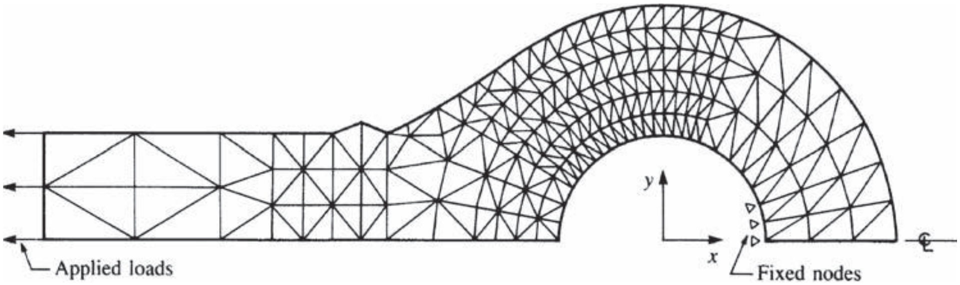


Figure 1–5 Two-dimensional analysis of a hydraulic cylinder rod end (120 nodes, 297 plane strain triangular elements)

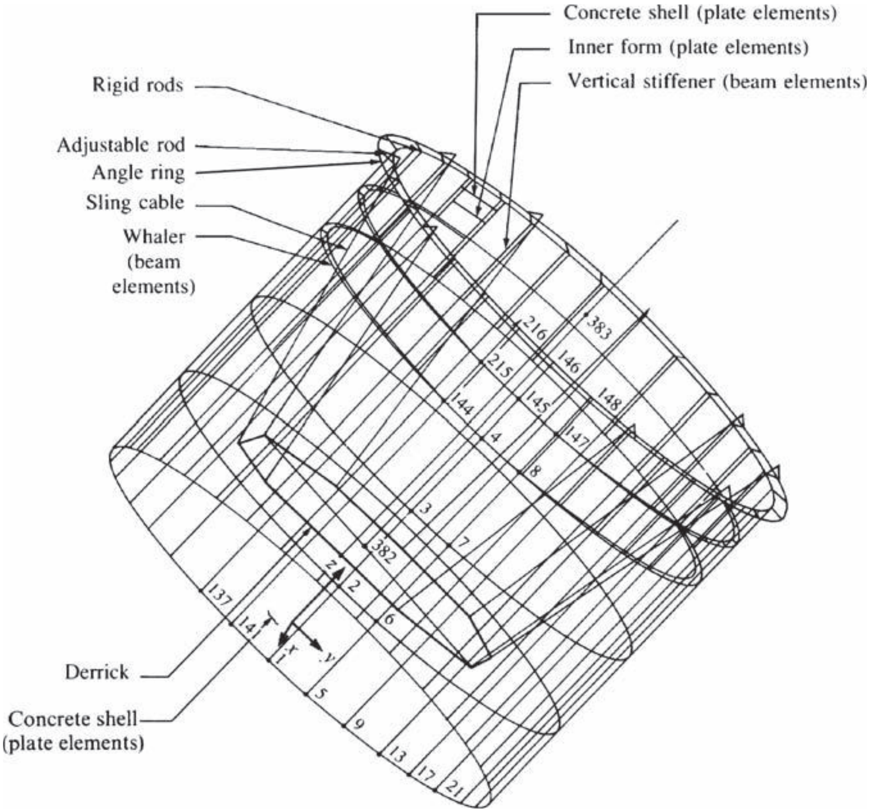
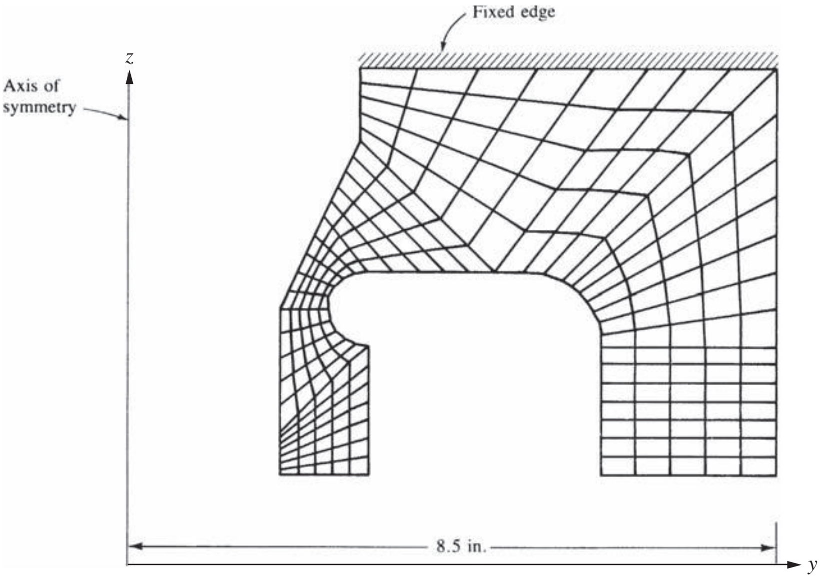
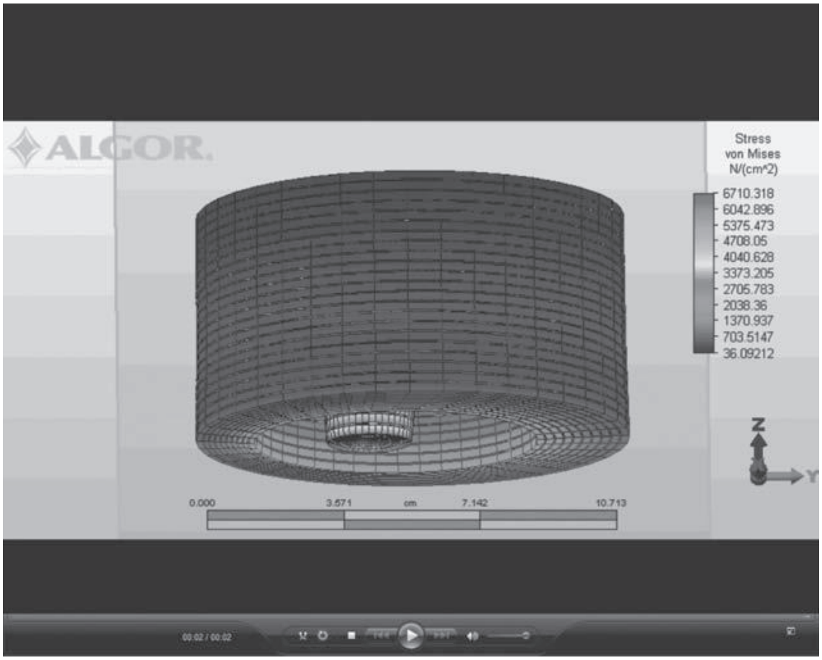


Figure 1–6 Finite element model of a chimney stack section (end view rotated 45°) (584 beam and 252 flat-plate elements) (By Daryl L. Logan)

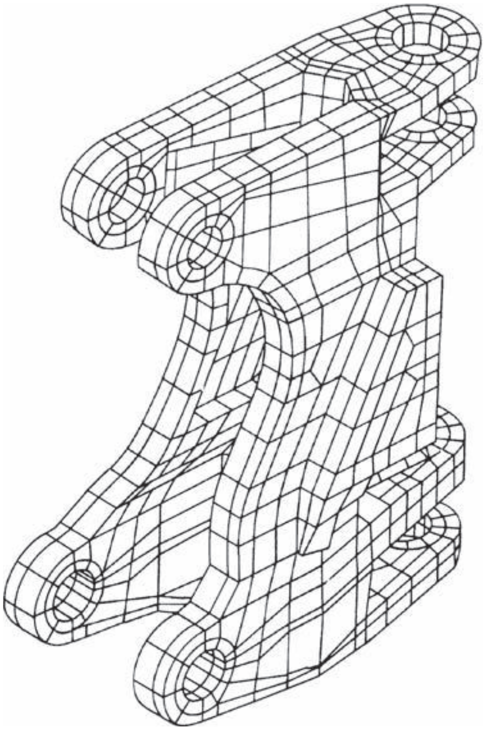


(a)



(b)

■ **Figure 1–7** (a) Model of a high-strength steel die (240 axisymmetric elements) used in the plastic film industry (By Daryl L. Logan) and (b) the three-dimensional visual of the die as the elements in the plane are rotated through 360° around the z-axis of symmetry (See the full-color insert for a color version of this figure.) (By Daryl L. Logan)



■ **Figure 1–8** Three-dimensional solid element model of a swing casting for a backhoe frame

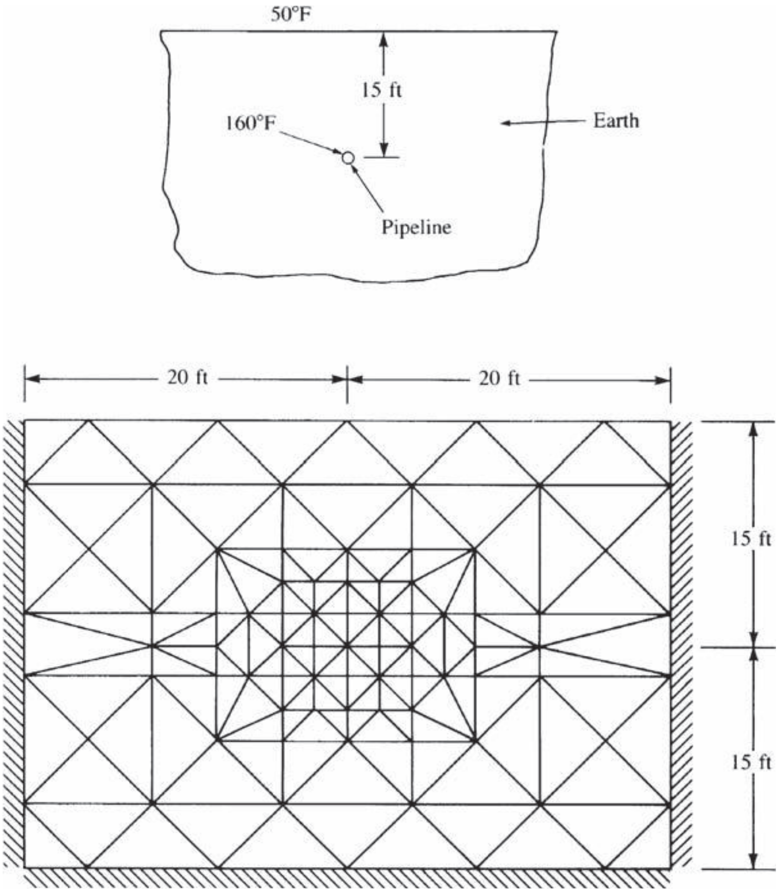
Finally, the field of computational fluid dynamics (CFD) using finite element analysis has recently been used to design ventilation systems, such as in large sports arenas, and to study air flow around race cars and around golf balls when suddenly struck by a golf club [63].

These illustrations suggest the kinds of problems that can be solved by the finite element method. Additional guidelines concerning modeling techniques will be provided in Chapter 7.

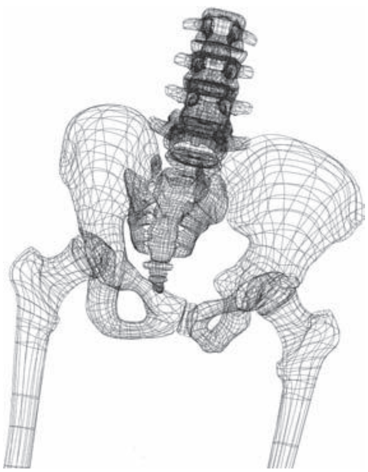
1.6 Advantages of the Finite Element Method

As previously mentioned, the finite element method has been applied to numerous problems, both structural and nonstructural. This method has a number of advantages over conventional approximate methods, such as presented by traditional courses in mechanics of material and heat transfer, and for modeling and determining physical quantities, such as displacements, stresses, temperatures, pressures, and electric currents that have made it very popular. They include the ability to

1. Model irregularly shaped bodies quite easily
2. Handle general load conditions without difficulty
3. Model bodies composed of several different materials because the element equations are evaluated individually
4. Handle unlimited numbers and kinds of boundary conditions
5. Vary the size of the elements to make it possible to use small elements where necessary



■ **Figure 1–9** Finite element model for a two-dimensional temperature distribution in the earth



■ **Figure 1–10** Finite element model of a human pelvis (© Studio MacBeth/Photo Researchers, Inc.)

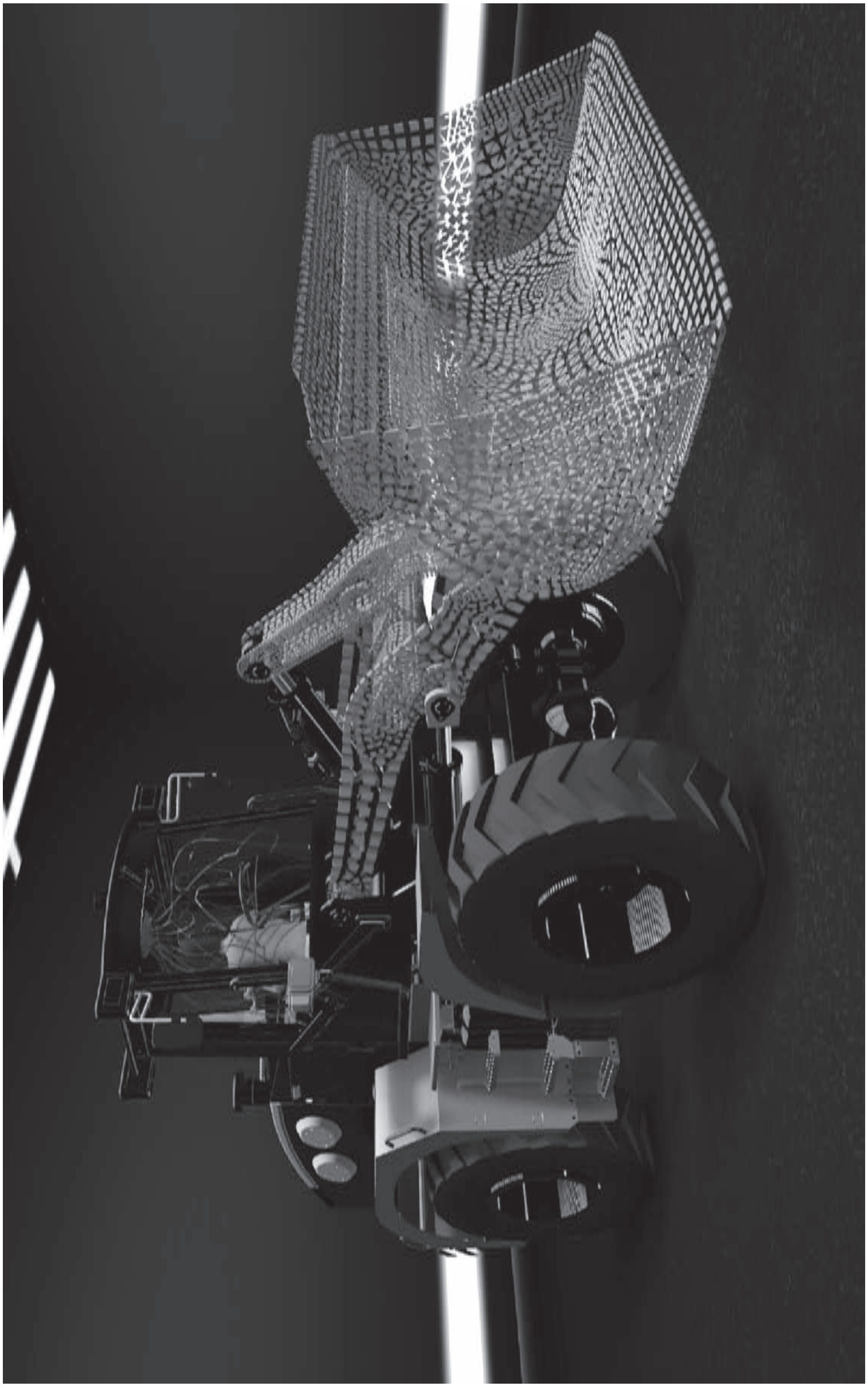
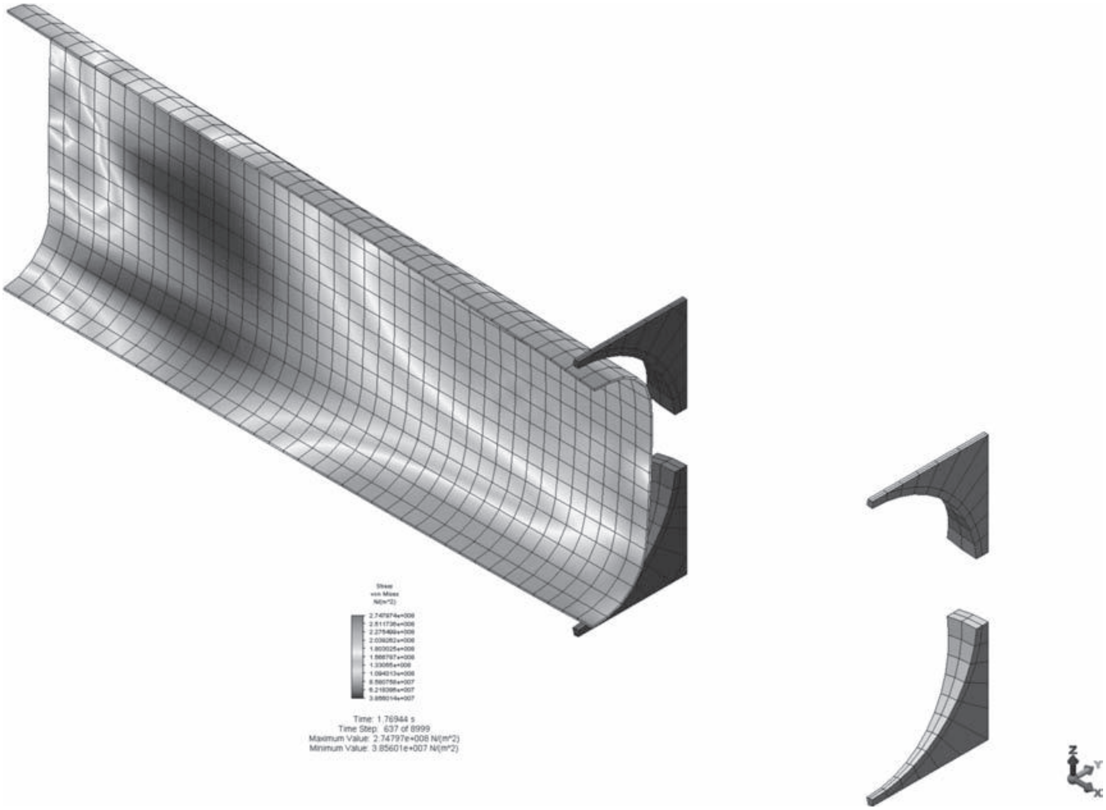


Figure 1–11 Finite element model of a 710G bucket with 169,595 elements and 185,026 nodes used (including 78,566 thin-shell linear quadrilateral elements for the bucket and coupler, 83,104 solid linear brick elements to model the bosses, and 212 beam elements to model lift arms, lift arm cylinders, and guide links) (Courtesy of Yousif Omer, Structural Design Engineer, Construction and Forestry Division, John Deere Dubuque Works) (See the full-color insert for a color version of this figure.)

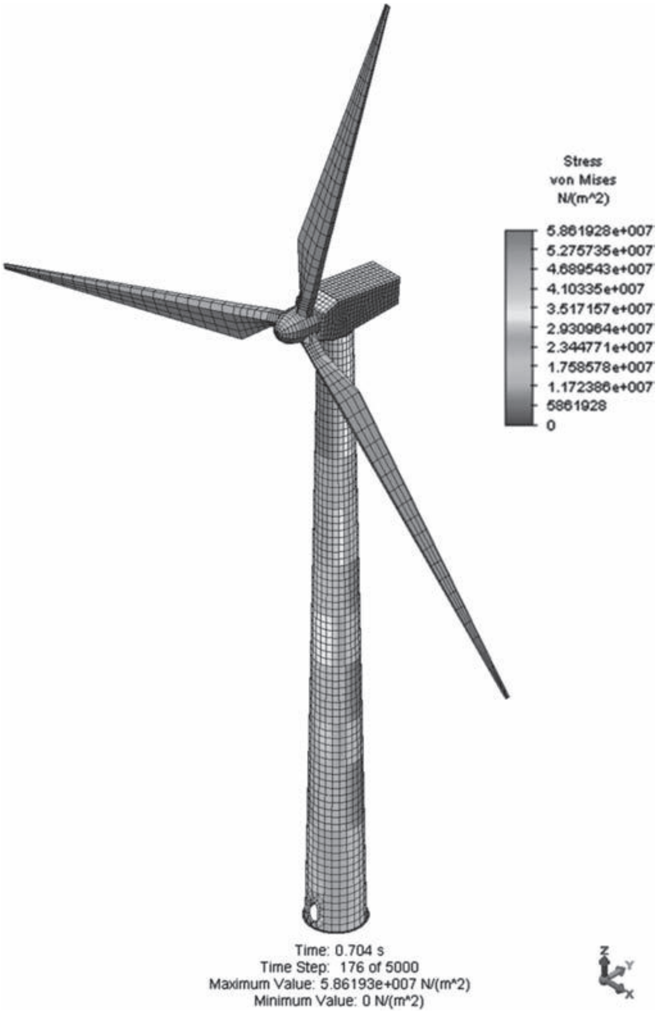


■ **Figure 1–12** Finite element model of contour roll forming or cold roll forming process (Courtesy of Valmont West Coast Engineering) (See the full-color insert for a color version of this figure.)

6. Alter the finite element model relatively easily and cheaply
7. Include dynamic effects
8. Handle nonlinear behavior existing with large deformations and nonlinear materials

The finite element method of structural analysis enables the designer to detect stress, vibration, and thermal problems during the design process and to evaluate design changes *before* the construction of a possible prototype. Thus confidence in the acceptability of the prototype is enhanced. Moreover, if used properly, the method can reduce the number of prototypes that need to be built.

Even though the finite element method was initially used for structural analysis, it has since been adapted to many other disciplines in engineering and mathematical physics, such as fluid flow, heat transfer, electromagnetic potentials, soil mechanics, and acoustics [22–24, 27, 42–44].



■ **Figure 1–13** Finite element model showing the von Mises stress plot of a wind mill tower at a critical time step using a nonlinear finite element simulation (Courtesy of Valmont West Coast Engineering)

1.7 Computer Programs for the Finite Element Method

There are two general computer methods of approach to the solution of problems by the finite element method. One is to use large commercial programs, many of which have been configured to run on personal computers (PCs); these general-purpose programs are designed to solve many types of problems. The other is to develop many small, special-purpose programs to solve specific problems. In this section, we will discuss the advantages and disadvantages of both methods. We will then list some of the available general-purpose programs and discuss some of their standard capabilities.

Some advantages of general-purpose programs:

1. The input is well organized and is developed with user ease in mind. Users do not need special knowledge of computer software or hardware. Preprocessors are readily available to help create the finite element model.
2. The programs are large systems that often can solve many types of problems of large or small size with the same input format.
3. Many of the programs can be expanded by adding new modules for new kinds of problems or new technology. Thus they may be kept current with a minimum of effort.
4. With the increased storage capacity and computational efficiency of PCs, many general-purpose programs can now be run on PCs.
5. Many of the commercially available programs have become very attractive in price and can solve a wide range of problems [45–56].

Some disadvantages of general-purpose programs:

1. The initial cost of developing general-purpose programs is high.
2. General-purpose programs are less efficient than special-purpose programs because the computer must make many checks for each problem, some of which would not be necessary if a special-purpose program were used.
3. Many of the programs are proprietary. Hence the user has little access to the logic of the program. If a revision must be made, it often has to be done by the developers.

Some advantages of special-purpose programs:

1. The programs are usually relatively short, with low development costs.
2. Small computers are able to run the programs.
3. Additions can be made to the program quickly and at a low cost.
4. The programs are efficient in solving the problems they were designed to solve.

The major disadvantage of special-purpose programs is their inability to solve different classes of problems. Thus one must have as many programs as there are different classes of problems to be solved. A list of special-purpose, public-domain finite-element programs is given in the website [60].

There are numerous vendors supporting finite element programs, and the interested user should carefully consult the vendor before purchasing any software. However, to give you an idea about the various commercial personal computer programs now available for solving problems by the finite element method, we present a partial list of existing programs.

1. Autodesk Simulation Multiphysics [46]
2. Abaqus [47]
3. ANSYS [48]
4. COSMOS/M [49]
5. GT-STRUDL [50]
6. LS-DYNA [59]
7. MARC [51]
8. MSC/NASTRAN [52]

9. NISA [53]
10. Pro/MECHANICA [54]
11. SAP2000 [55]
12. STARDYNE [56]

Standard capabilities of many of the listed programs are provided in the preceding references and in Reference [45]. These capabilities include information on

1. Element types available, such as beam, plane stress, and three-dimensional solid
2. Type of analysis available, such as static and dynamic
3. Material behavior, such as linear-elastic and nonlinear
4. Load types, such as concentrated, distributed, thermal, and displacement (settlement)
5. Data generation, such as automatic generation of nodes, elements, and restraints (most programs have preprocessors to generate the mesh for the model)
6. Plotting, such as original and deformed geometry and stress and temperature contours (most programs have postprocessors to aid in interpreting results in graphical form)
7. Displacement behavior, such as small and large displacement and buckling
8. Selective output, such as at selected nodes, elements, and maximum or minimum values

All programs include at least the bar, beam, plane stress, plate-bending, and three-dimensional solid elements, and most now include heat-transfer analysis capabilities.

Complete capabilities of the programs and their cost are best obtained through program reference manuals and websites, such as References [46–56, 59].

References

[1] Hrennikoff, A., “Solution of Problems in Elasticity by the Frame Work Method,” *Journal of Applied Mechanics*, Vol. 8, No. 4, pp. 169–175, Dec. 1941.

[2] McHenry, D., “A Lattice Analogy for the Solution of Plane Stress Problems,” *Journal of Institution of Civil Engineers*, Vol. 21, pp. 59–82, Dec. 1943.

[3] Courant, R., “Variational Methods for the Solution of Problems of Equilibrium and Vibrations,” *Bulletin of the American Mathematical Society*, Vol. 49, pp. 1–23, 1943.

[4] Levy, S., “Computation of Influence Coefficients for Aircraft Structures with Discontinuities and Sweepback,” *Journal of Aeronautical Sciences*, Vol. 14, No. 10, pp. 574–560, Oct. 1947.

[5] Levy, S., “Structural Analysis and Influence Coefficients for Delta Wings,” *Journal of Aeronautical Sciences*, Vol. 20, No. 7, pp. 449–454, July 1953.

[6] Argyris, J. H., “Energy Theorems and Structural Analysis,” *Aircraft Engineering*, Oct., Nov., Dec. 1954 and Feb., Mar., Apr., May 1955.

[7] Argyris, J. H., and Kelsey, S., *Energy Theorems and Structural Analysis*, Butterworths, London, 1960 (collection of papers published in *Aircraft Engineering* in 1954 and 1955).

[8] Turner, M. J., Clough, R. W., Martin, H. C., and Topp, L. J., “*Stiffness and Deflection Analysis of Complex Structures*,” *Journal of Aeronautical Sciences*, Vol. 23, No. 9, pp. 805–824, Sept. 1956.

[9] Clough, R. W., “The Finite Element Method in Plane Stress Analysis,” *Proceedings*, American Society of Civil Engineers, 2nd Conference on Electronic Computation, Pittsburgh, PA, pp. 345–378, Sept. 1960.

[10] Melosh, R. J., “A Stiffness Matrix for the Analysis of Thin Plates in Bending,” *Journal of the Aerospace Sciences*, Vol. 28, No. 1, pp. 34–42, Jan. 1961.