QUANTITATIVE INVESTMENT ANALYSIS

CFA Institute is the premier association for investment professionals around the world, with over 130,000 members in 151 countries and territories. Since 1963 the organization has developed and administered the renowned Chartered Financial Analyst® Program. With a rich history of leading the investment profession, CFA Institute has set the highest standards in ethics, education, and professional excellence within the global investment community and is the foremost authority on investment profession conduct and practice. Each book in the CFA Institute Investment Series is geared toward industry practitioners along with graduate-level finance students and covers the most important topics in the industry. The authors of these cutting-edge books are themselves industry professionals and academics and bring their wealth of knowledge and expertise to this series.

QUANTITATIVE INVESTMENT ANALYSIS

Third Edition

Richard A. DeFusco, CFA

Dennis W. McLeavey, CFA

Jerald E. Pinto, CFA

David E. Runkle, CFA

WILEY

Cover image: © r.nagy/Shutterstock
Cover design: Wiley

Published by John Wiley & Sons, Inc., Hoboken, New Jersey.
Published simultaneously in Canada.

ISBN 978-1-119-10422-3 (Hardcover)
ISBN 978-1-119-10459-9 (ePDF)
ISBN 978-1-119-10460-5 (ePub)

Printed in the United States of America.
10 9 8 7 6 5 4 3 2 1

CONTENTS

CHAPTER 10
Time-Series Analysis 459

CHAPTER 11
Introduction to Multifactor Models 525

FOREWORD

"Central limits," "probability distributions," "hypothesis test"— investors have a bit of trouble generating enthusiasm for such terms. Yet, they should be enthusiastic because every investor needs these tools to analyze, compete, and succeed in today's economic environment. The financial markets and the participants in them become increasingly sophisticated every year. So, at times, it seems like you need a PhD in mathematics just to keep up with the markets.

Fortunately, a PhD is not necessary to succeed. In fact, the financial market battlefield is littered with the credentials of highly educated individuals who have failed spectacularly despite their intense education. Nonetheless, the better equipped you are with the basic tools of financial calculus, the better your chance of success.

Quantitative Investment Analysis provides the necessary utensils for success. In this volume, you will find all the statistical gadgets you need to be a confident and knowledgeable investor. Math need not be a four letter word. It can make your wealth analysis sharper, your investment theme more precise, your portfolio construction more successful.

Furthermore, this book is chock full of examples, practice problems (with answers!), charts, tables, and graphs that bring home in clear detail the concepts and tools of financial calculus. Whether you are a novice investor or an experienced practitioner, this book has something for you. In fact, as I read the book in preparation for writing this foreword, I kept getting unconsciously pulled into the examples; unwittingly, I became engaged in the book before I knew it. But that effect is part of the beauty of this book: It is an easy-to-read and easy-to-use handbook. I wanted to know more with each example I read. I know that you, too, will find that this book stimulates your curiosity while having the same ease of use that I found. Enjoy!

Mark J. P. Anson, PhD, CFA, CAIA, CPA
President & Chief Investment Officer
Acadia Capital
Bass Family Office

PREFACE

We are pleased to bring you *Quantitative Investment Analysis, Third Edition*, which focuses on key tools that are needed for today's professional investor. In addition to classic time value of money, discounted cash flow applications, and probability material, the text covers advanced concepts such as correlation and regression that ultimately figure into the formation of hypotheses for purposes of testing. The text teaches critical skills that challenge many professionals, including the ability to distinguish useful information from the overwhelming quantity of available data.

The content was developed in partnership by a team of distinguished academics and practitioners, chosen for their acknowledged expertise in the field, and guided by CFA Institute. It is written specifically with the investment practitioner in mind and is replete with examples and practice problems that reinforce the learning outcomes and demonstrate real-world applicability.

The CFA Program Curriculum, from which the content of this book was drawn, is subjected to a rigorous review process to assure that it is:

- Faithful to the findings of our ongoing industry practice analysis
- Valuable to members, employers, and investors
- Globally relevant
- Generalist (as opposed to specialist) in nature
- Replete with sufficient examples and practice opportunities
- Pedagogically sound

The accompanying workbook is a useful reference that provides Learning Outcome Statements, which describe exactly what readers will learn and be able to demonstrate after mastering the accompanying material. Additionally, the workbook has summary overviews and practice problems for each chapter.

We hope you will find this and other books in the CFA Institute Investment Series helpful in your efforts to grow your investment knowledge, whether you are a relatively new entrant or an experienced veteran striving to keep up to date in the ever-changing market environment. CFA Institute, as a long-term committed participant in the investment profession and a not-for-profit global membership association, is pleased to provide you with this opportunity.

ACKNOWLEDGMENTS

We would like to thank Eugene L. Podkaminer, CFA, for his contribution to revising coverage of multifactor models. We are indebted to Professor Sanjiv Sabherwal for his painstaking work in updating self-test examples in all chapters except the chapter on multifactor models. His contribution was essential to the fresh look of this third edition.

We are indebted to Wendy L. Pirie, CFA, Gregory Siegel, CFA, and Stephen E. Wilcox, CFA, for their help in verifying the accuracy of the text. Margaret Hill, Wanda Lauziere, and Julia MacKesson and the production team at CFA Institute provided essential support through the various stages of production. We thank Robert E. Lamy, CFA, and Christopher B. Wiese, CFA, for their encouragement and oversight of the production of a third edition.

ABOUT THE CFA INSTITUTE INVESTMENT SERIES

CFA Institute is pleased to provide you with the CFA Institute Investment Series, which covers major areas in the field of investments. We provide this best-in-class series for the same reason we have been chartering investment professionals for more than 45 years: to lead the investment profession globally by setting the highest standards of ethics, education, and professional excellence.

The books in the CFA Institute Investment Series contain practical, globally relevant material. They are intended both for those contemplating entry into the extremely competitive field of investment management as well as for those seeking a means of keeping their knowledge fresh and up to date. This series was designed to be user friendly and highly relevant.

We hope you find this series helpful in your efforts to grow your investment knowledge, whether you are a relatively new entrant or an experienced veteran ethically bound to keep up to date in the ever-changing market environment. As a long-term, committed participant in the investment profession and a not-for-profit global membership association, CFA Institute is pleased to provide you with this opportunity.

THE TEXTS

Corporate Finance: A Practical Approach is a solid foundation for those looking to achieve lasting business growth. In today's competitive business environment, companies must find innovative ways to enable rapid and sustainable growth. This text equips readers with the foundational knowledge and tools for making smart business decisions and formulating strategies to maximize company value. It covers everything from managing relationships between stakeholders to evaluating merger and acquisition bids, as well as the companies behind them. Through extensive use of real-world examples, readers will gain critical perspective into interpreting corporate financial data, evaluating projects, and allocating funds in ways that increase corporate value. Readers will gain insights into the tools and strategies used in modern corporate financial management.

Equity Asset Valuation is a particularly cogent and important resource for anyone involved in estimating the value of securities and understanding security pricing. A well-informed professional knows that the common forms of equity valuation—dividend discount modeling, free cash flow modeling, price/earnings modeling, and residual income modeling—can all be reconciled with one another under certain assumptions. With a deep understanding of the underlying assumptions, the professional investor can better understand what other investors assume when calculating their valuation estimates. This text has a global orientation, including emerging markets.

Fixed Income Analysis has been at the forefront of new concepts in recent years, and this particular text offers some of the most recent material for the seasoned professional who is not a fixed-income specialist. The application of option and derivative technology to the once staid province of fixed income has helped contribute to an explosion of thought in this area. Professionals have been challenged to stay up to speed with credit derivatives, swaptions, collateralized mortgage securities, mortgage-backed securities, and other vehicles, and this explosion of products has strained the world's financial markets and tested central banks to provide sufficient oversight. Armed with a thorough grasp of the new exposures, the professional investor is much better able to anticipate and understand the challenges our central bankers and markets face.

International Financial Statement Analysis is designed to address the ever-increasing need for investment professionals and students to think about financial statement analysis from a global perspective. The text is a practically oriented introduction to financial statement analysis that is distinguished by its combination of a true international orientation, a structured presentation style, and abundant illustrations and tools covering concepts as they are introduced in the text. The authors cover this discipline comprehensively and with an eye to ensuring the reader's success at all levels in the complex world of financial statement analysis.

Investments: Principles of Portfolio and Equity Analysis provides an accessible yet rigorous introduction to portfolio and equity analysis. Portfolio planning and portfolio management are presented within a context of up-to-date, global coverage of security markets, trading, and market-related concepts and products. The essentials of equity analysis and valuation are explained in detail and profusely illustrated. The book includes coverage of practitioner-important but often neglected topics, such as industry analysis. Throughout, the focus is on the practical application of key concepts with examples drawn from both emerging and developed markets. Each chapter affords the reader many opportunities to self-check his or her understanding of topics.

One of the most prominent texts over the years in the investment management industry has been Maginn and Tuttle's *Managing Investment Portfolios: A Dynamic Process*. The third edition updates key concepts from the 1990 second edition. Some of the more experienced members of our community own the prior two editions and will add the third edition to their libraries. Not only does this seminal work take the concepts from the other readings and put them in a portfolio context, but it also updates the concepts of alternative investments, performance presentation standards, portfolio execution, and, very importantly, individual investor portfolio management. Focusing attention away from institutional portfolios and toward the individual investor makes this edition an important and timely work.

The New Wealth Management: The Financial Advisor's Guide to Managing and Investing Client Assets is an updated version of Harold Evensky's mainstay reference guide for wealth managers. Harold Evensky, Stephen Horan, and Thomas Robinson have updated the core text of the 1997 first edition and added an abundance of new material to fully reflect today's investment challenges. The text provides authoritative coverage across the full spectrum of wealth management and serves as a comprehensive guide for financial advisors. The book expertly blends investment theory and real-world applications and is written in the same thorough but highly accessible style as the first edition.

All books in the CFA Institute Investment Series are available through all major booksellers. And, all titles are available on the Wiley Custom Select platform at http://customselect.wiley.com/ where individual chapters for all the books may be mixed and matched to create custom textbooks for the classroom.

CHAPTER 1

THE TIME VALUE OF MONEY

Richard A. DeFusco, CFA
Dennis W. McLeavey, CFA
Jerald E. Pinto, PhD, CFA
David E. Runkle, PhD, CFA

LEARNING OUTCOMES

After completing this chapter, you will be able to do the following:

- interpret interest rates as required rates of return, discount rates, or opportunity costs;
- explain an interest rate as the sum of a real risk-free rate and premiums that compensate investors for bearing distinct types of risk;
- calculate and interpret the effective annual rate, given the stated annual interest rate and the frequency of compounding;
- solve time value of money problems for different frequencies of compounding;
- calculate and interpret the future value (FV) and present value (PV) of a single sum of money, an ordinary annuity, an annuity due, a perpetuity (PV only), and a series of unequal cash flows;
- demonstrate the use of a time line in modeling and solving time value of money problems.

1. INTRODUCTION

As individuals, we often face decisions that involve saving money for a future use, or borrowing money for current consumption. We then need to determine the amount we need to invest, if we are saving, or the cost of borrowing, if we are shopping for a loan. As investment analysts, much of our work also involves evaluating transactions with present and future cash flows.

When we place a value on any security, for example, we are attempting to determine the worth of a stream of future cash flows. To carry out all the above tasks accurately, we must understand the mathematics of time value of money problems. Money has time value in that individuals value a given amount of money more highly the earlier it is received. Therefore, a smaller amount of money now may be equivalent in value to a larger amount received at a future date. The **time value of money** as a topic in investment mathematics deals with equivalence relationships between cash flows with different dates. Mastery of time value of money concepts and techniques is essential for investment analysts.

The reading[1] is organized as follows: Section 2 introduces some terminology used throughout the reading and supplies some economic intuition for the variables we will discuss. Section 3 tackles the problem of determining the worth at a future point in time of an amount invested today. Section 4 addresses the future worth of a series of cash flows. These two sections provide the tools for calculating the equivalent value at a future date of a single cash flow or series of cash flows. Sections 5 and 6 discuss the equivalent value today of a single future cash flow and a series of future cash flows, respectively. In Section 7, we explore how to determine other quantities of interest in time value of money problems.

2. INTEREST RATES: INTERPRETATION

In this reading, we will continually refer to interest rates. In some cases, we assume a particular value for the interest rate; in other cases, the interest rate will be the unknown quantity we seek to determine. Before turning to the mechanics of time value of money problems, we must illustrate the underlying economic concepts. In this section, we briefly explain the meaning and interpretation of interest rates.

Time value of money concerns equivalence relationships between cash flows occurring on different dates. The idea of equivalence relationships is relatively simple. Consider the following exchange: You pay $10,000 today and in return receive $9,500 today. Would you accept this arrangement? Not likely. But what if you received the $9,500 today and paid the $10,000 one year from now? Can these amounts be considered equivalent? Possibly, because a payment of $10,000 a year from now would probably be worth less to you than a payment of $10,000 today. It would be fair, therefore, to **discount** the $10,000 received in one year; that is, to cut its value based on how much time passes before the money is paid. An **interest rate**, denoted r, is a rate of return that reflects the relationship between differently dated cash flows. If $9,500 today and $10,000 in one year are equivalent in value, then $10,000 – $9,500 = $500 is the required compensation for receiving $10,000 in one year rather than now. The interest rate—the required compensation stated as a rate of return—is $500/$9,500 = 0.0526 or 5.26 percent.

Interest rates can be thought of in three ways. First, they can be considered required rates of return—that is, the minimum rate of return an investor must receive in order to accept the investment. Second, interest rates can be considered discount rates. In the example above, 5.26 percent is that rate at which we discounted the $10,000 future amount to find its value today. Thus, we use the terms "interest rate" and "discount rate" almost interchangeably. Third, interest rates can be considered opportunity costs. An **opportunity cost** is the value that investors forgo by choosing a particular course of action. In the example, if the party who supplied

[1] Examples in this reading and other readings in quantitative methods at Level I were updated in 2013 by Professor Sanjiv Sabherwal of the University of Texas, Arlington.

$9,500 had instead decided to spend it today, he would have forgone earning 5.26 percent on the money. So we can view 5.26 percent as the opportunity cost of current consumption.

Economics tells us that interest rates are set in the marketplace by the forces of supply and demand, where investors are suppliers of funds and borrowers are demanders of funds. Taking the perspective of investors in analyzing market-determined interest rates, we can view an interest rate *r* as being composed of a real risk-free interest rate plus a set of four premiums that are required returns or compensation for bearing distinct types of risk:

$$r = \text{Real risk-free interest rate} + \text{Inflation premium} + \text{Default risk premium} + \text{Liquidity premium} + \text{Maturity premium}$$

- The **real risk-free interest rate** is the single-period interest rate for a completely risk-free security if no inflation were expected. In economic theory, the real risk-free rate reflects the time preferences of individuals for current versus future real consumption.
- The **inflation premium** compensates investors for expected inflation and reflects the average inflation rate expected over the maturity of the debt. Inflation reduces the purchasing power of a unit of currency—the amount of goods and services one can buy with it. The sum of the real risk-free interest rate and the inflation premium is the **nominal risk-free interest rate**.[2] Many countries have governmental short-term debt whose interest rate can be considered to represent the nominal risk-free interest rate in that country. The interest rate on a 90-day US Treasury bill (T-bill), for example, represents the nominal risk-free interest rate over that time horizon.[3] US T-bills can be bought and sold in large quantities with minimal transaction costs and are backed by the full faith and credit of the US government.
- The **default risk premium** compensates investors for the possibility that the borrower will fail to make a promised payment at the contracted time and in the contracted amount.
- The **liquidity premium** compensates investors for the risk of loss relative to an investment's fair value if the investment needs to be converted to cash quickly. US T-bills, for example, do not bear a liquidity premium because large amounts can be bought and sold without affecting their market price. Many bonds of small issuers, by contrast, trade infrequently after they are issued; the interest rate on such bonds includes a liquidity premium reflecting the relatively high costs (including the impact on price) of selling a position.
- The **maturity premium** compensates investors for the increased sensitivity of the market value of debt to a change in market interest rates as maturity is extended, in general (holding all else equal). The difference between the interest rate on longer-maturity, liquid Treasury debt and that on short-term Treasury debt reflects a positive maturity premium for the longer-term debt (and possibly different inflation premiums as well).

[2]Technically, 1 plus the nominal rate equals the product of 1 plus the real rate and 1 plus the inflation rate. As a quick approximation, however, the nominal rate is equal to the real rate plus an inflation premium. In this discussion we focus on approximate additive relationships to highlight the underlying concepts.

[3]Other developed countries issue securities similar to US Treasury bills. The French government issues BTFs or negotiable fixed-rate discount Treasury bills (*Bons du Trésor àtaux fixe et à intérêts précomptés*) with maturities of up to one year. The Japanese government issues a short-term Treasury bill with maturities of 6 and 12 months. The German government issues at discount both Treasury financing paper (*Finanzierungsschätze des Bundes* or, for short, *Schätze*) and Treasury discount paper (*Bubills*) with maturities up to 24 months. In the United Kingdom, the British government issues gilt-edged Treasury bills with maturities ranging from 1 to 364 days. The Canadian government bond market is closely related to the US market; Canadian Treasury bills have maturities of 3, 6, and 12 months.

Using this insight into the economic meaning of interest rates, we now turn to a discussion of solving time value of money problems, starting with the future value of a single cash flow.

3. THE FUTURE VALUE OF A SINGLE CASH FLOW

In this section, we introduce time value associated with a single cash flow or lump-sum investment. We describe the relationship between an initial investment or **present value (PV)**, which earns a rate of return (the interest rate per period) denoted as r, and its **future value (FV)**, which will be received N years or periods from today.

The following example illustrates this concept. Suppose you invest \$100 (PV = \$100) in an interest-bearing bank account paying 5 percent annually. At the end of the first year, you will have the \$100 plus the interest earned, 0.05 × \$100 = \$5, for a total of \$105. To formalize this one-period example, we define the following terms:

PV = present value of the investment
FV_N = future value of the investment N periods from today
r = rate of interest per period

For $N = 1$, the expression for the future value of amount PV is

$$FV_1 = PV(1 + r) \tag{1}$$

For this example, we calculate the future value one year from today as FV_1 = \$100(1.05) = \$105.

Now suppose you decide to invest the initial \$100 for two years with interest earned and credited to your account annually (annual compounding). At the end of the first year (the beginning of the second year), your account will have \$105, which you will leave in the bank for another year. Thus, with a beginning amount of \$105 (PV = \$105), the amount at the end of the second year will be \$105(1.05) = \$110.25. Note that the \$5.25 interest earned during the second year is 5 percent of the amount invested at the beginning of Year 2.

Another way to understand this example is to note that the amount invested at the beginning of Year 2 is composed of the original \$100 that you invested plus the \$5 interest earned during the first year. During the second year, the original principal again earns interest, as does the interest that was earned during Year 1. You can see how the original investment grows:

Original investment	\$100.00
Interest for the first year (\$100 × 0.05)	5.00
Interest for the second year based on original investment (\$100 × 0.05)	5.00
Interest for the second year based on interest earned in the first year (0.05 × \$5.00 interest on interest)	0.25
Total	\$110.25

The \$5 interest that you earned each period on the \$100 original investment is known as **simple interest** (the interest rate times the principal). **Principal** is the amount of funds originally invested. During the two-year period, you earn \$10 of simple interest. The extra \$0.25 that you have at the end of Year 2 is the interest you earned on the Year 1 interest of \$5 that you reinvested.

The interest earned on interest provides the first glimpse of the phenomenon known as **compounding**. Although the interest earned on the initial investment is important, for a given interest rate it is fixed in size from period to period. The compounded interest earned on reinvested interest is a far more powerful force because, for a given interest rate, it grows in size each period. The importance of compounding increases with the magnitude of the interest rate. For example, \$100 invested today would be worth about \$13,150 after 100 years if compounded annually at 5 percent, but worth more than \$20 million if compounded annually over the same time period at a rate of 13 percent.

To verify the \$20 million figure, we need a general formula to handle compounding for any number of periods. The following general formula relates the present value of an initial investment to its future value after N periods:

$$FV_N = PV(1 + r)^N \tag{2}$$

where r is the stated interest rate per period and N is the number of compounding periods. In the bank example, $FV_2 = \$100(1 + 0.05)^2 = \$110.25$. In the 13 percent investment example, $FV_{100} = \$100(1.13)^{100} = \$20,316,287.42$.

The most important point to remember about using the future value equation is that the stated interest rate, r, and the number of compounding periods, N, must be compatible. Both variables must be defined in the same time units. For example, if N is stated in months, then r should be the one-month interest rate, unannualized.

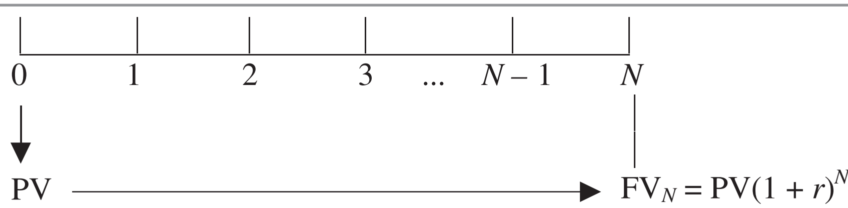
A time line helps us to keep track of the compatibility of time units and the interest rate per time period. In the time line, we use the time index t to represent a point in time a stated number of periods from today. Thus the present value is the amount available for investment today, indexed as $t = 0$. We can now refer to a time N periods from today as $t = N$. The time line in Figure 1 shows this relationship.

In Figure 1, we have positioned the initial investment, PV, at $t = 0$. Using Equation 2, we move the present value, PV, forward to $t = N$ by the factor $(1 + r)^N$. This factor is called a future value factor. We denote the future value on the time line as FV and position it at $t = N$. Suppose the future value is to be received exactly 10 periods from today's date ($N = 10$). The present value, PV, and the future value, FV, are separated in time through the factor $(1 + r)^{10}$.

The fact that the present value and the future value are separated in time has important consequences:

- We can add amounts of money only if they are indexed at the same point in time.
- For a given interest rate, the future value increases with the number of periods.
- For a given number of periods, the future value increases with the interest rate.

FIGURE 1 The Relationship between an Initial Investment, PV, and Its Future Value, FV



To better understand these concepts, consider three examples that illustrate how to apply the future value formula.

EXAMPLE 1 The Future Value of a Lump Sum with Interim Cash Reinvested at the Same Rate

You are the lucky winner of your state's lottery of $5 million after taxes. You invest your winnings in a five-year certificate of deposit (CD) at a local financial institution. The CD promises to pay 7 percent per year compounded annually. This institution also lets you reinvest the interest at that rate for the duration of the CD. How much will you have at the end of five years if your money remains invested at 7 percent for five years with no withdrawals?

Solution: To solve this problem, compute the future value of the $5 million investment using the following values in Equation 2:

$$\begin{aligned}
\text{PV} &= \$5{,}000{,}000 \\
r &= 7\% = 0.07 \\
N &= 5 \\
\text{FV}_N &= \text{PV}(1+r)^N \\
&= \$5{,}000{,}000(1.07)^5 \\
&= \$5{,}000{,}000(1.402552) \\
&= \$7{,}012{,}758.65
\end{aligned}$$

At the end of five years, you will have $7,012,758.65 if your money remains invested at 7 percent with no withdrawals.

In this and most examples in this reading, note that the factors are reported at six decimal places but the calculations may actually reflect greater precision. For example, the reported 1.402552 has been rounded up from 1.40255173 (the calculation is actually carried out with more than eight decimal places of precision by the calculator or spreadsheet). Our final result reflects the higher number of decimal places carried by the calculator or spreadsheet.[4]

[4] We could also solve time value of money problems using tables of interest rate factors. Solutions using tabled values of interest rate factors are generally less accurate than solutions obtained using calculators or spreadsheets, so practitioners prefer calculators or spreadsheets.

EXAMPLE 2 The Future Value of a Lump Sum with No Interim Cash

An institution offers you the following terms for a contract: For an investment of ¥2,500,000, the institution promises to pay you a lump sum six years from now at an 8 percent annual interest rate. What future amount can you expect?

Solution: Use the following data in Equation 2 to find the future value:

$$\begin{aligned} \text{PV} &= ¥2{,}500{,}000 \\ r &= 8\% = 0.08 \\ N &= 6 \\ \text{FV}_N &= \text{PV}(1+r)^N \\ &= ¥2{,}500{,}000(1.08)^6 \\ &= ¥2{,}500{,}000(1.586874) \\ &= ¥3{,}967{,}186 \end{aligned}$$

You can expect to receive ¥3,967,186 six years from now.

Our third example is a more complicated future value problem that illustrates the importance of keeping track of actual calendar time.

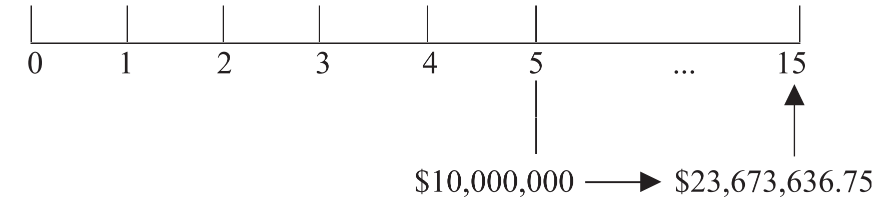
EXAMPLE 3 The Future Value of a Lump Sum

A pension fund manager estimates that his corporate sponsor will make a $10 million contribution five years from now. The rate of return on plan assets has been estimated at 9 percent per year. The pension fund manager wants to calculate the future value of this contribution 15 years from now, which is the date at which the funds will be distributed to retirees. What is that future value?

Solution: By positioning the initial investment, PV, at $t = 5$, we can calculate the future value of the contribution using the following data in Equation 2:

$$\begin{aligned} \text{PV} &= \$10 \text{ million} \\ r &= 9\% = 0.09 \\ N &= 10 \\ \text{FV}_N &= \text{PV}(1+r)^N \\ &= \$10{,}000{,}000(1.09)^{10} \\ &= \$10{,}000{,}000(2.367364) \\ &= \$23{,}673{,}636.75 \end{aligned}$$

FIGURE 2 The Future Value of a Lump Sum, Initial Investment Not at $t = 0$



This problem looks much like the previous two, but it differs in one important respect: its timing. From the standpoint of today ($t = 0$), the future amount of $23,673,636.75 is 15 years into the future. Although the future value is 10 years from its present value, the present value of $10 million will not be received for another five years.

As Figure 2 shows, we have followed the convention of indexing today as $t = 0$ and indexing subsequent times by adding 1 for each period. The additional contribution of $10 million is to be received in five years, so it is indexed as $t = 5$ and appears as such in the figure. The future value of the investment in 10 years is then indexed at $t = 15$; that is, 10 years following the receipt of the $10 million contribution at $t = 5$. Time lines like this one can be extremely useful when dealing with more complicated problems, especially those involving more than one cash flow.

In a later section of this reading, we will discuss how to calculate the value today of the $10 million to be received five years from now. For the moment, we can use Equation 2. Suppose the pension fund manager in Example 3 above were to receive $6,499,313.86 today from the corporate sponsor. How much will that sum be worth at the end of five years? How much will it be worth at the end of 15 years?

$$
\begin{aligned}
\mathrm{PV} &= \$6{,}499{,}313.86 \\
r &= 9\% = 0.09 \\
N &= 5 \\
\mathrm{FV}_N &= \mathrm{PV}(1+r)^N \\
&= \$6{,}499{,}313.86(1.09)^5 \\
&= \$6{,}499{,}313.86(1.538624) \\
&= \$10{,}000{,}000 \text{ at the five-year mark}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{PV} &= \$6{,}499{,}313.86 \\
r &= 9\% = 0.09 \\
N &= 15 \\
\mathrm{FV}_N &= \mathrm{PV}(1+r)^N \\
&= \$6{,}499{,}313.86(1.09)^{15} \\
&= \$6{,}499{,}313.86(3.642482) \\
&= \$23{,}673{,}636.74 \text{ at the 15-year mark}
\end{aligned}
$$

These results show that today's present value of about \$6.5 million becomes \$10 million after five years and \$23.67 million after 15 years.

3.1. The Frequency of Compounding

In this section, we examine investments paying interest more than once a year. For instance, many banks offer a monthly interest rate that compounds 12 times a year. In such an arrangement, they pay interest on interest every month. Rather than quote the periodic monthly interest rate, financial institutions often quote an annual interest rate that we refer to as the **stated annual interest rate** or **quoted interest rate**. We denote the stated annual interest rate by r_s. For instance, your bank might state that a particular CD pays 8 percent compounded monthly. The stated annual interest rate equals the monthly interest rate multiplied by 12. In this example, the monthly interest rate is 0.08/12 = 0.0067 or 0.67 percent.[5] This rate is strictly a quoting convention because $(1 + 0.0067)^{12} = 1.083$, not 1.08; the term $(1 + r_s)$ is not meant to be a future value factor when compounding is more frequent than annual.

With more than one compounding period per year, the future value formula can be expressed as

$$FV_N = PV\left(1+\frac{r_s}{m}\right)^{mN} \tag{3}$$

where r_s is the stated annual interest rate, m is the number of compounding periods per year, and N now stands for the number of years. Note the compatibility here between the interest rate used, r_s/m, and the number of compounding periods, mN. The periodic rate, r_s/m, is the stated annual interest rate divided by the number of compounding periods per year. The number of compounding periods, mN, is the number of compounding periods in one year multiplied by the number of years. The periodic rate, r_s/m, and the number of compounding periods, mN, must be compatible.

EXAMPLE 4 The Future Value of a Lump Sum with Quarterly Compounding

Continuing with the CD example, suppose your bank offers you a CD with a two-year maturity, a stated annual interest rate of 8 percent compounded quarterly, and a feature allowing reinvestment of the interest at the same interest rate. You decide to invest \$10,000. What will the CD be worth at maturity?

[5] To avoid rounding errors when using a financial calculator, divide 8 by 12 and then press the %*i* key, rather than simply entering 0.67 for %*i*, so we have $(1 + 0.08/12)^{12} = 1.083000$.

Solution: Compute the future value with Equation 3 as follows:

$$\begin{aligned}
\text{PV} &= \$10{,}000 \\
r_s &= 8\% = 0.08 \\
m &= 4 \\
r_s/m &= 0.08/4 = 0.02 \\
N &= 2 \\
mN &= 4(2) = 8 \text{ interest periods} \\
\text{FV}_N &= \text{PV}\left(1 + \frac{r_s}{m}\right)^{mN} \\
&= \$10{,}000(1.02)^8 \\
&= \$10{,}000(1.171659) \\
&= \$11{,}716.59
\end{aligned}$$

At maturity, the CD will be worth $11,716.59.

The future value formula in Equation 3 does not differ from the one in Equation 2. Simply keep in mind that the interest rate to use is the rate per period and the exponent is the number of interest, or compounding, periods.

EXAMPLE 5 The Future Value of a Lump Sum with Monthly Compounding

An Australian bank offers to pay you 6 percent compounded monthly. You decide to invest A$1 million for one year. What is the future value of your investment if interest payments are reinvested at 6 percent?

Solution: Use Equation 3 to find the future value of the one-year investment as follows:

$$\begin{aligned}
\text{PV} &= \text{A}\$1{,}000{,}000 \\
r_s &= 6\% = 0.06 \\
m &= 12 \\
r_s/m &= 0.06/12 = 0.0050 \\
N &= 1 \\
mN &= 12(1) = 12 \text{ interest periods} \\
\text{FV}_N &= \text{PV}\left(1 + \frac{r_s}{m}\right)^{mN} \\
&= \text{A}\$1{,}000{,}000(1.005)^{12} \\
&= \text{A}\$1{,}000{,}000(1.061678) \\
&= \text{A}\$1{,}061{,}677.81
\end{aligned}$$

If you had been paid 6 percent with annual compounding, the future amount would be only A$1,000,000(1.06) = A$1,060,000 instead of A$1,061,677.81 with monthly compounding.

3.2. Continuous Compounding

The preceding discussion on compounding periods illustrates discrete compounding, which credits interest after a discrete amount of time has elapsed. If the number of compounding periods per year becomes infinite, then interest is said to compound continuously. If we want to use the future value formula with continuous compounding, we need to find the limiting value of the future value factor for $m \to \infty$ (infinitely many compounding periods per year) in Equation 3. The expression for the future value of a sum in N years with continuous compounding is

$$\text{FV}_N = \text{PV}e^{r_s N} \tag{4}$$

The term $e^{r_s N}$ is the transcendental number $e \approx 2.7182818$ raised to the power $r_s N$. Most financial calculators have the function e^x.

EXAMPLE 6 The Future Value of a Lump Sum with Continuous Compounding

Suppose a $10,000 investment will earn 8 percent compounded continuously for two years. We can compute the future value with Equation 4 as follows:

$$\begin{aligned}
\text{PV} &= \$10{,}000 \\
r_s &= 8\% = 0.08 \\
N &= 2 \\
\text{FV}_N &= \text{PV}e^{r_s N} \\
&= \$10{,}000e^{0.08(2)} \\
&= \$10{,}000(1.173511) \\
&= \$11{,}735.11
\end{aligned}$$

With the same interest rate but using continuous compounding, the $10,000 investment will grow to $11,735.11 in two years, compared with $11,716.59 using quarterly compounding as shown in Example 4.

Table 1 shows how a stated annual interest rate of 8 percent generates different ending dollar amounts with annual, semiannual, quarterly, monthly, daily, and continuous compounding for an initial investment of $1 (carried out to six decimal places).

TABLE 1 The Effect of Compounding Frequency on Future Value

Frequency	r_s/m	mN	Future Value of \$1	
Annual	8%/1 = 8%	1 × 1 = 1	\$1.00(1.08)	= \$1.08
Semiannual	8%/2 = 4%	2 × 1 = 2	\$1.00$(1.04)^2$	= \$1.081600
Quarterly	8%/4 = 2%	4 × 1 = 4	\$1.00$(1.02)^4$	= \$1.082432
Monthly	8%/12 = 0.6667%	12 × 1 = 12	\$1.00$(1.006667)^{12}$	= \$1.083000
Daily	8%/365 = 0.0219%	365 × 1 = 365	\$1.00$(1.000219)^{365}$	= \$1.083278
Continuous			\$1.00$e^{0.08(1)}$	= \$1.083287

As Table 1 shows, all six cases have the same stated annual interest rate of 8 percent; they have different ending dollar amounts, however, because of differences in the frequency of compounding. With annual compounding, the ending amount is \$1.08. More frequent compounding results in larger ending amounts. The ending dollar amount with continuous compounding is the maximum amount that can be earned with a stated annual rate of 8 percent.

Table 1 also shows that a \$1 investment earning 8.16 percent compounded annually grows to the same future value at the end of one year as a \$1 investment earning 8 percent compounded semiannually. This result leads us to a distinction between the stated annual interest rate and the **effective annual rate** (EAR).[6] For an 8 percent stated annual interest rate with semiannual compounding, the EAR is 8.16 percent.

3.3. Stated and Effective Rates

The stated annual interest rate does not give a future value directly, so we need a formula for the EAR. With an annual interest rate of 8 percent compounded semiannually, we receive a periodic rate of 4 percent. During the course of a year, an investment of \$1 would grow to $\$1(1.04)^2 = \$1.0816$, as illustrated in Table 1. The interest earned on the \$1 investment is \$0.0816 and represents an effective annual rate of interest of 8.16 percent. The effective annual rate is calculated as follows:

$$\text{EAR} = (1 + \text{Periodic interest rate})^m - 1 \tag{5}$$

The periodic interest rate is the stated annual interest rate divided by m, where m is the number of compounding periods in one year. Using our previous example, we can solve for EAR as follows: $(1.04)^2 - 1 = 8.16$ percent.

[6]Among the terms used for the effective annual return on interest-bearing bank deposits are annual percentage yield (APY) in the United States and equivalent annual rate (EAR) in the United Kingdom. By contrast, the **annual percentage rate** (APR) measures the cost of borrowing expressed as a yearly rate. In the United States, the APR is calculated as a periodic rate times the number of payment periods per year and, as a result, some writers use APR as a general synonym for the stated annual interest rate. Nevertheless, APR is a term with legal connotations; its calculation follows regulatory standards that vary internationally. Therefore, "stated annual interest rate" is the preferred general term for an annual interest rate that does not account for compounding within the year.

The concept of EAR extends to continuous compounding. Suppose we have a rate of 8 percent compounded continuously. We can find the EAR in the same way as above by finding the appropriate future value factor. In this case, a \$1 investment would grow to $\$1e^{0.08(1.0)} = \$1.0833$. The interest earned for one year represents an effective annual rate of 8.33 percent and is larger than the 8.16 percent EAR with semiannual compounding because interest is compounded more frequently. With continuous compounding, we can solve for the effective annual rate as follows:

$$\text{EAR} = e^{r_s} - 1 \tag{6}$$

We can reverse the formulas for EAR with discrete and continuous compounding to find a periodic rate that corresponds to a particular effective annual rate. Suppose we want to find the appropriate periodic rate for a given effective annual rate of 8.16 percent with semiannual compounding. We can use Equation 5 to find the periodic rate:

$$\begin{aligned}
0.0816 &= (1 + \text{Periodic rate})^2 - 1 \\
1.0816 &= (1 + \text{Periodic rate})^2 \\
(1.0816)^{1/2} - 1 &= \text{Periodic rate} \\
(1.04) - 1 &= \text{Periodic rate} \\
4\% &= \text{Periodic rate}
\end{aligned}$$

To calculate the continuously compounded rate (the stated annual interest rate with continuous compounding) corresponding to an effective annual rate of 8.33 percent, we find the interest rate that satisfies Equation 6:

$$\begin{aligned}
0.0833 &= e^{r_s} - 1 \\
1.0833 &= e^{r_s}
\end{aligned}$$

To solve this equation, we take the natural logarithm of both sides. (Recall that the natural log of e^{r_s} is $\ln e^{r_s} = r_s$.) Therefore, $\ln 1.0833 = r_s$, resulting in $r_s = 8$ percent. We see that a stated annual rate of 8 percent with continuous compounding is equivalent to an EAR of 8.33 percent.

4. THE FUTURE VALUE OF A SERIES OF CASH FLOWS

In this section, we consider series of cash flows, both even and uneven. We begin with a list of terms commonly used when valuing cash flows that are distributed over many time periods.

- An **annuity** is a finite set of level sequential cash flows.
- An **ordinary annuity** has a first cash flow that occurs one period from now (indexed at $t = 1$).
- An **annuity due** has a first cash flow that occurs immediately (indexed at $t = 0$).
- A **perpetuity** is a perpetual annuity, or a set of level never-ending sequential cash flows, with the first cash flow occurring one period from now.

4.1. Equal Cash Flows—Ordinary Annuity

Consider an ordinary annuity paying 5 percent annually. Suppose we have five separate deposits of \$1,000 occurring at equally spaced intervals of one year, with the first payment occurring at $t = 1$. Our goal is to find the future value of this ordinary annuity after the last deposit at $t = 5$. The increment in the time counter is one year, so the last payment occurs five years from now. As the time line in Figure 3 shows, we find the future value of each \$1,000 deposit as of $t = 5$ with Equation 2, $FV_N = PV(1 + r)^N$. The arrows in Figure 3 extend from the payment date to $t = 5$. For instance, the first \$1,000 deposit made at $t = 1$ will compound over four periods. Using Equation 2, we find that the future value of the first deposit at $t = 5$ is $\$1,000(1.05)^4 = \$1,215.51$. We calculate the future value of all other payments in a similar fashion. (Note that we are finding the future value at $t = 5$, so the last payment does not earn any interest.) With all values now at $t = 5$, we can add the future values to arrive at the future value of the annuity. This amount is \$5,525.63.

We can arrive at a general annuity formula if we define the annuity amount as A, the number of time periods as N, and the interest rate per period as r. We can then define the future value as

$$FV_N = A\left[(1+r)^{N-1} + (1+r)^{N-2} + (1+r)^{N-3} + \ldots + (1+r)^1 + (1+r)^0\right]$$

which simplifies to

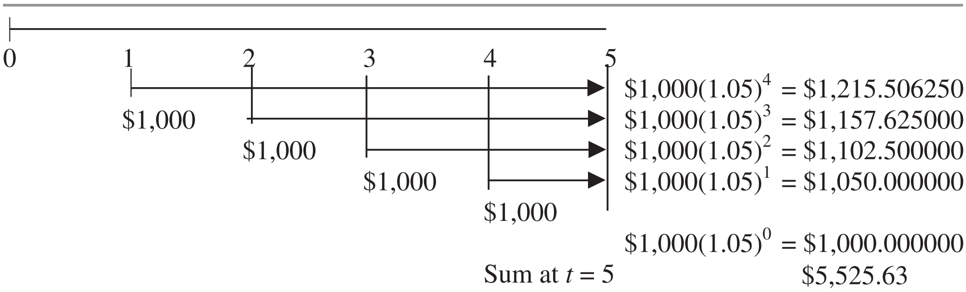
$$FV_N = A\left[\frac{(1+r)^N - 1}{r}\right] \tag{7}$$

The term in brackets is the future value annuity factor. This factor gives the future value of an ordinary annuity of \$1 per period. Multiplying the future value annuity factor by the annuity amount gives the future value of an ordinary annuity. For the ordinary annuity in Figure 3, we find the future value annuity factor from Equation 7 as

$$\left[\frac{(1.05)^5 - 1}{0.05}\right] = 5.525631$$

With an annuity amount $A = \$1,000$, the future value of the annuity is $\$1,000(5.525631) = \$5,525.63$, an amount that agrees with our earlier work.

FIGURE 3 The Future Value of a Five-Year Ordinary Annuity



The next example illustrates how to find the future value of an ordinary annuity using the formula in Equation 7.

EXAMPLE 7 The Future Value of an Annuity

Suppose your company's defined contribution retirement plan allows you to invest up to €20,000 per year. You plan to invest €20,000 per year in a stock index fund for the next 30 years. Historically, this fund has earned 9 percent per year on average. Assuming that you actually earn 9 percent a year, how much money will you have available for retirement after making the last payment?

Solution: Use Equation 7 to find the future amount:

$$A = €20{,}000$$
$$r = 9\% = 0.09$$
$$N = 30$$
$$\text{FV annuity factor} = \frac{(1+r)^N - 1}{r} = \frac{(1.09)^{30} - 1}{0.09} = 136.307539$$
$$\text{FV}_N = €20{,}000(136.307539)$$
$$= €2{,}726{,}150.77$$

Assuming the fund continues to earn an average of 9 percent per year, you will have €2,726,150.77 available at retirement.

4.2. Unequal Cash Flows

In many cases, cash flow streams are unequal, precluding the simple use of the future value annuity factor. For instance, an individual investor might have a savings plan that involves unequal cash payments depending on the month of the year or lower savings during a planned vacation. One can always find the future value of a series of unequal cash flows by compounding the cash flows one at a time. Suppose you have the five cash flows described in Table 2, indexed relative to the present ($t = 0$).

TABLE 2 A Series of Unequal Cash Flows and Their Future Values at 5 Percent

Time	Cash Flow ($)	Future Value at Year 5	
$t = 1$	1,000	$\$1{,}000(1.05)^4$ =	$1,215.51
$t = 2$	2,000	$\$2{,}000(1.05)^3$ =	$2,315.25
$t = 3$	4,000	$\$4{,}000(1.05)^2$ =	$4,410.00
$t = 4$	5,000	$\$5{,}000(1.05)^1$ =	$5,250.00
$t = 5$	6,000	$\$6{,}000(1.05)^0$ =	$6,000.00
		Sum =	$19,190.76

All of the payments shown in Table 2 are different. Therefore, the most direct approach to finding the future value at $t = 5$ is to compute the future value of each payment as of $t = 5$ and then sum the individual future values. The total future value at Year 5 equals \$19,190.76, as shown in the third column. Later in this reading, you will learn shortcuts to take when the cash flows are close to even; these shortcuts will allow you to combine annuity and single-period calculations.

5. THE PRESENT VALUE OF A SINGLE CASH FLOW

5.1. Finding the Present Value of a Single Cash Flow

Just as the future value factor links today's present value with tomorrow's future value, the present value factor allows us to discount future value to present value. For example, with a 5 percent interest rate generating a future payoff of \$105 in one year, what current amount invested at 5 percent for one year will grow to \$105? The answer is \$100; therefore, \$100 is the present value of \$105 to be received in one year at a discount rate of 5 percent.

Given a future cash flow that is to be received in N periods and an interest rate per period of r, we can use the formula for future value to solve directly for the present value as follows:

$$
\begin{aligned}
\mathrm{FV}_N &= \mathrm{PV}(1+r)^N \\
\mathrm{PV} &= \mathrm{FV}_N\left[\frac{1}{(1+r)^N}\right] \\
\mathrm{PV} &= \mathrm{FV}_N(1+r)^{-N}
\end{aligned}
\tag{8}
$$

We see from Equation 8 that the present value factor, $(1 + r)^{-N}$, is the reciprocal of the future value factor, $(1 + r)^N$.

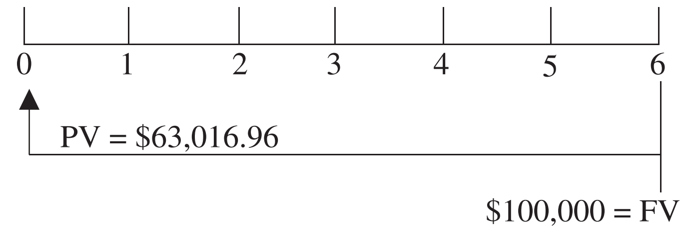
EXAMPLE 8 The Present Value of a Lump Sum

An insurance company has issued a Guaranteed Investment Contract (GIC) that promises to pay \$100,000 in six years with an 8 percent return rate. What amount of money must the insurer invest today at 8 percent for six years to make the promised payment?

Solution: We can use Equation 8 to find the present value using the following data:

$$
\begin{aligned}
\mathrm{FV}_N &= \$100{,}000 \\
r &= 8\% = 0.08 \\
N &= 6 \\
\mathrm{PV} &= \mathrm{FV}_N(1+r)^{-N} \\
&= \$100{,}000\left[\frac{1}{(1.08)^6}\right] \\
&= \$100{,}000(0.6301696) \\
&= \$63{,}016.96
\end{aligned}
$$

FIGURE 4 The Present Value of a Lump Sum to Be Received at Time $t = 6$



We can say that \$63,016.96 today, with an interest rate of 8 percent, is equivalent to \$100,000 to be received in six years. Discounting the \$100,000 makes a future \$100,000 equivalent to \$63,016.96 when allowance is made for the time value of money. As the time line in Figure 4 shows, the \$100,000 has been discounted six full periods.

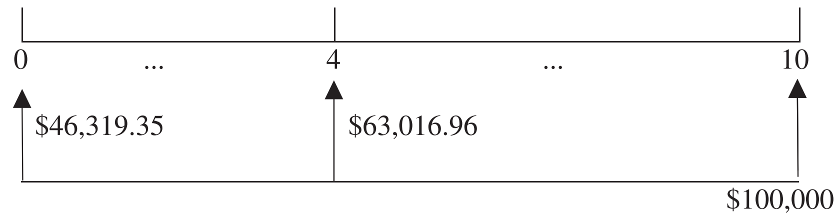
EXAMPLE 9 The Projected Present Value of a More Distant Future Lump Sum

Suppose you own a liquid financial asset that will pay you \$100,000 in 10 years from today. Your daughter plans to attend college four years from today, and you want to know what the asset's present value will be at that time. Given an 8 percent discount rate, what will the asset be worth four years from today?

Solution: The value of the asset is the present value of the asset's promised payment. At $t = 4$, the cash payment will be received six years later. With this information, you can solve for the value four years from today using Equation 8:

$$\begin{aligned}
FV_N &= \$100{,}000 \\
r &= 8\% = 0.08 \\
N &= 6 \\
PV &= FV_N(1+r)^{-N} \\
&= \$100{,}000\frac{1}{(1.08)^6} \\
&= \$100{,}000(0.6301696) \\
&= \$63{,}016.96
\end{aligned}$$

FIGURE 5 The Relationship between Present Value and Future Value



The time line in Figure 5 shows the future payment of $100,000 that is to be received at $t = 10$. The time line also shows the values at $t = 4$ and at $t = 0$. Relative to the payment at $t = 10$, the amount at $t = 4$ is a projected present value, while the amount at $t = 0$ is the present value (as of today).

Present value problems require an evaluation of the present value factor, $(1 + r)^{-N}$. Present values relate to the discount rate and the number of periods in the following ways:

- For a given discount rate, the further in the future the amount to be received, the smaller that amount's present value.
- Holding time constant, the larger the discount rate, the smaller the present value of a future amount.

5.2. The Frequency of Compounding

Recall that interest may be paid semiannually, quarterly, monthly, or even daily. To handle interest payments made more than once a year, we can modify the present value formula (Equation 8) as follows. Recall that r_s is the quoted interest rate and equals the periodic interest rate multiplied by the number of compounding periods in each year. In general, with more than one compounding period in a year, we can express the formula for present value as

$$PV = FV_N \left(1 + \frac{r_s}{m}\right)^{-mN} \tag{9}$$

where

m = number of compounding periods per year
r_s = quoted annual interest rate
N = number of years

The formula in Equation 9 is quite similar to that in Equation 8. As we have already noted, present value and future value factors are reciprocals. Changing the frequency of compounding does not alter this result. The only difference is the use of the periodic interest rate and the corresponding number of compounding periods.

The following example illustrates Equation 9.

EXAMPLE 10 The Present Value of a Lump Sum with Monthly Compounding

The manager of a Canadian pension fund knows that the fund must make a lump-sum payment of C$5 million 10 years from now. She wants to invest an amount today in a GIC so that it will grow to the required amount. The current interest rate on GICs is 6 percent a year, compounded monthly. How much should she invest today in the GIC?

Solution: Use Equation 9 to find the required present value:

$$\begin{aligned}
\text{FV}_N &= \text{C\$5,000,000} \\
r_s &= 6\% = 0.06 \\
m &= 12 \\
r_s/m &= 0.06/12 = 0.005 \\
N &= 10 \\
mN &= 12(10) = 120 \\
\text{PV} &= \text{FV}_N\left(1+\frac{r_s}{m}\right)^{-mN} \\
&= \text{C\$5,000,000}(1.005)^{-120} \\
&= \text{C\$5,000,000}(0.549633) \\
&= \text{C\$2,748,163.67}
\end{aligned}$$

In applying Equation 9, we use the periodic rate (in this case, the monthly rate) and the appropriate number of periods with monthly compounding (in this case, 10 years of monthly compounding, or 120 periods).

6. THE PRESENT VALUE OF A SERIES OF CASH FLOWS

Many applications in investment management involve assets that offer a series of cash flows over time. The cash flows may be highly uneven, relatively even, or equal. They may occur over relatively short periods of time, longer periods of time, or even stretch on indefinitely. In this section, we discuss how to find the present value of a series of cash flows.

6.1. The Present Value of a Series of Equal Cash Flows

We begin with an ordinary annuity. Recall that an ordinary annuity has equal annuity payments, with the first payment starting one period into the future. In total, the annuity makes N payments, with the first payment at $t = 1$ and the last at $t = N$. We can express the present value of an ordinary annuity as the sum of the present values of each individual annuity payment, as follows:

$$PV = \frac{A}{(1+r)} + \frac{A}{(1+r)^2} + \frac{A}{(1+r)^3} + \ldots + \frac{A}{(1+r)^{N-1}} + \frac{A}{(1+r)^N} \tag{10}$$

where

A = the annuity amount

r = the interest rate per period corresponding to the frequency of annuity payments (for example, annual, quarterly, or monthly)

N = the number of annuity payments

Because the annuity payment (A) is a constant in this equation, it can be factored out as a common term. Thus the sum of the interest factors has a shortcut expression:

$$PV = A\left[\frac{1 - \frac{1}{(1+r)^N}}{r}\right] \tag{11}$$

In much the same way that we computed the future value of an ordinary annuity, we find the present value by multiplying the annuity amount by a present value annuity factor (the term in brackets in Equation 11).

EXAMPLE 11 The Present Value of an Ordinary Annuity

Suppose you are considering purchasing a financial asset that promises to pay €1,000 per year for five years, with the first payment one year from now. The required rate of return is 12 percent per year. How much should you pay for this asset?

Solution: To find the value of the financial asset, use the formula for the present value of an ordinary annuity given in Equation 11 with the following data:

A = €1,000

r = 12% = 0.12

N = 5

$$PV = A\left[\frac{1 - \frac{1}{(1+r)^N}}{r}\right]$$

$$= €1{,}000\left[\frac{1 - \frac{1}{(1.12)^5}}{0.12}\right]$$

$= €1{,}000(3.604776)$
$= €3{,}604.78$

The series of cash flows of €1,000 per year for five years is currently worth €3,604.78 when discounted at 12 percent.

FIGURE 6 An Annuity Due of $100 per Period

Keeping track of the actual calendar time brings us to a specific type of annuity with level payments: the annuity due. An annuity due has its first payment occurring today ($t = 0$). In total, the annuity due will make N payments. Figure 6 presents the time line for an annuity due that makes four payments of $100.

As Figure 6 shows, we can view the four-period annuity due as the sum of two parts: a $100 lump sum today and an ordinary annuity of $100 per period for three periods. At a 12 percent discount rate, the four $100 cash flows in this annuity due example will be worth $340.18.[7]

Expressing the value of the future series of cash flows in today's dollars gives us a convenient way of comparing annuities. The next example illustrates this approach.

EXAMPLE 12 An Annuity Due as the Present Value of an Immediate Cash Flow Plus an Ordinary Annuity

You are retiring today and must choose to take your retirement benefits either as a lump sum or as an annuity. Your company's benefits officer presents you with two alternatives: an immediate lump sum of $2 million or an annuity with 20 payments of $200,000 a year with the first payment starting today. The interest rate at your bank is 7 percent per year compounded annually. Which option has the greater present value? (Ignore any tax differences between the two options.)

Solution: To compare the two options, find the present value of each at time $t = 0$ and choose the one with the larger value. The first option's present value is $2 million,

[7] There is an alternative way to calculate the present value of an annuity due. Compared to an ordinary annuity, the payments in an annuity due are each discounted one less period. Therefore, we can modify Equation 11 to handle annuities due by multiplying the right-hand side of the equation by $(1 + r)$:

$$\text{PV}(\text{Annuity due}) = A\left\{\left[1-(1+r)^{-N}\right]/r\right\}(1+r)$$

already expressed in today's dollars. The second option is an annuity due. Because the first payment occurs at $t = 0$, you can separate the annuity benefits into two pieces: an immediate \$200,000 to be paid today ($t = 0$) and an ordinary annuity of \$200,000 per year for 19 years. To value this option, you need to find the present value of the ordinary annuity using Equation 11 and then add \$200,000 to it.

$$A = \$200{,}000$$
$$N = 19$$
$$r = 7\% = 0.07$$
$$\text{PV} = A\left[\frac{1 - \dfrac{1}{(1+r)^N}}{r}\right]$$
$$= \$200{,}000\left[\frac{1 - \dfrac{1}{(1.07)^{19}}}{0.07}\right]$$
$$= \$200{,}000(10.335595)$$
$$= \$2{,}067{,}119.05$$

The 19 payments of \$200,000 have a present value of \$2,067,119.05. Adding the initial payment of \$200,000 to \$2,067,119.05, we find that the total value of the annuity option is \$2,267,119.05. The present value of the annuity is greater than the lump sum alternative of \$2 million.

We now look at another example reiterating the equivalence of present and future values.

EXAMPLE 13 The Projected Present Value of an Ordinary Annuity

A German pension fund manager anticipates that benefits of €1 million per year must be paid to retirees. Retirements will not occur until 10 years from now at time $t = 10$. Once benefits begin to be paid, they will extend until $t = 39$ for a total of 30 payments. What is the present value of the pension liability if the appropriate annual discount rate for plan liabilities is 5 percent compounded annually?

Solution: This problem involves an annuity with the first payment at $t = 10$. From the perspective of $t = 9$, we have an ordinary annuity with 30 payments. We can compute the present value of this annuity with Equation 11 and then look at it on a time line.

$$A = €1{,}000{,}000$$
$$r = 5\% = 0.05$$

$$N = 30$$

$$PV = A\left[\frac{1-\frac{1}{(1+r)^N}}{r}\right]$$

$$= €1{,}000{,}000\left[\frac{1-\frac{1}{(1.05)^{30}}}{0.05}\right]$$

$$= €1{,}000{,}000(15.372451)$$
$$= €15{,}372{,}451.03$$

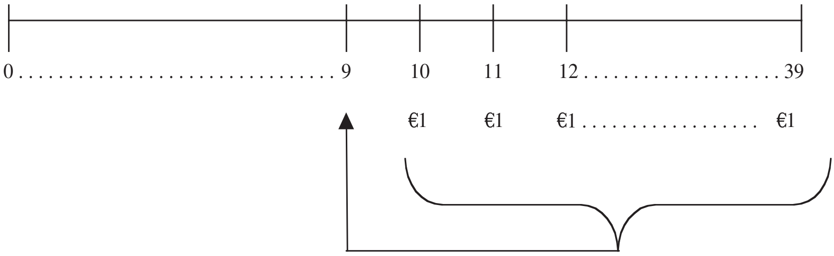
On the time line, we have shown the pension payments of €1 million extending from $t = 10$ to $t = 39$. The bracket and arrow indicate the process of finding the present value of the annuity, discounted back to $t = 9$. The present value of the pension benefits as of $t = 9$ is €15,372,451.03. The problem is to find the present value today (at $t = 0$).

Now we can rely on the equivalence of present value and future value. As Figure 7 shows, we can view the amount at $t = 9$ as a future value from the vantage point of $t = 0$. We compute the present value of the amount at $t = 9$ as follows:

$$FV_N = €15{,}372{,}451.03 \text{ (the present value at } t = 9\text{)}$$
$$N = 9$$
$$r = 5\% = 0.05$$
$$PV = FV_N(1 + r)^{-N}$$
$$= €15{,}372{,}451.03(1.05)^{-9}$$
$$= €15{,}372{,}451.03(0.644609)$$
$$= €9{,}909{,}219.00$$

The present value of the pension liability is €9,909,219.00.

FIGURE 7 The Present Value of an Ordinary Annuity with First Payment at Time $t = 10$ (in Millions)



Example 13 illustrates three procedures emphasized in this reading:

- finding the present or future value of any cash flow series;
- recognizing the equivalence of present value and appropriately discounted future value; and
- keeping track of the actual calendar time in a problem involving the time value of money.

6.2. The Present Value of an Infinite Series of Equal Cash Flows—Perpetuity

Consider the case of an ordinary annuity that extends indefinitely. Such an ordinary annuity is called a perpetuity (a perpetual annuity). To derive a formula for the present value of a perpetuity, we can modify Equation 10 to account for an infinite series of cash flows:

$$\text{PV} = A\sum_{t=1}^{\infty}\left[\frac{1}{(1+r)^t}\right] \tag{12}$$

As long as interest rates are positive, the sum of present value factors converges and

$$\text{PV} = \frac{A}{r} \tag{13}$$

To see this, look back at Equation 11, the expression for the present value of an ordinary annuity. As N (the number of periods in the annuity) goes to infinity, the term $1/(1 + r)^N$ approaches 0 and Equation 11 simplifies to Equation 13. This equation will reappear when we value dividends from stocks because stocks have no predefined life span. (A stock paying constant dividends is similar to a perpetuity.) With the first payment a year from now, a perpetuity of \$10 per year with a 20 percent required rate of return has a present value of \$10/0.2 = \$50.

Equation 13 is valid only for a perpetuity with level payments. In our development above, the first payment occurred at $t = 1$; therefore, we compute the present value as of $t = 0$.

Other assets also come close to satisfying the assumptions of a perpetuity. Certain government bonds and preferred stocks are typical examples of financial assets that make level payments for an indefinite period of time.

EXAMPLE 14 The Present Value of a Perpetuity

The British government once issued a type of security called a consol bond, which promised to pay a level cash flow indefinitely. If a consol bond paid £100 per year in perpetuity, what would it be worth today if the required rate of return were 5 percent?

Solution: To answer this question, we can use Equation 13 with the following data:

$$\begin{aligned} A &= £100 \\ r &= 5\% = 0.05 \\ \text{PV} &= A/r \\ &= £100/0.05 \\ &= £2{,}000 \end{aligned}$$

The bond would be worth £2,000.

6.3. Present Values Indexed at Times Other than $t = 0$

In practice with investments, analysts frequently need to find present values indexed at times other than $t = 0$. Subscripting the present value and evaluating a perpetuity beginning with \$100 payments in Year 2, we find PV_1 = \$100/0.05 = \$2,000 at a 5 percent discount rate. Further, we can calculate today's PV as PV_0 = \$2,000/1.05 = \$1,904.76.

Consider a similar situation in which cash flows of \$6 per year begin at the end of the fourth year and continue at the end of each year thereafter, with the last cash flow at the end of the 10th year. From the perspective of the end of the third year, we are facing a typical seven-year ordinary annuity. We can find the present value of the annuity from the perspective of the end of the third year and then discount that present value back to the present. At an interest rate of 5 percent, the cash flows of \$6 per year starting at the end of the fourth year will be worth \$34.72 at the end of the third year ($t = 3$) and \$29.99 today ($t = 0$).

The next example illustrates the important concept that an annuity or perpetuity beginning sometime in the future can be expressed in present value terms one period prior to the first payment. That present value can then be discounted back to today's present value.

EXAMPLE 15 The Present Value of a Projected Perpetuity

Consider a level perpetuity of £100 per year with its first payment beginning at $t = 5$. What is its present value today (at $t = 0$), given a 5 percent discount rate?

Solution: First, we find the present value of the perpetuity at $t = 4$ and then discount that amount back to $t = 0$. (Recall that a perpetuity or an ordinary annuity has its first payment one period away, explaining the $t = 4$ index for our present value calculation.)

i. Find the present value of the perpetuity at $t = 4$:

$$\begin{aligned} A &= £100 \\ r &= 5\% = 0.05 \\ PV &= A/r \\ &= £100/0.05 \\ &= £2{,}000 \end{aligned}$$

ii. Find the present value of the future amount at $t = 4$. From the perspective of $t = 0$, the present value of £2,000 can be considered a future value. Now we need to find the present value of a lump sum:

$$\begin{aligned} FV_N &= £2{,}000 \text{ (the present value at } t = 4) \\ r &= 5\% = 0.05 \\ N &= 4 \\ PV &= FV_N (1+r)^{-N} \\ &= £2{,}000(1.05)^{-4} \\ &= £2{,}000(0.822702) \\ &= £1{,}645.40 \end{aligned}$$

Today's present value of the perpetuity is £1,645.40.

As discussed earlier, an annuity is a series of payments of a fixed amount for a specified number of periods. Suppose we own a perpetuity. At the same time, we issue a perpetuity obligating us to make payments; these payments are the same size as those of the perpetuity we own. However, the first payment of the perpetuity we issue is at $t = 5$; payments then continue on forever. The payments on this second perpetuity exactly offset the payments received from the perpetuity we own at $t = 5$ and all subsequent dates. We are left with level nonzero net cash flows at $t = 1, 2, 3$, and 4. This outcome exactly fits the definition of an annuity with four payments. Thus we can construct an annuity as the difference between two perpetuities with equal, level payments but differing starting dates. The next example illustrates this result.

EXAMPLE 16 The Present Value of an Ordinary Annuity as the Present Value of a Current Minus Projected Perpetuity

Given a 5 percent discount rate, find the present value of a four-year ordinary annuity of £100 per year starting in Year 1 as the difference between the following two level perpetuities:

Perpetuity 1 £100 per year starting in Year 1 (first payment at $t = 1$)
Perpetuity 2 £100 per year starting in Year 5 (first payment at $t = 5$)

Solution: If we subtract Perpetuity 2 from Perpetuity 1, we are left with an ordinary annuity of £100 per period for four years (payments at $t = 1, 2, 3, 4$). Subtracting the present value of Perpetuity 2 from that of Perpetuity 1, we arrive at the present value of the four-year ordinary annuity:

$$\begin{aligned}
PV_0(\text{Perpetuity 1}) &= £100 / 0.05 = £2{,}000 \\
PV_4(\text{Perpetuity 2}) &= £100 / 0.05 = £2{,}000 \\
PV_0(\text{Perpetuity 2}) &= £2{,}000 / (1.05)^4 = £1{,}645.40 \\
PV_0(\text{Annuity}) &= PV_0(\text{Perpetuity 1}) - PV_0(\text{Perpetuity 2}) \\
&= £2{,}000 - £1{,}645.40 \\
&= £354.60
\end{aligned}$$

The four-year ordinary annuity's present value is equal to £2,000 – £1,645.40 = £354.60.

6.4. The Present Value of a Series of Unequal Cash Flows

When we have unequal cash flows, we must first find the present value of each individual cash flow and then sum the respective present values. For a series with many cash flows, we usually use a spreadsheet. Table 3 lists a series of cash flows with the time periods in the first column, cash flows in the second column, and each cash flow's present value in the third column. The last row of Table 3 shows the sum of the five present values.

TABLE 3 A Series of Unequal Cash Flows and Their Present Values at 5 Percent

Time Period	Cash Flow ($)	Present Value at Year 0	
1	1,000	$\$1{,}000(1.05)^{-1}$ =	\$952.38
2	2,000	$\$2{,}000(1.05)^{-2}$ =	\$1,814.06
3	4,000	$\$4{,}000(1.05)^{-3}$ =	\$3,455.35
4	5,000	$\$5{,}000(1.05)^{-4}$ =	\$4,113.51
5	6,000	$\$6{,}000(1.05)^{-5}$ =	\$4,701.16
		Sum =	\$15,036.46

We could calculate the future value of these cash flows by computing them one at a time using the single-payment future value formula. We already know the present value of this series, however, so we can easily apply time-value equivalence. The future value of the series of cash flows from Table 2, \$19,190.76, is equal to the single \$15,036.46 amount compounded forward to $t = 5$:

$$\begin{aligned} \text{PV} &= \$15{,}036.46 \\ N &= 5 \\ r &= 5\% = 0.05 \\ \text{FV}_N &= \text{PV}(1+r)^N \\ &= \$15{,}036.46(1.05)^5 \\ &= \$15{,}036.46(1.276282) \\ &= \$19{,}190.76 \end{aligned}$$

7. SOLVING FOR RATES, NUMBER OF PERIODS, OR SIZE OF ANNUITY PAYMENTS

In the previous examples, certain pieces of information have been made available. For instance, all problems have given the rate of interest, r, the number of time periods, N, the annuity amount, A, and either the present value, PV, or future value, FV. In real-world applications, however, although the present and future values may be given, you may have to solve for either the interest rate, the number of periods, or the annuity amount. In the subsections that follow, we show these types of problems.

7.1. Solving for Interest Rates and Growth Rates

Suppose a bank deposit of €100 is known to generate a payoff of €111 in one year. With this information, we can infer the interest rate that separates the present value of €100 from the future value of €111 by using Equation 2, $\text{FV}_N = \text{PV}(1 + r)^N$, with $N = 1$. With PV, FV, and N known, we can solve for r directly:

$$\begin{aligned} 1 + r &= \text{FV}/\text{PV} \\ 1 + r &= €111/€100 = 1.11 \\ r &= 0.11, \text{ or } 11\% \end{aligned}$$

The interest rate that equates €100 at $t = 0$ to €111 at $t = 1$ is 11 percent. Thus we can state that €100 grows to €111 with a growth rate of 11 percent.

As this example shows, an interest rate can also be considered a growth rate. The particular application will usually dictate whether we use the term "interest rate" or "growth rate." Solving Equation 2 for r and replacing the interest rate r with the growth rate g produces the following expression for determining growth rates:

$$g = (\mathrm{FV}_N/\mathrm{PV})^{1/N} - 1 \tag{14}$$

Below are two examples that use the concept of a growth rate.

EXAMPLE 17 Calculating a Growth Rate (1)

Hyundai Steel, the first Korean steelmaker, was established in 1953. Hyundai Steel's sales increased from ₩10,503.0 billion in 2008 to ₩14,146.4 billion in 2012. However, its net profit declined from ₩822.5 billion in 2008 to ₩796.4 billion in 2012. Calculate the following growth rates for Hyundai Steel for the four-year period from the end of 2008 to the end of 2012:

1. Sales growth rate.
2. Net profit growth rate.

Solution to 1: To solve this problem, we can use Equation 14, $g = (\mathrm{FV}_N/\mathrm{PV})^{1/N} - 1$. We denote sales in 2008 as PV and sales in 2012 as FV_4. We can then solve for the growth rate as follows:

$$\begin{aligned} g &= \sqrt[4]{₩14{,}146.4/₩10{,}503.0} - 1 \\ &= \sqrt[4]{1.346891} - 1 \\ &= 1.077291 - 1 \\ &= 0.077291 \text{ or about } 7.7\% \end{aligned}$$

The calculated growth rate of about 7.7 percent a year shows that Hyundai Steel's sales grew substantially during the 2008–2012 period.

Solution to 2: In this case, we can speak of a positive compound rate of decrease or a negative compound growth rate. Using Equation 14, we find

$$\begin{aligned} g &= \sqrt[4]{₩796.4/₩822.5} - 1 \\ &= \sqrt[4]{0.968267} - 1 \\ &= 0.991971 - 1 \\ &= -0.008029 \text{ or about } -0.80\% \end{aligned}$$

In contrast to the positive sales growth, the rate of growth in net profit was approximately –0.80 percent during the 2008–2012 period.

EXAMPLE 18 Calculating a Growth Rate (2)

Toyota Motor Corporation, one of the largest automakers in the world, had consolidated vehicle sales of 7.35 million units in 2012. This is substantially less than consolidated vehicle sales of 8.52 million units five years earlier in 2007. What was the growth rate in number of vehicles sold by Toyota from 2007 to 2012?

Solution: Using Equation 14, we find

$$\begin{aligned} g &= \sqrt[5]{7.35/8.52} - 1 \\ &= \sqrt[5]{0.862676} - 1 \\ &= 0.970889 - 1 \\ &= -0.029111 \text{ or about } -2.9\% \end{aligned}$$

The rate of growth in vehicles sold was approximately –2.9 percent during the 2007–2012 period. Note that we can also refer to –2.9 percent as the compound annual growth rate because it is the single number that compounds the number of vehicles sold in 2007 forward to the number of vehicles sold in 2012. Table 4 lists the number of vehicles sold by Toyota from 2007 to 2012.

Table 4 also shows 1 plus the one-year growth rate in number of vehicles sold. We can compute the 1 plus five-year cumulative growth in number of vehicles sold from 2007 to 2012 as the product of quantities (1 + one-year growth rate). We arrive at the same result as when we divide the ending number of vehicles sold, 7.35 million, by the beginning number of vehicles sold, 8.52 million:

$$\begin{aligned} \frac{7.35}{8.52} &= \left(\frac{8.91}{8.52}\right)\left(\frac{7.57}{8.91}\right)\left(\frac{7.24}{7.57}\right)\left(\frac{7.31}{7.24}\right)\left(\frac{7.35}{7.31}\right) \\ &= (1+g_1)(1+g_2)(1+g_3)(1+g_4)(1+g_5) \\ 0.862676 &= (1.045775)(0.849607)(0.956407)(1.009669)(1.005472) \end{aligned}$$

TABLE 4 Number of Vehicles Sold, 2007–2012

Year	Number of Vehicles Sold (Millions)	$(1 + g)_t$	t
2007	8.52		0
2008	8.91	8.91/8.52 = 1.045775	1
2009	7.57	7.57/8.91 = 0.849607	2
2010	7.24	7.24/7.57 = 0.956407	3
2011	7.31	7.31/7.24 = 1.009669	4
2012	7.35	7.35/7.31 = 1.005472	5

Source: www.toyota.com.

The right-hand side of the equation is the product of 1 plus the one-year growth rate in number of vehicles sold for each year. Recall that, using Equation 14, we took the fifth root of 7.35/8.52 = 0.862676. In effect, we were solving for the single value of *g* which, when compounded over five periods, gives the correct product of 1 plus the one-year growth rates.[8]

In conclusion, we do not need to compute intermediate growth rates as in Table 4 to solve for a compound growth rate *g*. Sometimes, however, the intermediate growth rates are interesting or informative. For example, at first (from 2007 to 2008), Toyota Motors increased its number of vehicles sold. We can also analyze the variability in growth rates when we conduct an analysis as in Table 4. Most of the decline in Toyota Motor's sales occurred in 2009. Elsewhere in Toyota Motor's disclosures, the company noted that the substantial decline in vehicle sales in 2009 was due to the steep downturn in the global economy. Sales declined further in 2010 as the market conditions remained difficult. Each of the next two years saw a slight increase in sales.

The compound growth rate is an excellent summary measure of growth over multiple time periods. In our Toyota Motors example, the compound growth rate of −2.9 percent is the single growth rate that, when added to 1, compounded over five years, and multiplied by the 2007 number of vehicles sold, yields the 2012 number of vehicles sold.

7.2. Solving for the Number of Periods

In this section, we demonstrate how to solve for the number of periods given present value, future value, and interest or growth rates.

EXAMPLE 19 The Number of Annual Compounding Periods Needed for an Investment to Reach a Specific Value

You are interested in determining how long it will take an investment of €10,000,000 to double in value. The current interest rate is 7 percent compounded annually. How many years will it take €10,000,000 to double to €20,000,000?

Solution: Use Equation 2, $FV_N = PV(1 + r)^N$, to solve for the number of periods, N, as follows:

$$
\begin{aligned}
(1+r)^N &= FV_N/PV = 2 \\
N \ln(1+r) &= \ln(2) \\
N &= \ln(2)/\ln(1+r) \\
&= \ln(2)/\ln(1.07) = 10.24
\end{aligned}
$$

[8]The compound growth rate that we calculate here is an example of a geometric mean, specifically the geometric mean of the growth rates. We define the geometric mean in the reading on statistical concepts.

With an interest rate of 7 percent, it will take approximately 10 years for the initial €10,000,000 investment to grow to €20,000,000. Solving for N in the expression $(1.07)^N = 2.0$ requires taking the natural logarithm of both sides and using the rule that $\ln(x^N) = N \ln(x)$. Generally, we find that $N = [\ln(\text{FV}/\text{PV})]/\ln(1 + r)$. Here, $N = \ln(€20{,}000{,}000/ €10{,}000{,}000)/\ln(1.07) = \ln(2)/\ln(1.07) = 10.24$.[9]

7.3. Solving for the Size of Annuity Payments

In this section, we discuss how to solve for annuity payments. Mortgages, auto loans, and retirement savings plans are classic examples of applications of annuity formulas.

EXAMPLE 20 Calculating the Size of Payments on a Fixed-Rate Mortgage

You are planning to purchase a $120,000 house by making a down payment of $20,000 and borrowing the remainder with a 30-year fixed-rate mortgage with monthly payments. The first payment is due at $t = 1$. Current mortgage interest rates are quoted at 8 percent with monthly compounding. What will your monthly mortgage payments be?

Solution: The bank will determine the mortgage payments such that at the stated periodic interest rate, the present value of the payments will be equal to the amount borrowed (in this case, $100,000). With this fact in mind, we can use Equation 11,

$$\text{PV} = A\left[\frac{1-\frac{1}{(1+r)^N}}{r}\right]$$

, to solve for the annuity amount, A, as the present value divided by the present value annuity factor:

[9]To quickly approximate the number of periods, practitioners sometimes use an ad hoc rule called the **Rule of 72**: Divide 72 by the stated interest rate to get the approximate number of years it would take to double an investment at the interest rate. Here, the approximation gives 72/7 = 10.3 years. The Rule of 72 is loosely based on the observation that it takes 12 years to double an amount at a 6 percent interest rate, giving 6 × 12 = 72. At a 3 percent rate, one would guess it would take twice as many years, 3 × 24 = 72.

$$\text{PV} = \$100{,}000$$
$$r_s = 8\% = 0.08$$
$$m = 12$$
$$r_s/m = 0.08/12 = 0.006667$$
$$N = 30$$
$$mN = 12 \times 30 = 360$$
$$\text{Present value annuity factor} = \frac{1 - \frac{1}{\left[1 + \left(r_s/m\right)\right]^{mN}}}{r_s/m} = \frac{1 - \frac{1}{(1.006667)^{360}}}{0.006667}$$
$$= 136.283494$$
$$A = \text{PV/Present value annuity factor}$$
$$= \$100{,}000/136.283494$$
$$= \$733.76$$

The amount borrowed, $100,000, is equivalent to 360 monthly payments of $733.76 with a stated interest rate of 8 percent. The mortgage problem is a relatively straightforward application of finding a level annuity payment.

Next, we turn to a retirement-planning problem. This problem illustrates the complexity of the situation in which an individual wants to retire with a specified retirement income. Over the course of a life cycle, the individual may be able to save only a small amount during the early years but then may have the financial resources to save more during later years. Savings plans often involve uneven cash flows, a topic we will examine in the last part of this reading. When dealing with uneven cash flows, we take maximum advantage of the principle that dollar amounts indexed at the same point in time are additive—the **cash flow additivity principle**.

EXAMPLE 21 The Projected Annuity Amount Needed to Fund a Future-Annuity Inflow

Jill Grant is 22 years old (at $t = 0$) and is planning for her retirement at age 63 (at $t = 41$). She plans to save $2,000 per year for the next 15 years ($t = 1$ to $t = 15$). She wants to have retirement income of $100,000 per year for 20 years, with the first retirement payment starting at $t = 41$. How much must Grant save each year from $t = 16$ to $t = 40$ in order to achieve her retirement goal? Assume she plans to invest in a diversified stock-and-bond mutual fund that will earn 8 percent per year on average.

Solution: To help solve this problem, we set up the information on a time line. As Figure 8 shows, Grant will save $2,000 (an outflow) each year for Years 1 to 15. Starting in Year 41, Grant will start to draw retirement income of $100,000 per year for 20 years.

FIGURE 8 Solving for Missing Annuity Payments (in Thousands)

0	1	2 ...	15	16	17 ...	40	41	42 ...	60
	(\$2)	(\$2) ...	(\$2)	(*X*)	(*X*) ...	(*X*)	\$100	\$100 ...	\$100

In the time line, the annual savings is recorded in parentheses (\$2) to show that it is an outflow. The problem is to find the savings, recorded as X, from Year 16 to Year 40.

Solving this problem involves satisfying the following relationship: the present value of savings (outflows) equals the present value of retirement income (inflows). We could bring all the dollar amounts to $t = 40$ or to $t = 15$ and solve for X.

Let us evaluate all dollar amounts at $t = 15$ (we encourage the reader to repeat the problem by bringing all cash flows to $t = 40$). As of $t = 15$, the first payment of X will be one period away (at $t = 16$). Thus we can value the stream of Xs using the formula for the present value of an ordinary annuity.

This problem involves three series of level cash flows. The basic idea is that the present value of the retirement income must equal the present value of Grant's savings. Our strategy requires the following steps:

1. Find the future value of the savings of \$2,000 per year and index it at $t = 15$. This value tells us how much Grant will have saved.
2. Find the present value of the retirement income at $t = 15$. This value tells us how much Grant needs to meet her retirement goals (as of $t = 15$). Two substeps are necessary. First, calculate the present value of the annuity of \$100,000 per year at $t = 40$. Use the formula for the present value of an annuity. (Note that the present value is indexed at $t = 40$ because the first payment is at $t = 41$.) Next, discount the present value back to $t = 15$ (a total of 25 periods).
3. Now compute the difference between the amount Grant has saved (Step 1) and the amount she needs to meet her retirement goals (Step 2). Her savings from $t = 16$ to $t = 40$ must have a present value equal to the difference between the future value of her savings and the present value of her retirement income.

Our goal is to determine the amount Grant should save in each of the 25 years from $t = 16$ to $t = 40$. We start by bringing the \$2,000 savings to $t = 15$, as follows:

$$
\begin{aligned}
A &= \$2{,}000 \\
r &= 8\% = 0.08 \\
N &= 15 \\
\text{FV} &= A\left[\frac{(1+r)^N - 1}{r}\right] \\
&= \$2{,}000\left[\frac{(1.08)^{15} - 1}{0.08}\right] \\
&= \$2{,}000(27.152114) \\
&= \$54{,}304.23
\end{aligned}
$$