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This second edition retains the positive features of being clearly written, well organized, and incorporating calculus in the text while adding expanded coverage on game theory, experimental economics, and behavioral economics. It remains more focused and manageable than similar textbooks, and provides a concise yet comprehensive treatment of the core topics of microeconomics, including theories of the consumer and of the firm, market structure, partial and general equilibrium, and market failures caused by public goods, externalities, and asymmetric information. It includes helpful solved problems in all the substantive chapters, as well as over seventy new mathematical exercises and enhanced versions of the ones in the first edition. The authors make use of the book's full color with sharp and helpful graphs and illustrations. This mathematically rigorous textbook is meant for students at the intermediate level who have already had an introductory course in microeconomics, and a calculus course.

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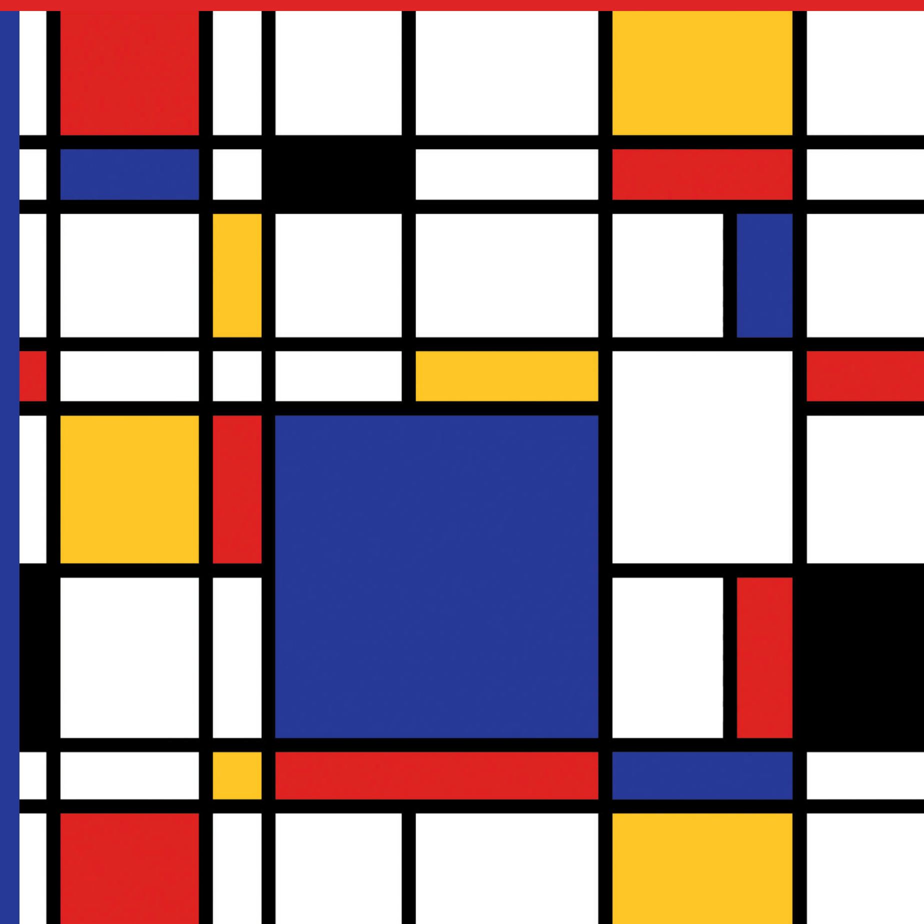

Roberto Serrano • Allan M. Feldman

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Roberto Serrano is the Harrison S. Kravis University Professor of Economics at Brown University. He received his BA in economics from Universidad Complutense de Madrid in 1987, and a PhD in economics from Harvard University in 1992, when he began teaching at Brown. He is the co-author (with Feldman) of *Welfare Economics and Social Choice Theory*, 2nd edition. He was elected Fellow of the Game Theory Society in 2017, member of its Council from 2005 to 2011, Fellow of the Econometric Society in 2013, Alfred P. Sloan Foundation Fellow in 1998, and received the Fundacion Banco Herrero Prize in 2004 (awarded to the best Spanish economist under 40). He has delivered a high number of plenary and keynote lectures at economic theory and game theory meetings, and his editorial responsibilities include being an Associate Editor of *Games and Economic Behavior* and the *International Journal of Game Theory*, and having been the Editor-in-Chief of *Economics Letters*.

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A Short Course in Intermediate Microeconomics with Calculus

Second edition

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BRIEF CONTENTS

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PREFACE

How We Started

This textbook grew out of lecture notes that Roberto Serrano developed to teach the Intermediate Microeconomics course at Brown University. The notes were shared with other instructors at Brown over the years. One of these instructors, Amy Serrano (Roberto's wife), first suggested turning the notes into a book: "This looks like a good skeleton of something; perhaps flesh can be put around these bones." Following this suggestion, Roberto and Allan Feldman began work on the book project. Our first edition saw the light in 2013, published by Cambridge University Press. We are now happy to introduce the second edition.

Main Features of This Book

When we conceived this book project, we saw an opportunity to offer something new in the market for intermediate microeconomics textbooks. Here are what we think are some key strengths of this text. (Happily, colleagues in the profession who have adopted the book for their courses seem to agree.)

Clear, concise, and uncluttered approach. We try to be short and to the point, to cover what is essential but leave out what is not.

Integration of calculus in the main body of the text. When our first edition was published, other intermediate level microeconomics textbooks typically had no calculus, or stuck the calculus in footnotes. Most students have now taken at least one calculus course, and we take advantage of their preparation by using the math where it should be used. Much of microeconomics is about maximizing or minimizing something, and calculus provides the tools for solving maximization and minimization problems.

End-of-chapter solved problems. At the end of each chapter except the first, we include at least one long problem with a thoroughly explained solution.

End-of-chapter student exercises. At the end of each chapter except the first, we now include an improved and expanded section consisting of ten exercises for students.

Narrative cohesion. We think our book is well organized and its parts are well connected; it covers all the basics in intermediate level economics but it leaves out topics that are tangential.

Affordably priced. The price is right!

What Is New in this Second Edition

To produce this second edition, we have modified our first edition in the following ways:

In response to the most popular suggestion from reviewers, we have increased the number of exercises at the end of each chapter from six to ten. The addition of 76 new exercises is a significant improvement for students and instructors. We continue to have two files which provide solutions to all the exercises. "Solutions for Instructors" has detailed solutions with explanations. It will be sent to instructors who contact us directly. (Also, Cambridge University Press will make it available to adopters.) The second solutions file is "Solutions for Students"– this file only has bottom line solutions for each exercise. It omits intermediate steps and explanations. This file is available on Roberto Serrano's website. It is useful for students who want to work out exercises, assigned or unassigned, and check to see if their answers are right.

Several appendices to chapters have been added or extended in order to cover material that is more mathematical than the rest of the text. Topics include coverage of optimization with inequality constraints, the Slutsky equation, Kuhn–Tucker conditions, the competitive limit of large oligopolies, and a simple proof of the first theorem of welfare economics.

We have expanded our coverage of game theory. The expansion includes a longer section devoted to experimental economics and to notions of behavioral economics.

Figures have been redrawn where appropriate, and color has been added to all the figures. The figures have also been enlarged.

Some of the material in later chapters (that is, in the market failures part of the book) has been rewritten to improve clarity. All errata and typos that we knew about have been corrected.

Course Prerequisites and their Rationale

A student in a course that uses this book should already have taken an introductory economics class, exposing him or her to the main ideas of the two parts of economic theory, microeconomics and macroeconomics. The concise style of this text assumes familiarity with basic economic jargon.

In addition, the student should also have taken a calculus course. Calculus is basic to microeconomics, much of which is about maximizing something (for instance, utility, or output, or profit), or about minimizing something else (for instance, costs). Calculus is the area of mathematics most connected to maximization and minimization problems, and using it makes microeconomics straightforward, transparent, and precise.

The Contents of the Book

Microeconomics begins with the study of how economic agents (*consumers* and *firms*) in the economy's private sector make their decisions. We start this course with a brief introduction in Chapter 1. Then we turn to the main events: Part I of our course (Chapters 2 through 7) is about the theory of the consumer, and Part II (Chapters 8 through 10) is about the theory of the producer; that is, the firm. Part I provides a foundation for the *demand curves* seen in a principles course, and Part II provides a foundation for the *supply curves*.

Economic decisions are mostly made in the private sector, but governments also make many important economic decisions. We touch on these throughout the course, particularly when we discuss taxes, monopolies, externalities, and public goods. Our main focus, though, is the private sector, since in *market economies* the private sector is the main protagonist.

Next, Part III (Chapters 11 through 13) combines theories of the consumer and the producer into theories of *markets*. Here, our focus is on different types of market structure, depending on the market power of the firms producing the goods. Market power is related to the number of firms in the market. We begin, in Chapter 11, with the case of *perfect competition*, where each firm is powerless to affect the price of the good it sells; this is usually a consequence of there being many firms selling the same good. In Chapter 12, we analyze the polar opposite case, called *monopoly*, where only one firm sells the good. We also consider intermediate cases between these extremes: in Chapter 13, we analyze *duopoly*, where two firms compete in the market. One important point that we emphasize is the strong connection between competition and the welfare of a society. This is the connection that was first described in 1776 by Adam Smith in *The Wealth of Nations*. Smith famously argued that the *invisible hand* of market competition leads self-interested buyers and sellers to an outcome that is beneficial to society as a whole.

Our analysis in Part III is sometimes called *partial equilibrium* analysis, because it focuses on *one market in isolation*. In Part IV (Chapters 15 and 16), we develop models that look at *all markets simultaneously;* this is called *general equilibrium* analysis. The general equilibrium approach is useful to understand the implications of interactions among the different markets. These interactions are, of course, essential in the economy. A main theme in Part IV is the generalization of the invisible hand idea that market competition leads to the social good. We shall see that under certain conditions there are strong connections between competition in markets and the efficient allocation of resources. These connections, or *fundamental theorems of welfare economics* as economists call them, are important both to people interested in economic ideas, and to people simply interested in what kind of economic world they want to inhabit.

Finally, Part V (Chapters 17, 18, and 20) focus on the circumstances under which even competitive markets, left by themselves, fail to allocate resources efficiently. This is a very important area of study because these *market failures* are common, and, when they occur, governments, policy makers, and informed citizens must consider what *policy interventions* would best improve the performance of the unregulated market.

Two Special Chapters and Suggestions about What to Teach

Our course includes two chapters that are not really part of the building blocks flow from consumer theory through market failure. Chapter 14 is a basic introduction to game theory. The use of game theory is so prevalent in economics today that we think it is important to provide a treatment here, even if the theories of the consumer, of the firm, of competitive markets, and of market failure could get along without it. A similar comment applies to Chapter 19 on uncertainty and expected utility. While most of this course describes decision problems and markets under complete information, the presence of uncertainty is crucial in much of economic life, and much modern microeconomic analysis centers around it. Some instructors may choose to ignore these chapters in their intermediate microeconomics courses, but others may want to cover them. In order to free up some time to do that, we offer some suggestions:

We include two alternative treatments of the theory of the firm in this book. The first is contained in Chapter 8, the single-input model of the firm, which abstracts from the cost-minimization problem. The second is contained in Chapters 9 and 10, the multiple-input model of the firm, which includes the cost-minimization problem. Chapter 8 can be viewed as a quick route to the supply curve. An instructor looking for time to teach some of the newer topics covered in Chapter 14 or Chapter 19 might cover Chapter 8 and omit Chapters 9 and/or 10. Also, our chapters on market failure generally contain basic theory in their first sections and applications in later sections. Instructors might choose to include or omit some of the theory or some of the applications, depending on time.

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We are grateful to thousands of intermediate microeconomics students at Brown University who helped us develop and present this material. Martin Besfamille, Dror Brenner, Pedro Dal Bó, EeCheng Ong, Amy Serrano, and Rajiv Vohra were kind enough to try out different preliminary versions of our manuscript in their sections of the course at Brown. Many other colleagues, at Brown and at other universities, have provided us with useful feedback. We thank all of them and their students for all the helpful comments that they provided. Amy also provided many comments that improved the exposition throughout, and her input was especially important in Chapter 7. EeCheng provided superb assistance with the exercises and their solutions. Elise Fishelson gave us detailed comments on each chapter at a preliminary stage; Omer Ozak and Xu Zhang helped with some graphs and TeX issues; and Rachel Bell helped with some graphs. Michelle Turcotte, of Brown University Graphic Services, colorized the graphs for the second edition, and Paula Feldman (Allan's daughter) helped us improve its figure captions. Barbara Feldman (Allan's wife) was patient and encouraging. We thank the anonymous reviewers used by Cambridge University Press for their helpful feedback, Scott Parris and Chris Harrison (our first edition editors at Cambridge University Press), and Karen Maloney, Stephen Acerra, and Lisa Pinto (editors of the second edition), for their encouragement and support of the project.

1 Introduction

Economists study the *economic problem*. The nature of the economic problem, however, has changed over time. For the *classical school* of economists (including Adam Smith (1723–1790), David Ricardo (1772–1823), Karl Marx (1818–1883), and John Stuart Mill (1806–1873)), the economic problem was to discover the laws which governed the production of goods and the distribution of goods among the different social classes: land owners, capitalists, workers. These laws were thought to be like the natural laws or physical laws, similar to Newton's law of gravitational attraction. Forces of history, and phenomena such as the industrial revolution, produce "universal constants," which govern the production of goods and the distribution of wealth.

Towards the end of the nineteenth century, however, there was a major shift in the orientation of economics, brought about by the *neoclassical school* of economists. This group includes William Stanley Jevons (1835–1882), Leon Walras (1834–1910), Francis Ysidro Edgeworth (1845–1926), Vilfredo Pareto (1848–1923), and Alfred Marshall (1842–1924). The neoclassical revolution was a shift in the emphasis of the discipline, away from a search for natural laws of production and distribution, and toward the analysis of decision making by individuals and firms.

In this book, we will describe *modern microeconomics*, which mostly follows the neoclassical path. For us, and for the majority of contemporary microeconomists, the economic problem is the problem of the "economic agent." He lives in a world of scarcity. Economists focus on the fact that resources are limited or *constrained*. These constraints apply to men, women, households, firms, governments, and even humanity. On the other hand, our wants and needs are unlimited. We want more and better material things, for ourselves, our families, our children, our friends. Even if we are not personally greedy, we want better education for our children, better culture, better health for people in our country, and longer lives for everyone. Economics is about how decision makers choose among all the things that they want, given that they cannot have everything. The economic world is the world of limited resources and unlimited needs, and the economic problem is how to best meet those needs given those limited resources.

The key assumption in microeconomics, which could be taken as our slogan, our credo, is this: economic agents are rational. This means that they will choose the best alternatives, given what's available, given the constraints. Of course, we know that (to paraphrase Abraham Lincoln) some of the people behave irrationally all the time, and all of the people behave irrationally some of the time. But we will take rationality as our basic assumption, especially when important goods and services, and money, are at stake.

Economics applies the scientific method to the investigation and understanding of the economic problem. As with the natural sciences, like biology, chemistry or physics, economics has theory, and it has empirical analysis. Modern economic theory usually involves the construction of abstract, often mathematical models, which are intended to help us understand some aspect of the economic world. A useful model makes simplifying assumptions about the world. (A completely realistic economic model would usually be too complicated to be useful.) The assumptions incorporated in a useful model should be plausible or reasonable, and not absurd on their face. For instance, it is reasonable to assume that firms want to maximize profits, even though some firms may not be concerned with profits in some circumstances. It is reasonable to assume that a typical consumer wants to eat some food, wear some clothing, and live in a house or an apartment. It would be unreasonable to assume that a typical consumer wants to spend all her income on housing, and eat no food. Once a model has assumptions, the economic analyst applies deductive reasoning and logic to it, in order to derive conclusions. This is where the use of mathematics is important.

Correct logical and mathematical arguments clarify the structure of a model and help us avoid mistaken conclusions. The aim is to have a model which sheds some light on the economic world. For example, we might have a logical result like this: if we assume A, B, and C, then D holds, where D = "when the price of ice cream rises, the consumer will eat less of it." If A, B, and C are very reasonable assumptions, then we feel confident that D will be true. On the other hand, if we do some empirical work and see that D is in fact false, then we are led to the conclusion that either A, B, or C must also be false. Either way, the logical proposition "A, B, and C together imply D" gives us insight into the way the economic world works.

Economics is divided between *microeconomics* and *macroeconomics*. Macroeconomics studies the economy from above, as if seen from space. It studies aggregate magnitudes, the big things like booms and busts, gross domestic product, rates of employment and unemployment, money supply, and inflation. In contrast, microeconomics takes the close-up approach to understand the workings of the economy. It begins by looking at how individuals, households, and firms make decisions, and how those decisions interact in markets. The individual decisions result in market variables, quantities demanded by buyers and supplied by sellers, and market prices.

When people, households, firms, and other economic agents make economic decisions, they alter the allocation of resources. For example, if many people suddenly want to buy some goods in large quantities, they may drive up the prices of those goods, they may drive up employment and wages of the workers who make those goods, they may drive up the profits of the firms that sell them, and they may drive down the wages of people making other goods and the profits of firms that supply the competing goods. When a microeconomist analyzes a market in isolation, assuming that no effects are taking place in other markets, he is doing what is called *partial equilibrium analysis*. Partial equilibrium analysis focuses on the market for one good, and assumes prices and quantities of other goods are fixed. *General equilibrium analysis* assumes that what goes on in one market does affect prices and quantities in other markets. All markets in the economy interact, and all prices and quantities are determined more-or-less simultaneously. Obviously, general equilibrium analysis is more difficult and complex than partial equilibrium analysis. Both types of analysis, however, are part of microeconomics, and we will do both in this book. Doing general equilibrium analysis allows the people who do microeconomics to connect to the aggregates of the economy, to see the "big picture." This creates a link between microeconomics and macroeconomics.

We will now move on to begin our study, and we do so by considering how individual households make consumption decisions. This is called the *theory of the consumer.*

Part I

Theory of the Consumer

2 Preferences and Utility

2.1 Introduction

Life is like a shopping center. The consumer enters it and sees lots of goods, in various quantities, that she might buy. A *consumption bundle*, or a *bundle* for short, is a combination of quantities of the various goods (and services) that are available. For instance, a consumption bundle might be 2 apples, 1 banana, 0 cookies, and 5 diet sodas. We would write this as $(2, 1, 0, 5)$. Of course, the consumer prefers some consumption bundles to others; that is, she has tastes or *preferences* regarding those bundles.

In this chapter, we will discuss the *economic theory of preferences* in some detail. We will make various assumptions about a consumer's feelings about alternative consumption bundles. We will assume that when given a choice between two alternative bundles, the consumer can make a comparison. (This assumption is called *completeness*.) We will assume that when looking at three alternatives, the consumer is rational in the sense that, if she says she likes the first better than the second and the second better than the third, she will also say that she likes the first better than the third. (This is part of what is called *transitivity*.) We will examine other basic assumptions that economists usually make about a consumer's preferences: one says that the consumer prefers more of each good to less (called *monotonicity*), and another says that a consumer's *indifference curves* (or sets of equally desirable consumption bundles) have a certain plausible curvature (called *convexity*). We will describe and discuss the consumer's rate of tradeoff of one good against another (called her *marginal rate of substitution*).

After discussing the consumer's preferences, we will turn to her *utility function*. A utility function is a numerical representation of how a consumer feels about alternative consumption bundles: if she likes the first bundle better than the second, then the utility function assigns a higher number to the first than to the second, and if she likes them equally well, then the utility function assigns the same number to both. We will analyze utility functions and describe *marginal utility*, which, loosely speaking, is the extra utility provided by one additional unit of a good. We will derive the relationship between the marginal utilities of two goods and the marginal rate of substitution of one of the goods for the other. We will provide various algebraic examples of utility functions, and, in the appendix, we will briefly review the calculus of derivatives and partial derivatives.

In this chapter and others to follow, we will often assume there are only two goods available, with x_1 and x_2 representing quantities of goods 1 and 2, respectively. Why only two goods? For two reasons: first, for simplicity (two goods gives a much simpler model than three goods or five thousand, often with no loss of generality); and, second, because we are often interested in one particular good, and we can easily focus on that good and call the second good "all other goods," or "everything else," or "other stuff." When there are two goods, any consumption bundle can easily be shown in a standard two-dimensional graph, with the quantity of the first good on the horizontal axis and the quantity of the second good on the vertical axis. All the figures in this chapter are drawn this way.

In this chapter, we will focus on the consumer's preferences about bundles of goods, or how she feels about various things that she might consume. But in the shopping center of life, some bundles are *feasible* or *affordable* for the consumer; these are the ones her budget will allow. Other bundles are *non-feasible* or *unaffordable*; these are the ones her budget won't allow. We will focus on the consumer's budget in Chapter 3.

2.2 The Consumer's Preference Relation

The consumer has preferences over consumption bundles. We represent consumption bundles with symbols like X and Y. If there are two goods, X is a vector (x_1, x_2), where x_1 is the quantity of good 1 and x_2 is the quantity of good 2. The consumer can compare any pair of bundles and decide which one is better, or decide they are equally good. If she decides one is better than the other, we represent her feelings with what is called a *preference relation*; we use the symbol $\succ$ to represent the preference relation. That is, $X \succ Y$ means the consumer prefers bundle X over bundle Y. Presented with the choice between X and Y, she would choose X. We assume that if $X \succ Y$, then $Y \succ X$ cannot be true; if the consumer likes X better than Y, then she had better not like Y better than X! Obviously, a consumer's preferences might change over time, and might change as she learns more about the consumption bundles. (The $\succ$ relation is sometimes called the *strict preference relation* rather than the *preference relation*, because $X \succ Y$ means the consumer definitely, unambiguously, prefers X to Y, or *strictly prefers* X to Y.)

If the consumer likes X and Y equally well, we say she is *indifferent* between them. We write $X \sim Y$ in this case, and $\sim$ is called the *indifference relation.* Sometimes, we will say that X and Y are indifferent bundles for the consumer. In this case, if presented with the choice between them, the consumer might choose X, might choose Y, might flip a coin, or might even ask us to choose for her. We assume that if $X \sim Y$, then $Y \sim X$ must be true; if the consumer likes X exactly as well as Y, then she had better like Y exactly as well as X!

The reader might notice that the symbols for preference and for indifference are a little like the mathematical symbols $>$ and $=$, for *greater than* and *equal to*, respectively. This is no accident. And, just as there is a mathematical relation that combines these two, $\geq$ for *greater than or equal to*, there is also a preference relation symbol $\succeq$, for *preferred or indifferent to*. That is, we write $X \succeq Y$ to represent the consumer's either preferring X to Y, or being indifferent between the two. (The $\succeq$ relation is sometimes called the *weak preference relation*.)

Assumptions on preferences. At this point, we make some basic assumptions about the consumer's preference and indifference relations. Our intention is to model the behavior of what we would consider a rational consumer. In this section, we will assume the two goods are desirable to the consumer; we will touch on other possibilities (such as neutral goods or bads) in the Exercises.

Assumption 1. *Completeness*. For all consumption bundles X and Y, either $X \succ Y$, or $Y \succ X$, or $X \sim Y$. That is, the consumer must like one better than the other, or like them equally well. This may seem obvious, but sometimes it's not. For example, what if the consumer must choose what's behind the screen on the left, or the screen on the right, and she has no idea what might be hidden behind the screens? That is, what if she doesn't know what X and Y are? We force her to make a choice, or at least to say she is indifferent. Having a *complete ordering* of bundles is very important for our analysis throughout this book. (In Chapters 19 and 20, we will analyze consumer behavior under uncertainty, or incomplete information.)

Assumption 2. *Transitivity*. This assumption has four parts:

- First, transitivity of preference: if $X \succ Y$ and $Y \succ Z$, then $X \succ Z$.
- Second, transitivity of indifference: if $X \sim Y$ and $Y \sim Z$, then $X \sim Z$.
- Third, if $X \succ Y$ and $Y \sim Z$, then $X \succ Z$.
- Fourth and finally, if $X \sim Y$ and $Y \succ Z$, then $X \succ Z$.

The transitivity of preference assumption is meant to rule out irrational preference cycles. You would probably think your friend needs psychiatric help if she says she prefers Econ. 1 (the basic economics course) to Soc. 1 (the basic sociology course), and she prefers Soc. 1 to Psych. 1 (the basic psychology course), *and* she prefers Psych. 1 to Econ. 1. Cycles in preferences seem irrational. However, do not be too dogmatic about this assumption; there are interesting exceptions in the real world. We will provide one later on in the exercises.

The transitivity of indifference assumption (that is, if $X \sim Y$ and $Y \sim Z$, then $X \sim Z$) makes *indifference curves* possible.

An *indifference curve* is a set of consumption bundles (or, when there are two goods, points in a two-dimensional graph), which the consumer thinks are all equally good; she is indifferent among them. We will use indifference curves frequently throughout this book, starting in Figure 2.1 below. The figure shows two consumption bundles, X and Y, and an indifference curve. The two bundles are on the same indifference curve, and therefore the consumer likes them equally well.

Assumption 3. *Monotonicity*. We normally assume that goods are *desirable*, which means the consumer prefers consuming more of a good to consuming less. That is, suppose X and Y are two bundles of goods such that (1) X has more of one good

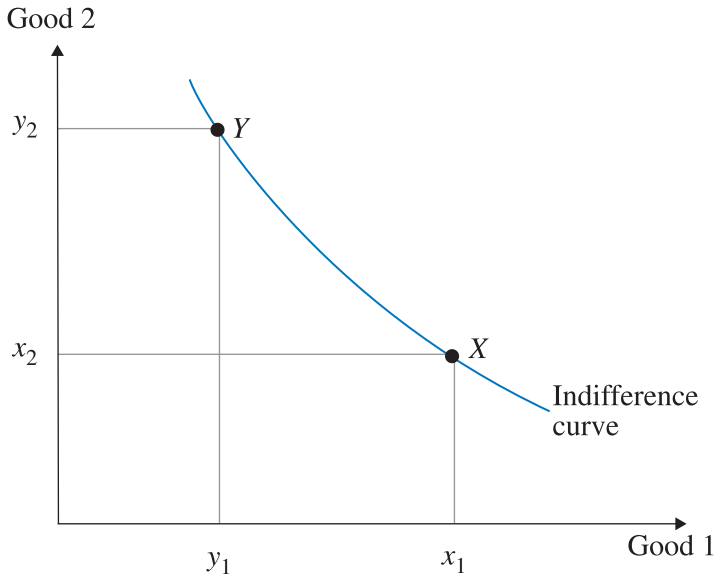

Figure 2.1 At bundle X, the consumer is consuming x_1 units of good 1 and x_2 units of good 2. Similarly, at bundle Y, she is consuming y_1 units of good 1 and y_2 units of good 2. Since X and Y are on one indifference curve, the consumer is indifferent between them.

(or both) than Y does and (2) X has at least as much of both goods as Y has. Then $X \succ Y$. Of course, there are times when this assumption is inappropriate. For instance, suppose a bundle of goods is a quantity of cake and a quantity of ice cream, which you will eat *this evening*. After 3 slices of cake and 6 scoops of ice cream, more cake and more ice cream may not be welcome. But if the goods are more generally defined (for example, education, housing), monotonicity is a very reasonable assumption.

Some important consequences of monotonicity are the following: indifference curves representing preferences over two desirable goods cannot be thick or upward sloping. Nor can they be vertical or horizontal. This should be apparent from Figure 2.2 below, which shows an upward-sloping indifference curve, and a thick indifference curve. On any indifference curve, the consumer is indifferent between any pair of consumption bundles. A brief examination of the figure should convince the reader that the monotonicity assumption rules out both types of indifference curves shown, and similar arguments rule out vertical and horizontal indifference curves.

In Figure 2.3 below, we show a downward-sloping thin indifference curve, which is what the monotonicity assumption requires. The figure also shows the set of bundles which by the monotonicity assumption must be preferred to all the bundles on the indifference curve (the *more preferred set*), and the set of bundles which by the monotonicity assumption must be liked less than all the bundles on the indifference curve (the *less preferred set*).

Another implication of the assumptions of transitivity (of indifference) and monotonicity is that two distinct indifference curves cannot cross. This is shown in Figure 2.4 below.

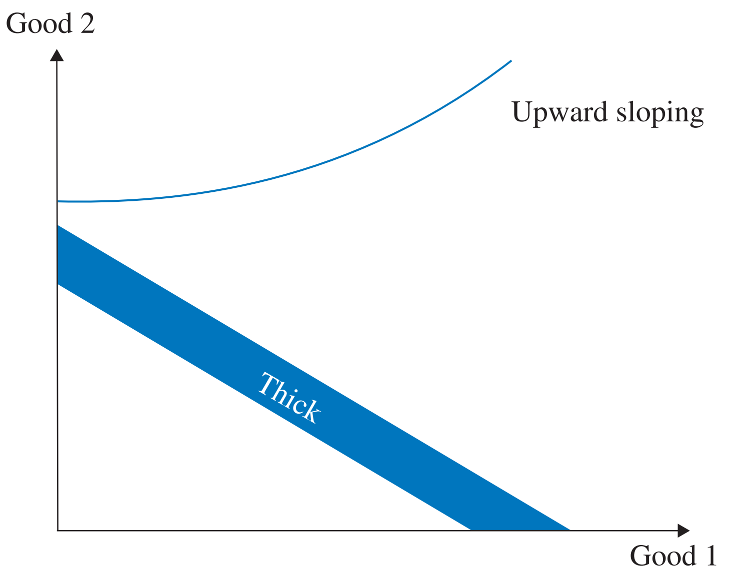

Figure 2.2 Each of the two indifference curves shown is a set of equally desirable consumption bundles – for example, for any pair of bundles X and Y on the upward-sloping curve, $X \sim Y$. Can you see why the monotonicity assumption makes the upward-sloping indifference curve impossible? How about the thick indifference curve?

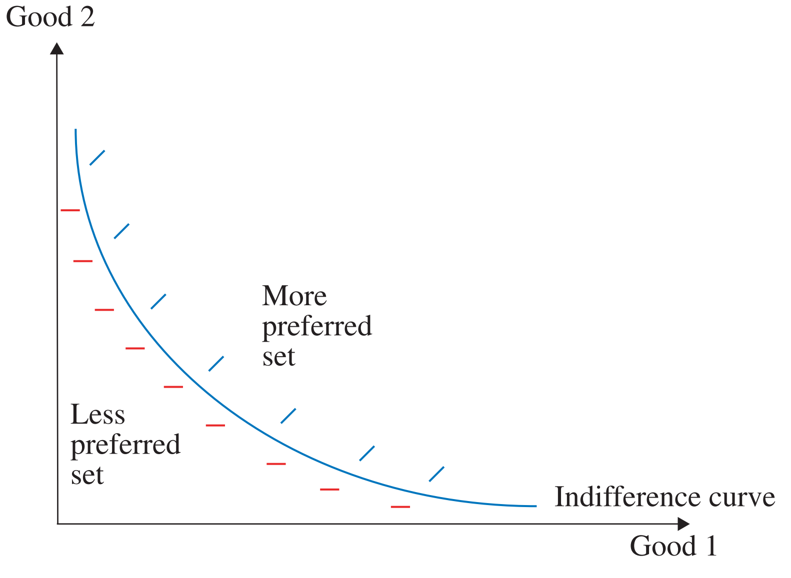

Figure 2.3 The only graph compatible with monotonic preferences is a downward-sloping thin indifference curve. The consumer prefers points above this curve.

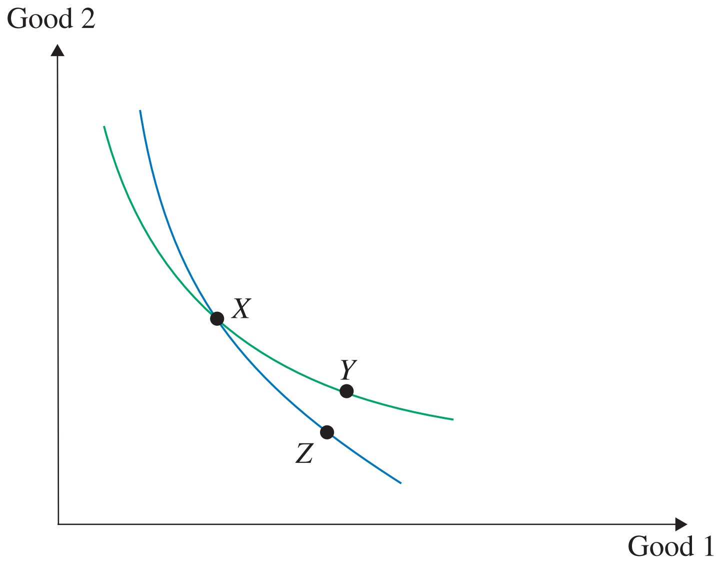

Figure 2.4 Two distinct indifference curves cannot cross. Here is why. Suppose the curves did cross at the point X. Because Y and X are on the same (green) indifference curve, $Y \sim X$. Because X and Z are on the same (blue) indifference curve, $X \sim Z$. Then by transitivity of indifference, $Y \sim Z$. But by monotonicity, $Y \succ Z$. Therefore, having the indifference curves cross leads to a contradiction.

Assumption 4. *Convexity for indifference curves*. This assumption means that averages of consumption bundles are preferred to extremes. Consider two distinct points on one indifference curve. The (arithmetic) average of the two points would be found by connecting them with a straight-line segment, and then taking the midpoint of that segment. This is the standard average, which gives equal weight to the two extreme points. A *weighted average* gives possibly unequal weights to the two points; geometrically, a weighted average would be any point on the line segment connecting the two original points, not just the midpoint. The assumption of convexity for indifference curves means this: for any two distinct points on the same indifference curve, the line segment connecting them (excepting its end points) lies above the indifference curve. In other words, if we take a weighted average of two distinct points, between which the consumer is indifferent, she prefers the weighted average to the original points. We show this in Figure 2.5 below.

We call preferences *well behaved* when indifference curves are downward sloping and convex.

In reality, of course, indifference curves are sometimes concave. There are many examples we can think of in which a consumer might like two goods, but not in combination. You may like sushi and chocolate ice cream, but not together in the same dish; you may like classical music and hip-hop, but not in the same evening; you may like pink clothing and orange clothing, but not in the same outfit. Again, if the goods are defined generally enough, like classical music consumption per year, hip-hop consumption per year, pink and orange clothing worn this year, the assumption of

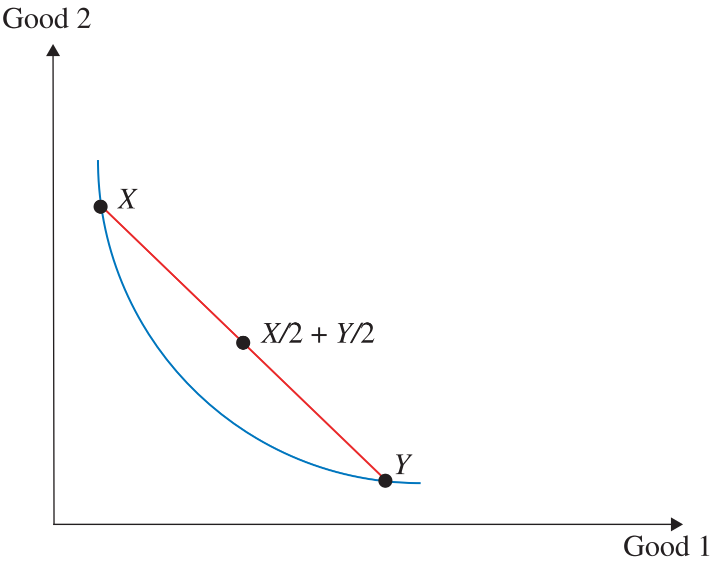

Figure 2.5 Convexity of preferences means that indifference curves are convex, as in the figure, rather than concave. This means that the consumer prefers averaged bundles over extreme bundles. For example, the bundle made up of 1/2 times X plus 1/2 times Y; that is, $X/2 + Y/2$ is preferred to either X or Y. This is what we normally assume to be the case.

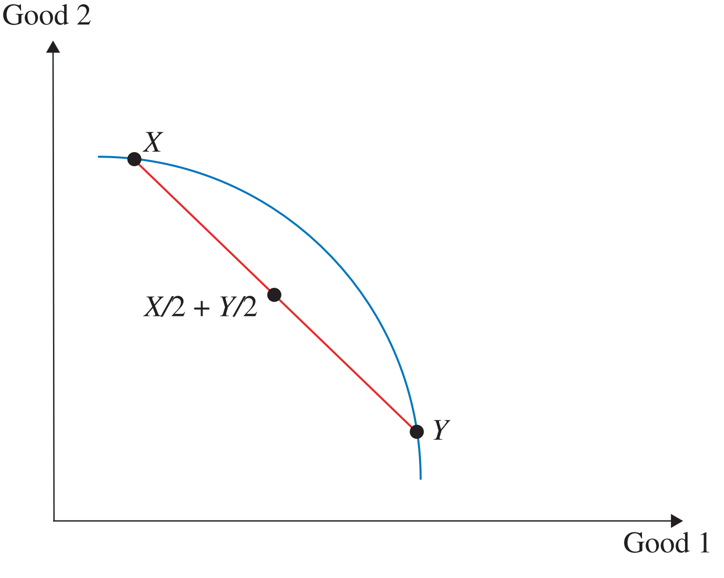

Figure 2.6 A concave indifference curve. This consumer prefers the extreme points X and Y to the average $X/2 + Y/2$.

convexity of indifference becomes very reasonable. We show a concave indifference curve in Figure 2.6 above.

2.3 The Marginal Rate of Substitution

The *marginal rate of substitution* is an important and useful concept because it describes the consumer's willingness to trade consumption of one good for consumption of the other. Consider this thought experiment. The consumer gives up a unit of good 1 in exchange for getting some amount of good 2. How much good 2 does she need to get in order to end up on the same indifference curve? This is the quantity of good 2 that she needs to replace one unit of good 1.

Or, consider a slightly different thought experiment. The consumer gets a unit of good 1 in exchange for giving up some amount of good 2. How much good 2 can she give up and end up on the same indifference curve? This is the quantity of good 2 that she is willing to give up in exchange for a unit of good 1.

The answer to either of these questions is a measure of *her valuation of a unit of good 1, in terms of units of good 2*. This is the intuitive idea of the marginal rate of substitution of good 2 for good 1. It is her rate of tradeoff between the two goods, the rate at which she can substitute good 2 for good 1 and remain as well off as she was before the substitution.

Now let Δx_1 represent a change in her consumption of good 1, and Δx_2 represent a change in her consumption of good 2, and suppose the two changes move her from a point on an indifference curve to another point *on the same indifference curve*. Remember that for well-behaved preferences, indifference curves are downward sloping, and therefore one of the Δs will be positive and the other negative. If $\Delta x_i > 0$, she's getting some good i; if $\Delta x_i < 0$, she's giving up some good i. In the first thought experiment above, we let $\Delta x_1 = -1$; in the second, we let $\Delta x_1 = +1$. In both, we were really interested in the magnitude of the resulting Δx_2. This is the amount of good 2 needed to replace a unit of good 1, or the amount of good 2 that she would be willing to give up to get another unit of good 1.

At this point, rather than thinking about the consumer swapping a unit of good 1 in exchange for some amount of good 2, we consider the ratio $\Delta x_2/\Delta x_1$. This ratio is the rate at which the consumer has to get good 2 in exchange for giving up good 1 (if $\Delta x_1 < 0$ and $\Delta x_2 > 0$), or the rate at which she has to give up good 2 in exchange for getting good 1 (if $\Delta x_1 > 0$ and $\Delta x_2 < 0$). Also, we assume that the Δs are very small, or infinitesimal. More formally, we take the limit as Δx_1 and Δx_2 approach 0.

Because we are assuming that Δx_1 and Δx_2 are small moves from a point on an indifference curve that leave the consumer on the same indifference curve, *the ratio* $\Delta x_2/\Delta x_1$ *represents the slope of that indifference curve at that point*. Since the indifference curves are downward sloping,

$$\Delta x_2/\Delta x_1 = \text{Indifference Curve Slope} < 0.$$

The definition of the *marginal rate of substitution of good 2 for good 1*, which we will write as MRS_{x_1,x_2}, or just *MRS* for short, is

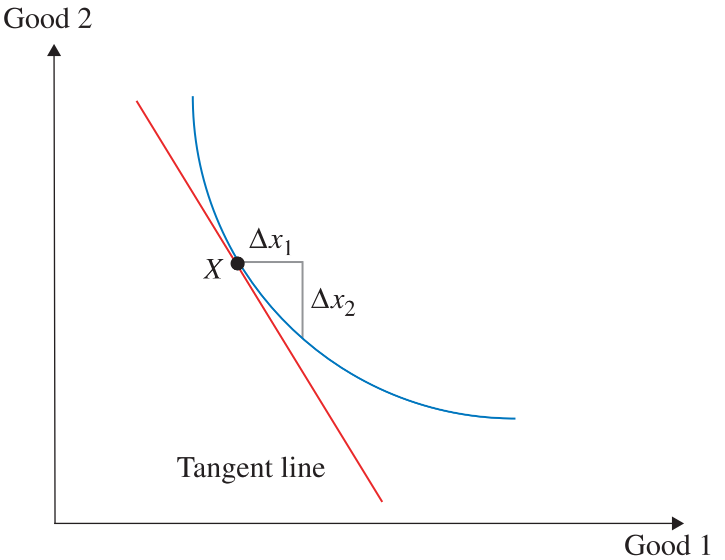

Figure 2.7 Intuitively, the marginal rate of substitution is an answer to one of these questions: "If I take away Δx_1 units of good 1, how much good 2 do I need to give you for you to remain indifferent?" or "If I give you Δx_1 of units of good 1, how much good 2 can I take away from you and have you remain indifferent?" The second question is illustrated here.

$$MRS_{x_1,x_2} = MRS = -\Delta x_2/\Delta x_1 = -\text{Indifference Curve Slope.}$$

More formally,

$$MRS = \lim_{\Delta x_1, \Delta x_2 \to 0} -\Delta x_2/\Delta x_1 = -\text{Indifference Curve Slope.}$$

In Figure 2.7, we show a downward-sloping indifference curve, and a tangent line at a point X on the indifference curve. We show two increments from X, Δx_1 and Δx_2, that get the consumer back to the same indifference curve. Note that $\Delta x_1 > 0$ and $\Delta x_2 < 0$ in the figure. If the consumer gets Δx_1 units of good 1, she is willing to give up $-\Delta x_2$ units of good 2. Her marginal rate of substitution is the limit of $-\Delta x_2/\Delta x_1$, as Δx_1 and Δx_2 approach 0. That is, her marginal rate of substitution is -1 times the slope of the indifference curve at X, or -1 times the slope of the tangent line at X.

For well-behaved preferences, the *MRS* decreases as you move down and to the right along an indifference curve. This makes good sense. It means that if a consumer consumes more and more of a good, while staying on the same indifference curve, she values an additional unit of that good less and less. To convince yourself that this is plausible, consider the following story.

A well-off woman (Ms. Well-Off) is lost in the middle of a desert. She is *so* thirsty, almost dying of thirst. She has no water (good 1), but she does have \$100 (good 2) in her pocket. A profit-seeking local trader (Mr. Rip-Off), carrying water, offers her a drink, and asks her: "How much are you willing to pay me for your first glass of

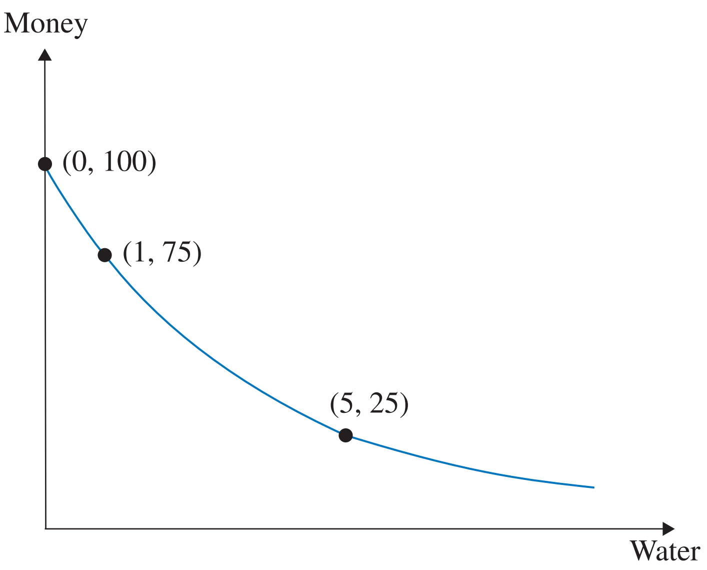

Figure 2.8 The *MRS* is decreasing because the consumer gets satiated with water as she consumes more of it. She is willing to pay less and less for the incremental drink.

water?" (That is, "What is your *MRS* of money for water when you have no water, but \$100?") Honest to a fault, she answers \$25. Mr. Rip-Off immediately proposes this trade, and the first glass of water is sold for \$25. At this point, Mr. Rip-Off asks again: "You are probably still thirsty, aren't you? How much are you willing to pay for a second glass of water?" (That is, "What is your *MRS* of money for water when you already have had a glass of water, and you have \$75 left?") She now answers: "Yes, I am still thirsty. I would pay you \$10 for a second glass." They make this trade also. Her valuation of the second glass of water, her *MRS* of money for water, has dropped by more than half. This process continues for a while. By the time Ms. Well-Off has had nine or ten glasses of water, her *MRS* has dropped to 0, because at this point her need for water is much less pressing than her need for a bathroom. See Figure 2.8.

2.4 The Consumer's Utility Function

Mathematically, it is much easier to work with functions than with relations, such as the preference relation and the indifference relation. Our goal now is to construct a function that will represent the preferences of a consumer. Such a function is called a *utility function*.

Imagine that we assign a number to each bundle. For example, we assign the number $u(X) = u(x_1, x_2) = 5$, to the bundle $X = (x_1, x_2)$; we assign the number $u(Y) = u(y_1, y_2) = 4$, to $Y = (y_1, y_2)$; and so on.

We say that such an assignment of numbers to bundles is a *consumer's utility function* if:

- First, $u(X) > u(Y)$ whenever $X \succ Y$.
- And second, $u(X) = u(Y)$ whenever $X \sim Y$.

Note how this assignment of numbers to bundles is a faithful translation of the consumer's preferences. It gives a higher utility number to the preferred bundle, and it gives the same number to two bundles that the consumer likes equally well. This is the sense in which this function accurately represents the preferences of the consumer.

Our consumer's utility function is said to be an "ordinal" utility function rather than a "cardinal" utility function.

An *ordinal* statement only gives information about relative magnitudes; for instance, "I like Tiffany more than Jennifer." A *cardinal* statement provides information about magnitudes that can be added, subtracted, and so on. For instance, "Billy weighs 160 lbs and Johnny weighs 120 lbs." We can conclude from the latter statement that Billy weighs 40 lbs more than Johnny, that the ratio of their weights is exactly 4/3, and that the sum of their weights is 280 lbs. Is utility an ordinal or a cardinal concept? The utilitarians, led by the English philosopher Jeremy Bentham (1748–1832), believed that utility is a cardinal magnitude, perhaps as measurable as length, weight, and so on. For them, statements like these would make sense: "I get three times as much utility from my consumption bundle as you get from your consumption bundle" or "I like a vacation cruise in the West Indies twice as much as you do." Today, for the most part, we treat utility simply as an ordinal magnitude. All we care about is whether an individual's utility number from one consumption bundle is larger than, equal to, or smaller than the same individual's utility number from another bundle. For one individual, differences or ratios of utility numbers from different bundles generally do not matter, and comparisons of utilities across different individuals have no meaning.

Under the ordinal interpretation of utility numbers, if we start with any utility function representing my preferences, and we transform it by adding a constant, it still represents my preferences perfectly well. Or, if we multiply it by a positive number, it still works perfectly well. Or, assuming all my utility numbers are positive, if we square all of them, or raise them all to a positive power, we are left with a modified utility function that still represents my preferences perfectly well. In short, if we start with a utility function representing my preferences, and modify it with what's called an *order-preserving transformation*, then it still represents my preferences. All this is summed up in the following statement:

If $u(X) = u(x_1, x_2)$ is a utility function that represents the preferences of a consumer, and f is any order-preserving transformation of u, the transformed function $f(u(X)) = f(u(x_1, x_2))$ is another utility function that also represents those preferences.

What is the connection between indifference curves and utility functions? The answer is that we use indifference curves to represent constant levels of utility. Remember that we are assuming the consumer's utility level depends on her consumption of two goods, measured as variables x_1 and x_2. We need one axis to represent the amount of x_1, and a second axis to represent the amount of x_2. If we were to show utility in the same picture as quantities of the two goods, we would need a third axis to represent the utility level u that corresponds to the consumption bundle (x_1, x_2). A utility function in such a three-dimensional picture looks like a

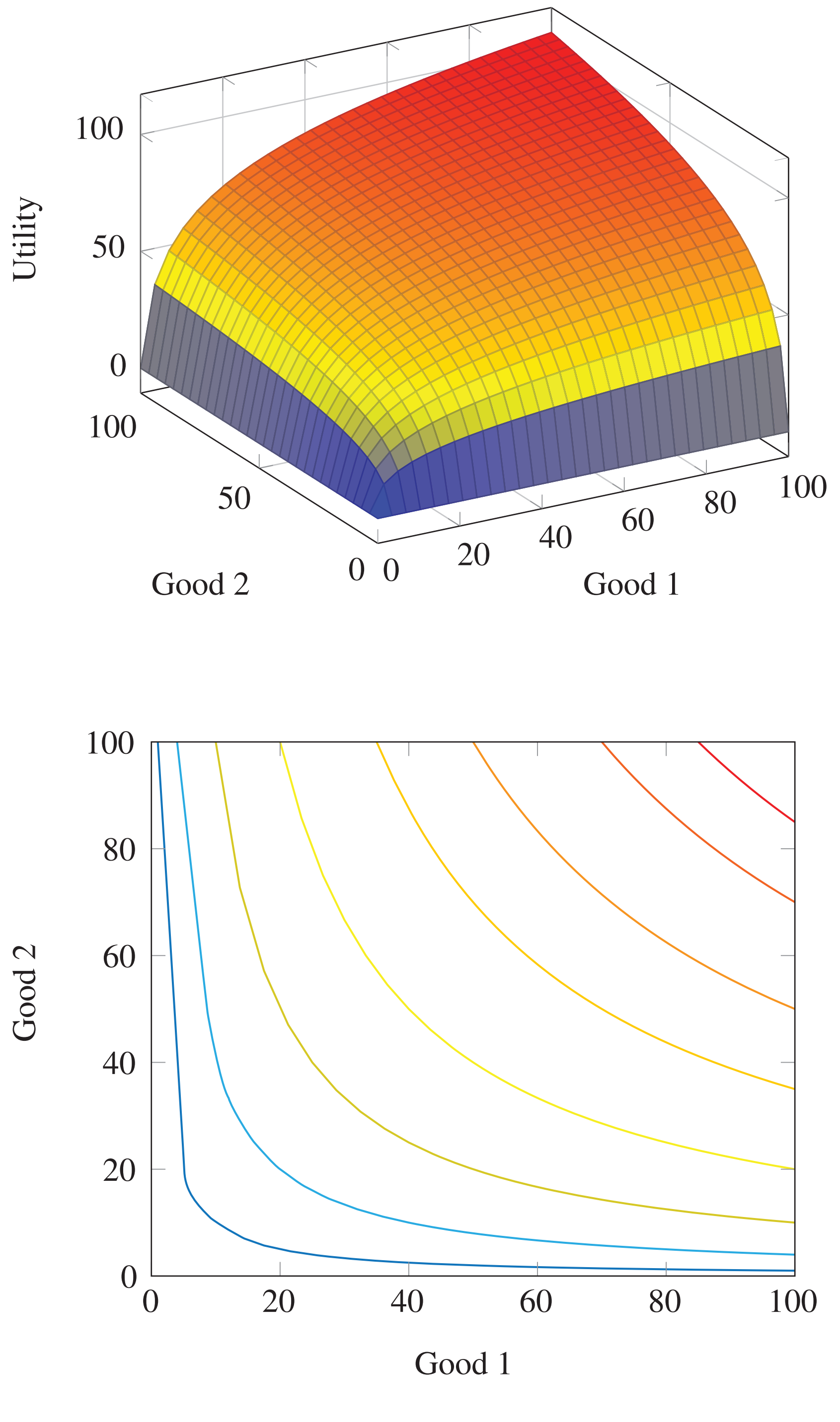

Figure 2.9 The indifference curves (bottom) are the level curves of the utility function (top).

hillside. But three-dimensional pictures are hard to draw. It is much easier to draw two-dimensional graphs with level curves.

A *level curve* for a function is a set of points in the function's domain, over which the function takes a constant value. If you've hiked or climbed mountains with the help of a topographical map, you have used a picture with level curves; an elevation contour on the map is a level curve. Similarly, a weather map has level curves; the isobar lines represent sets of points with the same barometric pressure. (*Isobar* means: the *same bar*ometric pressure.)

An indifference curve is a set of points in the consumption bundle picture, among which the consumer is indifferent. Since she is indifferent among these points, they all give her the same utility. Hence, the indifference curve is a level curve for her utility function. Therefore, in order to represent a consumer's utility function, we will simply draw its level curves, its indifference curves, in the (x_1, x_2) quadrant. This is like transforming a three-dimensional picture of a mountain into a two-dimensional topographical map, with elevation contours. See Figure 2.9 above.

2.5 Utility Functions and the Marginal Rate of Substitution

Next, we explain the connection between the marginal rate of substitution, and the utility function that represents the consumer's preferences. Figure 2.10 below is similar to Figure 2.7. The marginal rate of substitution of good 2 for good 1, at the point X, is $-\Delta x_2/\Delta x_1$, roughly speaking. (And precisely speaking, in the limit.) How does this relate to a utility function for this consumer?

The *marginal utility* of good 1 is the rate at which the consumer's utility increases as good 1 increases, while we hold the quantity of good 2 constant. Loosely speaking, it is the *extra utility from an extra unit of good 1*. More formally, let Δx_1 represent an increment of good 1. The marginal utility of good 1, which we write MU_1, is defined as

$$MU_1 = \lim_{\Delta x_1 \to 0} \frac{u(x_1 + \Delta x_1, x_2) - u(x_1, x_2)}{\Delta x_1}.$$

If it weren't for the presence of the variable x_2, students would recognize this as the derivative of the function $u(x_1)$. And this is almost exactly what it is, except the function $u(x_1, x_2)$ is really a function of two variables, the second of which, x_2, is being held constant. The derivative of a function of two variables, with respect to x_1 while x_2 is being held constant, is called the *partial derivative* of the function $u(x_1, x_2)$ with respect to x_1. A derivative is commonly shown with a d symbol, as in $df(x)/dx$. A partial derivative is commonly shown with a ∂ symbol instead of a d, and so the marginal utility of good 1 can be written as

$$MU_1 = \frac{\partial u(x_1, x_2)}{\partial x_1} = \frac{\partial u}{\partial x_1}.$$

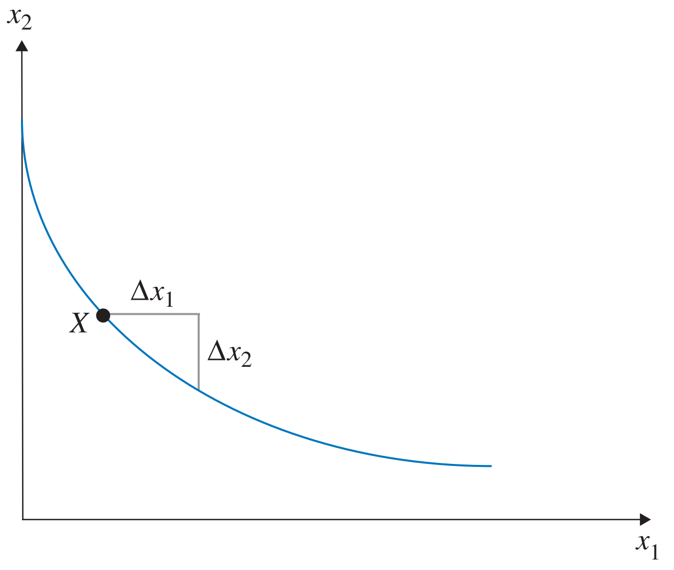

Figure 2.10 Marginal utility and the marginal rate of substitution.

The marginal utility of good 2, which we write MU_2 is defined as

$$MU_2 = \lim_{\Delta x_2 \to 0} \frac{u(x_1, x_2 + \Delta x_2) - u(x_1, x_2)}{\Delta x_2} = \frac{\partial u}{\partial x_2}.$$

Since marginal utility is derived from the utility function, which is ordinal, it shouldn't be interpreted as a cardinal measure. That is, we don't attach any meaning to a statement like "My marginal utility from an additional apple is 3." We do attach meaning to a statement like "My marginal utility from an additional apple is 3, and my marginal utility from an additional banana is 2." This simply means "I prefer an additional apple."

Our main use of the marginal utility concept at this point is to calculate the consumer's *MRS*. Consider Figure 2.10 again. From the bundle $X = (x_1, x_2)$, we increase good 1 by Δx_1, and simultaneously decrease good 2 by Δx_2, to get back to the original indifference curve. If we evaluate the change in utility along the way (keeping in mind that we are really thinking of very small moves), we have the following: utility increases because of the increase in good 1, by an amount equal to the marginal utility of good 1 times Δx_1. At the same time, utility decreases because of the decrease in good 2, by an amount equal to the marginal utility of good 2 times Δx_2. The sum of the increase and the decrease is 0, since the consumer ends up on the original indifference curve. This gives the following equation (note that Δx_1 is positive and Δx_2 is negative):

$$MU_1 \Delta x_1 + MU_2 \Delta x_2 = 0.$$

From this we easily get

$$-\frac{\Delta x_2}{\Delta x_1} = \frac{MU_1}{MU_2}.$$

But $MRS = -\Delta x_2/\Delta x_1$. We conclude that

$$MRS = \frac{MU_1}{MU_2}.$$

This gives us a convenient tool for calculating the consumer's marginal rate of substitution, either as a function of (x_1, x_2), or as a numerical value at a given point.

2.6 A Solved Problem

The Problem

For each of the following utility functions, find the marginal rate of substitution function, or *MRS*.

(a) $u(x_1, x_2) = x_1 x_2$
(b) $u(x_1, x_2) = 2x_2$
(c) $u(x_1, x_2) = x_1 + x_2$
(d) $u(x_1, x_2) = \min\{x_1, 2x_2\}$
(e) $u(x_1, x_2) = x_2 - x_1^2$.

The Solution

We use the fact that the *MRS* equals the ratio of the marginal utilities, or $MRS = \frac{MU_1}{MU_2}$. In each case, we first calculate the marginal utilities, and then we find their ratio.

(a) Assume $u(x_1, x_2) = x_1 x_2$.

$$MU_1 = \frac{\partial(x_1 x_2)}{\partial x_1} = x_2 \text{ and } MU_2 = \frac{\partial(x_1 x_2)}{\partial x_2} = x_1.$$

Therefore,

$$MRS = \frac{MU_1}{MU_2} = \frac{x_2}{x_1}.$$

(b) Assume $u(x_1, x_2) = 2x_2$.

$$MU_1 = \frac{\partial(2x_2)}{\partial x_1} = 0 \text{ and } MU_2 = \frac{\partial(2x_2)}{\partial x_2} = 2.$$

Therefore,

$$MRS = \frac{MU_1}{MU_2} = \frac{0}{2} = 0.$$

(c) Assume $u(x_1, x_2) = x_1 + x_2$.

$$MU_1 = \frac{\partial(x_1 + x_2)}{\partial x_1} = 1 \text{ and } MU_2 = \frac{\partial(x_1 + x_2)}{\partial x_2} = 1.$$

Therefore,

$$MRS = \frac{MU_1}{MU_2} = \frac{1}{1} = 1.$$

(d) Assume $u(x_1,x_2) = \min\{x_1, 2x_2\}$. The marginal utilities depend on whether $x_1 < 2x_2$, or $x_1 > 2x_2$.

If $x_1 < 2x_2$, then

$$MU_1 = \frac{\partial(\min\{x_1, 2x_2\})}{\partial x_1} = 1 \text{ and } MU_2 = \frac{\partial(\min\{x_1, 2x_2\})}{\partial x_2} = 0.$$

Therefore,

$$MRS = \frac{MU_1}{MU_2} = \frac{1}{0} = \infty.$$

If $x_1 > 2x_2$, then

$$MU_1 = \frac{\partial(\min\{x_1, 2x_2\})}{\partial x_1} = 0 \text{ and } MU_2 = \frac{\partial(\min\{x_1, 2x_2\})}{\partial x_2} = 2.$$

Therefore,

$$MRS = \frac{MU_1}{MU_2} = \frac{0}{2} = 0.$$

Finally, if $x_1 = 2x_2$, then MRS is undefined.

(e) Assume $u(x_1,x_2) = x_2 - x_1^2$.

$$MU_1 = \frac{\partial(x_2 - x_1^2)}{\partial x_1} = -2x_1 \text{ and } MU_2 = \frac{\partial(x_2 - x_1^2)}{\partial x_2} = 1.$$

Therefore,

$$MRS = \frac{MU_1}{MU_2} = \frac{-2x_1}{1} = -2x_1.$$

Exercises

1. We assumed at the beginning of the chapter that a consumer's preferences must be transitive, but we hinted that there might be interesting exceptions. Here are two:

   (a) A consumer likes sugar in her coffee, but she simply cannot taste the difference between a cup of coffee with n grams of sugar in it and a cup of coffee with $n+1$ grams. Suppose a teaspoon of sugar is 10 grams, and suppose she takes her coffee with one teaspoon of sugar. Why does this violate transitivity?

   (b) Let's call a committee of three people a "consumer." (Groups of people often act together as "consumers.") Our committee makes decisions using majority voting. When they compare two alternatives x and y, they simply take a vote, and the winner is said to be "preferred" by the committee to the loser. Suppose that the preferences of the individuals are as follows: Person 1 likes x best, y second best, and z third best. We write this in the following way: Person 1: x, y, z. Assume the preferences of the other two people are: Person 2: y, z, x; and Person 3: z, x, y. Show that in this example the committee preferences produced by majority voting violate transitivity. (This is the famous "voting paradox" first described by the French philosopher and mathematician Marquis de Condorcet (1743–1794).)

2. Consider the utility function $u(x_1, x_2) = x_1 x_2$.

   (a) Graph the indifference curves for utility levels 1 and 2. (They are symmetric hyperbolas asymptotic to both axes.)
   (b) Graph the locus of points for which the *MRS* of good 2 for good 1 is equal to 1, and the locus of points for which the *MRS* is equal to 2.

3. Different students at World's Greatest University (W.G.U.) have different preferences about economics. Draw the indifference curves associated with each of the following statements. Measure "economics books" along the horizontal axis and "books about other subjects" along the vertical. Draw arrows indicating the direction in which utility is increasing.

   (a) "I care only about the total amount of knowledge I acquire. It is the same whether that is economics knowledge or of any other kind. That is, all books on all subjects are perfect substitutes for me."
   (b) "I hate the Serrano–Feldman textbook and all other economics books. On the other hand, I love everything else in the W.G.U. curriculum."
   (c) "I really like books about economics because I want to understand the economic world. Books about other subjects make no difference to me."
   (d) "I like all my courses and the liberal education that W.G.U. offers. That is, I prefer to read books on a variety of different subjects, rather than to read lots on one subject and little on the others."

4. Sketch indifference curves for utility levels 1 and 2 for each of the following utility functions. Describe in a sentence or two the consumer's preferences for the two goods.

   (a) $u(x_1, x_2) = 2x_2$
   (b) $u(x_1, x_2) = x_1 + x_2$
   (c) $u(x_1, x_2) = \min\{x_1, 2x_2\}$
   (d) $u(x_1, x_2) = x_2 - x_1^2$.

5. Donald likes fishing (x_1) and hanging out in his hammock (x_2). His utility function for these two activities is $u(x_1, x_2) = 3x_1^2 x_2^4$.

   (a) Calculate MU_1, the marginal utility of fishing.
   (b) Calculate MU_2, the marginal utility of hanging out in his hammock.
   (c) Calculate *MRS*, the rate at which he is willing to substitute hanging out in his hammock for fishing.
   (d) Last week, Donald fished two hours a day, and hung out in his hammock four hours a day. Using your formula for *MRS* from (c) above, find his *MRS* last week.
   (e) This week, Donald is fishing eight hours a day, and hanging out in his hammock two hours a day. Calculate his *MRS* this week. Has his *MRS* increased or decreased? Explain why.
   (f) Is Donald happier or sadder this week compared to last week? Explain.

6. Suppose you are choosing between hours of work (a bad measured on the *horizontal* axis) and money (a good measured on the *vertical* axis).

   (a) Explain the meaning of *MRS* in words.
   (b) Should your *MRS* be positive or negative in this case?
   (c) Is your *MRS* increasing, constant, or decreasing as you increase the hours of work along an indifference curve? Explain and draw some indifference curves for this example.

7. Suppose the consumer is choosing between hours of work (a bad measured on the *vertical* axis) and money (a good measured on the *horizontal* axis).

   (a) Explain why the *MRS* is negative in this case.
   (b) Explain in words the economic interpretation of the *MRS* in this case.
   (c) Is the *MRS* increasing, constant, or decreasing as you increase the pay for work along an indifference curve? Explain briefly and draw some indifference curves.

8. According to a recent survey, different students at NotSoGood University have different preferences about microeconomics. Draw the indifference curves associated with each of the following statements. Measure "microeconomics books" along the horizontal axis and "books about other subjects" along the vertical.

   (a) "I am a tools-oriented person. I appreciate microeconomics as a tool to shed light on the world, but I also appreciate other subjects to shed light on microeconomics. Thus, for me, to see value in an additional micro book, I would need additional books in other subjects, and vice versa. In a sense, all books are perfect complements for me."
   (b) "I hate the Serrano–Feldman book and all other books in all subjects. In fact, what the heck am I doing here? I'm not sure why I came to NotSoGood."
   (c) "I really like books about any subject other than microeconomics because I want to understand the non-microeconomics world. Microeconomics is a neutral for me."
   (d) "I like to buy only microeconomics books or only books in other subjects, but I am always reluctant to combine them in my Amazon basket."

9. Daisy watches two types of movies: Disney movies and even more horrible movies. She has a big ego and loves to watch by herself. In particular, evaluating movie bundles, she first looks at how many Disney movies the bundles include (the more the better). And if two bundles have the same number of Disney movies, she then looks at the number of even more horrible movies the bundle has (the more the better). Note: these preferences are called lexicographic, because they resemble how people look for words in a dictionary.

   (a) Suppose she is currently considering to watch one Disney movie and one even more horrible movie this weekend. Draw the indifference curve of movie bundles indifferent to her current situation.
   (b) Are these preferences complete and transitive? Do they satisfy monotonicity?

10. Blissa has the following utility function: $u(x_1, x_2) = -(x_1 - 1)^2 - (x_2 - 2)^2$ over goods 1 and 2. We often say that such preferences exhibit a bliss point.

    (a) Draw several indifference curves corresponding to this utility function.
    (b) What assumptions on preferences are violated by it?

Appendix: Differentiation of Functions

This short appendix is not meant to be a substitute for a calculus course. However, it may serve as a helpful review. Let's begin with functions of one variable. Consider a function $y = f(x)$. Its derivative is

$$y' = f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

The derivative of the function f is the rate at which f increases as we increase x, the infinitesimal increment in f divided by the infinitesimal increment in x.

Some examples of differentiation of functions of one variable are:

(1) $y = 4x, \quad y' = 4;$
(2) $y = 7x^2, \quad y' = 14x;$
(3) $y = \ln x; \quad y' = 1/x.$

What about functions of several variables? Consider a function $u(x_1, x_2)$, like our utility function. We define two partial derivatives of u, with respect to x_1 and with respect to x_2:

$$\frac{\partial u}{\partial x_1} = \lim_{\Delta x_1 \to 0} \frac{u(x_1 + \Delta x_1, x_2) - u(x_1, x_2)}{\Delta x_1}$$

and

$$\frac{\partial u}{\partial x_2} = \lim_{\Delta x_2 \to 0} \frac{u(x_1, x_2 + \Delta x_2) - u(x_1, x_2)}{\Delta x_2}.$$

The first is the rate at which u increases as we increase x_1, while holding x_2 constant. The second is the rate at which u increases as we increase x_2, while holding x_1 constant.

How do we partially differentiate a function of several variables? Almost exactly the same way we differentiate a function of one variable, except that we must remember that if we are differentiating with respect to variable x_i, we treat any other variable x_j as a constant.

Some examples are:

(1) $u(x_1, x_2) = x_1 x_2, \quad \partial u/\partial x_1 = x_2, \quad \partial u/\partial x_2 = x_1;$
(2) $u(x_1, x_2) = x_1^2 x_2^3, \quad \partial u/\partial x_1 = 2x_1 x_2^3, \quad \partial u/\partial x_2 = 3x_1^2 x_2^2;$
(3) $u(x_1, x_2) = \ln x_1 + 2 \ln x_2, \quad \partial u/\partial x_1 = 1/x_1, \quad \partial u/\partial x_2 = 2/x_2.$

3 The Budget Constraint and the Consumer's Optimal Choice

3.1 Introduction

In Chapter 2, we described the consumer's preferences and utility function. Now we turn to what constrains him, and what he should do to achieve the best outcome given his constraint. The consumer prefers some bundles to other bundles. He wants to get to the most-preferred bundle, or the highest possible utility level, but he cannot afford everything. He has a *budget constraint*. The consumer wants to make the best choice possible, the *optimal choice*, or the *utility-maximizing choice*, subject to his budget constraint.

In this chapter, we will describe the consumer's standard budget constraint. We will give some examples of special budget constraints created by non-market rationing devices, such as coupon rationing. We will also analyze budget constraints involving consumption over time.

After describing various budget constraints, we will turn to the consumer's basic economic problem: how to find the best consumption bundle, or how to maximize his utility, subject to the budget constraint. We will do this graphically using indifference curves, and we will do it analytically with utility functions. In the appendix to this chapter, we will describe the Lagrange function method for maximizing a function subject to a constraint.

3.2 The Standard Budget Constraint, the Budget Set, and the Budget Line

A consumer cannot spend more money than he has. (We know about credit and will discuss it in a later section of this chapter.) We call what he has his *income*, written M, for "money". He wants to spend it on goods 1 and 2. Each has a price, represented by p_1 and p_2, respectively. The consumer's *standard budget constraint*, or *budget constraint* for short, says that the amount he spends (the sum of price times quantity for each of the two goods) must be less than or equal to the money he has! This gives

$$p_1x_1 + p_2x_2 \leq M.$$

The *budget set* is the set of all bundles that satisfy the budget constraint; that is, all the bundles the consumer can afford. Of course, there will generally be many bundles available in the budget set.

The *budget line* is the set of bundles where the consumer is spending exactly what he has. That is, it is the set of bundles (x_1, x_2) satisfying the equation

$$p_1 x_1 + p_2 x_2 = M.$$

The figure below represents the consumer's budget line.

The horizontal intercept of the budget line is the amount of good 1 the consumer would have if he spent all his money on that good; that is, if he consumed $x_2 = 0$. This is $x_1 = M/p_1$ units of good 1. Similarly, he would have M/p_2 units of good 2 if he spent all his money on that good. Since the price per unit of each good is a constant, the budget line is a straight line connecting these two intercepts.

The slope of the budget line is obviously negative. The absolute value of the slope, p_1/p_2, is sometimes called the *relative price* of good 1. This is the amount Δx_2 of good 2 that the consumer *must give up*, if he wants to consume an additional amount Δx_1 of good 1. (Compare this with the *MRS* of good 2 for good 1 – the amount Δx_2 of good 2 that the consumer *is just willing to* give up, in order to consume an additional amount Δx_1 of good 1.)

The budget line defines a tradeoff for the consumer who wants to increase his consumption of good 1 and simultaneously decrease his consumption of good 2. Note that in Figure 3.1, Δx_1 is a positive number (good 1 is increasing) and Δx_2 is a negative

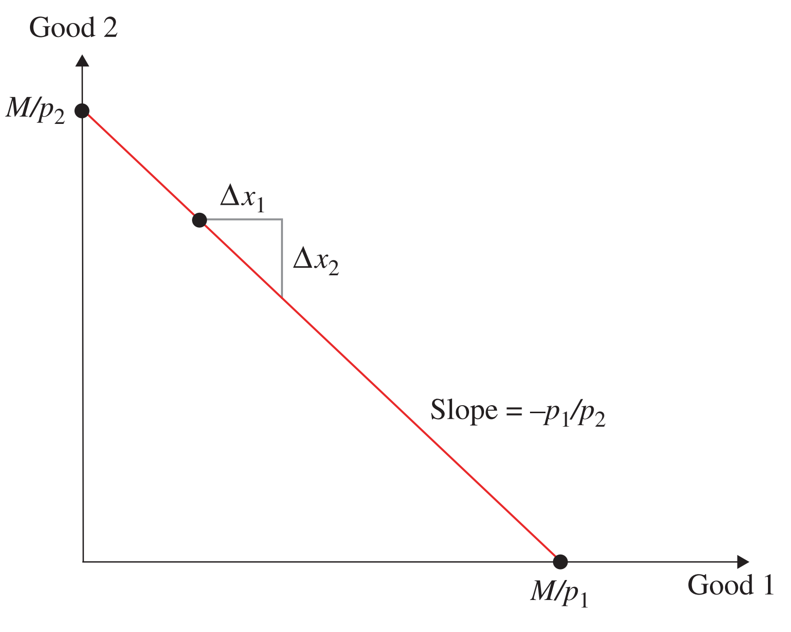

Figure 3.1 The budget line (in red) is a downward-sloping straight line. The intercepts are M/p_1 and M/p_2, and the slope is $-p_1/p_2$.

number (good 2 is decreasing). If the amount spent on the two goods remains constant, the sum of the increase in money spent on good 1 and the decrease in money spent on good 2 must be 0, or

$$p_1 \Delta x_1 + p_2 \Delta x_2 = 0.$$

This gives

$$-\frac{\Delta x_2}{\Delta x_1} = \frac{p_1}{p_2}.$$

3.3 Shifts of the Budget Line

If the consumer's income changes, or if the prices of the goods change, the budget line moves. Figure 3.2 shows how the budget line shifts if income increases while prices stay constant.

If both prices decrease by the same proportion, the same kind of shift occurs. Suppose the new prices are $p_1' = kp_1$ and $p_2' = kp_2$, where $k < 1$ is the same factor for both prices. Then the new budget line has slope $-p_1'/p_2' = -[kp_1]/[kp_2] = -p_1/p_2$. The new intercept on the horizontal axis is $M/p_1' = M/kp_1 = (1/k)M/p_1$, which is farther out the axis because $k < 1$.

If income decreases while both prices stay the same, or if both prices rise by the same proportion while income stays constant, the budget line shifts inwards.

Now consider what happens when one price, say p_1, rises, while the other price and income stay the same. Let p_1' be the new price and p_1 the old, with $p_1' > p_1$.

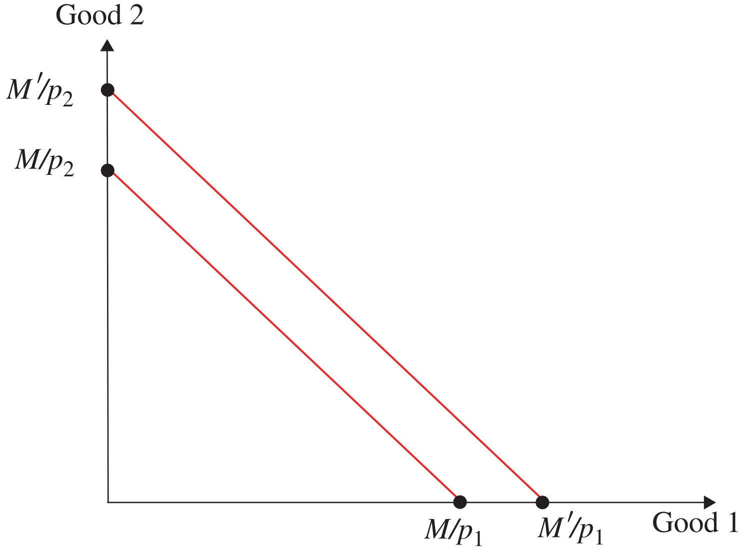

Figure 3.2 In this figure, income increases from M to M'. The budget line shifts out, parallel to itself. The new intercepts are M'/p_1 and M'/p_2 on the horizontal and vertical axes, respectively.

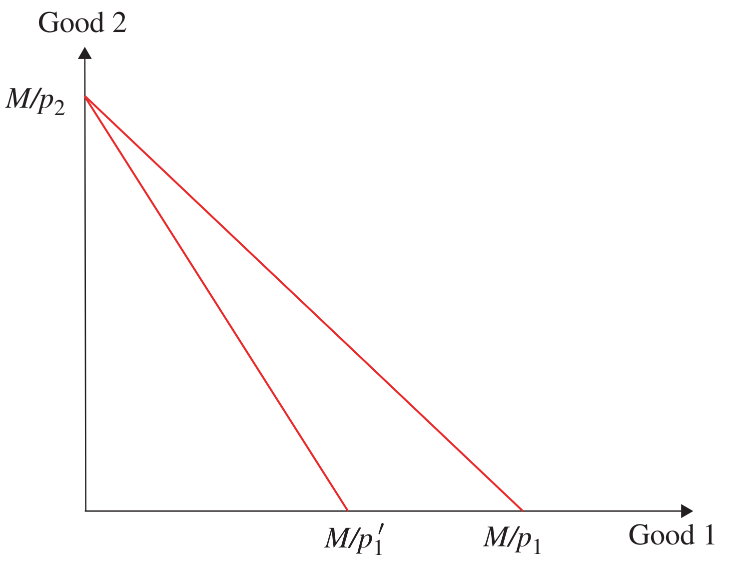

Figure 3.3 The price of good 1 rises, while the price of good 2, and income, stay the same.

If the consumer spends all his income on good 1, he will consume less, because the new intercept M/p_1' is smaller than the old M/p_1. The intercept on the good 2 axis doesn't move. The budget line gets steeper, because the absolute value of the new slope, p_1'/p_2, is greater than the absolute value of the old slope, p_1/p_2. Figure 3.3 shows this important type of budget line shift.

3.4 Odd Budget Constraints

The standard budget constraint described above assumes, first, that prices are constant for any quantities of the goods the consumer might want to consume. Second, it assumes that prices don't depend on income. Third, it assumes that nothing constrains the consumer except prices and the money in his pocket. However, the real world often doesn't follow these assumptions. The real world is full of non-standard budget constraints; here are two examples:

Example 1. A 2-for-1 store coupon

The consumer has one (and only one) coupon from a grocery store, allowing him to buy up to two units of good 1 at half price. The regular price for good 1, charged for units beyond two, is \$1 per unit. Also, the consumer's income is $M = 5$, and the price of good 2 is $p_2 = 1$. It follows that $p_1 = 1/2$ if $x_1 \leq 2$, and $p_1 = 1$ for $x_1 > 2$. Figure 3.4 below illustrates this case. The intercept on the good 2 axis is obviously $M/p_2 = 5$, while the intercept on the horizontal axis is somewhat less obviously 6.

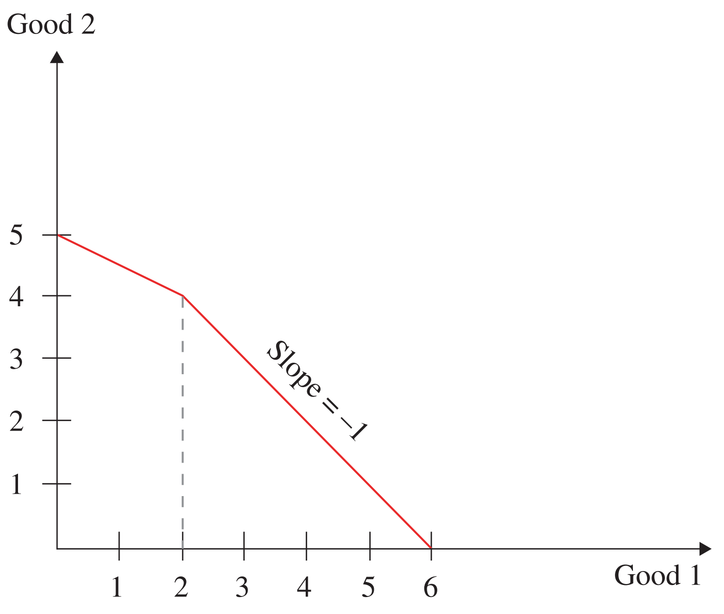

Figure 3.4 The case of a 2-for-the-price-of-1 promotional coupon.

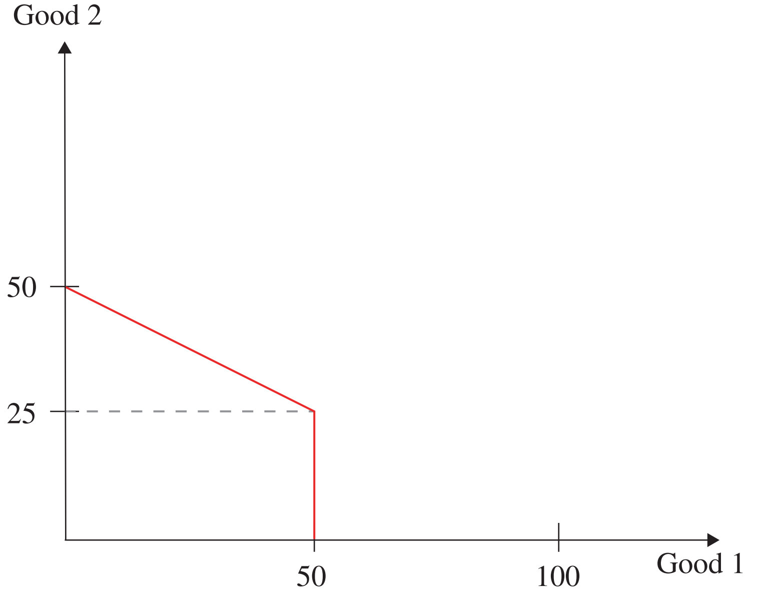

Figure 3.5 The case of wartime coupon rationing.

Example 2. Ration coupons

In times of war (and other emergencies, real or imagined), governments will sometimes ration scarce commodities (including food, fuel, and so on). This might mean that goods 1 and 2 sell for money at prices p_1 and p_2, but that the purchaser *also* needs a government coupon for each unit of the rationed good (say good 1) that he buys.

Suppose the consumer has income of $M = 100$; let $p_1 = 1$, and $p_2 = 2$, and suppose that the consumer has ration coupons for 50 units of good 1. See Figure 3.5. The vertical intercept is $x_2 = 50$ and the slope of the budget line for $x_1 < 50$ is $-1/2$. At the point $(x_1, x_2) = (50, 25)$, the budget line becomes vertical; the consumer cannot buy more than 50 units of good 1 since he only has 50 coupons.

3.5 Income and Consumption Over Time

One very crucial type of budget constraint shows the consumer's choices over time. This is called an *intertemporal budget constraint*. For this purpose, we start by assuming there are two time periods ("this year" and "next year"), and we let x_1 represent consumption this year, and x_2 represent consumption next year. For simplicity, we assume that a unit of the consumption good, called "stuff," has a price of 1, *both this year and next year*. (Assuming that a unit of a good that has a price of 1 is sometimes called *normalizing* the price. A good with a price of 1 is sometimes called a *numeraire* good.) Since we are assuming the price of a unit of the good is the same this year and next year, we are assuming *no price inflation*. (We will add inflation to the mix in some exercises in this chapter, and again when we revisit this topic in Chapter 5.)

We assume that the consumer has income M_1 this year, and will have income M_2 next year. He could obviously choose $x_1 = M_1$ and $x_2 = M_2$. In this case, he's spending everything that he gets this year on his consumption this year, and spending everything that he gets next year on his consumption next year. He's neither borrowing nor saving.

Alternatively, he could save some of this year's income. In this case, he spends some of M_1 on consumption this year, and he sets some aside until next year, when he spends all that remains from this year, plus his income from next year. (We assume that he has monotonic preferences; he always prefers more stuff to less, and will therefore end up spending everything available by the end of next year.) Assume for now that what the consumer doesn't spend this year he hides under his mattress for next year. In other words, he puts the money he doesn't spend away in a safe place, but he doesn't get any interest on his *savings*. His budget constraint now says that what he consumes next year (x_2) must equal what he saved and put under his mattress this year ($M_1 - x_1$), plus his income next year (M_2). This gives

$$x_2 = (M_1 - x_1) + M_2 \qquad \text{or} \qquad x_1 + x_2 = M_1 + M_2.$$

Note that we have written the budget constraint as an equation, rather than as an inequality, since the consumer ultimately spends all that he has.

Next, let's assume that the consumer doesn't hide his money under his mattress. Instead, whatever he doesn't spend this year he puts into a bank account (or an investment) that pays a fixed and certain rate of return i (i is for "interest," expressed as a decimal). Now what he saves and puts away in the first year ($M_1 - x_1$), he gets back with interest (multiply by $(1 + i)$), causing it to grow to $(1 + i)(M_1 - x_1)$ next year. The consumer's budget constraint now becomes

$$x_2 = (1+i)(M_1 - x_1) + M_2 \qquad \text{or} \qquad (1+i)x_1 + x_2 = (1+i)M_1 + M_2.$$

Finally, dividing both sides of the equation by $1 + i$ gives

$$x_1 + \left(\frac{1}{1+i}\right) x_2 = M_1 + \left(\frac{1}{1+i}\right) M_2.$$

Economists call the term $1/(1+i)$ the *discount factor*. In general, the term *present value* means that some future amount – or amounts, or some series of amounts over time – is being converted to the *current time, or current year* equivalent. The term $x_2/(1+i)$ is called the (year 1) *present value of year 2 consumption*. The term $M_2/(1+i)$ is called the (year 1) *present value of year 2 income*. The left-hand side of the budget equation, or $x_1 + x_2/(1+i)$, is called the *present value of the consumer's consumption stream*, and the right-hand side of the equation, or $M_1 + M_2/(1+i)$, is called the *present value of the consumer's income stream*. Therefore, the budget equation we just derived says that the present value of the consumption stream equals the present value of the income stream.

In the analysis above, we assumed the consumer saves some of his first-year income M_1 in order to be able to consume more in the second year than his second-year income M_2. Now let's assume he does the reverse. That is, we now assume that he *borrows against next year's income, in order to increase this year's consumption.* For simplicity, we will assume that the interest rate i is the same for savers and borrowers. (This is, of course, quite unrealistic; in reality the interest rate *paid to* savers is normally much less than the interest rate *paid by* borrowers. To appreciate the difference, compare the interest rate applied to balances on your credit card to the interest rate paid to savers at your bank.)

If the consumer intends to spend less than his income in the second year, then $x_2 < M_2$, or $M_2 - x_2 > 0$. Suppose the consumer goes to his banker in the first year and asks this question: Next year I can pay you back $M_2 - x_2$. How much can you lend me this year, based on this anticipated repayment? The banker reasons to himself: If I make a loan of L this year, I must get all my money back next year, plus interest, or a total of $(1+i)L$. Therefore, I require $(1+i)L = M_2 - x_2$. Solving for L then gives $L = (M_2 - x_2)/(1+i)$. (Of course, this process may be more complicated in the real world. In reality, bankers either require collateral or security for loans – as with real estate mortgages – or, for unsecured loans, they charge interest rates high enough to compensate for defaults.)

We can now lay out the consumer's budget constraint in the case where he is a borrower. His consumption this year (x_1), is equal to his income this year (M_1), plus the loan he gets from his banker $(M_2 - x_2)/(1+i)$. This gives

$$x_1 = M_1 + (M_2 - x_2)/(1+i),$$

or, rearranging terms,

$$x_1 + \left(\frac{1}{1+i}\right) x_2 = M_1 + \left(\frac{1}{1+i}\right) M_2.$$

But this is exactly the same budget equation as in the saver case!

To summarize, we have looked at a consumer who has income this year and income next year, and who will consume some stuff this year and more stuff next year. We assumed the consumer can save or borrow, and that the interest rate is the same for savers and for borrowers. We have shown that the consumer has a simple budget constraint involving consumption quantities this year and next, income this year and next, and the interest rate. We have shown that there is a simple and intuitive interpretation of the budget constraint: *The present value of the consumer's consumption stream must equal the present value of his income stream*. This is a crucial result for the theory of intertemporal choice. Moreover, the budget constraint we found, and more generally, the methodology of present values, are crucial in the theory and practice of finance.

3.6 The Consumer's Optimal Choice: Graphical Analysis

As we said when we began with the theory of the consumer, the consumer will choose the bundle that he most prefers among those that he can afford. This is his *optimal choice*. To put it another way, he will find *the highest indifference curve that's consistent with his budget*. Figure 3.6 illustrates.

What conditions must be satisfied by the consumer's optimal choice?

First, at the optimal point, the consumer's indifference curve and budget line are just touching, as we can plainly see in Figure 3.6.

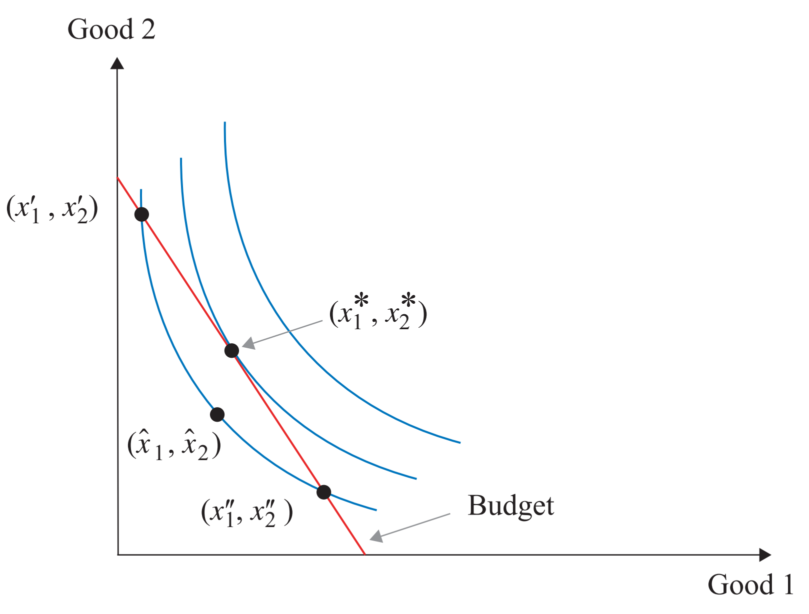

Figure 3.6 The standard case. The consumer's optimal choice is at (x^*_1, x^*_2), where his budget line (in red) is tangent to one of his indifference curves (in blue).

The figure actually shows more than that; it shows the standard case where the indifference curve and the budget line are in fact *tangent* at (x_1^*, x_2^*). That is, both slopes are well-defined and equal. (We will consider some examples where the slopes are either not defined, or not equal, below.) Since the slopes are equal in the figure, the absolute values of the slopes are also equal. The absolute value of the slope of an indifference curve at a point is equal to the *MRS*, and the absolute value of the slope of the budget line is p_1/p_2. Therefore, for the standard case illustrated in Figure 3.6 we have

$$MRS = p_1/p_2.$$

Recall that the marginal rate of substitution is interpreted as the amount of good 2 the consumer is *just willing* to give up in exchange for getting an increment of good 1, and the price ratio p_1/p_2 is the amount of good 2 that the *market demands* the consumer give up in exchange for an increment of good 1. At the optimal point (x_1^*, x_2^*) of the figure, what the consumer is just willing to do is exactly equal to what the market demands that he do.

Now consider a point where $MRS \neq p_1/p_2$, for instance the bundle (x_1', x_2') in Figure 3.6. At that bundle, consider the possibility of the consumer giving up some of good 2, and getting some of good 1 in exchange. For a given increment of good 1, the consumer would be willing to give up much more of good 2 than the market requires that he give up (the indifference curve is relatively steep and the budget line is relatively flat). Therefore, he would trade according to market prices, move down and to the right on the budget line, and *make himself better off*. The opposite adjustment would happen at the bundle (x_1'', x_2''). And no such adjustment can happen at the optimal bundle (x_1^*, x_2^*).

And second, the optimal point must be on the budget line. That is, it must be the case that

$$p_1x_1 + p_2x_2 = M.$$

This is because we are assuming monotonic preferences; the consumer always prefers more to less, and will spend all of his income. A bundle like $(\hat{x}_1, \hat{x}_2)$ in Figure 3.6 is not optimal and would not be chosen by the consumer because he prefers, and can afford, bundles above and to the right of $(\hat{x}_1, \hat{x}_2)$; that is, bundles with more of both goods.

The principle behind the consumer's optimal choice is always the same: he wants to buy the most-preferred bundle, or get to the highest indifference curve, that he can afford. However, our marginal rate of substitution condition assumes that the *MRS* of good 2 for good 1 is well defined, and that the optimal bundle has positive amounts of both goods. What happens to the consumer's optimal choice without these assumptions? Consider the following examples:

Marginal rate of substitution not defined. Suppose $u(x_1, x_2) = \min\{x_1, x_2\}$; $p_1 = 2$ and $p_2 = 1$. When the utility function has this form, we call x_1 and x_2 *perfect complements.*

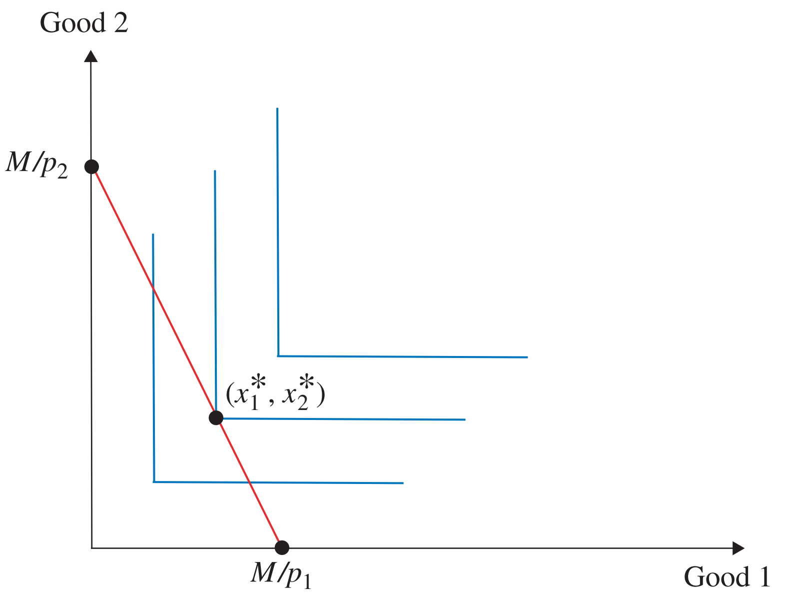

Figure 3.7 Perfect complements.

This means a unit of good 1 is always consumed with exactly one unit of good 2. Think, for example, of a left shoe and a right shoe. Unless you are missing a limb, you always want to consume exactly one right shoe with each left shoe. You can check to see that the *MRS* is not defined when $x_1 = x_2$ because the utility function is not differentiable there. However, it's easy to graph the consumer's choice problem in this case. See Figure 3.7.

Clearly, the budget line equation $p_1x_1 + p_2x_2 = M$ still holds because of monotonicity of preferences. And the second equation we need is $x_1 = x_2$ in this case. It's pointless for this consumer to choose a bundle where this condition is not met, given his preferences.

Corner solution. Assume $u(x_1, x_2) = x_1 + x_2$. When the utility function has this form, we call x_1 and x_2 *perfect substitutes*. This is like Coke and Pepsi for a consumer who (strangely) cannot taste the difference; a bottle of one soda can be freely substituted for a bottle of the other, with no effect on utility. Clearly, if $p_1 < p_2$, this consumer will spend all his income on good 1. The utility function is differentiable and the *MRS* is equal to 1 everywhere. If the price ratio $p_1/p_2 \neq 1$, it is impossible to have a tangency of an indifference curve with the budget line. But this only means that the optimal choice must be at an end point of the budget line; that is, on the good 1 axis or the good 2 axis. (Such an optimal choice is called a corner solution.) This consumer would drink only Coke, or only Pepsi, whichever is cheaper. See Figure 3.8 bleow.

In most of this book, we construct examples of optimal choices where indifference curves and budget lines are tangent. We do it this way to make the explanations simpler.

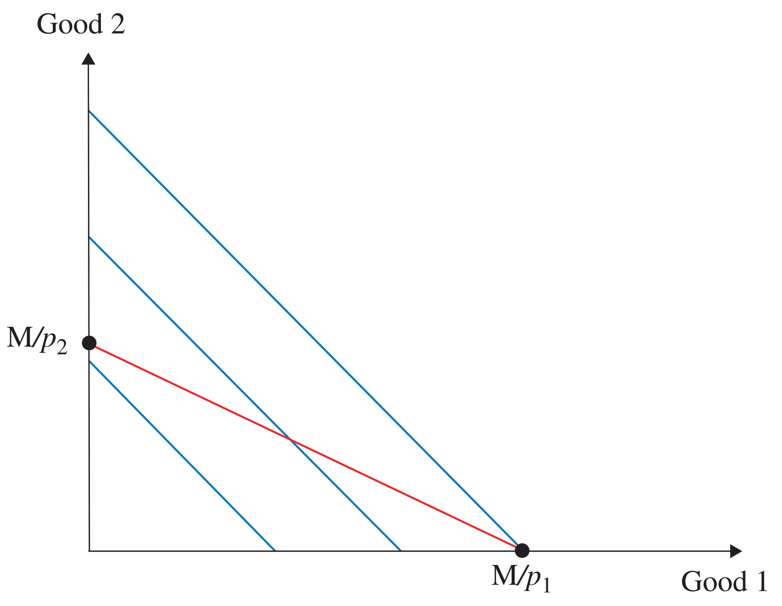

Figure 3.8 Perfect substitutes. Note that p_1/p_2, the absolute value of slope of the budget line, is now less than 1. This consumer only buys good 1.

3.7 The Consumer's Optimal Choice: Utility Maximization Subject to the Budget Constraint

As we have said, the consumer will choose the bundle that he most prefers among those that he can afford. This is the consumer's optimal choice. In the last section, we thought of this as finding the highest indifference curve that's consistent with the consumer's budget line. Now we think of the same problem, but this time we think of it as *maximizing the consumer's utility function subject to his budget constraint*. We will assume in this section that the consumer's optimal choice is at a point where an indifference curve is tangent to the budget line.

The consumer's optimal choice (x_1^*, x_2^*) is the solution to this problem:

$$\text{Maximize } u(x_1, x_2)$$

subject to

$$p_1x_1 + p_2x_2 = M.$$

This is a special type of calculus problem; the objective function $u(x_1, x_2)$ is being maximized *subject to a constraint*. If the constraint were not there, there would be no maximum, given our assumption of monotonicity. Therefore, we cannot try to solve the problem by first maximizing $u(x_1, x_2)$ and then worrying about the constraint. There are three ways we can solve the problem.

1. **Brute force method.** We could use the constraint to solve for one of the variables, plug the result back into the objective function, and then maximize the objective function, which has been reduced to a function of just one variable. (This function does have a maximum.) That is, we use the constraint to solve for x_2:

$$x_2 = \frac{M - p_1 x_1}{p_2}.$$

We plug this into $u(x_1, x_2)$ giving

$$u\left(x_1, \frac{M - p_1 x_1}{p_2}\right).$$

Note that x_2 has disappeared from the utility function. We differentiate this function with respect to x_1 and set the result equal to 0. Solving the resulting equation gives x_1^*. We then plug this back into the budget equation, $p_1 x_1^* + p_2 x_2 = M$, and use this to solve for x_2^*.

The brute force method, as the name suggests, may be rather ugly and difficult, and we will try to avoid using it in what follows.

2. **Use-the-graphs method.** We could rely on what we learned from the graphs; at a consumer optimum, where an indifference curve is tangent to the budget line, it must be the case that

$$MRS = p_1/p_2.$$

We then combine this equation with the budget constraint equation $p_1 x_1 + p_2 x_2 = M$ to solve for the two unknowns (x_1^*, x_2^*). This is the method that we use most often in this book.

3. **The Lagrange function method.** The standard mathematical method for solving a constrained maximization problem is the following. First, set up a special function, called the *Lagrange function*, that incorporates both the objective function *and* the constraint. In our case, the Lagrange function would be

$$L = u(x_1, x_2) + \lambda(M - (p_1 x_1 + p_2 x_2)).$$

In this function, λ is a special variable called the *Lagrange multiplier*. Next, we proceed to find the first-order conditions for the maximization of L with respect to x_1, x_2, and λ; these boil down to $MRS = p_1/p_2$ and $p_1 x_1 + p_2 x_2 = M$. Finally, we use the first-order conditions to solve for the optimal quantities of the goods (x_1^*, x_2^*), and for the optimal λ^*. This method is more elegant than methods 1 and 2 above, and the Lagrange multiplier has a nice economic interpretation in terms of how much the consumer would value a \$1 increase in his income. In general, however, we will stick to the use-the-graphs method in this book, since it is simpler that Lagrange function method. We do describe the Lagrange method in more detail in the appendix to this chapter.

3.8 Two Solved Problems

Problem 1

Part 1. Assume $p_1 = 1$ and $p_2 = 2$, and the consumer has income $M = 10$. Find the consumer's budget constraint. Find utility-maximizing consumption bundles for the following utility functions:

(a) $u(x_1, x_2) = x_1 + x_2$
(b) $u(x_1, x_2) = x_1 x_2$

Part 2. Now assume the prices change to $p_1 = 2$ and $p_2 = 1$. What is his new budget constraint? What happens to the utility-maximizing consumption bundles in the two cases?

Solution to Problem 1

First note that with these utility functions, the consumer will want to spend all his income. He will want to be on his budget line, not below it. The relevant budget constraint is an equation, not an inequality.

Part 1. In general, the budget constraint is $p_1x_1 + p_2x_2 = M$. With prices $(p_1, p_2) = (1, 2)$, this gives $x_1 + 2x_2 = 10$. His budget line has slope $p_1/p_2 = 1/2$ in absolute value, going from intercept $M/p_1 = 10$ on the good 1 (horizontal) axis to intercept $M/p_2 = 5$ on the good 2 (vertical) axis.

(a) If $u(x_1, x_2) = x_1 + x_2$, his indifference curves are straight lines, with slope equal to $MRS = MU_1/MU_2 = 1/1 = 1$ in absolute value. There is no indifference curve/budget line tangency possible, because the indifference curves have slope 1 and the budget line has slope 1/2 (both in absolute value). To find the corner solution, we can use a sketch as in Figure 3.8 above, or we can simply calculate utility levels at the ends of the budget line. If he puts all his income into buying 10 units of good 1, $u(10, 0) = 10$; if he puts all his income into buying 5 units of good 2, $u(0, 5) = 5$. His optimal consumption bundle is, therefore, $(x_1^*, x_2^*) = (10, 0)$.
(b) If $u(x_1, x_2) = x_1x_2$, his indifference curves are hyperbolas. The tangency condition is $MRS = p_1/p_2$ or

$$MRS = \frac{MU_1}{MU_2} = \frac{x_2}{x_1} = \frac{p_1}{p_2} = \frac{1}{2}.$$

This gives $x_1 = 2x_2$. His budget constraint is $x_1 + 2x_2 = 10$, and substituting for x_1 gives $4x_2 = 10$. It follows that the solution is $x_1^* = 5$ and $x_2^* = 2.5$.

Part 2. Now suppose the prices change to $(p_1, p_2) = (2, 1)$. His budget constraint becomes $2x_1 + x_2 = 10$. His budget line now has slope $p_1/p_2 = 2/1 = 2$ in absolute value, going from intercept $M/p_1 = 5$ on the good 1 (horizontal) axis to intercept $M/p_2 = 10$ on the good 2 (vertical) axis.