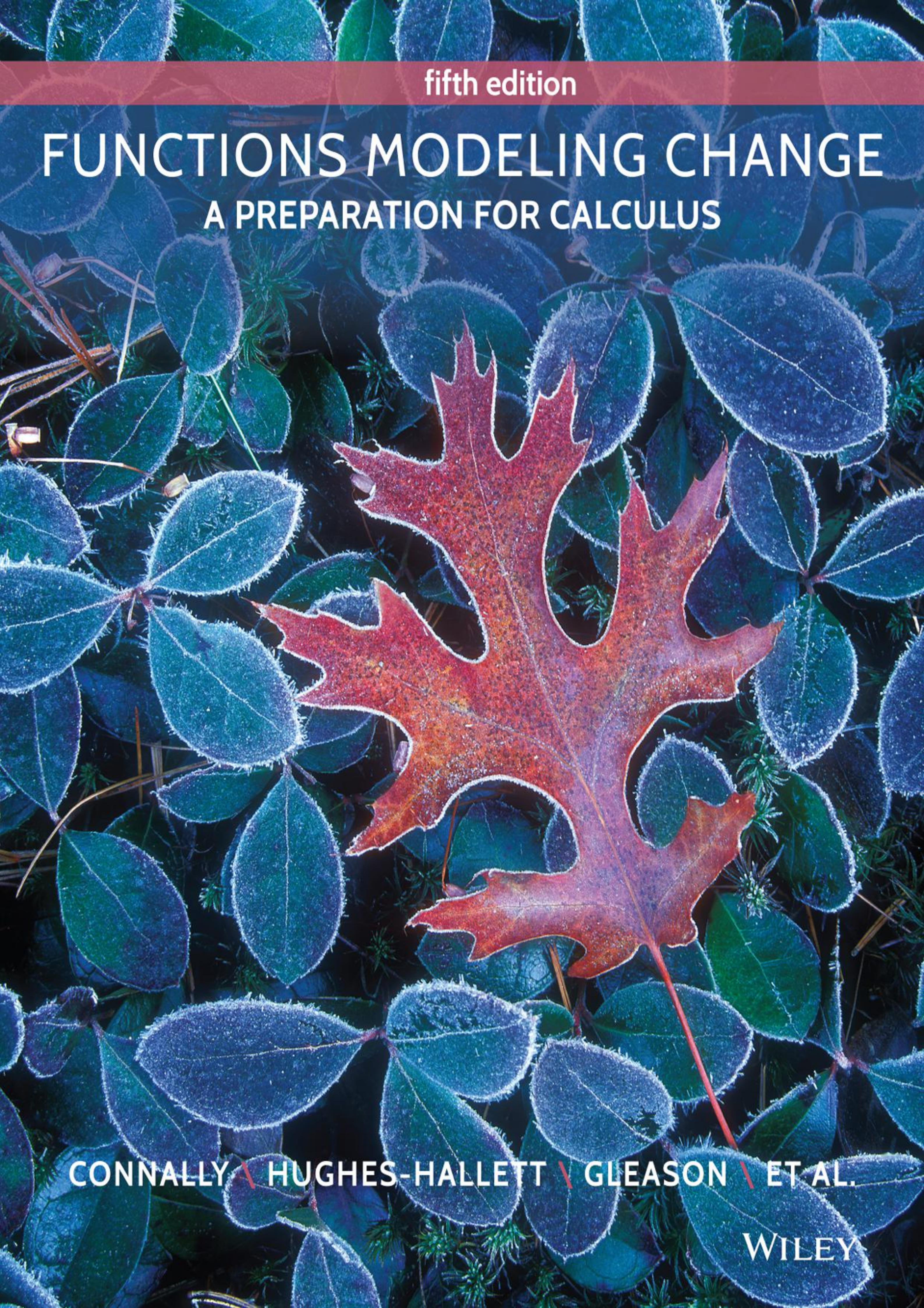
fifth edition

FUNCTIONS MODELING CHANGE

A PREPARATION FOR CALCULUS



CONNALLY \ HUGHES-HALLETT \ GLEASON \ ET AL.

WILEY

FUNCTIONS MODELING CHANGE:
A Preparation for Calculus

Fifth Edition

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Eric Connally
Harvard University Extension

Deborah Hughes-Hallett
The University of Arizona

Andrew M. Gleason
Harvard University

Philip Cheifetz
Nassau Community College

Ann Davidian
Gen. Douglas MacArthur HS

Daniel E. Flath
Macalester College

Selin Kalaycıoğlu
New York University

Brigitte Lahme
Sonoma State University

Patti Frazer Lock
St. Lawrence University

Guadalupe I. Lozano
The University of Arizona

William G. McCallum
The University of Arizona

Jerry Morris
Sonoma State University

Karen Rhea
University of Michigan

Ellen Schmierer
Nassau Community College

Pat Shure
University of Michigan

Adam H. Spiegler
Loyola University Chicago

Carl Swenson
Seattle University

Aaron D. Wootton
University of Portland

Elliot J. Marks

with the assistance of

Frank Avenoso
Nassau Community College

Douglas Quinney
University of Keele

Katherine Yoshiwara
Los Angeles Pierce College

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Dedicated to Ben, Jonah, and Isabel

PUBLISHER	Laurie Rosatone
ACQUISITIONS EDITOR	Joanna Dingle
ASSOCIATE EDITOR	Jacqueline Sinacori
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COVER DESIGNER	Madelyn Lesure
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PREFACE

In the 21st century, students need the ability to solve problems; innovation depends on citizens who can think critically. The fifth edition of *Functions Modeling Change: A Preparation for Calculus* reaffirms our effort to refocus the teaching of mathematics in a balanced blend of concepts and procedures. Our approach encourages students to develop their problem-solving skills while acquiring the mathematical background needed to learn calculus or pursue their careers.

Fifth Edition: Focus

This edition of Precalculus continues to stress conceptual understanding and the connections among mathematical ideas. Understanding exponential change, for example, means being able to relate the size of a percent growth to the shape of a function's graph, as well as assessing the reasonability of a model.

Functions are the foundation of much of mathematics. Our approach encourages a robust understanding of functions in traditional and novel contexts. Skills are introduced and reinforced throughout the book. This balance of skills and understanding promotes students' critical thinking.

Fifth Edition: Flexibility

Precalculus courses are taken by a wide range of students and are taught in a wide variety of styles. As instructors ourselves, we know that the balance we choose depends on the students we have: sometimes a focus on conceptual understanding is best; sometimes more skill-building is needed.

To enable instructors to select the balance appropriate for their students, the fifth edition expands the options available for customizing material. For example, we have integrated examples involving linear inequalities in our treatment of linear equations, and offered more examples on rates of change in several chapters. The sections involving limit notation and phase shift have been restructured so that these topics can be easily skipped. Transformations of functions are now introduced in Chapter 2, while the full treatment remains in Chapter 6, providing instructors with more freedom about when to introduce them and how deeply to cover this topic.

Origin of Text: The Calculus Consortium for Higher Education

This book is the work of faculty at a diverse consortium of institutions, and was originally generously supported by the National Science Foundation. It represents the first consensus among such a diverse group of faculty to have shaped a mainstream precalculus text. Bringing together the results of research and experience with the views of many users, this text is designed to be used in a wide range of institutions.

Guiding Principles: Varied Problems and the Rule of Four

Conceptual understanding is enhanced when students engage in non-procedural problem-solving. Problems classified into clear-cut types tend to develop proficiency at finding answers, but not necessarily a robust conceptual understanding. Strong problem-solvers feel capable of making progress on problems they have not seen before, not just those of known type. Consequently, we are guided by the following principles:

- Problems are varied and often challenging. Many of our problems cannot be answered by following a template in the text.
- Text examples are diverse and represent the natural integration of skills and concepts.
- The Rule of Four: each concept and function is represented symbolically, numerically, graphically, and verbally. This principle, originally introduced by the Consortium, promotes multiple representations.

- We provide students and instructors quick means of assessing comprehension before moving on by using the true-false Strengthen Your Understanding problems at the end of each chapter.
- Our central theme, functions as models of change, links the components of our precalculus curriculum. Algebra is integrated where appropriate.
- Topics are fewer in number than it is customary so they can be treated in greater depth. The topics are those essential to the study of calculus.
- Problems involving real data give students practice with modeling.
- We include both classic and current applications to prepare students to model with mathematics in a variety of contexts.
- Our problems allow students to become proficient in the use of technology, including symbolic manipulators, computers, tablets, and online software, as appropriate.
- Our precalculus materials allow for a broad range of teaching styles. They are flexible enough for use in large lecture halls, small classes, or in group or lab settings.

Changes in the Fifth Edition

The fifth edition reflects the many helpful suggestions from users while preserving the focus and guiding principles of previous editions. We have made the following changes:

- **Many examples and problems** are new or have been rewritten. Data has been updated and new data introduced.
- **The three chapters on trigonometry** have been reorganized and rewritten:
  - **Trigonometric functions on the circle** are introduced both in radians and degrees in Chapter 7, highlighting the natural relationship between radians and the unit circle as soon as periodic functions are introduced.
  - **Sinusoidal functions** modeling periodic phenomena and **simple trigonometric equations** are also introduced in Chapter 7, while more involved trigonometric models and equations requiring relationships such as **double-angle identities** are treated in Chapter 9.
  - **Phase shift** has been made an optional topic in Chapter 7.
  - **Triangle trigonometry** is discussed separately in Chapter 8, after trigonometric and sinusoidal functions are treated in Chapter 7.
- A brief exploration of **linear inequalities** has been integrated with the material on solving linear equations in Chapter 1.
- **Vertical and horizontal shifts** are introduced in Chapter 2, and referenced in Chapter 3 when introducing the vertex form of a quadratic equation. Shifts are reviewed at the beginning of Chapter 6 and considered in combination with other transformations.
- **Odd and even functions** are introduced by looking at invariance of certain functions under reflections, and thus better integrated with the transformation focus of Chapter 6.
- The effect of **changing the order of transformations**, the last section in Chapter 6 in the fourth edition, has been shortened and included in the previous section.
- The section on **power functions** in Chapter 11 has been rewritten to increase the focus on graphical behavior and proportionality.
- **Examples on average rate of change** have been added throughout the book, such as for exponential functions (in Chapter 5) and periodic functions (in Chapters 7 and 9).
- **WileyPLUS**, the primary online resource suite paired with the textbook, has been updated with improved problems and hints, including many new problems from the fifth edition.

What Student Background is Expected?

Students using this book should have successfully completed a course in intermediate algebra or high school algebra II. The book is thought-provoking for well-prepared students while still accessible to students with weaker backgrounds. Providing numerical, graphical, and algebraic approaches builds on different student strengths and provides students with a variety of ways to master the material. Multiple representations give students tools to persist, lowering failure rates.

Our Experiences

The first four editions of this book were used at hundreds of schools around the country in a wide variety of settings. It has been used successfully in both semester and quarter systems, in large lectures and small classes as well as in full-year courses in secondary schools. It has also been used in computer labs and small groups, often with the integration of a number of different technologies.

Content

The central theme of this book is functions as models of change. We emphasize that functions can be grouped into families and that functions can be used as models. We explore how function characteristics connect to difference quotients and rates of change, naturally previewing key calculus ideas.

Because linear, quadratic, exponential, power, and periodic functions are most frequently used to model physical phenomena, they are introduced before polynomial and rational functions. Once introduced, a family of functions is compared and contrasted with other families of functions.

A large number of the examples that students see in this precalculus course are real-world problems. By the end of the course, we hope that students will use functions to help them understand the world in which they live. We include non-routine problems to emphasize that such problems are not only part of mathematics, but in some sense are the reason for doing mathematics.

Technology

The book does not require any specific software or technology. Instructors have used the material with graphing calculators, graphing software, or scientific calculators.

Chapter 1: Linear Functions and Change

This chapter introduces the concept of a function as well as graphical, tabular, symbolic, and verbal representations of functions, discussing the advantages and disadvantages of each representation. It introduces rates of change and uses them to characterize linear functions. Examples on modeling with linear functions, including interpreting linear inequalities, are discussed. A section on fitting a linear function to data is included.

The **Skills Refresher** section for Chapter 1 reviews linear equations, linear inequalities, and the coordinate plane.

Chapter 2: Functions

This chapter studies functions in more detail. It introduces domain, range, previews function shifts and the concepts of composite and inverse functions, and investigates the idea of concavity using rates of change. A section on piecewise functions is included.

Chapter 3: Quadratic Functions

This chapter introduces the standard, factored, and vertex forms of a quadratic function and explores their relationship to graphs, including shifts. The family of quadratic functions provides an opportunity to see the effect of parameters on functional behavior.

The **Skills Refresher** section for Chapter 3 reviews factoring, completing the square, and quadratic equations.

Chapter 4: Exponential Functions

This chapter introduces the family of exponential functions and the number *e*. It compares exponential and linear functions, solves exponential equations graphically, and gives applications to compound interest.

The **Skills Refresher** section for Chapter 4 reviews the properties of exponents.

Chapter 5: Logarithmic Functions

This chapter introduces logarithmic functions with base 10 and base *e*, both in order to solve exponential equations and as inverses of exponential functions. After discussing manipulations with logarithms, the chapter focuses on modeling with exponential functions and logarithms. Logarithmic scales and a section on linearizing data conclude the chapter.

The **Skills Refresher** section for Chapter 5 reviews the properties of logarithms.

Chapter 6: Transformations of Functions and Their Graphs

This chapter investigates transformations. It revisits shifts, introduces reflections and stretches, and explores even and odd symmetry using transformations. The effect of ordering when combining transformations is explored.

Chapter 7: Trigonometry and Periodic Functions

This chapter introduces trigonometric functions in both radians and degrees as models for periodic motion. After exploring graphs and formulas of sine, cosine, and tangent, general sinusoidal functions are introduced. This chapter introduces the inverse trigonometric functions to solve trigonometric equations.

The **Skills Refresher** section for Chapter 7 reviews sine and cosine values of special angles measured in both radians and degrees.

Chapter 8: Triangle Trigonometry and Polar Coordinates

This chapter develops right-triangle trigonometry, and introduces the Law of Sines and the Law of Cosines. It also defines polar coordinates, explores their relationship to Cartesian coordinates, and considers the graphs of polar inequalities.

Note: Chapter 8 can be skipped by instructors who prefer not to emphasize triangle trigonometry.

Chapter 9: Trigonometric Identities, Models and Complex Numbers

This chapter opens with a discussion of trigonometric equations, and then explores trigonometric identities and their role in trigonometric models, such as damped oscillations and acoustic beats. The chapter closes with complex numbers, including Euler's Formula and de Moivre's theorem.

Chapter 10: Compositions, Inverses, and Combinations of Functions

This chapter discusses combinations of functions. It investigates composite and inverse functions, which were introduced in Chapter 2, in more detail.

Chapter 11: Polynomial and Rational Functions

This chapter discusses power functions, polynomials, and rational functions. The chapter explores dominance and long-run behavior and concludes by comparing polynomial and exponential functions, and by fitting functions to data.

The **Skills Refresher** section for Chapter 11 reviews algebraic fractions.

Chapter 12: Vectors

This chapter contains material on vectors and operations involving vectors. An introduction to matrices is included in the last section.

Chapter 13: Sequences and Series

This chapter introduces arithmetic and geometric sequences and series and their applications.

Chapter 14: Parametric Equations and Conic Sections

The concluding chapter looks at parametric equations, implicit functions, hyperbolic functions, and conic sections: circles, ellipses, and hyperbolas. The chapter includes a section on the geometrical properties of the conic sections and their applications to orbits.

Note: Chapter 14 is available online only.

Supplementary Materials

The following supplementary materials are available for the fifth edition:

- **The Instructor's Manual** contains teaching tips, lesson plans, syllabi, and worksheets. It has been expanded and revised to include worksheets, identification of technology-oriented problems, and new syllabi. (ISBN 978-1-119-01383-9)
- **The Printed Test Bank** contains test questions arranged by section.
- **The Instructor's Solution Manual** has complete solutions to all problems. (ISBN 978-1-118-94162-1)
- **The Student Solution Manual** has complete solutions to half of the odd-numbered problems. (ISBN 978-1-118-94163-8)
- **The Computerized Test Bank**, available in both PC and Macintosh formats, allows instructors to create, customize, and print a test containing any combinations of questions from a large bank of questions. Instructors can also customize the questions or create their own.
- **Classroom Activities** are posted at the book companion website. These activities were developed to facilitate in-class group work as well as to introduce new concepts and to practice skills. In addition to the blank copies for each activity that can be handed out to the students, a copy of the activity with fully worked out solutions is also available.
- **The Book Companion Site** at www.wiley.com/college/connally contains all instructor supplements.
- **WileyPLUS** is a powerful online suite of teaching and learning resources tightly integrated with the text. WileyPLUS enables instructors to assign, deliver and grade individually customized homework assignments using exercises and problems from the text. Students receive immediate feedback on their homework and access to full solutions to assigned problems electronically. Students may also access hints to the problems. In addition to online homework, WileyPLUS provides student tutorials, an instructor gradebook, integrated links to the electronic version of the text, and all of the text's supplemental materials. For more information, visit www.wiley.com/college/wileyplus or contact your local Wiley representative for more details.
- **Mini-lecture Videos** linked with examples in the WileyPLUS student version of the text provide greater detail to the solution of examples in each section of the text. These may assist students in reading the text prior to class or in reviewing material after class.
- **The Faculty Network** is a peer-to-peer network of academic faculty dedicated to the effective use of technology in the classroom. This group can help you apply innovative classroom techniques, implement specific software packages, and tailor the technology experience to the specific needs of each individual class. Visit www.wherefacultyconnect.com or ask your Wiley representative for details.

ConcepTests

ConcepTests, modeled on the pioneering work of Harvard physicist Eric Mazur, are questions designed to promote active learning during class, particularly (but not exclusively) in large lectures. Our evaluation data show students taught with ConcepTests outperformed students taught by traditional lecture methods 73% versus 17% on conceptual questions, and 63% versus 54% on computational problems. ConcepTests arranged by section are available in PowerPoint and Classroom Response System-ready formats from your Wiley representative. (ISBN 978-1-118-94161-4)

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Jerry Morris	Karen Rhea	Ellen Schmierer	Pat Shure
Adam H. Spiegler	Carl Swenson	Aaron D. Wootton	

To Students: How to Learn from this Book

- This book may be different from other math textbooks that you have used, so it may be helpful to know about some of the differences in advance. At every stage, this book emphasizes the *meaning* (in practical, graphical or numerical terms) of the symbols you are using. There is much less emphasis on "plug-and-chug" and using formulas, and much more emphasis on the interpretation of these formulas than you may expect. You will often be asked to explain your ideas in words or to explain an answer using graphs.
- The book contains the main ideas of precalculus in plain English. Success in using this book will depend on reading, questioning, and thinking hard about the ideas presented. It will be helpful to read the text in detail, not just the worked examples.
- There are few examples in the text that are exactly like the homework problems, so homework problems can't be done by searching for similar-looking "worked out" examples. Success with the homework will come by grappling with the ideas of precalculus.
- Many of the problems in the book are open-ended. This means that there is more than one correct approach and more than one correct solution. Sometimes, solving a problem relies on common-sense ideas that are not stated in the problem explicitly but which you know from everyday life.
- This book assumes that you have access to a calculator or computer that can graph functions and find (approximate) roots of equations. There are many situations where you may not be able to find an exact solution to a problem, but can use a calculator or computer to get a reasonable approximation. An answer obtained this way can be as useful as an exact one. However, the problem does not always state that a calculator is required, so use your own judgment.
- This book attempts to give equal weight to four methods for describing functions: graphical (a picture), numerical (a table of values), algebraic (a formula) and verbal (words). Sometimes it's easier to translate a problem given in one form into another. For example, you might replace the graph of a parabola with its equation, or plot a table of values to see its behavior. It is important to be flexible about your approach: if one way of looking at a problem doesn't work, try another.
- Students using this book have found discussing these problems in small groups helpful. There are a great many problems that are not cut-and-dried; it can help to attack them with the other perspectives your colleagues can provide. If group work is not feasible, see if your instructor can organize a discussion session in which additional problems can be worked on.
- You are probably wondering what you'll get from the book. The answer is, if you put in a solid effort, you will get a real understanding of functions as well as a real sense of how mathematics is used in the age of technology.

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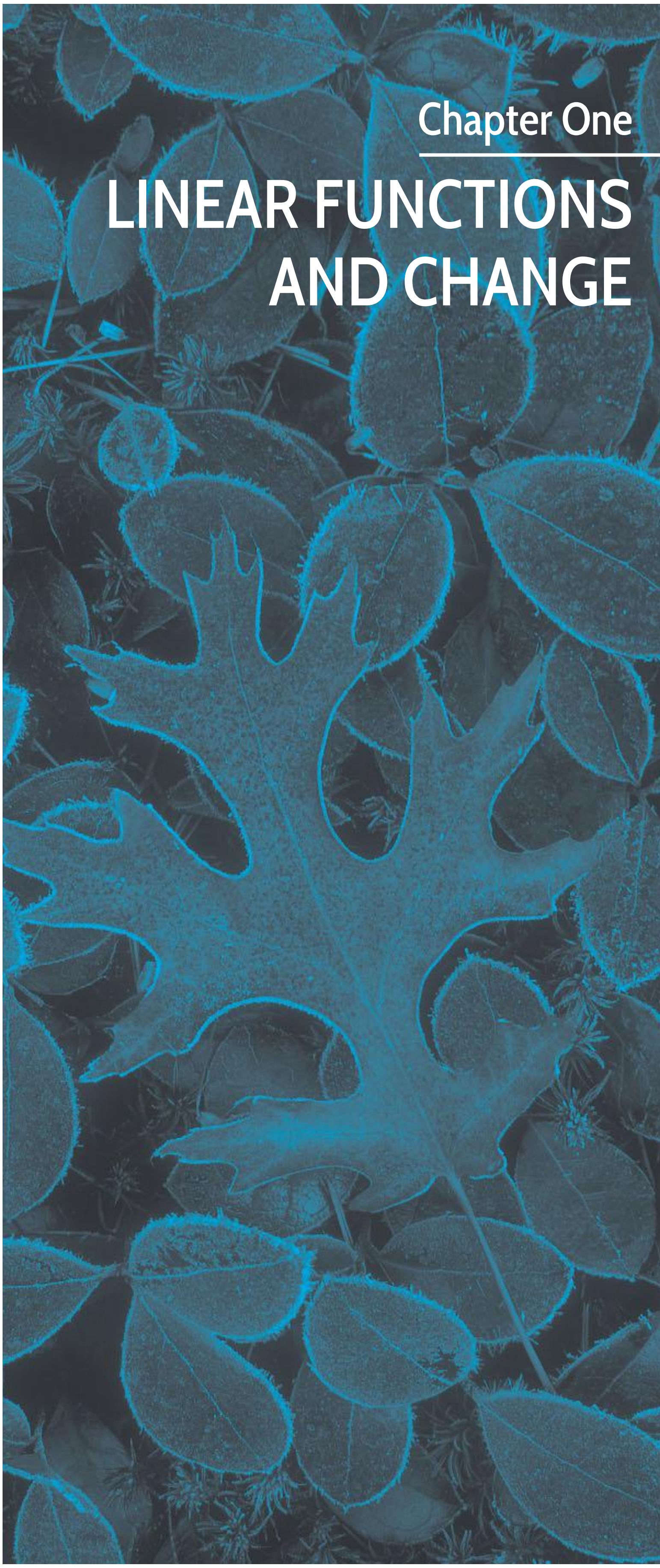
Chapter One

LINEAR FUNCTIONS AND CHANGE



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Contents

1.1 FUNCTIONS AND FUNCTION NOTATION

In everyday language, the word *function* expresses the notion of dependence. For example, a person might say that election results are a function of the economy, meaning that the winner of an election is determined by how the economy is doing. Someone else might claim that car sales are a function of the weather, meaning that the number of cars sold on a given day is affected by the weather.

In mathematics, the meaning of the word *function* is more precise, but the basic idea is the same. A function is a relationship between two quantities. If the value of the first quantity determines exactly one value of the second quantity, we say the second quantity is a function of the first. We make the following definition:

> A **function** is a rule that takes certain numbers as inputs and assigns to each input number exactly one output number. The output is a function of the input.

The inputs and outputs are also called *variables*.

Representing Functions: Words, Tables, Graphs, and Formulas

A function can be described using words, data in a table, points on a graph, or a formula.

Example 1 It is a surprising biological fact that most crickets chirp at a rate that increases as the temperature increases. For the snowy tree cricket (*Oecanthus fultoni*), the relationship between temperature and chirp rate is so reliable that this type of cricket is called the thermometer cricket. We can estimate the temperature (in degrees Fahrenheit) by counting the number of times a snowy tree cricket chirps in 15 seconds and adding 40. For instance, if we count 20 chirps in 15 seconds, then a good estimate of the temperature is $20 + 40 = 60°$F.[1]

The rule used to find the temperature T (in °F) from the chirp rate R (in chirps per minute) is an example of a function. The input is chirp rate and the output is temperature. Describe this function using words, a table, a graph, and a formula.

Solution

- **Words**: To estimate the temperature, we count the number of chirps in fifteen seconds and add forty. Alternatively, we can count R chirps per minute, divide R by four and add forty. This is because there are one-fourth as many chirps in fifteen seconds as there are in sixty seconds. For instance, 80 chirps per minute works out to $\frac{1}{4} \cdot 80 = 20$ chirps every 15 seconds, giving an estimated temperature of $20 + 40 = 60°$F.
- **Table**: Table 1.1 gives the estimated temperature, T, as a function of R, the number of chirps per minute. Notice the pattern in Table 1.1: each time the chirp rate, R, goes up by 20 chirps per minute, the temperature, T, goes up by 5°F.

Table 1.1 *Chirp rate and temperature*

R, chirp rate (chirps/minute)	T, predicted temperature (°F)
20	45
40	50
60	55
80	60
100	65
120	70
140	75
160	80

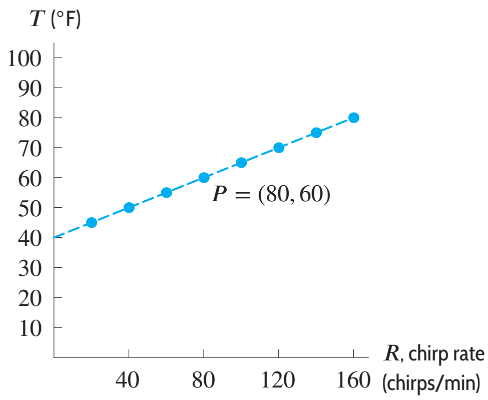


Figure 1.1: Chirp rate and temperature

[1] This relationship is often called Dolbear's Law, as it was first proposed in Amos Dolbear, "The Cricket as a Thermometer," in *The American Naturalist,* 31(1897), pp. 970–971.

- **Graph**: The data from Table 1.1 are plotted in Figure 1.1. For instance, the pair of values $R = 80$, $T = 60$ is plotted as the point P, which is 80 units along the horizontal axis and 60 units up the vertical axis. Data represented in this way are said to be plotted on the *Cartesian plane*. The precise position of P is shown by its coordinates, written $P = (80, 60)$.
- **Formula**: A formula is an equation giving T in terms of R. Dividing the chirp rate by four and adding forty gives the estimated temperature, so:

$$\underbrace{\text{Estimated temperature (in °F)}}_{T} = \frac{1}{4} \cdot \underbrace{\text{Chirp rate (in chirps/min)}}_{R} + 40.$$

Rewriting this using the variables T and R gives the formula:

$$T = \frac{1}{4}R + 40.$$

Let's check the formula. Substituting $R = 80$, we have

$$T = \frac{1}{4} \cdot 80 + 40 = 60,$$

which agrees with point $P = (80, 60)$ in Figure 1.1. The formula $T = \frac{1}{4}R + 40$ also tells us that if $R = 0$, then $T = 40$. Thus, the dashed line in Figure 1.1 crosses (or intersects) the T-axis at $T = 40$; we say the T*-intercept* is 40.

All the descriptions given in Example 1 provide the same information, but each description has a different emphasis. A relationship between variables is often given in words, as at the beginning of Example 1. Table 1.1 is useful because it shows the predicted temperature for various chirp rates. Figure 1.1 is more suggestive of a trend than the table, although it is harder to read exact values of the function. For example, you might have noticed that every point in Figure 1.1 falls on a straight line that slopes up from left to right. In general, a graph can reveal a pattern that might otherwise go unnoticed. Finally, the formula has the advantage of being both compact and precise. However, this compactness can also be a disadvantage since it may be harder to gain as much insight from a formula as from a table or a graph.

Mathematical Models

When we use a function to describe an actual situation, the function is referred to as a **mathematical model**. The formula $T = \frac{1}{4}R + 40$ is a mathematical model of the relationship between the temperature and the cricket's chirp rate. Such models can be powerful tools for understanding phenomena and making predictions. For example, this model predicts that when the chirp rate is 80 chirps per minute, the temperature is 60°F. In addition, since $T = 40$ when $R = 0$, the model predicts that the chirp rate is 0 at 40°F. Whether the model's predictions are accurate for chirp rates down to 0 and temperatures as low as 40°F is a question that mathematics alone cannot answer; an understanding of the biology of crickets is needed. However, we can safely say that the model does not apply for temperatures below 40°F, because the chirp rate would then be negative. For the range of chirp rates and temperatures in Table 1.1, the model is remarkably accurate.

In everyday language, saying that T is a function of R suggests that making the cricket chirp faster would somehow make the temperature change. Clearly, the cricket's chirping does not cause the temperature to be what it is. In mathematics, saying that the temperature "depends" on the chirp rate means only that knowing the chirp rate is sufficient to tell us the temperature.

Function Notation

To indicate that a quantity Q is a function of a quantity t, we abbreviate

$$Q \text{ is a function of } t \quad \text{to} \quad Q \text{ equals "} f \text{ of } t\text{"}$$

and, using function notation, to

$$Q = f(t).$$

Thus, applying the rule f to the input value, t, gives the output value, $f(t)$, which is a value of Q. Here Q is called the *dependent variable* and t is called the *independent variable*. In other words,

$$\text{Output} = f(\text{Input})$$

or

$$\text{Dependent} = f(\text{Independent}).$$

We could have used any letter, not just f, to represent the rule.

The expressions "Q depends on t" or "Q is a function of t" do *not* imply a cause-and-effect relationship, as the snowy tree cricket example illustrates.

Example 2 Example 1 gives the following formula for estimating air temperature, T, based on the chirp rate, R, of the snowy tree cricket:

$$T = \frac{1}{4}R + 40.$$

In this formula, T depends on R. Writing $T = f(R)$ indicates that the relationship is a function.

Example 3 The number of gallons of paint needed to paint a house depends on the size of the house. A gallon of paint typically covers 250 square feet. Thus, the number of gallons of paint, n, is a function of the area to be painted, A ft^2. We write $n = f(A)$.

(a) Find a formula for f.
(b) Explain in words what the statement $f(10{,}000) = 40$ tells us about painting houses.

Solution

(a) If $A = 250$, the house requires one gallon of paint. If $A = 500$, it requires $500/250 = 2$ gallons of paint, if $A = 750$ it requires $750/250 = 3$ gallons of paint, and so on. We see that a house of area A requires $A/250$ gallons of paint, so n and A are related by the formula

$$n = f(A) = \frac{A}{250}.$$

(b) The input of the function $n = f(A)$ is an area and the output is an amount of paint. The statement $f(10{,}000) = 40$ tells us that an area of $A = 10{,}000$ ft^2 requires $n = 40$ gallons of paint.

Functions Don't Have to Be Defined by Formulas

People sometimes think that functions are always represented by formulas. However, other representations, such as tables or graphs, can be useful.

Example 4 The average monthly rainfall, R, at Chicago's O'Hare airport is given in Table 1.2, where time, t, is in months and $t = 1$ is January, $t = 2$ is February, and so on. The rainfall is a function of the month, so we write $R = f(t)$. However, there is no formula that gives R when t is known. Evaluate $f(1)$ and $f(11)$. Explain what your answers mean.

Table 1.2 *Average monthly rainfall at Chicago's O'Hare airport*

Month, t	1	2	3	4	5	6	7	8	9	10	11	12
Rainfall, R (inches)	1.8	1.8	2.7	3.1	3.5	3.7	3.5	3.4	3.2	2.5	2.4	2.1

Solution The value of $f(1)$ is the average rainfall in inches at Chicago's O'Hare airport in a typical January. From the table, $f(1) = 1.8$ inches. Similarly, $f(11) = 2.4$ means that in a typical November, there are 2.4 inches of rain at O'Hare.

When Is a Relationship Not a Function?

It is possible for two quantities to be related and yet for neither quantity to be a function of the other.

Example 5 A national park contains foxes that prey on rabbits. Table 1.3 gives the two populations, F and R, over a 12-month period, where $t = 0$ means January 1, $t = 1$ means February 1, and so on.

Table 1.3 *Number of foxes and rabbits in a national park, by month*

t, month	0	1	2	3	4	5	6	7	8	9	10	11
R, rabbits	1000	750	567	500	567	750	1000	1250	1433	1500	1433	1250
F, foxes	150	143	125	100	75	57	50	57	75	100	125	143

(a) Is F a function of t? Is R a function of t?
(b) Is F a function of R? Is R a function of F?

Solution

(a) Remember that for a relationship to be a function, an input can only give a single output. Both F and R are functions of t. For each value of t, there is exactly one value of F and exactly one value of R. For example, Table 1.3 shows that if $t = 5$, then $R = 750$ and $F = 57$. This means that on June 1 there are 750 rabbits and 57 foxes in the park. If we write $R = f(t)$ and $F = g(t)$, then $f(5) = 750$ and $g(5) = 57$.

(b) No, F is not a function of R. For example, suppose $R = 750$, meaning there are 750 rabbits. This happens both at $t = 1$ (February 1) and at $t = 5$ (June 1). In the first instance, there are 143 foxes; in the second instance, there are 57 foxes. Since there are R-values which correspond to more than one F-value, F is not a function of R.

Similarly, R is not a function of F. At time $t = 5$, we have $R = 750$ when $F = 57$, while at time $t = 7$, we have $R = 1250$ when $F = 57$ again. Thus, the value of F does not uniquely determine the value of R.

How to Tell if a Graph Represents a Function: Vertical Line Test

What does it mean graphically for y to be a function of x? Look at a graph of y against x, with y on the vertical axis and x on the horizontal axis. For a function, each x-value corresponds to exactly one y-value. This means that the graph intersects any vertical line at most once (either once or not at all). If a vertical line cuts the graph twice, the graph contains two points with different y-values but the same x-value; this violates the definition of a function. Thus, we have the following criterion:

> **Vertical Line Test**: If there is a vertical line that intersects a graph in more than one point, then the graph does not represent a function.

Example 6 In which of the graphs in Figures 1.2 and 1.3 could y be a function of x?

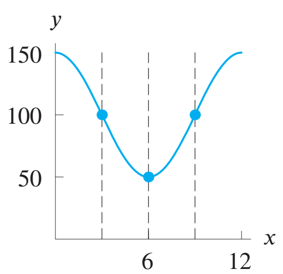


Figure 1.2: Since no vertical line intersects this curve at more than one point, y could be a function of x

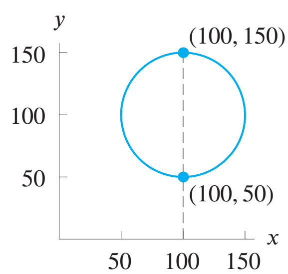


Figure 1.3: Since one vertical line intersects this curve at more than one point, y is not a function of x

Solution The graph in Figure 1.2 could represent y as a function of x because no vertical line intersects this curve in more than one point. The graph in Figure 1.3 does not represent a function because the vertical line shown intersects the curve at two points.

A graph fails the vertical line test if at least one vertical line cuts the graph more than once, as in Figure 1.3. However, if a graph represents a function, then *every* vertical line must intersect the graph at no more than one point.

Exercises and Problems for Section 1.1

Skill Refresher

In Exercises S1–S4, simplify each expression.

S1. $c + \frac{1}{2}c$

S2. $P + 0.07P + 0.02P$

S3. $2\pi r^2 + 2\pi r \cdot 2r$

S4. $\dfrac{12\pi - 2\pi}{6\pi}$

In Exercises S5–S8, find the value of the expressions for the given value of x and y.

S5. $x - 5y$ for $x = \frac{1}{2}$, $y = -5$.

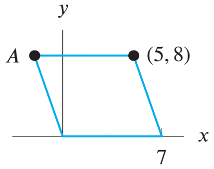
S6. $1 - 12x + x^2$ for $x = 3$.

S7. $\dfrac{3}{2 - x^3}$ for $x = -1$.

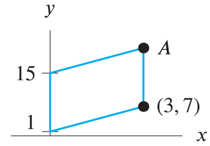
S8. $\dfrac{4}{1 + 1/x}$ for $x = -\frac{3}{4}$.

The figures in Exercises S9–S10 are parallelograms. Find the coordinates of the point A.

S9.



S10.



Exercises

In Exercises 1–2, write the relationship using function notation (that is, y is a function of x is written $y = f(x)$).

1. Number of molecules, m, in a gas, is a function of the volume of the gas, v.

2. Weight, w, is a function of caloric intake, c.

In Exercises 3–6, label the axes for a sketch to illustrate the given statement.

3. "Over the past century we have seen changes in the population, P (in millions), of the city. . ."

4. "Sketch a graph of the cost of manufacturing q items. . ."

5. "Graph the pressure, p, of a gas as a function of its volume, v, where p is in pounds per square inch and v is in cubic inches."

6. "Graph D in terms of y. . ."

7. Figure 1.4 gives the depth of the water at Montauk Point, New York, for a day in November.

   (a) How many high tides took place on this day?
   (b) How many low tides took place on this day?
   (c) How much time elapsed in between high tides?

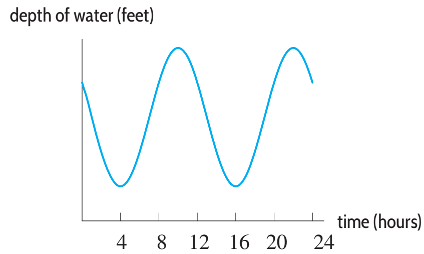


Figure 1.4

8. Using Table 1.4, graph $n = f(A)$, the number of gallons of paint needed to cover walls of area A. Identify the independent and dependent variables.

Table 1.4

A	0	250	500	750	1000	1250	1500
n	0	1	2	3	4	5	6

9. Use Figure 1.5 to fill in the missing values:

(a) $f(0) = ?$ **(b)** $f(?) = 0$

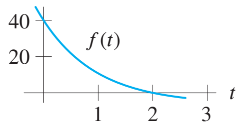


Figure 1.5

10. Use Table 1.5 to fill in the missing values. (There may be more than one answer.)

(a) $f(0) = ?$ **(b)** $f(?) = 0$
(c) $f(1) = ?$ **(d)** $f(?) = 1$

Table 1.5

x	0	1	2	3	4
$f(x)$	4	2	1	0	1

Exercises 11–14 use Figure 1.6.

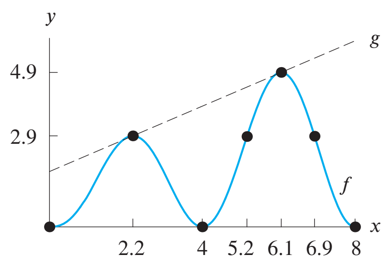


Figure 1.6

11. Find $f(6.9)$.

12. Give the coordinates of two points on the graph of g.

13. Solve $f(x) = 0$ for x.

14. Solve $f(x) = g(x)$ for x.

15. **(a)** You are going to graph $p = f(w)$. Which variable goes on the horizontal axis?
(b) If $10 = f(-4)$, give the coordinates of a point on the graph of f.
(c) If 6 is a solution of the equation $f(w) = 1$, give a point on the graph of f.

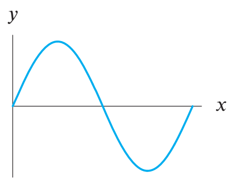
In Exercises 16–19 a relationship is given between two quantities. Are both quantities functions of the other one, or is one or neither a function of the other? Explain.

16. $7w^2 + 5 = z^2$ **17.** $y = x^4 - 1$ **18.** $m = \sqrt{t}$

19. The number of gallons of gas, g, at \$3 per gallon and the number of pounds of coffee, c, at \$10 per pound that can be bought for a total of \$100.

In Exercises 20–25, could the graph represent y as a function of x?

20.



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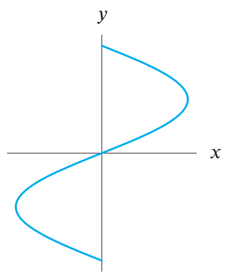

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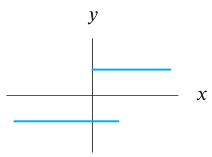

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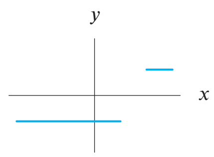

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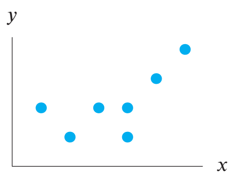

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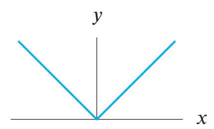


Problems

26. At the end of a semester, students' math grades are listed in a table which gives each student's ID number in the left column and the student's grade in the right column. Let N represent the ID number and G represent the grade. Which quantity, N or G, must necessarily be a function of the other?

27. A person's blood sugar level at a particular time of the day is partially determined by the time of the most recent meal. After a meal, blood sugar level increases rapidly, then slowly comes back down to a normal level. Sketch a person's blood sugar level as a function of time over the course of a day. Label the axes to indicate normal blood sugar level and the time of each meal.

28. When a parachutist jumps out of a plane, the speed of her fall increases until she opens her parachute, at which time her falling speed suddenly decreases and stays constant until she reaches the ground. Sketch a possible graph of the height H of the parachutist as a function of time t, from the time when she jumps from the plane to the time when she reaches the ground.

29. A buzzard is circling high overhead when it spies some road kill. It swoops down, lands, and eats. Later it takes off sluggishly, and resumes circling overhead, but at a lower altitude. Sketch a possible graph of the height of the buzzard as a function of time.

30. Table 1.6 gives the ranking r for three different names—Hannah, Alexis, and Madison. Of the three names, which was most popular and which was least popular in

(a) 1995? **(b)** 2004?

Table 1.6 *Ranking of names—Hannah (r_h), Alexis (r_a), and Madison (r_m)—for girls born between 1995 ($t = 0$) and 2004 ($t = 9$)*[2]

t	0	1	2	3	4	5	6	7	8	9
r_h	7	7	5	2	2	2	3	3	4	5
r_a	14	8	8	6	3	6	5	5	7	11
r_m	29	15	10	9	7	3	2	2	3	3

31. Table 1.6 gives information about the popularity of the names Hannah, Madison, and Alexis. Describe in words what your answers to parts (a)–(c) tell you about these names.

(a) Evaluate $r_m(0) - r_h(0)$.
(b) Evaluate $r_m(9) - r_h(9)$.
(c) Solve $r_m(t) < r_a(t)$.

32. Figure 1.7 shows the fuel consumption (in miles per gallon, mpg) of a car traveling at various speeds (in mph).

(a) How much gas is used on a 300-mile trip at 40 mph?
(b) How much gas is saved by traveling 60 mph instead of 70 mph on a 200-mile trip?
(c) According to this graph, what is the most fuel-efficient speed to travel? Explain.

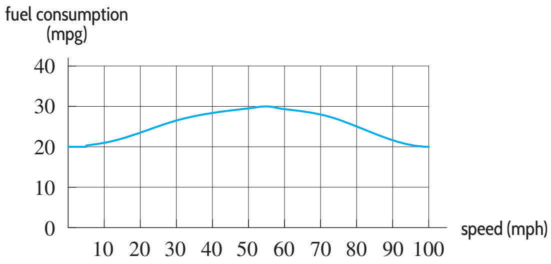
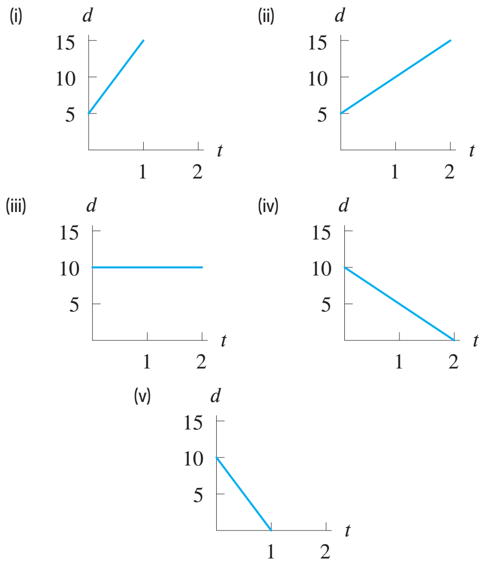


Figure 1.7

33. Match each story about a bike ride to one of the graphs (i)–(v), where d represents distance from home and t is time in hours since the start of the ride. (A graph may be used more than once.)

(a) Starts 5 miles from home and rides 5 miles per hour away from home.
(b) Starts 5 miles from home and rides 10 miles per hour away from home.
(c) Starts 10 miles from home and arrives home one hour later.
(d) Starts 10 miles from home and is halfway home after one hour.
(e) Starts 5 miles from home and is 10 miles from home after one hour.



34. Figure 1.8 shows the mass of water in air, in grams of water per kilogram of air, as a function of air temperature in °C, for two different levels of relative humidity.

(a) Find the mass of water in 1 kg of air at 30°C if the relative humidity is

(a) 100% **(b)** 50% **(c)** 75%

(b) How much water is in a room containing 300 kg of air if the relative humidity is 50% and the temperature is 20°C?
(c) The density of air is approximately 1.2 kg/m^3. If the relative humidity in your classroom is 50% and the temperature is 20°C, estimate the amount of water in the air.

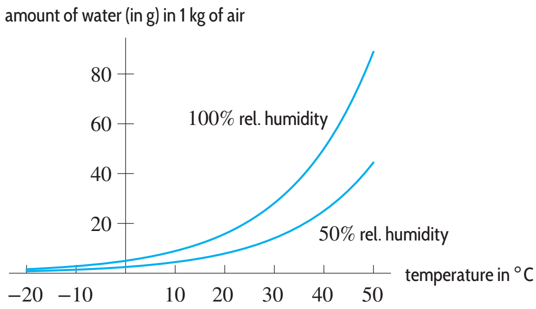


Figure 1.8

[2] Data from the SSA website at www.ssa.gov, accessed January 12, 2006.

35. Let $f(t)$ be the number of people, in millions, who own cell phones t years after 1990. Explain the meaning of the following statements.

(a) $f(10) = 100.3$ **(b)** $f(a) = 20$
(c) $f(20) = b$ **(d)** $n = f(t)$

36. **(a)** Ten inches of snow is equivalent to about one inch of rain.[3] Write an equation for the amount of precipitation, measured in inches of rain, $r = f(s)$, as a function of the number of inches of snow, s.
(b) Evaluate and interpret $f(5)$.
(c) Find s such that $f(s) = 5$ and interpret your result.

37. An 8-foot-tall cylindrical water tank has a base of diameter 6 feet.

(a) How much water can the tank hold?
(b) How much water is in the tank if the water is 5 feet deep?
(c) Write a formula for the volume of water as a function of its depth in the tank.

38. Table 1.7 gives $A = f(d)$, the amount of money in bills of denomination d circulating in US currency in 2013.[4] For example, there were \$74.5 billion worth of \$50 bills in circulation.

(a) Find $f(100)$. What does this tell you about money?
(b) Are there more \$1 bills or \$5 bills in circulation?

Table 1.7

Denomination (\$)	1	2	5	10	20	50	100
Circulation (\$bn)	10.6	2.1	12.7	18.5	155.0	74.5	924.7

39. Table 1.8 shows the daily low temperature for a one-week period in New York City during July.

(a) What was the low temperature on July 19?
(b) When was the low temperature 73°F?
(c) Is the daily low temperature a function of the date?
(d) Is the date a function of the daily low temperature?

Table 1.8

Date	17	18	19	20	21	22	23
Low temp (°F)	73	77	69	73	75	75	70

40. Use the data from Table 1.3 on page 5.

(a) Plot R on the vertical axis and t on the horizontal axis. Use this graph to explain why you believe that R is a function of t.
(b) Plot F on the vertical axis and t on the horizontal axis. Use this graph to explain why you believe that F is a function of t.
(c) Plot F on the vertical axis and R on the horizontal axis. From this graph show that F is not a function of R.
(d) Plot R on the vertical axis and F on the horizontal axis. From this graph show that R is not a function of F.

41. Since Roger Bannister broke the 4-minute mile on May 6, 1954, the record has been lowered by over sixteen seconds. Table 1.9 shows the year and times (as min:sec) of new world records for the one-mile run.[5] (Official records for the mile ended in 1999.)

(a) Is the time a function of the year? Explain.
(b) Is the year a function of the time? Explain.
(c) Let $y(r)$ be the year in which the world record, r, was set. Explain what is meant by the statement $y(3:47.33) = 1981$.
(d) Evaluate and interpret $y(3:51.1)$.

Table 1.9

Year	Time	Year	Time	Year	Time
1954	3:59.4	1966	3:51.3	1981	3:48.53
1954	3:58.0	1967	3:51.1	1981	3:48.40
1957	3:57.2	1975	3:51.0	1981	3:47.33
1958	3:54.5	1975	3:49.4	1985	3:46.32
1962	3:54.4	1979	3:49.0	1993	3:44.39
1964	3:54.1	1980	3:48.8	1999	3:43.13
1965	3:53.6				

42. The sales tax on an item is 6%. Express the total cost, C, in terms of the price of the item, P.

43. A price increases 5% due to inflation and is then reduced 10% for a sale. Express the final price as a function of the original price, P.

44. Write a formula for the area of a circle as a function of its radius and determine the percent increase in the area if the radius is increased by 10%.

45. There are x male job-applicants at a certain company and y female applicants. Suppose that 15% of the men are accepted and 18% of the women are accepted. Write an expression in terms of x and y representing each of the following quantities:

(a) The total number of applicants to the company.
(b) The total number of applicants accepted.
(c) The percentage of all applicants accepted.

[3] http://mo.water.usgs.gov/outreach/rain, accessed May 7, 2006.
[4] http://www.federalreserve.gov/paymentsystems/coin_currcircvalue.htm, accessed February 16, 2014.
[5] www.infoplease.com/ipsa/A0112924.html, accessed January 15, 2006.

46. A chemical company spends \$2 million to buy machinery before it starts producing chemicals. Then it spends \$0.5 million on raw materials for each million liters of chemical produced.

(a) The number of liters produced ranges from 0 to 5 million. Make a table showing the relationship between the number of million liters produced, l, and the total cost, C, in millions of dollars, to produce that number of million liters.

(b) Find a formula that expresses C as a function of l.

47. A person leaves home and walks due west for a time and then walks due north.

(a) The person walks 10 miles in total. If w represents the (variable) distance west she walks, and D represents her (variable) distance from home at the end of her walk, is D a function of w? Why or why not?

(b) Suppose now that x is the distance that she walks in total. Is D a function of x? Why or why not?

1.2 RATE OF CHANGE

The worldwide annual sales of smartphones have increased each year from when they were first introduced. To measure how fast sales increase, we calculate a *rate of change* in number of units sold

$$\frac{\text{Change in units sold}}{\text{Change in time}}.$$

Due to the rising popularity of smartphones, annual sales of regular cell phones, or *feature phones*, have been declining. See Table 1.10.

Let us calculate the rate of change of smartphone and feature phone sales between 2010 and 2013. Table 1.10 gives

$$\begin{array}{c}\text{Average rate of change of}\\ \text{smartphone sales from 2010 to 2013}\end{array} = \frac{\text{Change in sales}}{\text{Change in time}} = \frac{968-301}{2013-2010} \approx 222.3 \ \begin{array}{l}\text{mn /}\\ \text{year.}\end{array}$$

Thus, the number of smartphones sold has increased on average by 222.3 million units per year between 2010 and 2013. See Figure 1.9. Similarly, Table 1.10 gives

$$\begin{array}{c}\text{Average rate of change of feature phone}\\ \text{sales from 2010 to 2013}\end{array} = \frac{\text{Change in sales}}{\text{Change in time}} = \frac{838-1079}{2013-2010} \approx -80.3 \ \begin{array}{l}\text{mn/}\\ \text{year.}\end{array}$$

Thus, the number of feature phones sold has decreased on average by 80.3 million units per year between 2010 and 2013. See Figure 1.10.

Table 1.10 *Worldwide annual sales of smartphones and feature phones*[6]

Year	2010	2011	2012	2013
Smartphone sales (millions of units)	301	480	661	968
Feature phone sales (millions of units)	1079	1075	914	838

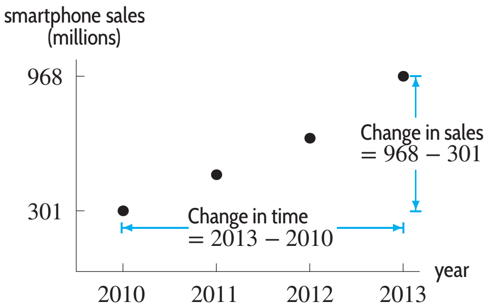


Figure 1.9: Smartphone sales

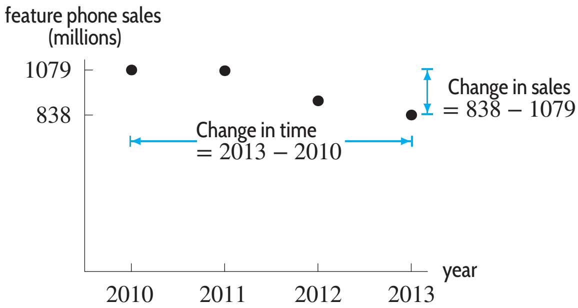


Figure 1.10: Feature phone sales

[6] Accessed Feb. 14 2014, www.fiercewireless.com/europe/story/gartner-smartphones-outsell-feature-phones-2013/2014-02-13 and www.fiercewireless.com/europe/special-reports/analyzing-worlds-14-biggest-handset-makers-q2-2013

Rate of Change of a Function

The rate of change of sales is an example of the rate of change of a function. In general, if $Q = f(t)$, we write ΔQ for a change in Q and Δt for a change in t. We define:[7]

> The **average rate of change**, or **rate of change**, of Q with respect to t over an interval is
>
> $$\begin{matrix}\text{Average rate of change}\\ \text{over an interval}\end{matrix} = \frac{\text{Change in } Q}{\text{Change in } t} = \frac{\Delta Q}{\Delta t}.$$

The average rate of change of the function $Q = f(t)$ over an interval tells us how much Q changes, on average, for each unit change in t within that interval. On some parts of the interval, Q may be changing rapidly, while on other parts Q may be changing slowly. The average rate of change evens out these variations.

Increasing and Decreasing Functions

In the previous example, the average rate of change of smartphone sales is positive on the interval from 2010 to 2013 since the number of smartphones sold increased over this interval. Similarly, the average rate of change of feature phone sales is negative on the same interval since the number of feature phones sold decreased over this interval. The annual sales of smartphones is an example of an *increasing function* and the annual sales of feature phones is an example of a *decreasing function*. In general we say the following:

> If $Q = f(t)$ for t in the interval $a \le t \le b$,
> - f is an **increasing function** if the values of f increase as t increases in this interval.
> - f is a **decreasing function** if the values of f decrease as t increases in this interval.

Looking at smartphone sales, we see that an increasing function has a positive rate of change. From the feature phone sales, we see that a decreasing function has a negative rate of change. In general:

> If $Q = f(t)$,
> - If f is an increasing function, then the average rate of change of Q with respect to t is positive on every interval.
> - If f is a decreasing function, then the average rate of change of Q with respect to t is negative on every interval.

Example 1 The function $A = q(r) = \pi r^2$ gives the area, A, of a circle as a function of its radius, r. Graph q. Explain how the fact that q is an increasing function can be seen on the graph.

Solution The area increases as the radius increases, so $A = q(r)$ is an increasing function. We can see this in Figure 1.11 because the graph climbs as we move from left to right and the average rate of change, $\Delta A/\Delta r$, is positive on every interval.

[7] The Greek letter Δ, delta, is often used in mathematics to represent change. In this book, we use rate of change to mean average rate of change across an interval. In calculus, rate of change means something called instantaneous rate of change.

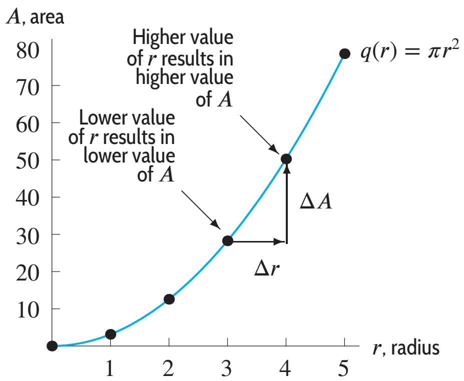


Figure 1.11: The graph of an increasing function, $A = q(r)$, rises when read from left to right

Example 2 Carbon-14 is a radioactive element that exists naturally in the atmosphere and is absorbed by living organisms. When an organism dies, the carbon-14 present at death begins to decay. Let $L = g(t)$ represent the quantity of carbon-14 (in micrograms, μg) in a tree t years after its death. See Table 1.11. Explain why we expect g to be a decreasing function of t and how the graph displays this.

Table 1.11 *Quantity of carbon-14 as a function of time*

t, time (years)	0	1000	2000	3000	4000	5000
L, quantity of carbon-14 (μg)	200	177	157	139	123	109

Solution Since the amount of carbon-14 is decaying over time, g is a decreasing function. In Figure 1.12, the graph falls as we move from left to right and the average rate of change in the level of carbon-14 with respect to time, $\Delta L/\Delta t$, is negative on every interval.

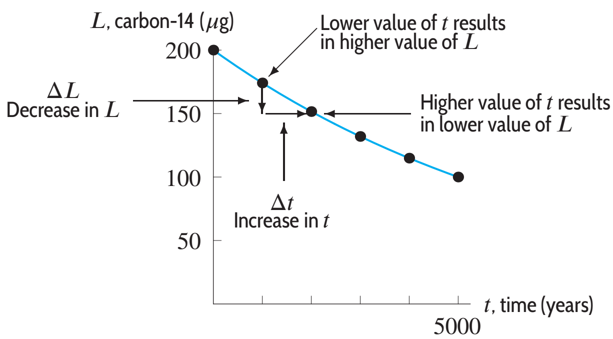


Figure 1.12: The graph of a decreasing function, $L = g(t)$, falls when read from left to right

In general, we can identify an increasing or decreasing function from its graph as follows:

- The graph of an increasing function rises when read from left to right.
- The graph of a decreasing function falls when read from left to right.

Many functions have some intervals on which they are increasing and other intervals on which they are decreasing. These intervals can often be identified from the graph.

Example 3 On what intervals is the function graphed in Figure 1.13 increasing? Decreasing?

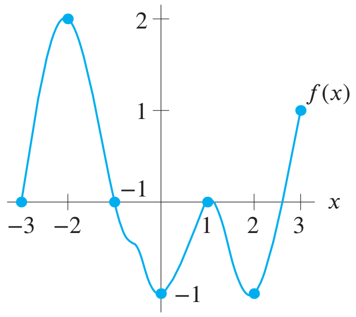


Figure 1.13: Graph of a function that is increasing on some intervals and decreasing on others

Solution The function appears to be increasing for values of x between -3 and -2, for x between 0 and 1, and for x between 2 and 3. The function appears to be decreasing for x between -2 and 0 and for x between 1 and 2. Using inequalities, we say that f is increasing for $-3 < x < -2$, for $0 < x < 1$, and for $2 < x < 3$. Similarly, f is decreasing for $-2 < x < 0$ and $1 < x < 2$.

Function Notation for the Average Rate of Change

Suppose we want to find the average rate of change of a function $Q = f(t)$ over the interval $a \le t \le b$. On this interval, the change in t is given by

$$\Delta t = b - a.$$

At $t = a$, the value of Q is $f(a)$, and at $t = b$, the value of Q is $f(b)$. Therefore, the change in Q is given by

$$\Delta Q = f(b) - f(a).$$

Using function notation, we express the average rate of change as follows:

$$\begin{array}{c}\text{Average rate of change of } Q = f(t) \\ \text{over the interval } a \le t \le b\end{array} = \frac{\text{Change in } Q}{\text{Change in } t} = \frac{\Delta Q}{\Delta t} = \frac{f(b) - f(a)}{b - a}.$$

In Figure 1.14, notice that the average rate of change is given by the ratio of the rise, $f(b) - f(a)$, to the run, $b - a$. This ratio is also called the *slope* of the dashed line segment.[8]

In the future, we may drop the word "average" and talk about the rate of change over an interval.

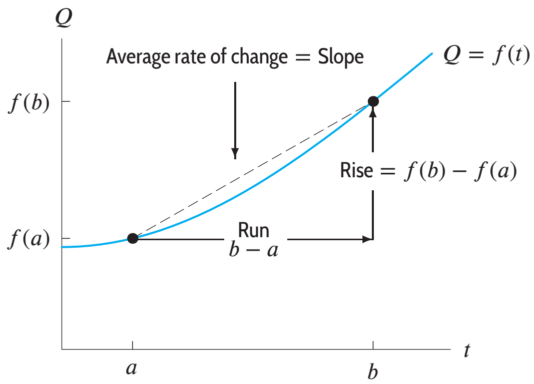


Figure 1.14: The average rate of change is the ratio Rise/Run

[8] See Section 1.3 for further discussion of slope.

In previous examples we calculated the average rate of change from data. We now calculate average rates of change for functions given by formulas.

Example 4 Calculate the average rates of change of the function $f(x) = x^2$ between $x = 1$ and $x = 3$ and between $x = -2$ and $x = 1$. Show your results on a graph.

Solution Between $x = 1$ and $x = 3$, we have

$$\begin{array}{c}\text{Average rate of change of } f(x)\\ \text{over the interval } 1 \le x \le 3\end{array} = \frac{\text{Change in } f(x)}{\text{Change in } x} = \frac{f(3) - f(1)}{3 - 1} = \frac{3^2 - 1^2}{3 - 1} = \frac{9 - 1}{2} = 4.$$

Between $x = -2$ and $x = 1$, we have

$$\begin{array}{c}\text{Average rate of change of } f(x)\\ \text{over the interval } -2 \le x \le 1\end{array} = \frac{\text{Change in } f(x)}{\text{Change in } x} = \frac{f(1) - f(-2)}{1 - (-2)} = \frac{1^2 - (-2)^2}{1 - (-2)} = \frac{1 - 4}{3} = -1.$$

The average rate of change between $x = 1$ and $x = 3$ is positive because $f(x)$ is increasing on this interval. See Figure 1.15. However, on the interval from $x = -2$ and $x = 1$, the function is partly decreasing and partly increasing. The average rate of change on this interval is negative because the decrease on the interval is larger than the increase.

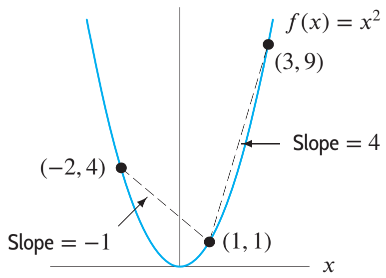


Figure 1.15: Average rate of change of $f(x)$ on an interval is the slope of the dashed line on that interval

Exercises and Problems for Section 1.2

Skill Refresher

In Exercises S1–S10, simplify each expression.

S1. $\dfrac{4 - 6}{3 - 2}$

S2. $\dfrac{1 - 3}{2^2 - (-3)^2}$

S3. $\dfrac{-3 - (-9)}{-1 - 2}$

S4. $\dfrac{(1 - 3^2) - (1 - 4^2)}{3 - 4}$

S5. $\dfrac{\frac{1}{2} - (-4)^2 - \left(\frac{1}{2} - (5^2)\right)}{-4 - 5}$

S6. $2(x + a) - 3(x - b)$

S7. $x^2 - (2x + a)^2$

S8. $4x^2 - (x - b)^2$

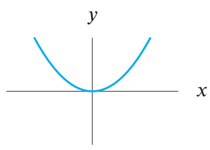
S9. $\dfrac{x^2 - \frac{3}{4} - \left(y^2 - \frac{3}{4}\right)}{x - y}$

S10. $\dfrac{2(x + h)^2 - 2x^2}{(x + h) - x}$

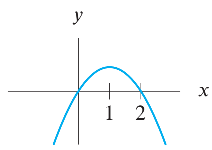
Exercises

In Exercises 1–4, on what intervals is the function increasing? Decreasing?

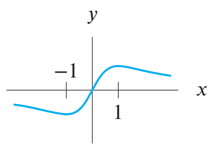
1.



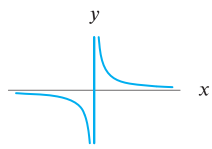
2.



3.



4.



5. Table 1.10 on page 10 gives the annual sales (in millions) of smartphones and feature phones. What was the average rate of change of annual sales of each of them between

(a) 2010 and 2012? **(b)** 2012 and 2013?
(c) Interpret these results in terms of sales.

6. Table 1.10 on page 10 shows that feature phone sales are a function of smartphone sales. Is this function increasing or decreasing?

7. In 2007, you have 40 songs in your favorite iTunes playlist. In 2010, you have 120 songs. In 2014, you have 40. What is the average rate of change per year in the number of songs in your favorite iTunes playlist between

(a) 2007 and 2010? **(b)** 2010 and 2014?
(c) 2007 and 2014?

8. Table 1.12 gives the populations of two cities (in thousands) over a 17-year period.

(a) Find the average rate of change of each population on the following intervals:

(i) 1996 to 2006 (ii) 1996 to 2013
(iii) 2001 to 2013

(b) What do you notice about the average rate of change of each population? Explain what the average rate of change tells you about each population.

Table 1.12

Year	1996	1998	2001	2006	2013
P_1	42	46	52	62	76
P_2	82	80	77	72	65

9. Figure 1.16 shows distance traveled as a function of time.

(a) Find ΔD and Δt between:

(i) $t = 2$ and $t = 5$ (ii) $t = 0.5$ and $t = 2.5$
(iii) $t = 1.5$ and $t = 3$

(b) Compute the rate of change, $\Delta D/\Delta t$, over each of the intervals in part (a), and interpret its meaning.

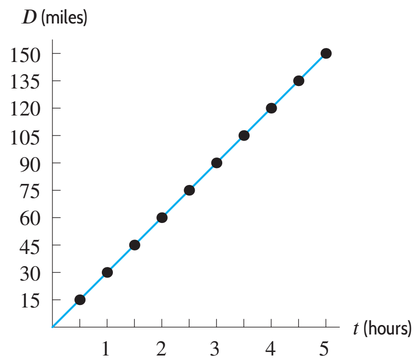


Figure 1.16

Exercises 10–14 use Figure 1.17.

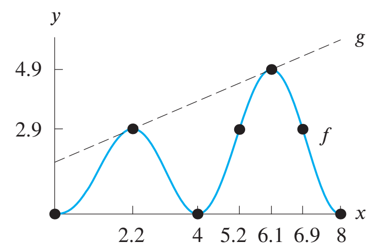


Figure 1.17

10. Find the average rate of change of f for $2.2 \leq x \leq 6.1$.

11. Give two different intervals on which $\Delta f(x)/\Delta x = 0$.

12. What is the average rate of change of g between $x = 2.2$ and $x = 6.1$?

13. What is the relation between the average rate of change of f and the average rate of change of g between $x = 2.2$ and $x = 6.1$?

14. Is the rate of change of f positive or negative on the following intervals?

(a) $2.2 \leq x \leq 4$ **(b)** $5 \leq x \leq 6$

15. If F is a decreasing function, what can you say about $F(-2)$ compared to $F(2)$?

16. If G is an increasing function, what can you say about $G(3) - G(-1)$?

Problems

17. Figure 1.18 shows the percent of the side of the moon toward the earth illuminated by the sun at different times during the year 2008. Use the figure to answer the following questions.

(a) Give the coordinates of the points A, B, C, D, E.
(b) Plot the point $F = (15, 60)$ and $G = (60, 15)$. Which point is on the graph?
(c) During which time intervals is the function increasing?
(d) During which time intervals is the function decreasing?

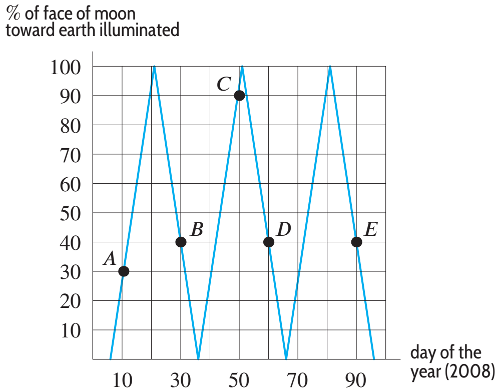


Figure 1.18: Moon phases

18. Figure 1.19 gives the population of two different towns over a 50-year period of time.

(a) Which town starts (in year $t = 0$) with the most people?
(b) Which town is growing faster over these 50 years?

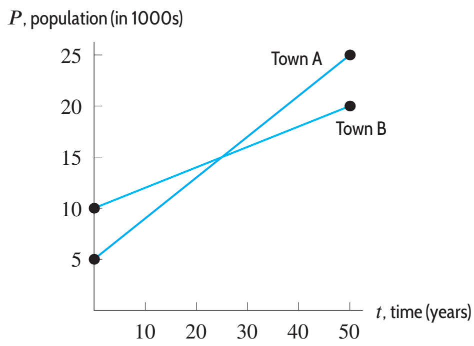


Figure 1.19

19. **(a)** What is the average rate of change of $g(x) = 2x - 3$ between the points $(-2, -7)$ and $(3, 3)$?
(b) The function g is either increasing or decreasing everywhere. Explain how your answer to part (a) tells you which.
(c) Graph the function.

20. **(a)** Let $f(x) = 16 - x^2$. Compute each of the following expressions, and interpret each as an average rate of change.

(i) $\dfrac{f(2) - f(0)}{2 - 0}$ (ii) $\dfrac{f(4) - f(2)}{4 - 2}$

(iii) $\dfrac{f(4) - f(0)}{4 - 0}$

(b) Graph $f(x)$. Illustrate each ratio in part (a) by sketching the line segment with the given slope. Over which interval is the average rate of decrease the greatest?

21. Imagine you constructed a list of the world record times for a particular event—such as the mile footrace, or the 100-meter freestyle swimming race—in terms of when they were established. Is the world record time a function of the date when it was established? If so, is this function increasing or decreasing? Explain. Could a world record be established twice in the same year? Is the world record time a function of the year it was established?

22. The function $P = f(t)$ gives the population of a town, in thousands, after t years. A graph of f is given in Figure 1.20.

(a) Find the average rate of change of the population of the town during the first 10 years.
(b) Does the population of the town grow more between $t = 5$ and $t = 10$ years, or between $t = 15$ and $t = 30$ years? Explain.
(c) Does the population of the town grow faster between $t = 5$ and $t = 10$ years, or between $t = 15$ and $t = 30$ years? Explain.

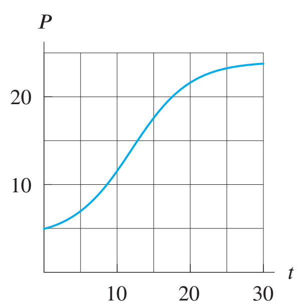


Figure 1.20

23. The most freakish change in temperature ever recorded was from −4°F to 45°F between 7:30 am and 7:32 am on January 22, 1943 at Spearfish, South Dakota.[9] What was the average rate of change of the temperature for this time period?

24. You have zero dollars now and the average rate of change in your net worth is $5000 per year. How much money will you have in forty years?

25. The surface of the sun has dark areas known as sunspots, that are cooler than the rest of the sun's surface. The number of sunspots[10] fluctuates with time, as shown in Figure 1.21.

(a) Explain how you know the number of sunspots, s, in year t is a function of t.
(b) Approximate the time intervals on which s is an increasing function of t.

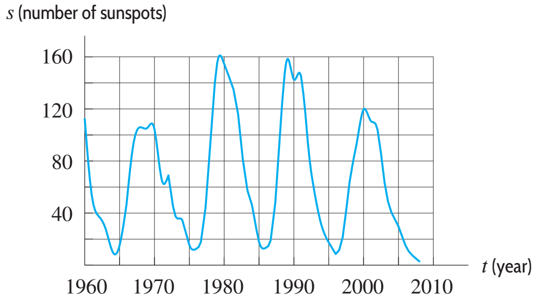


Figure 1.21

26. Table 1.13 shows the number of calories used per minute as a function of body weight for three sports.[11]

(a) Determine the number of calories that a 200-lb person uses in one half-hour of walking.
(b) Who uses more calories, a 120-lb person swimming for one hour or a 220-lb person bicycling for a half-hour?
(c) Does the number of calories used by a person walking increase or decrease as weight increases?

Table 1.13

Activity	100 lb	120 lb	150 lb	170 lb	200 lb	220 lb
Walking	2.7	3.2	4.0	4.6	5.4	5.9
Bicycling	5.4	6.5	8.1	9.2	10.8	11.9
Swimming	5.8	6.9	8.7	9.8	11.6	12.7

27. Because scientists know how much carbon-14 a living organism should have in its tissues, they can measure the amount of carbon-14 present in the tissue of a fossil and then calculate how long it took for the original amount to decay to the current level, thus determining the time of the organism's death. A tree fossil is found to contain 130 μg of carbon-14, and scientists determine from the size of the tree that it would have contained 200 μg of carbon-14 at the time of its death. Using Table 1.11 on page 12, approximately how long ago did the tree die?

28. Figure 1.22 shows the graph of the function $g(x)$.

(a) Estimate $\dfrac{g(4)-g(0)}{4-0}$.
(b) The ratio in part (a) is the slope of a line segment joining two points on the graph. Sketch this line segment on the graph.
(c) Estimate $\dfrac{g(b)-g(a)}{b-a}$ for $a=-9$ and $b=-1$.
(d) On the graph, sketch the line segment whose slope is given by the ratio in part (c).

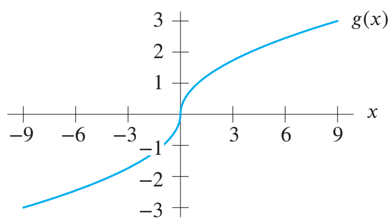


Figure 1.22

29. Find the average rate of change of $f(x)=3x^2+1$ between the points

(a) $(1,4)$ and $(2,13)$ **(b)** (j,k) and (m,n)
(c) $(x, f(x))$ and $(x+h, f(x+h))$

30. A water company employee measured the water level (in centimeters) in a reservoir during March and computed the average rates of change in Table 1.14.

(a) What are the units of the average rates of change in Table 1.14?
(b) Interpret the average rates of change in context.
(c) For each time interval, what was the total change in water level?
(d) Draw a possible graph of the water level as a function of time.

Table 1.14

Interval (days)	$1 \le t < 11$	$11 \le t < 16$	$16 \le t < 31$
Average rate of change	7	5	3

[9] *The Guinness Book of Records.* 1995.
[10] ftp://ftp.ngdc.noaa.gov/STP/SOLAR DATA/SUNSPOT NUMBERS/YEARLY.PLT, accessed November 30, 2009.
[11] From *1993 World Almanac.*

31. Table 1.15 gives the amount of garbage, G, in millions of tons, produced[12] in the US in year t.

(a) What is the value of Δt for consecutive entries in this table?
(b) Calculate the value of ΔG for each pair of consecutive entries in this table.
(c) Are all the values of ΔG you found in part (b) the same? What does this tell you?
(d) The function G changed from increasing to decreasing between 2005 and 2010. To what might this be attributed?

Table 1.15

t	1960	1970	1980	1990	2000	2005	2010
G	88.1	121.1	151.6	208.3	242.5	252.7	249.9

32. Table 1.16 shows the times, t, in sec, achieved every 10 meters by Carl Lewis in the 100-meter final of the World Championship in Rome in 1987.[13] Distance, d, is in meters.

(a) For each successive time interval, calculate the average rate of change of distance. What is a common name for the average rate of change of distance?
(b) Where did Carl Lewis attain his maximum speed during this race? Some runners are running their fastest as they cross the finish line. Does that seem to be true in this case?

Table 1.16

t	0.00	1.94	2.96	3.91	4.78	5.64
d	0	10	20	30	40	50
t	6.50	7.36	8.22	9.07	9.93	
d	60	70	80	90	100	

1.3 LINEAR FUNCTIONS

Constant Rate of Change

In the previous section, we introduced the average rate of change of a function on an interval. For many functions, the average rate of change is different on different intervals. For the remainder of this chapter, we consider functions that have the same average rate of change on every interval. Such a function has a graph that is a line and is called *linear*.

Population Growth

Mathematical models of population growth are used by city planners to project the growth of towns and states. Biologists model the growth of animal populations and physicians model the spread of an infection in the bloodstream. One possible model, a linear model, assumes that the population changes at the same average rate on every time interval.

Example 1 A town of 30,000 people grows by 2000 people every year. Since the population, P, is growing at the constant rate of 2000 people per year, P is a linear function of time, t, in years.

(a) What is the average rate of change of P over every time interval?
(b) Make a table that gives the town's population every five years over a 20-year period. Graph the population.
(c) Find a formula for P as a function of t.

Solution

(a) The average rate of change of population with respect to time is 2000 people per year.
(b) The initial population in year $t = 0$ is $P = 30{,}000$ people. Since the town grows by 2000 people every year, after five years it has grown by

$$\frac{2000 \text{ people}}{\text{year}} \cdot 5 \text{ years} = 10{,}000 \text{ people}.$$

Thus, in year $t = 5$ the population is given by

$$P = \text{Initial population} + \text{New population} = 30{,}000 + 10{,}000 = 40{,}000.$$

[12] http://http://www.epa.gov/osw/nonhaz/municipal/pubs/msw_2010_rev_factsheet.pdf, accessed January, 2013.
[13] W. G. Pritchard, "Mathematical Models of Running", *SIAM Review.* 35, 1993, pp. 359–379.

In year $t = 10$ the population is given by

$$P = 30{,}000 + \underbrace{2000 \text{ people/year} \cdot 10 \text{ years}}_{20{,}000 \text{ new people}} = 50{,}000.$$

Similar calculations for year $t = 15$ and year $t = 20$ give the values in Table 1.17. See Figure 1.23; the dashed line shows the trend in the data.

Table 1.17 *Population over 20 years*

t, years	P, population
0	30,000
5	40,000
10	50,000
15	60,000
20	70,000

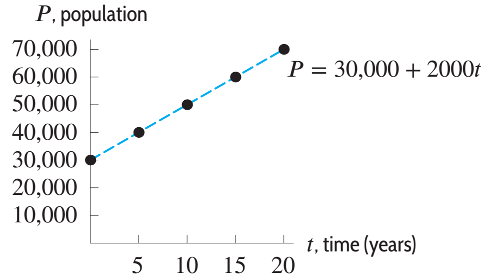


Figure 1.23: Town's population over 20 years

(c) From part (b), we see that the size of the population is given by

$$\begin{aligned} P &= \text{Initial population} + \text{Number of new people} \\ &= 30{,}000 + 2000 \text{ people/year} \cdot \text{Number of years,} \end{aligned}$$

so a formula for P in terms of t is

$$P = 30{,}000 + 2000t.$$

The graph of the population data in Figure 1.23 is a straight line. The average rate of change of the population over every interval is the same, namely 2000 people per year. Any linear function has the same average rate of change over every interval. Thus, we talk about *the* rate of change of a linear function. In general:

- A **linear function** has a constant rate of change.
- The graph of any linear function is a straight line.

Financial Models

Economists and accountants use linear functions for *straight-line depreciation*. For tax purposes, the value of certain equipment is considered to decrease, or depreciate, over time. For example, computer equipment may be state-of-the-art today, but after several years it is outdated. Straight-line depreciation assumes that the rate of change of value with respect to time is constant.

Example 2 A small business spends \$20,000 on new computer equipment and, for tax purposes, chooses to depreciate it to \$0 at a constant rate over a five-year period.

(a) Make a table and a graph showing the value of the equipment over the five-year period.
(b) Give a formula for value as a function of time.

Solution (a) After five years, the equipment is valued at \$0. If V is the value in dollars and t is the number of years, we see that

$$\begin{array}{c}\text{Rate of change of value} \\ \text{from } t = 0 \text{ to } t = 5\end{array} = \frac{\text{Change in value}}{\text{Change in time}} = \frac{\Delta V}{\Delta t} = \frac{-\$20{,}000}{5 \text{ years}} = -\$4000 \text{ per year.}$$

Thus, the value drops at the constant rate of \$4000 per year. (Notice that ΔV is negative because the value of the equipment decreases.) See Table 1.18 and Figure 1.24. Since V changes at a constant rate, $V = f(t)$ is a linear function and its graph is a straight line. The rate of change, $-\$4000$ per year, is negative because the function is decreasing and the graph slopes down.

Table 1.18 *Value of equipment depreciated over a 5-year period*

t, year	V, value (\$)
0	20,000
1	16,000
2	12,000
3	8000
4	4000
5	0

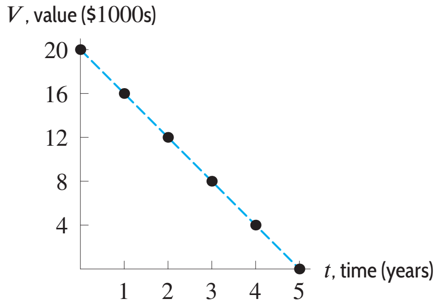


Figure 1.24: Value of equipment depreciated over a 5-year period

(b) After t years have elapsed,

$$\text{Decrease in value of equipment} = \$4000 \cdot \text{Number of years} = \$4000t.$$

The initial value of the equipment is \$20,000, so at time t,

$$V = 20{,}000 - 4000t.$$

A General Formula for the Family of Linear Functions

Example 1 involved a town whose population is growing at a constant rate with formula

$$\text{Current population} = \underbrace{\text{Initial population}}_{30{,}000 \text{ people}} + \underbrace{\text{Growth rate}}_{2000 \text{ people per year}} \times \underbrace{\text{Number of years}}_{t}$$

so

$$P = 30{,}000 + 2000t.$$

In Example 2, the value, V, as a function of t is given by

$$\text{Total cost} = \underbrace{\text{Initial value}}_{\$20{,}000} + \underbrace{\text{Change per year}}_{-\$4000 \text{ per year}} \times \underbrace{\text{Number of years}}_{t}$$

so

$$V = 20{,}000 + (-4000)t.$$

Using the symbols x, y, b, m, we see formulas for both of these linear functions follow the same pattern:

$$\underbrace{\text{Output}}_{y} = \underbrace{\text{Initial value}}_{b} + \underbrace{\text{Rate of change}}_{m} \times \underbrace{\text{Input}}_{x}.$$

Summarizing, we get the following results:

If $y = f(x)$ is a linear function, then for some constants b and m:

$$y = b + mx.$$

- m is called the **slope**, and gives the rate of change of y with respect to x. Thus,

$$m = \frac{\Delta y}{\Delta x}.$$

  If (x_0, y_0) and (x_1, y_1) are any two distinct points on the graph of f, then

$$m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}.$$

- b is called the **vertical intercept**, or ***y*-intercept**, and gives the value of y for $x = 0$. In mathematical models, b typically represents an initial, or starting, value of the output.

Every linear function can be written in the form $y = b + mx$. Different linear functions have different values for m and b. These constants are known as *parameters*; the set of all linear functions is called a *family*.

Example 3 In Example 1, the population function, $P = 30{,}000 + 2000t$, has slope $m = 2000$ and vertical intercept $b = 30{,}000$. In Example 2, the value of the computer equipment, $V = 20{,}000 - 4000t$, has slope $m = -4000$ and vertical intercept $b = 20{,}000$.

Tables for Linear Functions

A table of values could represent a linear function if the rate of change is constant, for all pairs of points in the table; that is,

$$\text{Rate of change of linear function} = \frac{\text{Change in output}}{\text{Change in input}} = \text{Constant.}$$

Thus, if the value of x goes up by equal steps in a table for a linear function, then the value of y goes up (or down) by equal steps as well. We say that changes in the value of y are *proportional* to changes in the value of x.

Example 4 Table 1.19 gives values of two functions, p and q. Could either of these functions be linear?

Table 1.19 *Values of two functions p and q*

x	50	55	60	65	70
$p(x)$	0.10	0.11	0.12	0.13	0.14
$q(x)$	0.01	0.03	0.06	0.14	0.15

Solution The value of x goes up by equal steps of $\Delta x = 5$. The value of $p(x)$ also goes up by equal steps of $\Delta p = 0.01$, so $\Delta p/\Delta x$ is a constant. See Table 1.20. Thus, p could be a linear function.

Table 1.20 *Values of $\Delta p/\Delta x$*

x	$p(x)$	Δp	$\Delta p/\Delta x$
50	0.10		
		0.01	0.002
55	0.11		
		0.01	0.002
60	0.12		
		0.01	0.002
65	0.13		
		0.01	0.002
70	0.14		

Table 1.21 *Values of $\Delta q/\Delta x$*

x	$q(x)$	Δq	$\Delta q/\Delta x$
50	0.01		
		0.02	0.004
55	0.03		
		0.03	0.006
60	0.06		
		0.08	0.016
65	0.14		
		0.01	0.002
70	0.15		

In contrast, the value of $q(x)$ does not go up by equal steps. The value climbs by 0.02, then by 0.03, and so on. See Table 1.21. This means that $\Delta q/\Delta x$ is not constant. Thus, q could not be a linear function.

It is possible to have data from a linear function in which neither the x-values nor the y-values go up by equal steps. However the rate of change must be constant, as in the following example.

Example 5 The former Republic of Yugoslavia exported cars called Yugos to the US between 1985 and 1989. The car is now a collector's item.[14] Table 1.22 gives the quantity of Yugos sold, Q, and the price, p, for each year from 1985 to 1988.

(a) Using Table 1.22, explain why Q could be a linear function of p.
(b) What does the rate of change of this function tell you about Yugos?

Table 1.22 *Price and sales of Yugos in the US*

Year	Price in \$, p	Number sold, Q
1985	3990	49,000
1986	4110	43,000
1987	4200	38,500
1988	4330	32,000

Solution (a) We are interested in Q as a function of p, so we plot Q on the vertical axis and p on the horizontal axis. The data points in Figure 1.25 appear to lie on a straight line, suggesting a linear function.

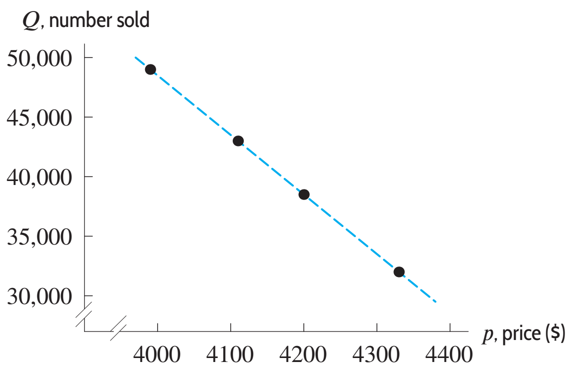


Figure 1.25: Since the data from Table 1.22 falls on a straight line, the table could represent a linear function

[14] www.inet.hr/~pauric/epov.htm, accessed January 16, 2006.

To provide further evidence that Q is a linear function, we check that the rate of change of Q with respect to p is constant for the points given. When the price of a Yugo rose from \$3990 to \$4110, sales fell from 49,000 to 43,000. Thus,

$$\Delta p = 4110 - 3990 = 120,$$

$$\Delta Q = 43{,}000 - 49{,}000 = -6000.$$

Since the number of Yugos sold decreased, ΔQ is negative. Thus, as the price increased from \$3990 to \$4110,

$$\text{Rate of change of quantity as price increases} = \frac{\Delta Q}{\Delta p} = \frac{-6000}{120} = -50 \text{ cars per dollar.}$$

Next, we calculate the rate of change as the price increased from \$4110 to \$4200 to see if the rate remains constant:

$$\text{Rate of change} = \frac{\Delta Q}{\Delta p} = \frac{38{,}500 - 43{,}000}{4200 - 4110} = \frac{-4500}{90} = -50 \text{ cars per dollar,}$$

and as the price increased from \$4200 to \$4330:

$$\text{Rate of change} = \frac{\Delta Q}{\Delta p} = \frac{32{,}000 - 38{,}500}{4330 - 4200} = \frac{-6500}{130} = -50 \text{ cars per dollar.}$$

Since the rate of change, -50, is constant, Q could be a linear function of p. Given additional data, $\Delta Q/\Delta p$ might not remain constant. However, based on the table, it appears that the function is linear.

(b) Since ΔQ is the change in the number of cars sold and Δp is the change in price, the rate of change is -50 cars per dollar. Thus the number of Yugos sold decreased by 50 each time the price increased by \$1.

Warning: Not All Graphs That Look Like Lines Represent Linear Functions

The graph of any linear function is a line. However, a function's graph can look like a line without actually being one. Consider the following example.

Example 6 The function $P = 100(1.02)^t$ approximates the population of Mexico in the early 2000s. Here P is the population (in millions) and t is the number of years since 2000. Table 1.23 and Figure 1.26 show values of P over a 5-year period. Is P a linear function of t?

Table 1.23 *Population of Mexico t years after 2000*

t (years)	P (millions)
0	100
1	102
2	104.04
3	106.12
4	108.24
5	110.41

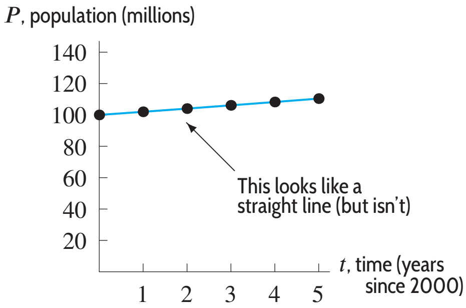


Figure 1.26: Graph of $P = 100(1.02)^t$ over 5-year period: Looks linear (but is not)

Solution The formula $P = 100(1.02)^t$ cannot be written in the form $P = b+mt$, so P is not a linear function of t. However, the graph of P in Figure 1.26 appears to be a straight line. We check P's rate of change in Table 1.23. When $t = 0$, $P = 100$ and when $t = 1$, $P = 102$. Thus, between 2000 and 2001,

$$\text{Rate of change of population} = \frac{\Delta P}{\Delta t} = \frac{102-100}{1-0} = 2.$$

For the interval from 2001 to 2002, we have

$$\text{Rate of change} = \frac{\Delta P}{\Delta t} = \frac{104.04-102}{2-1} = 2.04,$$

and for the interval from 2004 to 2005, we have

$$\text{Rate of change} = \frac{\Delta P}{\Delta t} = \frac{110.41-108.24}{5-4} = 2.17.$$

Thus, P's rate of change is not constant. In fact, P appears to be increasing at a faster and faster rate. Table 1.24 and Figure 1.27 show values of P over a longer (60-year) period. On this scale, these points do not appear to fall on a straight line. However, the graph of P curves upward so gradually at first that over the short interval shown in Figure 1.26, it barely curves at all. The graphs of many nonlinear functions, when viewed on a small scale, appear to be linear.

Table 1.24 *Population over 60 years*

t (years since 2000)	P (millions)
0	100
10	121.90
20	148.59
30	181.14
40	220.80
50	269.16
60	328.10

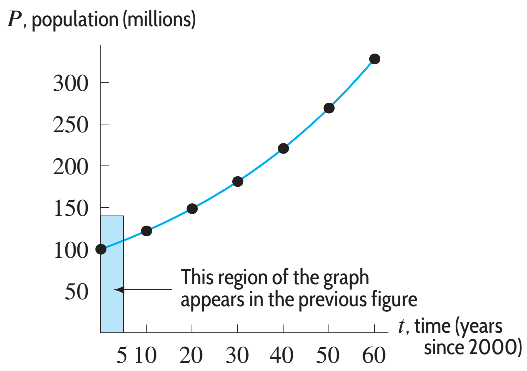


Figure 1.27: Graph of $P = 100(1.02)^t$ over 60 years: Not linear

Exercises and Problems for Section 1.3

Skill Refresher

In Exercises S1–S2, find $f(0)$ and $f(3)$.

S1. $f(x) = \frac{2}{3}x + 5$

S2. $f(t) = 17 - 4t$

In Exercises S3–S4, find $f(2) - f(0)$.

S3.

x	0	1	2	3
$f(x)$	−2	0	3	4

S4.

t	−1	0	1	2
$f(t)$	0	2	7	−1

In Exercises S5–S6, find the coordinates of the x and y intercepts.

S5. $y = -4x + 3$

S6. $5x - 2y = 4$

For each of the linear expressions in x in Exercises S7–S10, give the constant term and the coefficient of x.

S7. $3 - 2x + \frac{1}{2}$

S8. $4 - 3(x + 2) + 6(2x - 1)$

S9. $ax - ab - 3x + a + 3$

S10. $5(x - 1) + 3$

Exercises

In Exercises 1–3, which line has the greater

(a) Slope? **(b)** y-intercept?

1. $y = -1 + 2x$; $y = -2 + 3x$

2. $y = 3 + 4x$; $y = 5 - 2x$

3. $y = \frac{1}{4}x$; $y = 1 - 6x$

In Exercises 4–9, could the data represent a linear function? If so, give the rate of change.

4.

x	0	5	10	15
$f(x)$	10	20	30	40

5.

x	0	10	20	30
$h(x)$	20	40	50	55

6.

x	0	100	300	600
$g(x)$	50	100	150	200

7.

t	1	2	3	4	5
$g(t)$	5	4	5	4	5

8.

x	-3	-1	0	3
$j(x)$	5	1	-1	-7

9.

γ	9	8	7	6	5
$p(\gamma)$	42	52	62	72	82

In Exercises 10–13, identify the vertical intercept and the slope, and explain their meanings in practical terms.

10. The population of a town can be represented by the formula $P(t) = 54.25 - \frac{2}{7}t$, where $P(t)$ represents the population, in thousands, and t represents the time, in years, since 1970.

11. A stalactite grows according to the formula $L(t) = 17.75 + \frac{1}{250}t$, where $L(t)$ represents the length of the stalactite, in inches, and t represents the time, in years, since the stalactite was first measured.

12. The profit, in dollars, of selling n items is given by $P(n) = 0.98n - 3000$.

13. A phone company charges according to the formula $C(n) = 29.99 + 0.05n$, where n is the number of minutes, and $C(n)$ is the monthly phone charge, in dollars.

Problems

14. Table 1.25 shows the cost C, in dollars, of selling x cups of coffee per day from a cart.

(a) Using the table, show that the relationship appears to be linear.
(b) Plot the data in the table.
(c) Find the slope of the line. Explain what this means in the context of the given situation.
(d) Why should it cost \$50 to serve zero cups of coffee?

Table 1.25

x	0	5	10	50	100	200
C	50.00	51.25	52.50	62.50	75.00	100.00

15. Table 1.26 gives the proposed fine $r = f(v)$ to be imposed on a motorist for speeding, where v is the motorist's speed and 55 mph is the speed limit.

(a) Decide whether f appears to be linear.
(b) What would the rate of change represent in practical terms for the motorist?
(c) Plot the data points.

Table 1.26

v (mph)	60	65	70	75	80	85
r (dollars)	75	100	125	150	175	200

16. In 2003, the number, N, of cases of SARS (Severe Acute Respiratory Syndrome) reported in Hong Kong[15] was initially approximated by $N = 78.9 + 30.1t$, where t is the number of days since March 17. Interpret the constants 78.9 and 30.1.

17. A new Toyota RAV4 costs \$23,500. The car's value depreciates linearly to \$18,823 in three years time.[16] Write a formula which expresses its value, V, in terms of its age, t, in years.

18. In 2012, the population of a town was 21,510 and growing by 63 people per year. Find a formula for P, the town's population, in terms of t, the number of years since 2012.

19. A flight costs \$10,000 to operate, regardless of the number of passengers. Each ticket costs \$127. Express profit, π, as a linear function of the number of passengers, n, on the flight.

[15] World Health Organization, www.who.int/csr/sars/country/en, accessed September, 2005
[16] http://www.motortrend.com/used_cars/11/toyota/rav4/pricing/, accessed January 2013

20. Owners of an inactive quarry in Australia have decided to resume production. They estimate that it will cost them \$10,000 per month to maintain and insure their equipment and that monthly salaries will be \$30,000. It costs \$800 to mine a ton of rocks. Write a formula that expresses the total cost each month, c, as a function of r, the number of tons of rock mined per month.

21. In each case, graph a linear function with the given rate of change. Label and put scales on the axes.

(a) Increasing at 2.1 inches/day
(b) Decreasing at 1.3 gallons/mile

22. A small café sells coffee for \$3.50 per cup. On average, it costs the café \$0.50 to make a cup of coffee (for grounds, hot water, filters). The café also has a fixed daily cost of \$450 (for rent, wages, utilities).

(a) Let R, C, and P be the café's daily revenue, costs, and profit, respectively, for selling x cups of coffee in a day. Find formulas for R, C, and P as functions of x. [Hint: The revenue, R, is the total amount of money that the café brings in. The cost, C, includes the fixed daily cost as well as the cost for all x cups of coffee sold. P is the café's profit after costs have been accounted for.]
(b) Plot P against x. For what x-values is the graph of P below the x-axis? Above the x-axis? Interpret your results.
(c) Interpret the slope and both intercepts of your graph in practical terms.

23. Table 1.27 gives the area and perimeter of a square as a function of the length of its side.

(a) From the table, decide if either area or perimeter could be a linear function of side length.
(b) From the data make two graphs, one showing area as a function of side length, the other showing perimeter as a function of side length. Connect the points.
(c) If you find a linear relationship, give its corresponding rate of change and interpret its significance.

Table 1.27

Length of side	0	1	2	3	4	5	6
Area of square	0	1	4	9	16	25	36
Perimeter of square	0	4	8	12	16	20	24

24. Make two tables, one comparing the radius of a circle to its area, the other comparing the radius of a circle to its circumference. Repeat parts (a), (b), and (c) from Problem 23, this time comparing radius with circumference, and radius with area.

25. Sri Lanka is an island that experienced approximately linear population growth from 1950 to 2000. On the other hand, Afghanistan was torn by warfare in the 1980s and did not experience linear or near-linear growth.[17]

(a) Table 1.28 gives the population of these two countries, in millions. Which of these two countries is A and which is B? Explain.
(b) What is the approximate rate of change of the linear function? What does the rate of change represent in practical terms?
(c) Estimate the population of Sri Lanka in 1988.

Table 1.28

Year	1950	1960	1970	1980	1990	2000
Population of country A	8.2	9.8	12.4	15.1	14.7	23.9
Population of country B	7.5	9.9	12.5	14.9	17.2	19.2

26. Table 1.29 gives the average temperature, T, at a depth d, in a borehole in Belleterre, Quebec.[18] Evaluate $\Delta T/\Delta d$ on the following intervals, and explain what your answers tell you about borehole temperature.

(a) $25 \leq d \leq 150$
(b) $25 \leq d \leq 75$
(c) $100 \leq d \leq 200$

Table 1.29

d, depth (m)	25	50	75	100
T, temp (°C)	5.50	5.20	5.10	5.10
d, depth (m)	125	150	175	200
T, temp (°C)	5.30	5.50	5.75	6.00
d, depth (m)	225	250	275	300
T, temp (°C)	6.25	6.50	6.75	7.00

27. Table **??** gives the temperature-depth profile, $T = f(d)$, in a borehole in Belleterre, Quebec, where T is the average temperature at a depth d.

(a) Could f be linear?
(b) Graph f. What do you notice about the graph for $d \geq 150$?
(c) What can you say about the average rate of change of f for $d \geq 150$?

[17] www.census.gov/ipc/www/idbsusum.html, accessed January 12, 2006.
[18] Hugo Beltrami of St. Francis Xavier University and David Chapman of the University of Utah posted this data at http://geophysics.stfx.ca/public/borehole/borehole.html, accessed November 10, 2005.

28. A company finds that there is a linear relationship between the amount of money that it spends on advertising and the number of units it sells. If it spends no money on advertising, it sells 300 units. For each additional \$5000 spent, an additional 20 units are sold.

(a) If x is the amount of money that the company spends on advertising, find a formula for y, the number of units sold as a function of x.
(b) How many units does the firm sell if it spends \$25,000 on advertising? \$50,000?
(c) How much advertising money must be spent to sell 700 units?
(d) What is the slope of the line you found in part (a)? Give an interpretation of the slope that relates units sold and advertising costs.

29. Tuition cost T (in dollars) for part-time students at Stonewall College is given by $T = 300 + 200C$, where C represents the number of credits taken.

(a) Find the tuition cost for eight credits.
(b) How many credits were taken if the tuition was \$1700?
(c) Make a table showing costs for taking from one to twelve credits. For each value of C, give both the tuition cost, T, and the cost per credit, T/C. Round to the nearest dollar.
(d) Which of these values of C has the smallest cost per credit?
(e) What does the 300 represent in the formula for T?
(f) What does the 200 represent in the formula for T?

30. The summit of Africa's largest peak, Mt. Kilimanjaro, consists of the northern and southern ice fields and the Furtwangler glacier. An article in the Proceedings of the National Academy of Sciences[19] indicates that in 2000 ($t = 0$) the area of the ice cover at the peak of Mt. Kilimanjaro was approximately 1951 m^2. By 2007, the area had shrunk to approximately 1555 m^2.

(a) If this decline is modeled by a linear function, find $A = f(t)$, the equation of the ice-cover area as a function of time. Explain what the slope and A-intercept mean in terms of the ice cover.
(b) Evaluate $f(11)$.
(c) If this model is correct, when would you expect the ice cover to disappear?

31. When each of the following equations are written in the form $y = b + mx$, the result is $y = 5 + 4x$. Find the constants r, s, k, j in these equations.

(a) $y = 2r + x\sqrt{s}$
(b) $y = \dfrac{1}{k} - (j - 1)x$.

32. Graph the following function in the window $-10 \leq x \leq 10, -10 \leq y \leq 10$. Is this graph a line? Explain.

$$y = -x\left(\frac{x - 1000}{900}\right)$$

33. Graph $y = 2x + 400$ using the window $-10 \leq x \leq 10, -10 \leq y \leq 10$. Describe what happens, and how you can fix it by using a better window.

34. Graph $y = 200x + 4$ using the window $-10 \leq x \leq 10, -10 \leq y \leq 10$. Describe what happens and how you can fix it by using a better window.

35. Figure 1.28 shows the graph of $y = x^2/1000 + 5$ in the window $-10 \leq x \leq 10, -10 \leq y \leq 10$. Discuss whether this is a linear function.

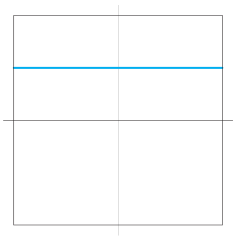


Figure 1.28

36. The cost of a cab ride is given by the function $C = 2.50 + 2d$, where d is the number of miles traveled and C is in dollars. Choose an appropriate window and graph the cost of a ride for a cab that travels no farther than a 10-mile radius from the center of the city.

1.4 FORMULAS FOR LINEAR FUNCTIONS

To find a formula for a linear function we find values for the slope, m, and the vertical intercept, b, in the formula $y = b + mx$.

Finding a Formula for a Linear Function from a Table of Data

If a table of data represents a linear function, we first calculate m and then determine b.

[19]http://www.pnas.org/content/early/2009/10/30/0906029106.full.pdf+html, accessed November 27, 2009.

Example 1 A grapefruit is thrown into the air. Its velocity, v, is a linear function of t, the time since it was thrown. A positive velocity indicates the grapefruit is rising and a negative velocity indicates it is falling. Check that the data in Table 1.30 corresponds to a linear function. Find a formula for v in terms of t.

Table 1.30 *Velocity of a grapefruit t seconds after being thrown into the air*

t, time (sec)	1	2	3	4
v, velocity (ft/sec)	48	16	−16	−48

Solution Figure 1.29 shows the data in Table 1.30. The points appear to fall on a line. To check that the velocity function is linear, calculate the rates of change of v and see that they are constant. From time $t = 1$ to $t = 2$, we have

$$\text{Average rate of change of velocity with time} = \frac{\Delta v}{\Delta t} = \frac{16 - 48}{2 - 1} = -32.$$

For the next second, from $t = 2$ to $t = 3$, we have

$$\text{Average rate of change} = \frac{\Delta v}{\Delta t} = \frac{-16 - 16}{3 - 2} = -32.$$

You can check that the rate of change from $t = 3$ to $t = 4$ is also -32.

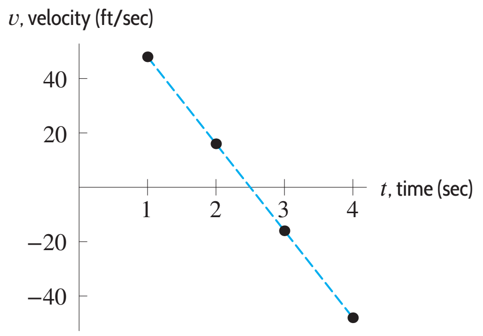


Figure 1.29: Velocity of a grapefruit is a linear function of time

A formula for v is of the form $v = b + mt$. Since m is the rate of change, we have $m = -32$ so $v = b - 32t$. The initial velocity (at $t = 0$) is represented by b. We are not given the value of v when $t = 0$, but we can use any data point to calculate b. For example, $v = 48$ when $t = 1$, so

$$48 = b - 32 \cdot 1,$$

which gives

$$b = 80.$$

Thus, a formula for the velocity is $v = 80 - 32t$.

What does the rate of change, m, in Example 1 tell us about the grapefruit? Think about the units:

$$m = \frac{\Delta v}{\Delta t} = \frac{\text{Change in velocity}}{\text{Change in time}} = \frac{-32 \text{ ft/sec}}{1 \text{ sec}} = -32 \text{ ft/sec per second.}$$

The value of m, -32 ft/sec per second, tells us that the grapefruit's velocity is decreasing by 32 ft/sec for every second that goes by. We say the grapefruit is accelerating at -32 ft/sec per second. (The units ft/sec per second are often written ft/sec^2. Negative acceleration is also called deceleration.[20])

[20]The notation ft/sec^2 is shorthand for ft/sec per second; it does not mean a "square second" in the same way that areas are measured in square feet or square meters.

Finding a Formula for a Linear Function from a Graph

We can calculate the slope, m, of a linear function using two points on its graph. Having found m, we can use either of the points to calculate b, the vertical intercept.

Example 2 Figure 1.30 shows oxygen consumption as a function of heart rate (that is, pulse) for two people.

(a) Assuming linearity, find formulas for these two functions.
(b) Interpret the slope of each graph in terms of oxygen consumption.

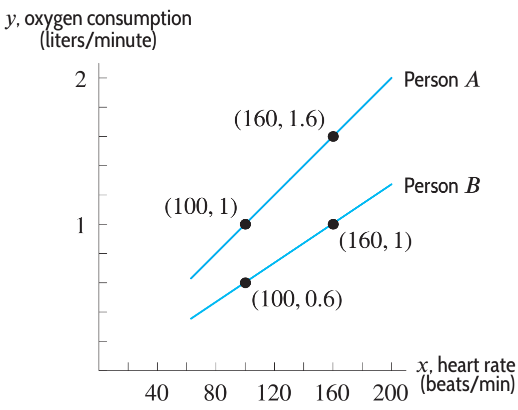


Figure 1.30: Oxygen consumption of two people running on treadmills

Solution (a) Let x be heart rate and let y be oxygen consumption. Since we are assuming linearity, $y = b + mx$. The two points on person A's line, (100, 1) and (160, 1.6), give

$$\text{Slope of A's line } = m = \frac{\Delta y}{\Delta x} = \frac{1.6 - 1}{160 - 100} = 0.01.$$

Thus $y = b + 0.01x$. To find b, use the fact that $y = 1$ when $x = 100$:

$$\begin{aligned} 1 &= b + 0.01(100) \\ 1 &= b + 1 \\ b &= 0, \end{aligned}$$

so $y = 0.01x$. Alternatively, b can be found using the fact that $x = 160$ if $y = 1.6$.

For person B, we again begin with the formula $y = b + mx$. In Figure 1.30, two points on B's line are (100, 0.6) and (160, 1), so

$$\text{Slope of B's line } = m = \frac{\Delta y}{\Delta x} = \frac{1 - 0.6}{160 - 100} = \frac{0.4}{60} \approx 0.0067.$$

To find b, use the fact that $y = 1$ when $x = 160$:

$$\begin{aligned} 1 &= b + (0.4/60) \cdot 160 \\ 1 &= b + 1.067 \\ b &= -0.067. \end{aligned}$$

Thus, for person B, we have $y = -0.067 + 0.0067x$.

(b) The slope for person A is $m = 0.01$, so

$$m = \frac{\text{Change in oxygen consumption}}{\text{Change in heart rate}} = \frac{\text{Change in liters/min}}{\text{Change in beats/min}} = 0.01 \frac{\text{liters}}{\text{heartbeat}}.$$

Every additional heartbeat (per minute) for person A translates to an additional 0.01 liters (per minute) of oxygen consumed.

The slope for person B is $m = 0.0067$. Thus, for every additional beat (per minute), person B consumes an additional 0.0067 liter of oxygen (per minute). Since the slope for person B is smaller than for person A, person B consumes less additional oxygen than person A for the same increase in pulse.

What do the y-intercepts of the functions in Example 2 say about oxygen consumption? Often the y-intercept of a function is a starting value. In this case, the y-intercept would be the oxygen consumption of a person whose pulse is zero (i.e. $x = 0$). Since a person running on a treadmill must have a pulse, in this case it makes no sense to interpret the y-intercept this way. The formula for oxygen consumption is useful only for realistic values of the pulse.

Finding a Formula for a Linear Function from a Verbal Description

Sometimes the verbal description of a linear function is less straightforward than those we saw in Section 1.3. Consider the following example.

Example 3 We have \$48 to spend on soda and chips for a party. A six-pack of soda costs \$6 and a bag of chips costs \$4. The number of six-packs we can afford, y, is a function of the number of bags of chips we decide to buy, x.

(a) Find an equation relating x and y.
(b) Graph the equation. Interpret the intercepts and the slope in the context of the party.

Solution (a) If we spend all \$48 on soda and chips, then we have the following equation:

$$\text{Amount spent on chips} + \text{Amount spent on soda} = \$48.$$

If we buy x bags of chips at \$4 per bag, then the amount spent on chips is $\$4x$. Similarly, if we buy y six-packs of soda at \$6 per six-pack, then the amount spent on soda is $\$6y$. Thus,

$$4x + 6y = 48.$$

We can solve for y, giving

$$6y = 48 - 4x$$
$$y = 8 - \frac{2}{3}x.$$

This is a linear function with slope $m = -2/3$ and y-intercept $b = 8$.

(b) The graph of this function is a discrete set of points, since the number of bags of chips and the number of six-packs of soda must be (nonnegative) integers.

To find the y-intercept, we set $x = 0$, giving

$$4 \cdot 0 + 6y = 48.$$

So $6y = 48$, giving $y = 8$.

Substituting $y = 0$ gives the x-intercept,

$$4x + 6 \cdot 0 = 48.$$

So $4x = 48$, giving $x = 12$. Thus the points $(0, 8)$ and $(12, 0)$ are on the graph.

The point $(0, 8)$ indicates that we can buy 8 six-packs of soda if we buy no chips. The point $(12, 0)$ indicates that we can buy 12 bags of chips if we buy no soda. The other points on the line describe affordable options between these two extremes. For example, the point $(6, 4)$ is on the line, because

$$4 \cdot 6 + 6 \cdot 4 = 48.$$

This means that if we buy 6 bags of chips, we can afford 4 six-packs of soda.

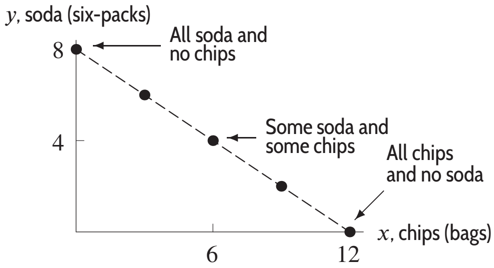


Figure 1.31: Relation between the number of six-packs, y, and the number of bags of chips, x

The points marked in Figure 1.31 represent affordable options. All affordable options lie on or below the line $4x + 6y = 48$. Not all points on the line are affordable options. For example, suppose we purchase one six-pack of soda for \$6. That leaves \$42 to spend on chips, meaning we would have to buy 10.5 bags of chips, which is not possible. Therefore, the point (10.5, 1) is not an option, although it is a point on the line $4x + 6y = 48$.

To interpret the slope, notice that

$$m = \frac{\Delta y}{\Delta x} = \frac{\text{Change in number of six-packs}}{\text{Change in number of bags of chips}},$$

so the units of m are six-packs of soda per bags of chips. The fact that $m = -2/3$ means that for each additional 3 bags of chips purchased, we can purchase 2 fewer six-packs of soda. This occurs because 2 six-packs cost \$12, the same as 3 bags of chips. Thus, $m = -2/3$ is the rate at which the amount of soda we can buy decreases as we buy more chips.

Alternative Forms for the Equation of a Line

In Example 3, the equation $2x + 3y = 24$ represents a linear relationship between x and y even though the equation is not in the form $y = b + mx$. The following equations represent lines.

- The *slope-intercept form* is
  $y = b + mx$ where m is the slope and b is the y-intercept.
- The *point-slope form* is
  $y - y_0 = m(x - x_0)$ where m is the slope and (x_0, y_0) is a point on the line.
- The *standard form* is
  $Ax + By + C = 0$ where A, B, and C are constants.

If we know the slope of a line and the coordinates of a point on the line, it is often convenient to use the point-slope form of the equation.

Example 4 Use the point-slope form to find the equation of the line for the oxygen consumption of person A in Example 2.

Solution In Example 2, we found the slope of person A's line to be $m = 0.01$. Since the point (100, 1) lies on the line, the point-slope form gives the equation

$$y - 1 = 0.01(x - 100).$$

To check that this gives the same equation we got in Example 2, we multiply out and simplify:

$$y - 1 = 0.01x - 1$$
$$y = 0.01x.$$

Alternatively, we could have used the point (160, 1.6) instead of (100, 1), giving

$$y - 1.6 = 0.01(x - 160).$$

Multiplying out again gives $y = 0.01x$.

Equations of Horizontal and Vertical Lines

The slope m of a line $y = b + mx$ gives the rate of change of y with respect to x. What about a line with slope $m = 0$? If the rate of change of a quantity is zero, then the quantity does not change. Thus, if the slope of a line is zero, the value of y must be constant. Such a line is horizontal.

Example 5 Explain why the equation $y = 4$ represents a horizontal line and the equation $x = 4$ represents a vertical line.

Solution The equation $y = 4$ represents a linear function with slope $m = 0$. To see this, notice that this equation can be rewritten as $y = 4 + 0 \cdot x$. Thus, the value of y is 4 no matter what the value of x is. See Figure 1.32. Similarly, the equation $x = 4$ means that x is 4 no matter what the value of y is. Every point on the line in Figure 1.33 has x equal to 4, so this line is the graph of $x = 4$.

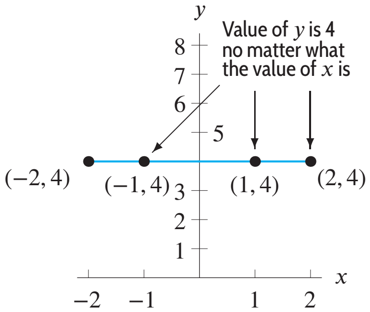


Figure 1.32: The horizontal line $y = 4$ has slope 0

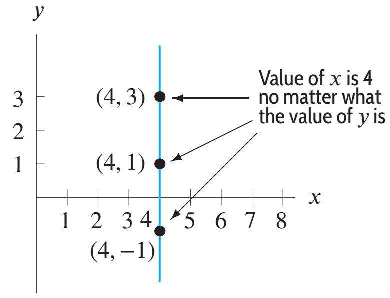


Figure 1.33: The vertical line $x = 4$ has an undefined slope

What is the slope of a vertical line? Figure 1.33 shows three points, $(4, -1)$, $(4, 1)$, and $(4, 3)$ on a vertical line. Calculating the slope, gives

$$m = \frac{\Delta y}{\Delta x} = \frac{3 - 1}{4 - 4} = \frac{2}{0}.$$

The slope is undefined because the denominator, Δx, is 0. The slope of every vertical line is undefined for the same reason. All the x-values on such a line are equal, so Δx is 0, and the denominator of the expression for the slope is 0. A vertical line is not the graph of a function, since it fails the vertical line test. It does not have an equation of the form $y = b + mx$.

In summary,

For any constant k:
- The graph of the equation $y = k$ is a horizontal line and its slope is zero.
- The graph of the equation $x = k$ is a vertical line and its slope is undefined.

Slopes of Parallel and Perpendicular Lines

Figure 1.34 shows two parallel lines. These lines are parallel because they have equal slopes.

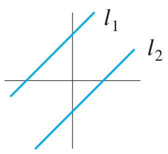


Figure 1.34: Parallel lines: l_1 and l_2 have equal slopes

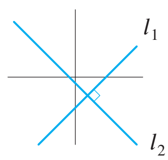


Figure 1.35: Perpendicular lines: Slopes of l_1 and l_2 are negative reciprocals of each other

What about perpendicular lines? Two perpendicular lines are graphed in Figure 1.35. We can see that if one line has a positive slope, then any perpendicular line must have a negative slope. In fact, we show that if l_1 and l_2 are two perpendicular lines with slopes, m_1 and m_2, then m_1 is the negative reciprocal of m_2. If m_1 and m_2 are not zero, we have the following result:

Let l_1 and l_2 be two lines having slopes m_1 and m_2, respectively. Then:

- These lines are parallel if and only if $m_1 = m_2$.
- These lines are perpendicular if and only if $m_1 = -\dfrac{1}{m_2}$.

In addition, any two horizontal lines are parallel and $m_1 = m_2 = 0$. Any two vertical lines are parallel and m_1 and m_2 are undefined. A horizontal line is perpendicular to a vertical line. See Figures 1.36–1.38.

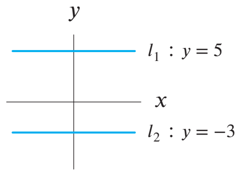


Figure 1.36: Any two horizontal lines are parallel

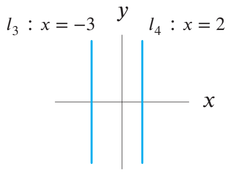


Figure 1.37: Any two vertical lines are parallel

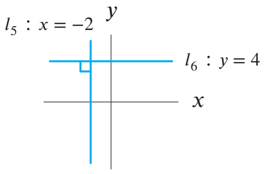


Figure 1.38: A horizontal line and a vertical line are perpendicular

Justification of Formula for Slopes of Perpendicular Lines

Figure 1.39 shows l_1 and l_2, two perpendicular lines with slopes m_1 and m_2. Neither line is horizontal or vertical, so m_1 and m_2 are both defined and nonzero. We will show that

$$m_2 = -\frac{1}{m_1}.$$

Using right triangle ΔPQR with side lengths a and b, we see that

$$m_1 = \frac{b}{a}.$$

Rotating ΔPQR by 90° about the point P produces triangle ΔPST. Using ΔPST, we see that

$$m_2 = -\frac{a}{b} = -\frac{1}{b/a} = -\frac{1}{m_1}.$$

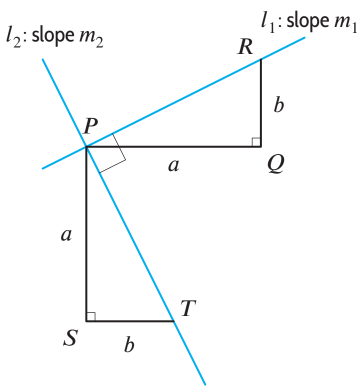


Figure 1.39: Perpendicular lines

Exercises and Problems for Section 1.4

Skill Refresher

Solve the equations in Exercises S1–S5.

S1. $y - 5 = 21$

S2. $2x - 5 = 13$

S3. $2x - 5 = 4x - 9$

S4. $17 - 28y = 13y + 24$

S5. $\frac{5}{3}(y + 2) = \frac{1}{2} - y$

In Exercises S6–S10, solve for the indicated variable.

S6. $I = Prt$, for P.

S7. $C = \frac{5}{9}(F - 32)$, for F.

S8. $C = 2\pi r$, for r.

S9. $ab + ax = c - ax$, for x.

S10. $by - d = ay + c$, for y.

Exercises

Find formulas for the linear functions in Exercises 1–8.

1. Slope -4 and x-intercept 7
2. Slope 3 and y-intercept 8
3. Passes through the points $(-1, 5)$ and $(2, -1)$
4. Slope $2/3$ and passes through the point $(5, 7)$
5. Has x-intercept 3 and y-intercept -5
6. Slope 0.1, passes through $(-0.1, 0.02)$
7. Function f has $f(0.3) = 0.8$ and $f(0.8) = -0.4$
8. Function f has $f(-2) = 7$ and $f(3) = -3$

Exercises 9–15 give data from a linear function. Find a formula for the function.

9.

Year, t	0	1	2
Value of computer, $\$V = f(t)$	2000	1500	1000

10.

Price per bottle, p (\$)	0.50	0.75	1.00
Number of bottles sold, $q = f(p)$	1500	1000	500

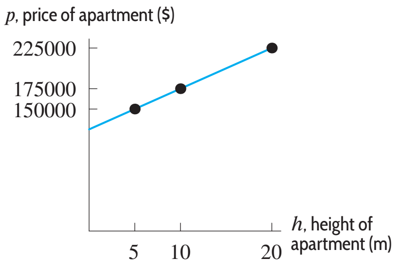
11.

Temperature, $y = f(x)$ (°C)	0	5	20
Temperature, x (°F)	32	41	68

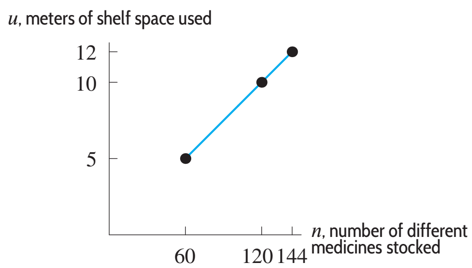
12.

Temperature, $y = f(x)$, (°R)	459.7	469.7	489.7
Temperature, x (°F)	0	10	30

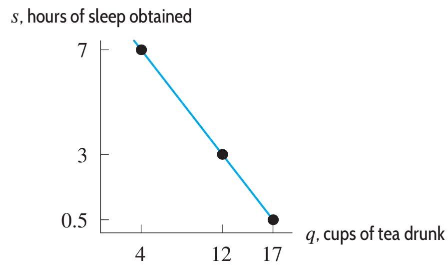
13.



14.



15.



In Exercises 16–24, if possible rewrite the equation in slope-intercept form, $y = b + mx$.

16. $5(x + y) = 4$

17. $3x + 5y = 20$

18. $0.1y + x = 18$

19. $5x - 3y + 2 = 0$

20. $y - 0.7 = 5(x - 0.2)$

21. $y = 5$

22. $3x + 2y + 40 = x - y$

23. $x = 4$

24. $\frac{x + y}{7} = 3$

In Exercises 25–30, is the function linear? If so, rewrite it in slope-intercept form.

25. $g(w) = -\frac{1 - 12w}{3}$

26. $F(P) = 13 - \frac{2^{-1}}{4}P$

27. $j(s) = 3s^{-1} + 7$

28. $C(r) = 2\pi r$

29. $h(x) = 3^x + 12$

30. $f(x) = m^2 x + n^2$

In Exercises 31–36, are the lines perpendicular? Parallel? Neither?

31. $y = 5x - 7;\quad y = 5x + 8$

32. $y = 4x + 3;\quad y = 13 - \frac{1}{4}x$

33. $y = 2x + 3;\quad y = 2x - 7$

34. $y = 4x + 7;\quad y = \frac{1}{4}x - 2$

35. $f(q) = 12q + 7;\quad g(q) = \frac{1}{12}q + 96$

36. $2y = 16 - x;\quad 4y = -8 - 2x$

Problems

Find formulas for the linear functions in Problems 37–40.

37. The graph of f contains $(-3, -8)$ and $(5, -20)$.

38. $g(100) = 2000$ and $g(400) = 3800$

39. $P = h(t)$ gives the size of a population that begins with 12,000 members and grows by 225 members each year.

40. The graph of h intersects the graph of $y = x^2$ at $x = -2$ and $x = 3$.

41. **(a)** By hand, graph $y = 3$ and $x = 3$.
(b) Can the equations in part (a) be written in slope-intercept form?

42. Find the equation of the line parallel to $3x + 5y = 6$ and passing through the point $(0, 6)$.

43. Find the equation of the line passing through the point $(2, 1)$ and perpendicular to the line $y = 5x - 3$.

44. Find the equations of the lines parallel to and perpendicular to the line $y + 4x = 7$, and through the point $(1, 5)$.

45. In Table 1.31, data for two functions f and g are given. One of the functions is linear, and the other is not.

(a) Which of the two functions is linear? Explain how you know.
(b) Find the equation of the linear function. Write your answer in slope-intercept form.

Table 1.31

x	1	1.5	2	2.5	3
$f(x)$	−1.22	−0.64	−0.06	0.52	1.10
$g(x)$	9.71	4.86	2.43	1.22	0.61

46. Line l is given by $y = 3 - \frac{2}{3}x$ and point P has coordinates $(6, 5)$.

(a) Find the equation of the line containing P and parallel to l.
(b) Find the equation of the line containing P and perpendicular to l.
(c) Graph the equations in parts (a) and (b).

47. An empty champagne bottle is tossed from a hot-air balloon. Its upward velocity is measured every second and recorded in Table 1.32.

(a) Describe the motion of the bottle in words. What do negative values of v represent?
(b) Find a formula for v in terms of t.
(c) Explain the physical significance of the slope of your formula.
(d) Explain the physical significance of the t-axis and v-axis intercepts.

Table 1.32

t (sec)	0	1	2	3	4	5
v (ft/sec)	40	8	−24	−56	−88	−120

Table 1.33 gives the cost, $C(n)$, of producing a certain good as a linear function of n, the number of units produced. Use the table to answer Problems 48–50.

Table 1.33

n (units)	100	125	150	175
$C(n)$ (dollars)	11000	11125	11250	11375

48. Evaluate the following expressions. Give economic interpretations for each.

(a) $C(175)$ **(b)** $C(175) - C(150)$
(c) $\dfrac{C(175) - C(150)}{175 - 150}$

49. Estimate $C(0)$. What is the economic significance of this value?

50. The *fixed cost* of production is the cost incurred before any goods are produced. The *unit cost* is the cost of producing an additional unit. Find a formula for $C(n)$ in terms of n, given that

$$\text{Total cost} = \text{Fixed cost} + \text{Unit cost} \cdot \text{Number of units}$$

51. John wants to buy a dozen rolls. The local bakery sells sesame and poppy-seed rolls for the same price.

(a) Make a table of all the possible combinations of rolls if he buys a dozen, where s is the number of sesame seed rolls and p is the number of poppy-seed rolls.
(b) Find a formula for p as a function of s.
(c) Graph this function.

52. In a college meal plan you pay a membership fee; then all your meals are at a fixed price per meal.

(a) If 90 meals cost \$1005 and 140 meals cost \$1205, write a linear function that describes the cost of a meal plan, C, in terms of the number of meals, n.
(b) What is the cost per meal and what is the membership fee?
(c) Find the cost for 120 meals.
(d) Find n in terms of C.
(e) Use part (d) to determine the maximum number of meals you can buy on a budget of \$1285.

53. The solid waste generated each year in the cities of the US is increasing.[21] The solid waste generated, in millions of tons, was 88.1 in 1960 and 249.9 in 2010. The trend appears linear during this time.

(a) Construct a formula for the amount of municipal solid waste generated in the US by finding the equation of the line through these two points.
(b) Use this formula to predict the amount of municipal solid waste generated in the US, in millions of tons, in the year 2020.

54. The demand for gasoline can be modeled as a linear function of price. If the price of gasoline is $p = \$3.10$ per gallon, the quantity demanded in a fixed period is $q = 65$ gallons. If the price rises to \$3.50 per gallon, the quantity demanded falls to 45 gallons in that period.

(a) Find a formula for q in terms of p.
(b) Explain the economic significance of the slope of your formula.
(c) Explain the economic significance of the q-axis and p-axis intercepts.

55. Find a formula for the linear function $h(t)$ whose graph intersects the graph of $j(t) = 30(0.2)^t$ at $t = -2$ and $t = 1$.

56. Find the equation of line l in Figure 1.40.

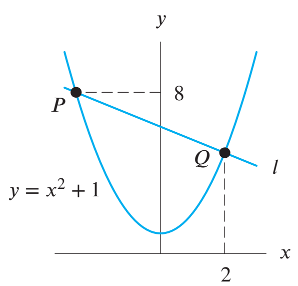


Figure 1.40

57. Find the equation of the line l, shown in Figure 1.41, if its slope is $m = 4$.

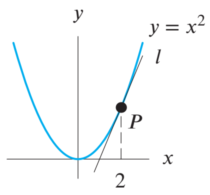


Figure 1.41

58. Find an equation for the line l in Figure 1.42 in terms of the constant A and values of the function f.

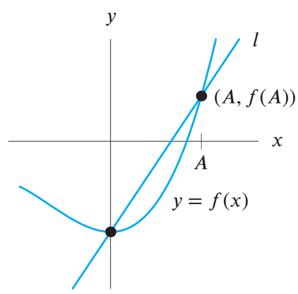


Figure 1.42

59. A dose-response function can be used to describe the increase in risk associated with the increase in exposure to various hazards. For example, the risk of contracting lung cancer depends, among other things, on the number of cigarettes a person smokes per day. This risk can be described by a linear dose-response function. For example, it is known that smoking 10 cigarettes per day increases a person's probability of contracting lung cancer by a factor of 25, while smoking 20 cigarettes a day increases the probability by a factor of 50.

(a) Find a formula for $i(x)$, the increase in the probability of contracting lung cancer for a person who smokes x cigarettes per day as compared to a non-smoker.
(b) Evaluate $i(0)$.
(c) Interpret the slope of the function i.

60. Wire is sold by gauge size, where the diameter of the wire is a decreasing linear function of gauge. Gauge 2 wire has a diameter of 0.2656 inches and gauge 8 wire has a diameter of 0.1719 inches. Find the diameter for wires of gauge 12.5 and gauge 0. What values of the gauge do not make sense in this model?

[21] http://http://www.epa.gov/osw/nonhaz/municipal/pubs/msw_2010_rev_factsheet.pdf, accessed January, 2013.

61. You can type four pages in 50 minutes and nine pages in an hour and forty minutes.

(a) Find a linear function for the number of pages typed, p, as a function of time, t. If time is measured in minutes, what values of t make sense in this example?
(b) How many pages can be typed in two hours?
(c) Interpret the slope of the function in practical terms.
(d) Use the result in part (a) to solve for time as a function of the number of pages typed.
(e) How long does it take to type a 15-page paper?
(f) Write a short paragraph explaining why it is useful to know both of the formulas obtained in part (a) and part (d).

In Problems 62–63, write the functions in slope-intercept form. Identify the values of b and m.

62. $v(s) = \pi x^2 - 3xr - 4rs - s\sqrt{x}$

63. $w(r) = \pi x^2 - 3xr - 4rs - s\sqrt{x}$

64. The development time, t, of an organism is the number of days required for the organism to mature, and the development rate is defined as $r = 1/t$. In cold-blooded organisms such as insects, the development rate depends on temperature: the colder it is, the longer the organism takes to develop. For such organisms, the degree-day model[22] assumes that the development rate r is a linear function of temperature H (in °C):

$$r = b + kH.$$

(a) According to the degree-day model, there is a minimum temperature H_{min} below which an organism never matures. Find a formula for H_{min} in terms of the constants b and k.
(b) Define S as $S = (H - H_{\text{min}})t$, where S is the number of degree-days. That is, S is the number of days t times the number of degrees between H and H_{min}. Use the formula for r to show that S is a constant. In other words, find a formula for S that does not involve H. Your formula will involve k.
(c) A certain organism requires $t = 25$ days to develop at a constant temperature of $H = 20$°C and has $H_{\text{min}} = 15$°C. Using the fact that S is a constant, how many days does it take for this organism to develop at a temperature of 25°C?
(d) In part (c) we assumed that the temperature H is constant throughout development. If the temperature varies from day to day, the number of degree-days can be accumulated until they total S, at which point the organism completes development. For instance, suppose on the first day the temperature is $H = 20$°C and that on the next day it is $H = 22$°C. Then for these first two days

Total number of degree days

$$= (20 - 15) \cdot 1 + (22 - 15) \cdot 1 = 12.$$

Based on Table 1.34, on what day does the organism reach maturity?

Table 1.34

Day	1	2	3	4	5	6	7	8	9	10	11	12
H (°C)	20	22	27	28	27	31	29	30	28	25	24	26

65. (Continuation of Problem 64.) Table 1.35 gives the development time t (in days) for an insect as a function of temperature H (in °C).

(a) Find a linear formula for r, the development rate, in terms of H.
(b) Find the value of S, the number of degree-days required for the organism to mature.

Table 1.35

H, °C	20	22	24	26	28	30
t, days	14.3	12.5	11.1	10.0	9.1	8.3

1.5 MODELING WITH LINEAR FUNCTIONS

Interpreting the Parameters of a Linear Function

The slope-intercept form for a linear function is $y = b + mx$, where b is the y-intercept and m is the slope. The parameters b and m can be used to compare linear functions.

[22] Information drawn from a web site created by Dr. Alexei A. Sharov at the Virginia Polytechnic Institute, http://www.ento.vt.edu/ sharov/PopEcol/popecol.html, accessed September 2002.

Example 1 With time, t, in years, the populations of four towns, P_A, P_B, P_C and P_D, are given by the following formulas:

$$P_A = 20{,}000 + 1600t, \quad P_B = 50{,}000 - 300t, \quad P_C = 650t + 45{,}000, \quad P_D = 15{,}000(1.07)^t.$$

(a) Which populations are represented by linear functions?
(b) Describe in words what each linear model tells you about that town's population. Of the towns that grow linearly, which town starts out with the most people? Which is growing fastest?

Solution

(a) The populations of towns A, B, and C are represented by linear functions because they are written in the form $P = b + mt$. Town D's population does not grow linearly since its formula, $P_D = 15{,}000(1.07)^t$, cannot be expressed in the form $P_D = b + mt$.
(b) For town A, we have

$$P_A = \underbrace{20{,}000}_{b} + \underbrace{1600}_{m} \cdot t,$$

so $b = 20{,}000$ and $m = 1600$. This means that in year $t = 0$, town A has 20,000 people. It grows by 1600 people per year.

For town B, we have

$$P_B = \underbrace{50{,}000}_{b} + \underbrace{(-300)}_{m} \cdot t,$$

so $b = 50{,}000$ and $m = -300$. This means that town B starts with 50,000 people. The negative slope indicates that the population is decreasing at the rate of 300 people per year.

For town C, we have

$$P_C = \underbrace{45{,}000}_{b} + \underbrace{650}_{m} \cdot t,$$

so $b = 45{,}000$ and $m = 650$. This means that town C begins with 45,000 people and grows by 650 people per year.

Town B starts out with the most people, 50,000, but town A, with a rate of change of 1600 people per year, grows the fastest of the three towns that grow linearly.

The Effect of the Parameters on the Graph of a Linear Function

The graph of a linear function is a line. Changing the values of b and m gives different members of the family of linear functions. In summary:

Let $y = b + mx$. Then the graph of y against x is a line.

- The y-intercept, b, tells us where the line crosses the y-axis.
- If the slope, m, is positive, the line climbs from left to right. If the slope, m, is negative, the line falls from left to right.
- The slope, m, tells us how fast the line is climbing or falling.
- The larger the magnitude of m (either positive or negative), the steeper the graph of f.

Example 2 (a) Graph the three linear functions P_A, P_B, P_C from Example 1 and show how to identify the values of b and m from the graph.
(b) Graph P_D from Example 1 and explain how the graph shows P_D is not a linear function.

Solution (a) Figure 1.43 gives graphs of the three functions:

$$P_A = 20{,}000 + 1600t, \qquad P_B = 50{,}000 - 300t, \qquad \text{and} \qquad P_C = 45{,}000 + 650t.$$

The values of b identified in Example 1 tell us the vertical intercepts. Figure 1.43 shows that the graph of P_A crosses the P-axis at $P = 20{,}000$, the graph of P_B crosses at $P = 50{,}000$, and the graph of P_C crosses at $P = 45{,}000$.

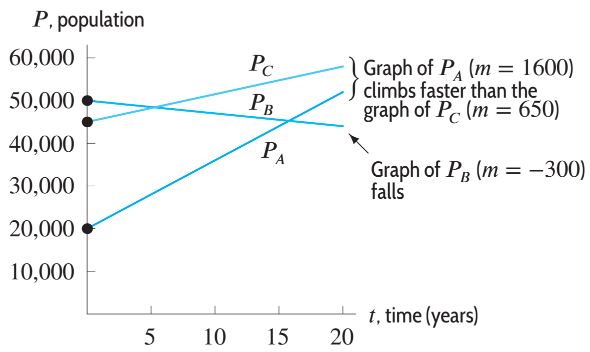


Figure 1.43: Graphs of three linear functions, P_A, P_B, and P_C, showing starting values and rates of climb

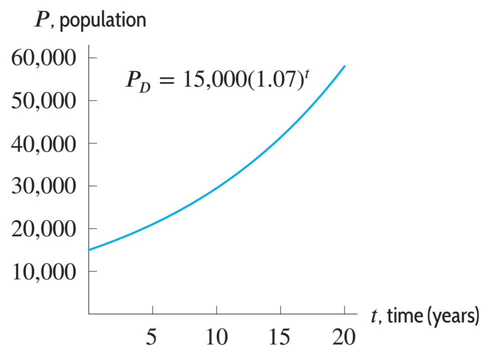


Figure 1.44: Graph of $P_D = 15{,}000(1.07)^t$ is not a line

Notice that the graphs of P_A and P_C are both climbing and that P_A climbs faster than P_C. This corresponds to the fact that the slopes of these two functions are positive ($m = 1600$ for P_A and $m = 650$ for P_C) and the slope of P_A is larger than the slope of P_C.

The graph of P_B falls when read from left to right, indicating that population decreases over time. This corresponds to the fact that the slope of P_C is negative ($m = -300$).

(b) Figure 1.44 gives a graph of P_D. Since it is not a line, P_D is not a linear function.

Intersection of Two Lines

The point (x, y) at which two lines intersect satisfies the equations for both lines. Thus, to find this point, we solve the equations simultaneously.[23] When modeling real phenomena, the point of intersection often has a practical meaning.

Example 3 The cost, in dollars, of renting a car for a day from three different rental agencies and driving it d miles is given by the following functions:

$$C_1 = 0.50d, \qquad C_2 = 30 + 0.20d, \qquad C_3 = 50 + 0.10d.$$

(a) For each agency, describe the rental agreement in words and graph the cost function. Use one set of axes for all three functions.
(b) For what driving distance does Agency 1 cost the same as Agency 2? If you drive 50 miles or less, which agency is cheapest?
(c) How do you determine which agency provides the cheapest car rental for any given driving distance?

[23]The algebra in this section is reviewed in the Skills Refresher on page 62.