Elementary Differential Equations and Boundary Value Problems

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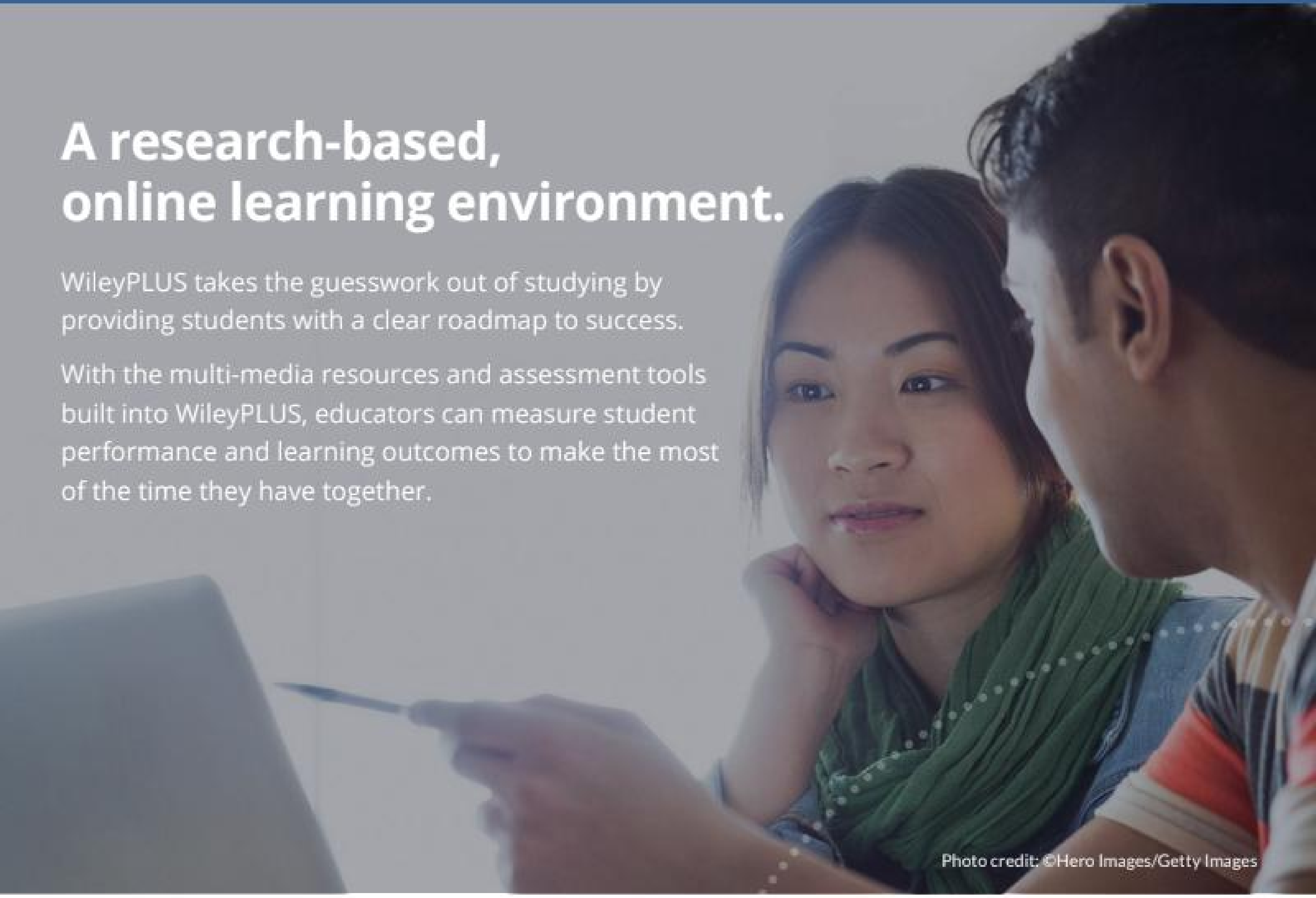

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Elementary Differential Equations and Boundary Value Problems

Eleventh Edition

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To Elsa, Betsy, and in loving memory of Maureen

To Siobhan, James, Richard Jr., Carolyn, Ann, Stuart,
Michael, Marybeth, and Bradley

And to the next generation:
Charles, Aidan, Stephanie, Veronica, and Deirdre

The Authors

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RICHARD C. DIPRIMA (deceased) received his B.S., M.S., and Ph.D. degrees in Mathematics from Carnegie-Mellon University. He joined the faculty of Rensselaer Polytechnic Institute after holding research positions at MIT, Harvard, and Hughes Aircraft. He held the Eliza Ricketts Foundation Professorship of Mathematics at Rensselaer, was a fellow of the American Society of Mechanical Engineers, the American Academy of Mechanics, and the American Physical Society. He was also a member of the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics. He served as the Chairman of the Department of Mathematical Sciences at Rensselaer, as President of the Society for Industrial and Applied Mathematics, and as Chairman of the Executive Committee of the Applied Mechanics Division of ASME. In 1980, he was the recipient of the William H. Wiley Distinguished Faculty Award given by Rensselaer. He received Fulbright fellowships in 1964–65 and 1983 and a Guggenheim fellowship in 1982–83. He was the author of numerous technical papers in hydrodynamic stability and lubrication theory and two texts on differential equations and boundary value problems. Professor DiPrima died on September 10, 1984.

DOUGLAS B. MEADE received B.S. degrees in Mathematics and Computer Science from Bowling Green State University, an M.S. in Applied Mathematics from Carnegie Mellon University, and a Ph.D. in mathematics from Carnegie Mellon University. After a two-year stint at Purdue University, he joined the mathematics faculty at the University of South Carolina, where he is currently an Associate Professor of mathematics and the Associate Dean for Instruction, Curriculum, and Assessment in the College of Arts and Sciences. He is a member of the American Mathematical Society, Mathematics Association of America, and Society for Industrial and Applied Mathematics; in 2016 he was named an ICTCM Fellow at the International Conference on Technology in Collegiate Mathematics (ICTCM). His primary research interests are in the numerical solution of partial differential equations arising from wave propagation problems in unbounded domains and from population models for infectious diseases. He is also well-known for his educational uses of computer algebra systems, particularly Maple. These include Getting Started with Maple (with M. May, C-K. Cheung, and G. E. Keough, Wiley, 2009, ISBN 978-0-470-45554-8), Engineer's Toolkit: Maple for Engineers (with E. Bourkoff, Addison-Wesley, 1998, ISBN 0-8053-6445-5), and numerous Maple supplements for numerous calculus, linear algebra, and differential equations textbooks - including previous editions of this book. He was a member of the MathDL New Collections Working Group for Single Variable Calculus, and chaired the Working Groups for Differential Equations and Linear Algebra. The NSF is partially supporting his work, together with Prof. Philip Yasskin (Texas A&M), on the Maplets for Calculus project.

Preface

As we have prepared an updated edition our first priorities are to preserve, and to enhance, the qualities that have made previous editions so successful. In particular, we adopt the viewpoint of an applied mathematician with diverse interests in differential equations, ranging from quite theoretical to intensely practical – and usually a combination of both. Three pillars of our presentation of the material are methods of solution, analysis of solutions, and approximations of solutions. Regardless of the specific viewpoint adopted, we have sought to ensure the exposition is simultaneously correct and complete, but not needlessly abstract.

The intended audience is undergraduate STEM students whose degree program includes an introductory course in differential equations during the first two years. The essential prerequisite is a working knowledge of calculus, typically a two- or three-semester course sequence or an equivalent. While a basic familiarity with matrices is helpful, Sections 7.2 and 7.3 provide an overview of the essential linear algebra ideas needed for the parts of the book that deal with systems of differential equations (the remainder of Chapter 7, Section 8.5, and Chapter 9).

A strength of this book is its appropriateness in a wide variety of instructional settings. In particular, it allows instructors flexibility in the selection of and the ordering of topics and in the use of technology. The essential core material is Chapter 1, Sections 2.1 through 2.5, and Sections 3.1 through 3.5. After completing these sections, the selection of additional topics, and the order and depth of coverage are generally at the discretion of the instructor. Chapters 4 through 11 are essentially independent of each other, except that Chapter 7 should precede Chapter 9, and Chapter 10 should precede Chapter 11.

A particularly appealing aspect of differential equations is that even the simplest differential equations have a direct correspondence to realistic physical phenomena: exponential growth and decay, spring-mass systems, electrical circuits, competitive species, and wave propagation. More complex natural processes can often be understood by combining and building upon simpler and more basic models. A thorough knowledge of these basic models, the differential equations that describe them, and their solutions – either explicit solutions or qualitative properties of the solution – is the first and indispensable step toward analyzing the solutions of more complex and realistic problems. The modeling process is detailed in Chapter 1 and Section 2.3. Careful constructions of models appear also in Sections 2.5, 3.7, 9.4, 10.5, and 10.7 (and the appendices to Chapter 10). Various problem sets throughout the book include problems that involve modeling to formulate an appropriate differential equation, and then to solve it or to determine some qualitative properties of its solution. The primary purposes of these applied problems are to provide students with hands-on experience in the derivation of differential equations, and to convince them that differential equations arise naturally in a wide variety of real-world applications.

Another important concept emphasized repeatedly throughout the book is the transportability of mathematical knowledge. While a specific solution method applies to only a particular class of differential equations, it can be used in any application in which that particular type of differential equation arises. Once this point is made in a convincing manner, we believe that it is unnecessary to provide specific applications of every method of solution or type of equation that we consider. This decision helps to keep this book to a reasonable size, and allows us to keep the primary emphasis on the development of more solution methods for additional types of differential equations.

From a student's point of view, the problems that are assigned as homework and that appear on examinations define the course. We believe that the most outstanding feature of this book is the number, and above all the variety and range, of the problems that it contains. Many problems are entirely straightforward, but many others are more challenging, and some are fairly open-ended and can even serve as the basis for independent student projects. The observant reader will notice that there are fewer problems in this edition than in previous editions; many of these problems remain available to instructors via the WileyPlus course. The remaining 1600 problems are still far more problems than any instructor can use in any given course, and this provides instructors with a multitude of choices in tailoring their course to meet their own goals and the needs of their students. The answers to almost all of these problems can be found in the pages at the back of the book; full solutions are in either the Student's Solution Manual or the Instructor's Solution Manual.

While we make numerous references to the use of technology, we do so without limiting instructor freedom to use as much, or as little, technology as they desire. Appropriate technologies include advanced graphing calculators (TI Nspire), a spreadsheet (Excel), web-based resources (applets), computer algebra systems, (Maple, Mathematica, Sage), scientific computation systems (MATLAB), or traditional programming (FORTRAN, Javascript, Python). Problems marked with a G are ones we believe are best approached with a graphical tool; those marked with a N are best solved with the use of a numerical tool. Instructors should consider setting their own policies, consistent with their interests and intents about student use of technology when completing assigned problems.

Many problems in this book are best solved through a combination of analytic, graphic, and numeric methods. Pencil-and-paper methods are used to develop a model that is best solved (or analyzed) using a symbolic or graphic tool. The quantitative results and graphs, frequently produced using computer-based resources, serve to illustrate and to clarify conclusions that might not be readily apparent from a complicated explicit solution formula. Conversely, the implementation of an efficient numerical method to obtain an approximate solution typically requires a good deal of preliminary analysis–to determine qualitative features of the solution as a guide to computation, to investigate limiting or special cases, or to discover ranges of the variables or parameters that require an appropriate combination of both analytic and numeric computation. Good judgment may well be required to determine the best choice of solution methods in each particular case. Within this context we point out that problems that request a "sketch" are generally intended to be completed without the use of any technology (except your writing device).

We believe that it is important for students to understand that (except perhaps in courses on differential equations) the goal of solving a differential equation is seldom simply to obtain the solution. Rather, we seek the solution in order to obtain insight into the behavior of the process that the equation purports to model. In other words, the solution is not an end in itself. Thus, we have included in the text a great many problems, as well as some examples, that call for conclusions to be drawn about the solution. Sometimes this takes the form of finding the value of the independent variable at which the solution has a certain property, or determining the long-term behavior of the solution. Other problems ask for the effect of variations in a parameter, or for the determination of all values of a parameter at which the solution experiences a substantial change. Such problems are typical of those that arise in the applications of differential equations, and, depending on the goals of the course, an instructor has the option of assigning as few or as many of these problems as desired.

Readers familiar with the preceding edition will observe that the general structure of the book is unchanged. The minor revisions that we have made in this edition are in many cases the result of suggestions from users of earlier editions. The goals are to improve the clarity and readability of our presentation of basic material about differential equations and their applications. More specifically, the most important revisions include the following:

1. Chapter 1 has been rewritten. Instead of a separate section on the History of Differential Equations, this material appears in three installments in the remaining three section.
2. Additional words of explanation and/or more explicit details in the steps in a derivation have been added throughout each chapter. These are too numerous and widespread to mention individually, but collectively they should help to make the book more readable for many students.
3. There are about forty new or revised problems scattered throughout the book. The total number of problems has been reduced by about 400 problems, which are still available through WileyPlus, leaving about 1600 problems in print.
4. There are new examples in Sections 2.1, 3.8, and 7.5.
5. The majority (is this correct?) of the figures have been redrawn, mainly by the use full color to allow for easier identification of critical properties of the solution. In addition, numerous captions have been expanded to clarify the purpose of the figure without requiring a search of the surrounding text.
6. There are several new references, and some others have been updated.

The authors have found differential equations to be a never-ending source of interesting, and sometimes surprising, results and phenomena. We hope that users of this book, both students and instructors, will share our enthusiasm for the subject.

William E. Boyce and Douglas B. Meade
Watervliet, New York and Columbia, SC
29 August 2016

Supplemental Resources for Instructors and Students

An Instructor's Solutions Manual, ISBN 978-1-119-16976-5, includes solutions for all problems not contained in the Student Solutions Manual.

A Student Solutions Manual, ISBN 978-1-119-16975-8, includes solutions for selected problems in the text.

A Book Companion Site, www.wiley.com/college/boyce, provides a wealth of resources for students and instructors, including

- PowerPoint slides of important definitions, examples, and theorems from the book, as well as graphics for presentation in lectures or for study and note taking.
- Chapter Review Sheets, which enable students to test their knowledge of key concepts. For further review, diagnostic feedback is provided that refers to pertinent sections in the text.
- Mathematica, Maple, and MATLAB data files for selected problems in the text providing opportunities for further exploration of important concepts.
- Projects that deal with extended problems normally not included among traditional topics in differential equations, many involving applications from a variety of disciplines. These vary in length and complexity, and they can be assigned as individual homework or as group assignments.

A series of supplemental guidebooks, also published by John Wiley & Sons, can be used with Boyce/DiPrima/Meade in order to incorporate computing technologies into the course. These books emphasize numerical methods and graphical analysis, showing how these methods enable us to interpret solutions of ordinary differential equations (ODEs) in the real world. Separate guidebooks cover each of the three major mathematical software formats, but the ODE subject matter is the same in each.

- Hunt, Lipsman, Osborn, and Rosenberg, *Differential Equations with MATLAB* , 3rd ed., 2012, ISBN 978-1-118-37680-5

- Hunt, Lardy, Lipsman, Osborn, and Rosenberg, *Differential Equations with Maple*, 3rd ed., 2008, ISBN 978-0-471-77317-7
- Hunt, Outing, Lipsman, Osborn, and Rosenberg, *Differential Equations with Mathematica*, 3rd ed., 2009, ISBN 978-0-471-77316-0

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- Guided Online (GO) Exercises, which prompt students to build solutions step-by-step. Rather than simply grading an exercise answer as wrong, GO problems show students precisely where they are making a mistake.
- Homework management tools, which enable instructors easily to assign and grade questions, as well as to gauge student comprehension.
- QuickStart pre-designed reading and homework assign ments. Use them as is, or customize them to fit the needs of your classroom.

Acknowledgments

It is a pleasure to express my appreciation to the many people who have generously assisted in various ways in the preparation of this book.

To the individuals listed below, who reviewed the manuscript and/or provided valuable suggestions for its improvement:

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WILLIAM E. BOYCE AND DOUGLAS B. MEADE

Brief Contents

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CHAPTER 1

Introduction

In this first chapter we provide a foundation for your study of differential equations in several different ways. First, we use two problems to illustrate some of the basic ideas that we will return to, and elaborate upon, frequently throughout the remainder of the book. Later, to provide organizational structure for the book, we indicate several ways of classifying differential equations.

The study of differential equations has attracted the attention of many of the world's greatest mathematicians during the past three centuries. On the other hand, it is important to recognize that differential equations remains a dynamic field of inquiry today, with many interesting open questions. We outline some of the major trends in the historical development of the subject and mention a few of the outstanding mathematicians who have contributed to it. Additional biographical information about some of these contributors will be highlighted at appropriate times in later chapters.

1.1 Some Basic Mathematical Models; Direction Fields

Before embarking on a serious study of differential equations (for example, by reading this book or major portions of it), you should have some idea of the possible benefits to be gained by doing so. For some students the intrinsic interest of the subject itself is enough motivation, but for most it is the likelihood of important applications to other fields that makes the undertaking worthwhile.

Many of the principles, or laws, underlying the behavior of the natural world are statements or relations involving rates at which things happen. When expressed in mathematical terms, the relations are equations and the rates are derivatives. Equations containing derivatives are **differential equations**. Therefore, to understand and to investigate problems involving the motion of fluids, the flow of current in electric circuits, the dissipation of heat in solid objects, the propagation and detection of seismic waves, or the increase or decrease of populations, among many others, it is necessary to know something about differential equations.

A differential equation that describes some physical process is often called a **mathematical model** of the process, and many such models are discussed throughout this book. In this section we begin with two models leading to equations that are easy to solve. It is noteworthy that even the simplest differential equations provide useful models of important physical processes.

EXAMPLE 1 | A Falling Object

Suppose that an object is falling in the atmosphere near sea level. Formulate a differential equation that describes the motion.

Solution:

We begin by introducing letters to represent various quantities that may be of interest in this problem. The motion takes place during a certain time interval, so let us use t to denote time. Also, let us use v to represent the velocity of the falling object. The velocity will presumably change with time, so we think of v as a function of t; in other words, t is the independent variable and v is the dependent variable. The choice of units of measurement is somewhat arbitrary, and there is nothing in the statement of the problem to suggest appropriate units, so we are free to make any choice that seems reasonable. To be specific, let us measure time t in seconds and velocity v in meters/second. Further, we will assume that v is positive in the downward direction—that is, when the object is falling.

The physical law that governs the motion of objects is **Newton's second law**, which states that the mass of the object times its acceleration is equal to the net force on the object. In mathematical terms this law is expressed by the equation

$$F = ma, \tag{1}$$

where m is the mass of the object, a is its acceleration, and F is the net force exerted on the object. To keep our units consistent, we will measure m in kilograms, a in meters/second^2, and F in newtons. Of course, a is related to v by $a = dv/dt$, so we can rewrite equation (1) in the form

$$F = m\frac{dv}{dt}. \tag{2}$$

Next, consider the forces that act on the object as it falls. Gravity exerts a force equal to the weight of the object, or mg, where g is the acceleration due to gravity. In the units we have chosen, g has been determined experimentally to be approximately equal to 9.8 m/s^2 near the earth's surface.

There is also a force due to air resistance, or drag, that is more difficult to model. This is not the place for an extended discussion of the drag force; suffice it to say that it is often assumed that the drag is proportional to the velocity, and we will make that assumption here. Thus the drag force has the magnitude γv, where γ is a constant called the drag coefficient. The numerical value of the drag coefficient varies widely from one object to another; smooth streamlined objects have much smaller drag coefficients than rough blunt ones. The physical units for γ are mass/time, or kg/s for this problem; if these units seem peculiar, remember that γv must have the units of force, namely, kg-m/s^2.

In writing an expression for the net force F, we need to remember that gravity always acts in the downward (positive) direction, whereas, for a falling object, drag acts in the upward (negative) direction, as shown in Figure 1.1.1. Thus

$$F = mg - \gamma v \tag{3}$$

and equation (2) then becomes

$$m\frac{dv}{dt} = mg - \gamma v. \tag{4}$$

Differential equation (4) is a mathematical model for the velocity v of an object falling in the atmosphere near sea level. Note that the model contains the three constants m, g, and γ. The constants m and γ depend very much on the particular object that is falling, and they are usually different for different objects. It is common to refer to them as parameters, since they may take on a range of values during the course of an experiment. On the other hand, g is a physical constant, whose value is the same for all objects.

FIGURE 1.1.1 Free-body diagram of the forces on a falling object.

To solve equation (4), we need to find a function $v = v(t)$ that satisfies the equation. It is not hard to do this, and we will show you how in the next section. For the present, however, let us see what we can learn about solutions without actually finding any of them. Our task is simplified slightly if we assign numerical values to m and γ, but the procedure is the same regardless of which values we choose. So, let us suppose that $m = 10$ kg and $\gamma = 2$ kg/s. Then equation (4) can be rewritten as

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}. \tag{5}$$

EXAMPLE 2 | A Falling Object (continued)

Investigate the behavior of solutions of equation (5) without solving the differential equation.

Solution:

First let us consider what information can be obtained directly from the differential equation itself. Suppose that the velocity v has a certain given value. Then, by evaluating the right-hand side of differential equation (5), we can find the corresponding value of dv/dt. For instance, if $v = 40$, then $dv/dt = 1.8$. This means that the slope of a solution $v = v(t)$ has the value 1.8 at any point where $v = 40$. We can display this information graphically in the tv-plane by drawing short line segments with slope 1.8 at several points on the line $v = 40$. (See Figure 1.1.2(a)). Similarly, when $v = 50$, then $dv/dt = -0.2$, and when $v = 60$, then $dv/dt = -2.2$, so we draw line segments with slope -0.2 at several points on the line $v = 50$ (see Figure 1.1.2(b)) and line segments with slope -2.2 at several points on the line $v = 60$ (see Figure 1.1.2(c)). Proceeding in the same way with other values of v we create what is called a **direction field**, or a **slope field**. The direction field for differential equation (5) is shown in Figure 1.1.3.

Remember that a solution of equation (5) is a function $v = v(t)$ whose graph is a curve in the tv-plane. The importance of Figure 1.1.3 is that each line segment is a tangent line to one of these solution curves. Thus, even though we have not found any solutions, and no graphs of solutions appear in the figure, we can nonetheless draw some qualitative conclusions about the behavior of solutions. For instance, if v is less than a certain critical value, then all the line segments have positive slopes, and the speed of the falling object increases as it falls. On the other hand, if v is greater than the critical value, then the line segments have negative slopes, and the falling object slows down as it falls. What is this critical value of v that separates objects whose speed is increasing from those whose speed is decreasing? Referring again to equation (5), we ask what value of v will cause dv/dt to be zero. The answer is $v = (5)(9.8) = 49$ m/s.

In fact, the constant function $v(t) = 49$ is a solution of equation (5). To verify this statement, substitute $v(t) = 49$ into equation (5) and observe that each side of the equation is zero. Because it does not change with time, the solution $v(t) = 49$ is called an **equilibrium solution**. It is the solution that corresponds to a perfect balance between gravity and drag. In Figure 1.1.3 we show the equilibrium solution $v(t) = 49$ superimposed on the direction field. From this figure we can draw another conclusion, namely, that all other solutions seem to be converging to the equilibrium solution as t increases. Thus, in this context, the equilibrium solution is often called the **terminal velocity**.

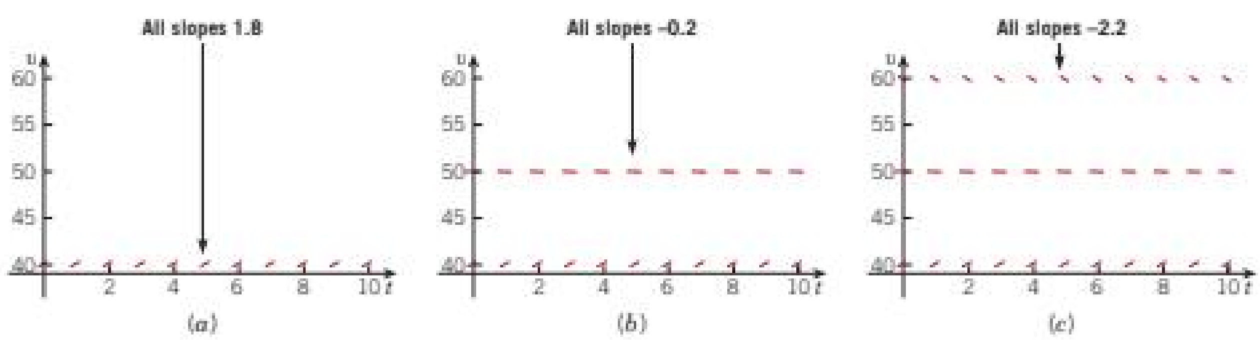

FIGURE 1.1.2 Assembling a direction field for equation (5): $dv/dt = 9.8 - v/5$. (a) when $v = 40$, $dv/dt = 1.8$, (b) when $v = 50$, $dv/dt = -0.2$, and (c) when $v = 60$, $dv/dt = -2.2$.

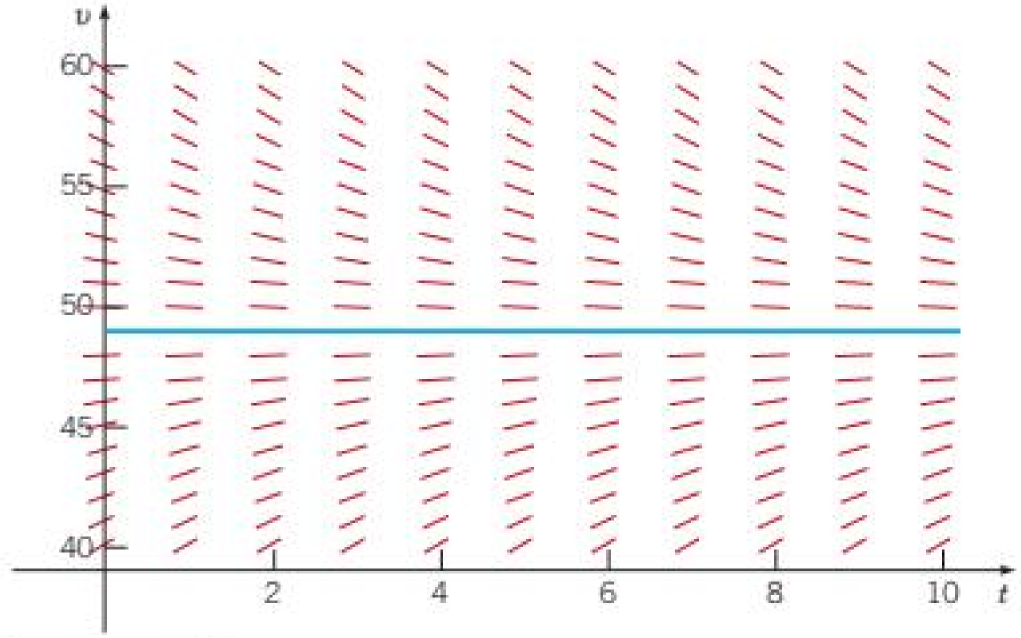

FIGURE 1.1.3 Direction field and equilibrium solution for equation (5): $dv/dt = 9.8 - v/5$.

The approach illustrated in Example 2 can be applied equally well to the more general differential equation (4), where the parameters m and γ are unspecified positive numbers. The results are essentially identical to those of Example 2. The equilibrium solution of equation (4) is the constant solution $v(t) = mg/\gamma$. Solutions below the equilibrium solution increase with time, and those above it decrease with time. As a result, we conclude that all solutions approach the equilibrium solution as t becomes large.

Direction Fields. Direction fields are valuable tools in studying the solutions of differential equations of the form

$$\frac{dy}{dt} = f(t, y), \tag{6}$$

where f is a given function of the two variables t and y, sometimes referred to as the **rate function**. A direction field for equations of the form (6) can be constructed by evaluating f at each point of a rectangular grid. At each point of the grid, a short line segment is drawn whose slope is the value of f at that point. Thus each line segment is tangent to the graph of the solution passing through that point. A direction field drawn on a fairly fine grid gives a good picture of the overall behavior of solutions of a differential equation. Usually a grid consisting of a few hundred points is sufficient. The construction of a direction field is often a useful first step in the investigation of a differential equation.

Two observations are worth particular mention. First, in constructing a direction field, we do not have to solve equation (6); we just have to evaluate the given function $f(t, y)$ many times. Thus direction fields can be readily constructed even for equations that may be quite difficult to solve. Second, repeated evaluation of a given function and drawing a direction field are tasks for which a computer or other computational or graphical aid are well suited. All the direction fields shown in this book, such as the one in Figures 1.1.2 and 1.1.3, were computer generated.

Field Mice and Owls. Now let us look at another, quite different example. Consider a population of field mice that inhabit a certain rural area. In the absence of predators we assume that the mouse population increases at a rate proportional to the current population. This assumption is not a well-established physical law (as Newton's law of motion is in Example 1), but it is a common initial hypothesis[1] in a study of population growth. If we denote time by t and the mouse population at time t by $p(t)$, then the assumption about population growth can be expressed by the equation

$$\frac{dp}{dt} = rp, \tag{7}$$

[1]A better model of population growth is discussed in Section 2.5.

where the proportionality factor r is called the **rate constant** or **growth rate**. To be specific, suppose that time is measured in months and that the rate constant r has the value 0.5/month. Then the two terms in equation (7) have the units of mice/month.

Now let us add to the problem by supposing that several owls live in the same neighborhood and that they kill 15 field mice per day. To incorporate this information into the model, we must add another term to the differential equation (7), so that it becomes

$$\frac{dp}{dt} = \frac{p}{2} - 450. \tag{8}$$

Observe that the predation term is -450 rather than -15 because time is measured in months, so the monthly predation rate is needed.

EXAMPLE 3

Investigate the solutions of differential equation (8) graphically.

Solution:

A direction field for equation (8) is shown in Figure 1.1.4. For sufficiently large values of p it can be seen from the figure, or directly from equation (8) itself, that dp/dt is positive, so that solutions increase. On the other hand, if p is small, then dp/dt is negative and solutions decrease. Again, the critical value of p that separates solutions that increase from those that decrease is the value of p for which dp/dt is zero. By setting dp/dt equal to zero in equation (8) and then solving for p, we find the equilibrium solution $p(t) = 900$, for which the growth term and the predation term in equation (8) are exactly balanced. The equilibrium solution is also shown in Figure 1.1.4.

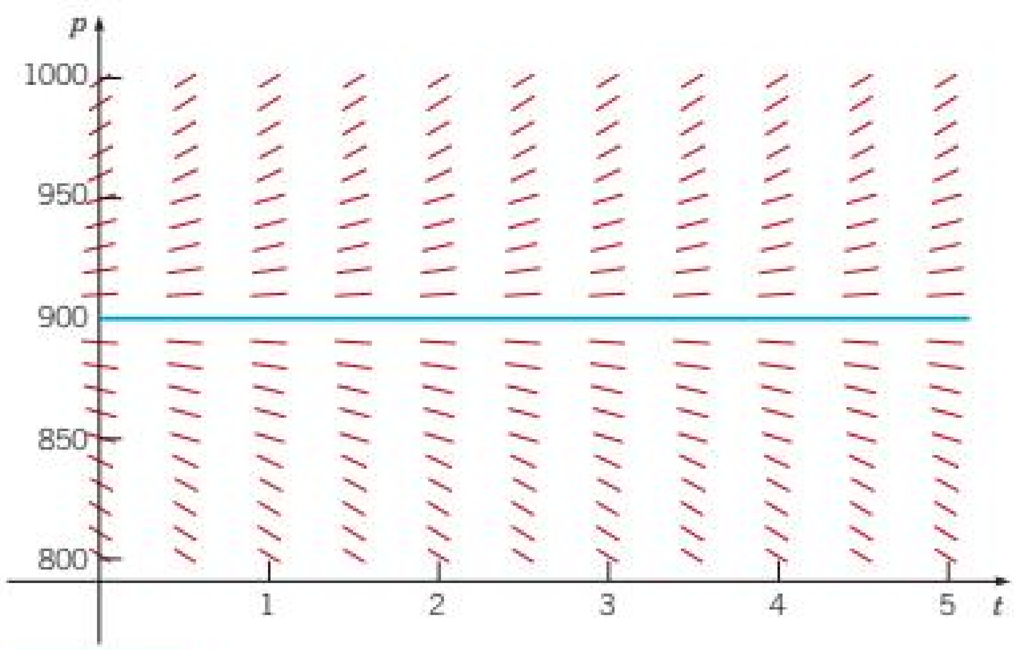

FIGURE 1.1.4 Direction field (red) and equilibrium solution (blue) for equation (8): $dp/dt = p/2 - 450$.

Comparing Examples 2 and 3, we note that in both cases the equilibrium solution separates increasing from decreasing solutions. In Example 2 other solutions converge to, or are attracted by, the equilibrium solution, so that after the object falls long enough, an observer will see it moving at very nearly the equilibrium velocity. On the other hand, in Example 3 other solutions diverge from, or are repelled by, the equilibrium solution. Solutions behave very differently depending on whether they start above or below the equilibrium solution. As time passes, an observer might see populations either much larger or much smaller than the equilibrium population, but the equilibrium solution itself will not, in practice, be observed. In both problems, however, the equilibrium solution is very important in understanding how solutions of the given differential equation behave.

A more general version of equation (8) is

$$\frac{dp}{dt} = rp - k, \tag{9}$$

where the growth rate r and the predation rate k are positive constants that are otherwise unspecified. Solutions of this more general equation are very similar to those of equation (8). The equilibrium solution of equation (9) is $p(t) = k/r$. Solutions above the equilibrium solution increase, while those below it decrease.

You should keep in mind that both of the models discussed in this section have their limitations. The model (5) of the falling object is valid only as long as the object is falling freely, without encountering any obstacles. If the velocity is large enough, the assumption that the frictional resistance is linearly proportional to the velocity has to be replaced with a nonlinear approximation (see Problem 21). The population model (8) eventually predicts negative numbers of mice (if $p < 900$) or enormously large numbers (if $p > 900$). Both of these predictions are unrealistic, so this model becomes unacceptable after a fairly short time interval.

Constructing Mathematical Models. In applying differential equations to any of the numerous fields in which they are useful, it is necessary first to formulate the appropriate differential equation that describes, or models, the problem being investigated. In this section we have looked at two examples of this modeling process, one drawn from physics and the other from ecology. In constructing future mathematical models yourself, you should recognize that each problem is different, and that successful modeling cannot be reduced to the observance of a set of prescribed rules. Indeed, constructing a satisfactory model is sometimes the most difficult part of the problem. Nevertheless, it may be helpful to list some steps that are often part of the process:

1. Identify the independent and dependent variables and assign letters to represent them. Often the independent variable is time.
2. Choose the units of measurement for each variable. In a sense the choice of units is arbitrary, but some choices may be much more convenient than others. For example, we chose to measure time in seconds for the falling-object problem and in months for the population problem.
3. Articulate the basic principle that underlies or governs the problem you are investigating. This may be a widely recognized physical law, such as Newton's law of motion, or it may be a more speculative assumption that may be based on your own experience or observations. In any case, this step is likely not to be a purely mathematical one, but will require you to be familiar with the field in which the problem originates.
4. Express the principle or law in step 3 in terms of the variables you chose in step 1. This may be easier said than done. It may require the introduction of physical constants or parameters (such as the drag coefficient in Example 1) and the determination of appropriate values for them. Or it may involve the use of auxiliary or intermediate variables that must then be related to the primary variables.
5. If the units agree, then your equation at least is dimensionally consistent, although it may have other shortcomings that this test does not reveal.
6. In the problems considered here, the result of step 4 is a single differential equation, which constitutes the desired mathematical model. Keep in mind, though, that in more complex problems the resulting mathematical model may be much more complicated, perhaps involving a system of several differential equations, for example.

Historical Background, Part I: Newton, Leibniz, and the Bernoullis. Without knowing something about differential equations and methods of solving them, it is difficult to appreciate the history of this important branch of mathematics. Further, the development of differential equations is intimately interwoven with the general development of mathematics and cannot be separated from it. Nevertheless, to provide some historical perspective, we indicate here some of the major trends in the history of the subject and identify the most prominent early contributors. The rest of the historical background in this section focuses on the earliest contributors from the seventeenth century. The story continues at the end of Section 1.2 with an overview of the contributions of Euler and other eighteenth-century (and early-nineteenth-century) mathematicians. More recent advances, including the use of computers and other technologies, are summarized at the end of Section 1.3. Additional historical information is contained in footnotes scattered throughout the book and in the references listed at the end of the chapter.

The subject of differential equations originated in the study of calculus by Isaac Newton (1643–1727) and Gottfried Wilhelm Leibniz (1646–1716) in the seventeenth century. Newton grew up in the English countryside, was educated at Trinity College, Cambridge, and became Lucasian Professor of Mathematics there in 1669. His epochal discoveries of calculus and of the fundamental laws of mechanics date to 1665. They were circulated privately among his friends, but Newton was extremely sensitive to criticism and did not begin to publish his results until 1687 with the appearance of his most famous book *Philosophiae Naturalis Principia Mathematica*. Although Newton did relatively little work in differential equations as such, his development of the calculus and elucidation of the basic principles of mechanics provided a basis for their applications in the eighteenth century, most notably by Euler (see Historical Background, Part II in Section 1.2). Newton identified three forms of first-order differential equations: $dy/dx = f(x)$, $dy/dx = f(y)$, and $dy/dx = f(x, y)$. For the latter equation he developed a method of solution using infinite series when $f(x, y)$ is a polynomial in x and y. Newton's active research in mathematics ended in the early 1690s, except for the solution of occasional "challenge problems" and the revision and publication of results obtained much earlier. He was appointed Warden of the British Mint in 1696 and resigned his professorship a few years later. He was knighted in 1705 and, upon his death in 1727, became the first scientist buried in Westminster Abbey.

Leibniz was born in Leipzig, Germany, and completed his doctorate in philosophy at the age of 20 at the University of Altdorf. Throughout his life he engaged in scholarly work in several different fields. He was mainly self-taught in mathematics, since his interest in this subject developed when he was in his twenties. Leibniz arrived at the fundamental results of calculus independently, although a little later than Newton, but was the first to publish them, in 1684. Leibniz was very conscious of the power of good mathematical notation and was responsible for the notation dy/dx for the derivative and for the integral sign. He discovered the method of separation of variables (Section 2.2) in 1691, the reduction of homogeneous equations to separable ones (Section 2.2, Problem 30) in 1691, and the procedure for solving first-order linear equations (Section 2.1) in 1694. He spent his life as ambassador and adviser to several German royal families, which permitted him to travel widely and to carry on an extensive correspondence with other mathematicians, especially the Bernoulli brothers. In the course of this correspondence many problems in differential equations were solved during the latter part of the seventeenth century.

The Bernoulli brothers, Jakob (1654–1705) and Johann (1667–1748), of Basel, Switzerland did much to develop methods of solving differential equations and to extend the range of their applications. Jakob became professor of mathematics at Basel in 1687, and Johann was appointed to the same position upon his brother's death in 1705. Both men were quarrelsome, jealous, and frequently embroiled in disputes, especially with each other. Nevertheless, both also made significant contributions to several areas of mathematics. With the aid of calculus, they solved a number of problems in mechanics by formulating them as differential equations. For example, Jakob Bernoulli solved the differential equation $y' = \left(a^3/(b^2y - a^3)\right)^{1/2}$ (see Problem 9 in Section 2.2) in 1690 and, in the same paper, first used the term "integral" in the modern sense. In 1694 Johann Bernoulli was able to solve the equation $dy/dx = y/(ax)$ (see Problem 10 in Section 2.2). One problem that both brothers solved, and that led to much friction between them, was the **brachistochrone problem** (see Problem 24 in Section 2.3). The brachistochrone problem was also solved by Leibniz, Newton, and the Marquis de l'Hôpital. It is said, perhaps apocryphally, that Newton learned of the problem late in the afternoon of a tiring day at the Mint and solved it that evening after dinner. He published the solution anonymously, but upon seeing it, Johann Bernoulli exclaimed, "Ah, I know the lion by his paw."

Daniel Bernoulli (1700–1782), son of Johann, migrated to St. Petersburg, Russia, as a young man to join the newly established St. Petersburg Academy, but returned to Basel in 1733 as professor of botany and, later, of physics. His interests were primarily in partial differential equations and their applications. For instance, it is his name that is associated with the Bernoulli equation in fluid mechanics. He was also the first to encounter the functions that a century later became known as Bessel functions (Section 5.7).

Problems

In each of Problems 1 through 4, draw a direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe the dependency.

G **1.** $y' = 3 - 2y$

G **2.** $y' = 2y - 3$

G **3.** $y' = -1 - 2y$

G **4.** $y' = 1 + 2y$

In each of Problems 5 and 6, write down a differential equation of the form $dy/dt = ay + b$ whose solutions have the required behavior as $t \to \infty$.

5. All solutions approach $y = 2/3$.

6. All other solutions diverge from $y = 2$.

In each of Problems 7 through 10, draw a direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency. Note that in these problems the equations are not of the form $y' = ay + b$, and the behavior of their solutions is somewhat more complicated than for the equations in the text.

G **7.** $y' = y(4 - y)$

G **8.** $y' = -y(5 - y)$

G **9.** $y' = y^2$

G **10.** $y' = y(y - 2)^2$

Consider the following list of differential equations, some of which produced the direction fields shown in Figures 1.1.5 through 1.1.10. In each of Problems 11 through 16, identify the differential equation that corresponds to the given direction field.

a. $y' = 2y - 1$
b. $y' = 2 + y$
c. $y' = y - 2$
d. $y' = y(y + 3)$
e. $y' = y(y - 3)$
f. $y' = 1 + 2y$
g. $y' = -2 - y$
h. $y' = y(3 - y)$
i. $y' = 1 - 2y$
j. $y' = 2 - y$

11. The direction field of Figure 1.1.5.

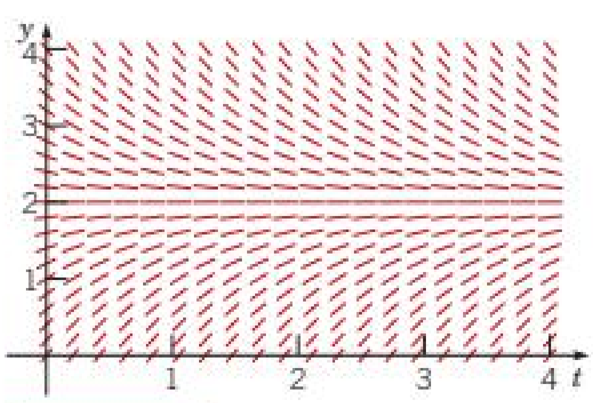

FIGURE 1.1.5 Problem 11.

12. The direction field of Figure 1.1.6.

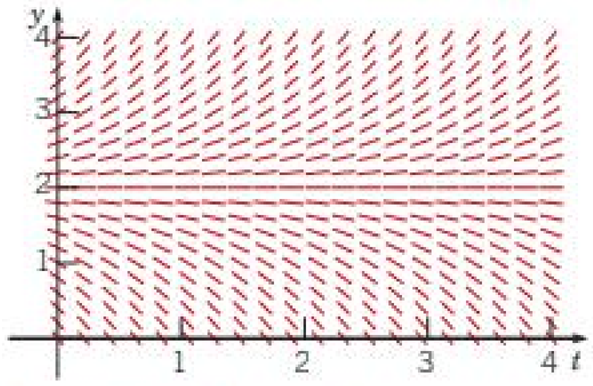

FIGURE 1.1.6 Problem 12.

13. The direction field of Figure 1.1.7.

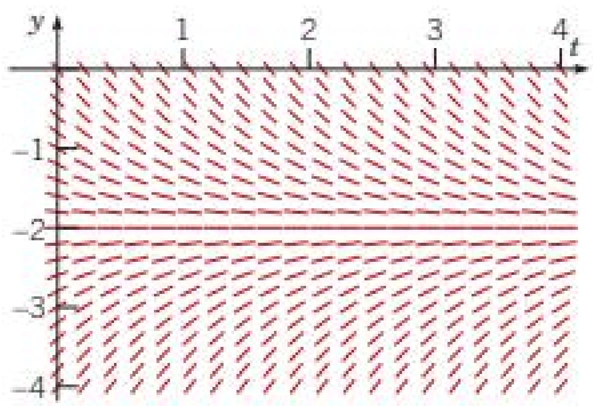

FIGURE 1.1.7 Problem 13.

14. The direction field of Figure 1.1.8.

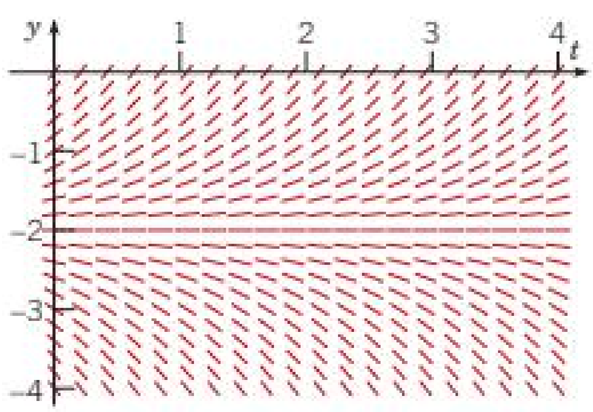

FIGURE 1.1.8 Problem 14.

15. The direction field of Figure 1.1.9.

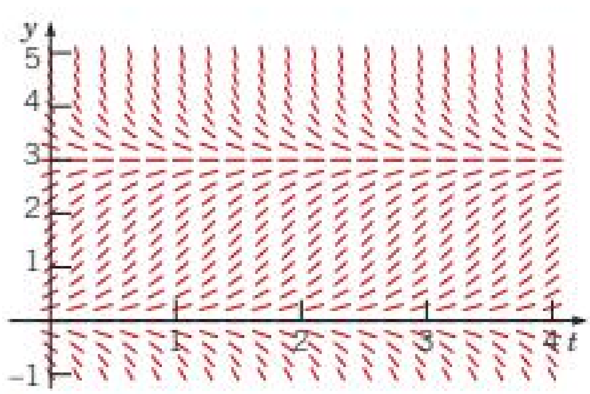

FIGURE 1.1.9 Problem 15.

16. The direction field of Figure 1.1.10.

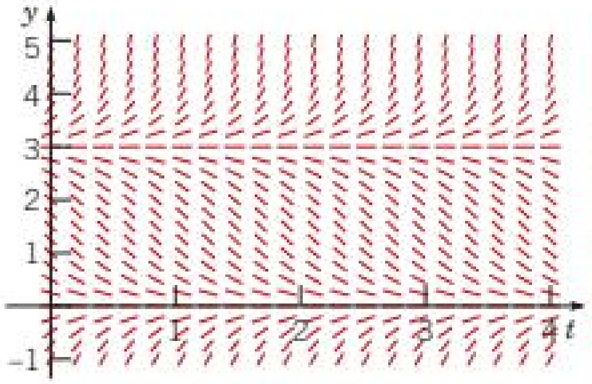

FIGURE 1.1.10 Problem 16.

17. A pond initially contains 1,000,000 gal of water and an unknown amount of an undesirable chemical. Water containing 0.01 grams of this chemical per gallon flows into the pond at a rate of 300 gal/h. The mixture flows out at the same rate, so the amount of water in the pond remains constant. Assume that the chemical is uniformly distributed throughout the pond.

a. Write a differential equation for the amount of chemical in the pond at any time.

b. How much of the chemical will be in the pond after a very long time? Does this limiting amount depend on the amount that was present initially?

c. Write a differential equation for the concentration of the chemical in the pond at time t. *Hint:* The concentration is $c = a/v = a(t)/10^6$.

18. A spherical raindrop evaporates at a rate proportional to its surface area. Write a differential equation for the volume of the raindrop as a function of time.

19. Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between the temperature of the object itself and the temperature of its surroundings (the ambient air temperature in most cases). Suppose that the ambient temperature is 70°F and that the rate constant is $0.05\ (\text{min})^{-1}$. Write a differential equation for the temperature of the object at any time. Note that the differential equation is the same whether the temperature of the object is above or below the ambient temperature.

20. A certain drug is being administered intravenously to a hospital patient. Fluid containing 5 mg/cm^3 of the drug enters the patient's bloodstream at a rate of 100 cm^3/h. The drug is absorbed by body tissues or otherwise leaves the bloodstream at a rate proportional to the amount present, with a rate constant of 0.4/h.

a. Assuming that the drug is always uniformly distributed throughout the bloodstream, write a differential equation for the amount of the drug that is present in the bloodstream at any time.

b. How much of the drug is present in the bloodstream after a long time?

N **21.** For small, slowly falling objects, the assumption made in the text that the drag force is proportional to the velocity is a good one. For larger, more rapidly falling objects, it is more accurate to assume that the drag force is proportional to the square of the velocity.[2]

a. Write a differential equation for the velocity of a falling object of mass m if the magnitude of the drag force is proportional to the square of the velocity and its direction is opposite to that of the velocity.

b. Determine the limiting velocity after a long time.

c. If $m = 10$ kg, find the drag coefficient so that the limiting velocity is 49 m/s.

N **d.** Using the data in part c, draw a direction field and compare it with Figure 1.1.3.

In each of Problems 22 through 25, draw a direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency. Note that the right-hand sides of these equations depend on t as well as y; therefore, their solutions can exhibit more complicated behavior than those in the text.

G **22.** $y' = -2 + t - y$

G **23.** $y' = e^{-t} + y$

G **24.** $y' = 3\sin t + 1 + y$

G **25.** $y' = -\dfrac{2t + y}{2y}$

[2] See Lyle N. Long and Howard Weiss, "The Velocity Dependence of Aerodynamic Drag: A Primer for Mathematicians," *American Mathematical Monthly 106* (1999), 2, pp. 127–135.

1.2 Solutions of Some Differential Equations

In the preceding section we derived the differential equations

$$m\frac{dv}{dt} = mg - \gamma v \tag{1}$$

and

$$\frac{dp}{dt} = rp - k. \tag{2}$$

Equation (1) models a falling object, and equation (2) models a population of field mice preyed on by owls. Both of these equations are of the general form

$$\frac{dy}{dt} = ay - b, \tag{3}$$

where a and b are given constants. We were able to draw some important qualitative conclusions about the behavior of solutions of equations (1) and (2) by considering the associated direction fields. To answer questions of a quantitative nature, however, we need to find the solutions themselves, and we now investigate how to do that.

EXAMPLE 1 | Field Mice and Owls (continued)

Consider the equation

$$\frac{dp}{dt} = 0.5p - 450, \tag{4}$$

which describes the interaction of certain populations of field mice and owls (see equation (8) of Section 1.1). Find solutions of this equation.

Solution:

To solve equation (4), we need to find functions $p(t)$ that, when substituted into the equation, reduce it to an obvious identity. Here is one way to proceed. First, rewrite equation (4) in the form

$$\frac{dp}{dt} = \frac{p - 900}{2}, \tag{5}$$

or, if $p \neq 900$,

$$\frac{dp/dt}{p - 900} = \frac{1}{2}. \tag{6}$$

By the chain rule the left-hand side of equation (6) is the derivative of $\ln|p - 900|$ with respect to t, so we have

$$\frac{d}{dt}\ln|p - 900| = \frac{1}{2}. \tag{7}$$

Then, by integrating both sides of equation (7), we obtain

$$\ln|p - 900| = \frac{t}{2} + C, \tag{8}$$

where C is an arbitrary constant of integration. Therefore, by taking the exponential of both sides of equation (8), we find that

$$|p - 900| = e^{t/2+C} = e^C e^{t/2}, \tag{9}$$

or

$$p - 900 = \pm e^C e^{t/2}, \tag{10}$$

and finally

$$p = 900 + ce^{t/2}, \tag{11}$$

where $c = \pm e^C$ is also an arbitrary (nonzero) constant. Note that the constant function $p = 900$ is also a solution of equation (5) and that it is contained in the expression (11) if we allow c to take the value zero. Graphs of equation (11) for several values of c are shown in Figure 1.2.1.

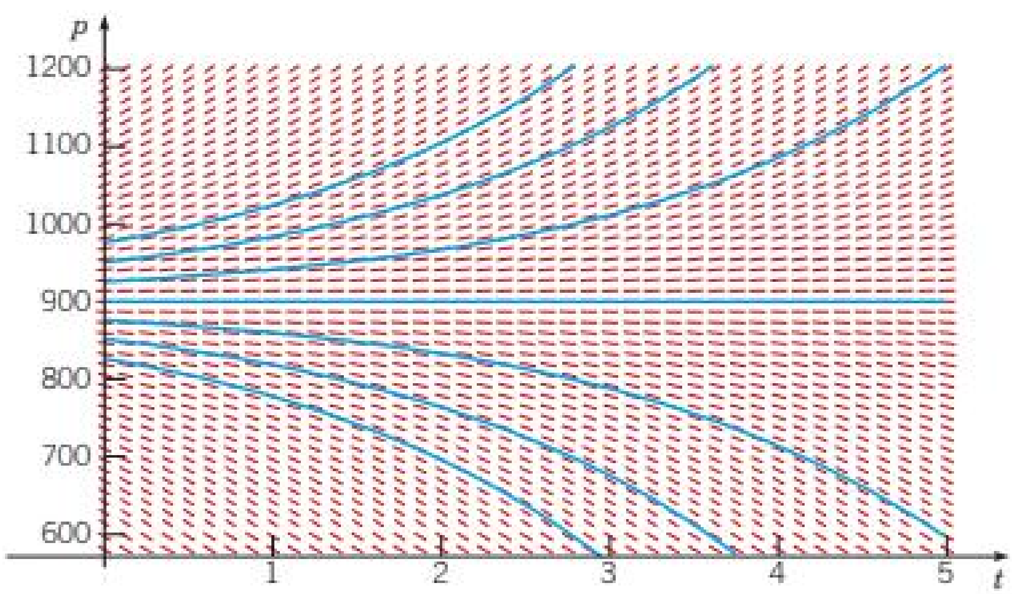

FIGURE 1.2.1 Graphs of $p = 900 + ce^{t/2}$ for several values of c. Each blue curve is a solution of $dp/dt = 0.5p - 450$.

Note that they have the character inferred from the direction field in Figure 1.1.4. For instance, solutions lying on either side of the equilibrium solution $p = 900$ tend to diverge from that solution.

In Example 1 we found infinitely many solutions of the differential equation (4), corresponding to the infinitely many values that the arbitrary constant c in equation (11) might have. This is typical of what happens when you solve a differential equation. The solution process involves an integration, which brings with it an arbitrary constant, whose possible values generate an infinite family of solutions.

Frequently, we want to focus our attention on a single member of the infinite family of solutions by specifying the value of the arbitrary constant. Most often, we do this indirectly by specifying instead a point that must lie on the graph of the solution. For example, to determine the constant c in equation (11), we could require that the population have a given value at a certain time, such as the value 850 at time $t = 0$. In other words, the graph of the solution must pass through the point $(0, 850)$. Symbolically, we can express this condition as

$$p(0) = 850. \tag{12}$$

Then, substituting $t = 0$ and $p = 850$ into equation (11), we obtain

$$850 = 900 + c.$$

Hence $c = -50$, and by inserting this value into equation (11), we obtain the desired solution, namely,

$$p = 900 - 50e^{t/2}. \tag{13}$$

The additional condition (12) that we used to determine c is an example of an **initial condition**. The differential equation (4) together with the initial condition (12) forms an **initial value problem**.

Now consider the more general problem consisting of the differential equation (3)

$$\frac{dy}{dt} = ay - b$$

and the initial condition

$$y(0) = y_0, \tag{14}$$

where y_0 is an arbitrary initial value. We can solve this problem by the same method as in Example 1. If $a \neq 0$ and $y \neq b/a$, then we can rewrite equation (3) as

$$\frac{dy/dt}{y - \frac{b}{a}} = a. \tag{15}$$

By integrating both sides, we find that

$$\ln\left|y(t) - \frac{b}{a}\right| = at + C, \tag{16}$$

where C is an arbitrary constant. Then, taking the exponential of both sides of equation (16) and solving for y, we obtain

$$y(t) = \frac{b}{a} + ce^{at}, \tag{17}$$

where $c = \pm e^C$ is also an arbitrary constant. Observe that $c = 0$ corresponds to the equilibrium solution $y(t) = b/a$. Finally, the initial condition (14) requires that $c = y_0 - (b/a)$, so the solution of the initial value problem (3), (14) is

$$y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right)e^{at}. \tag{18}$$

For $a \neq 0$ the expression (17) contains all possible solutions of equation (3) and is called the **general solution**.[3] The geometric representation of the general solution (17) is an infinite family of curves called **integral curves**. Each integral curve is associated with a particular

[3] If $a = 0$, then the solution of equation (3) is not given by equation (17). We leave it to you to find the general solution in this case.

value of c and is the graph of the solution corresponding to that value of c. Satisfying an initial condition amounts to identifying the integral curve that passes through the given initial point.

To relate the solution (18) to equation (2), which models the field mouse population, we need only replace a by the growth rate r and replace b by the predation rate k; we assume that $r > 0$ and $k > 0$. Then the solution (18) becomes

$$p(t) = \frac{k}{r} + \left(p_0 - \frac{k}{r}\right)e^{rt}, \tag{19}$$

where p_0 is the initial population of field mice. The solution (19) confirms the conclusions reached on the basis of the direction field and Example 1. If $p_0 = k/r$, then from equation (19) it follows that $p(t) = k/r$ for all t; this is the constant, or equilibrium, solution. If $p_0 \neq k/r$, then the behavior of the solution depends on the sign of the coefficient $p_0 - k/r$ of the exponential term in equation (19). If $p_0 > k/r$, then p grows exponentially with time t; if $p_0 < k/r$, then p decreases and becomes zero (at a finite time), corresponding to extinction of the field mouse population. Negative values of p, while possible for the expression (19), make no sense in the context of this particular problem.

To put the falling-object equation (1) in the form (3), we must identify a with $-\gamma/m$ and b with $-g$. Observe that assuming $\gamma > 0$ and $m > 0$ implies that $a < 0$ and $b < 0$. Making these substitutions in the solution (18), we obtain

$$v(t) = \frac{mg}{\gamma} + \left(v_0 - \frac{mg}{\gamma}\right)e^{-\gamma t/m}, \tag{20}$$

where v_0 is the initial velocity. Again, this solution confirms the conclusions reached in Section 1.1 on the basis of a direction field. There is an equilibrium, or constant, solution $v(t) = mg/\gamma$, and all other solutions tend to approach this equilibrium solution. The speed of convergence to the equilibrium solution is determined by the exponent $-\gamma/m$. Thus, for a given mass m, the velocity approaches the equilibrium value more rapidly as the drag coefficient γ increases.

EXAMPLE 2 | A Falling Object (continued)

Suppose that, as in Example 2 of Section 1.1, we consider a falling object of mass $m = 10$ kg and drag coefficient $\gamma = 2$ kg/s. Then the equation of motion (1) becomes

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}. \tag{21}$$

Suppose this object is dropped from a height of 300 m. Find its velocity at any time t. How long will it take to fall to the ground, and how fast will it be moving at the time of impact?

Solution:

The first step is to state an appropriate initial condition for equation (21). The word "dropped" in the statement of the problem suggests that the object starts from rest, that is, its initial velocity is zero, so we will use the initial condition

$$v(0) = 0. \tag{22}$$

The solution of equation (21) can be found by substituting the values of the coefficients into the solution (20), but we will proceed instead to solve equation (21) directly. First, rewrite the equation as

$$\frac{dv/dt}{v - 49} = -\frac{1}{5}. \tag{23}$$

By integrating both sides, we obtain

$$\ln\left|v(t) - 49\right| = -\frac{t}{5} + C, \tag{24}$$

and then the general solution of equation (21) is

$$v(t) = 49 + ce^{-t/5}, \tag{25}$$

where the constant c is arbitrary. To determine the particular value of c that corresponds to the initial condition (22), we substitute $t = 0$ and $v = 0$ into equation (25), with the result that $c = -49$. Then the solution of the initial value problem (21), (22) is

$$v(t) = 49\left(1 - e^{-t/5}\right). \tag{26}$$

Equation (26) gives the velocity of the falling object at any positive time after being dropped—until it hits the ground, of course.

Graphs of the solution (25) for several values of c are shown in Figure 1.2.2, with the solution (26) shown by the green curve. It is evident that, regardless of the initial velocity of the object, all solutions tend to approach the equilibrium solution $v(t) = 49$. This confirms the conclusions we reached in Section 1.1 on the basis of the direction fields in Figures 1.1.2 and 1.1.3.

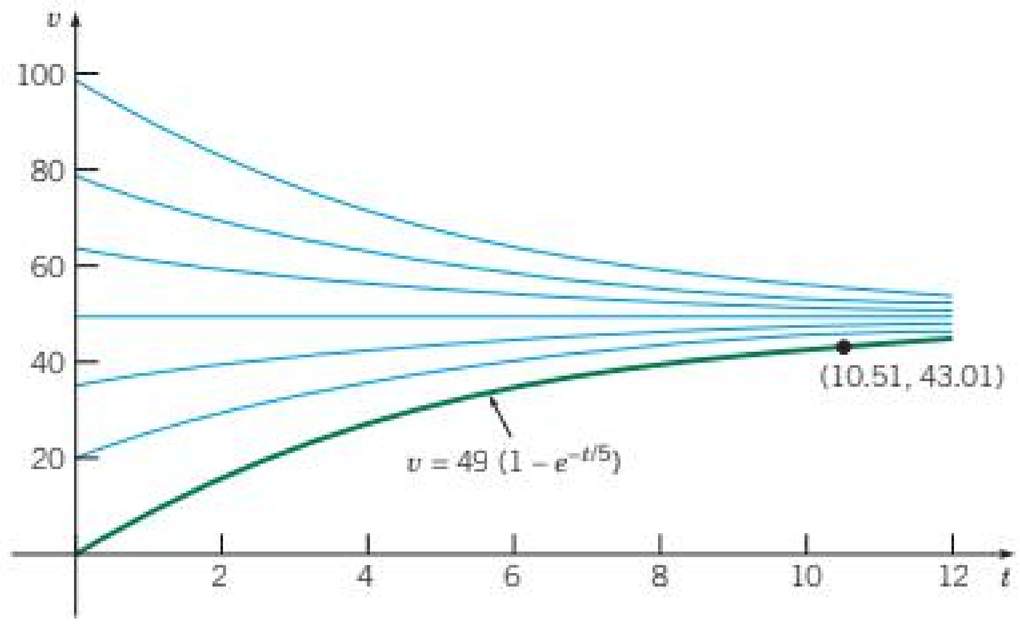

FIGURE 1.2.2 Graphs of the solution (25), $v = 49 + ce^{-t/5}$, for several values of c. The green curve corresponds to the initial condition $v(0) = 0$. The point $(10.51, 43.01)$ shows the velocity when the object hits the ground.

To find the velocity of the object when it hits the ground, we need to know the time at which impact occurs. In other words, we need to determine how long it takes the object to fall 300 m. To do this, we note that the distance x the object has fallen is related to its velocity v by the differential equation $v = dx/dt$, or

$$\frac{dx}{dt} = 49\left(1 - e^{-t/5}\right). \tag{27}$$

Consequently, by integrating both sides of equation (27) with respect to t, we have

$$x = 49t + 245e^{-t/5} + k, \tag{28}$$

where k is an arbitrary constant of integration. The object starts to fall when $t = 0$, so we know that $x = 0$ when $t = 0$. From equation (28) it follows that $k = -245$, so the distance the object has fallen at time t is given by

$$x = 49t + 245e^{-t/5} - 245. \tag{29}$$

Let T be the time at which the object hits the ground; then $x = 300$ when $t = T$. By substituting these values in equation (29), we obtain the equation

$$49T + 245e^{-T/5} - 245 = 300. \tag{30}$$

The value of T satisfying equation (30) can be approximated by a numerical process[4] using a calculator or other computational tool, with the result that $T \cong 10.51$ s. At this time, the corresponding velocity v_T is found from equation (26) to be $v_T \cong 43.01$ m/s. The point $(10.51, 43.01)$ is also shown in Figure 1.2.2.

[4]A computer algebra system provides this capability; many calculators also have built-in routines for solving such equations.

Further Remarks on Mathematical Modeling. Up to this point we have related our discussion of differential equations to mathematical models of a falling object and of a hypothetical relation between field mice and owls. The derivation of these models may have been plausible, and possibly even convincing, but you should remember that the ultimate test of any mathematical model is whether its predictions agree with observations or experimental results. We have no actual observations or experimental results to use for comparison purposes here, but there are several sources of possible discrepancies.

In the case of the falling object, the underlying physical principle (Newton's laws of motion) is well established and widely applicable. However, the assumption that the drag force is proportional to the velocity is less certain. Even if this assumption is correct, the determination of the drag coefficient γ by direct measurement presents difficulties. Indeed, sometimes one finds the drag coefficient indirectly—for example, by measuring the time of fall from a given height and then calculating the value of γ that predicts this observed time.

The model of the field mouse population is subject to various uncertainties. The determination of the growth rate r and the predation rate k depends on observations of actual populations, which may be subject to considerable variation. The assumption that r and k are constants may also be questionable. For example, a constant predation rate becomes harder to sustain as the field mouse population becomes smaller. Further, the model predicts that a population above the equilibrium value will grow exponentially larger and larger. This seems at variance with the behavior of actual populations; see the further discussion of population dynamics in Section 2.5.

If the differences between actual observations and a mathematical model's predictions are too great, then you need to consider refining the model, making more careful observations, or perhaps both. There is almost always a tradeoff between accuracy and simplicity. Both are desirable, but a gain in one usually involves a loss in the other. However, even if a mathematical model is incomplete or somewhat inaccurate, it may nevertheless be useful in explaining qualitative features of the problem under investigation. It may also give satisfactory results under some circumstances but not others. Thus you should always use good judgment and common sense in constructing mathematical models and in using their predictions.

Historical Background, Part II: Euler, Lagrange, and Laplace. The greatest mathematician of the eighteenth century, Leonhard Euler (1707–1783), grew up near Basel, Switzerland and was a student of Johann Bernoulli. He followed his friend Daniel Bernoulli to St. Petersburg in 1727. For the remainder of his life he was associated with the St. Petersburg Academy (1727–1741 and 1766–1783) and the Berlin Academy (1741–1766). Losing sight in his right eye in 1738, and in his left eye in 1766, did not stop Euler from being one of the most prolific mathematicians of all time. In addition to publishing more than 500 books and papers during his life, an additional 400 have appeared posthumously.

Of particular interest here is Euler's formulation of problems in mechanics in mathematical language and his development of methods of solving these mathematical problems. Lagrange said of Euler's work in mechanics, "The first great work in which analysis is applied to the science of movement." Among other things, Euler identified the condition for exactness of first-order differential equations (Section 2.6) in 1734–1735, developed the theory of integrating factors (Section 2.6) in the same paper, and gave the general solution of homogeneous linear differential equations with constant coefficients (Sections 3.1, 3.3, 3.4, and 4.2) in 1743. He extended the latter results to nonhomogeneous differential equations in 1750–1751. Beginning about 1750, Euler made frequent use of power series (Chapter 5) in solving differential equations. He also proposed a numerical procedure (Sections 2.7 and 8.1) in 1768–1769, made important contributions in partial differential equations, and gave the first systematic treatment of the calculus of variations.

Joseph-Louis Lagrange (1736–1813) became professor of mathematics in his native Turin, Italy, at the age of 19. He succeeded Euler in the chair of mathematics at the Berlin Academy in 1766 and moved on to the Paris Academy in 1787. He is most famous for his monumental work *Mécanique analytique*, published in 1788, an elegant and comprehensive treatise of Newtonian mechanics. With respect to elementary differential equations, Lagrange showed in 1762–1765 that the general solution of a homogeneous nth order linear differential equation is a linear combination of n independent solutions (Sections 3.2 and 4.1). Later, in 1774–1775, he offered a complete development of the method of variation of parameters (Sections 3.6 and 4.4). Lagrange is also known for fundamental work in partial differential equations and the calculus of variations.

Pierre-Simon de Laplace (1749–1827) lived in Normandy, France, as a boy but arrived in Paris in 1768 and quickly made his mark in scientific circles, winning election to the Académie des Sciences in 1773. He was preeminent in the field of celestial mechanics; his greatest work, *Traité de mécanique céleste*, was published in five volumes between 1799 and 1825. Laplace's equation is fundamental in many branches of mathematical physics, and Laplace studied it extensively in connection with gravitational attraction. The Laplace transform (Chapter 6) is also named for him, although its usefulness in solving differential equations was not recognized until much later.

By the end of the eighteenth century many elementary methods of solving ordinary differential equations had been discovered. In the nineteenth century interest turned more toward the investigation of theoretical questions of existence and uniqueness and to the development of less elementary methods such as those based on power series expansions (see Chapter 5). These methods find their natural setting in the complex plane. Consequently, they benefitted from, and to some extent stimulated, the more or less simultaneous development of the theory of complex analytic functions. Partial differential equations also began to be studied intensively, as their crucial role in mathematical physics became clear. In this connection a number of functions, arising as solutions of certain ordinary differential equations, occurred repeatedly and were studied exhaustively. Known collectively as higher transcendental functions, many of them are associated with the names of mathematicians, including Bessel (Section 5.7), Legendre (Section 5.3), Hermite (Section 5.2), Chebyshev (Section 5.3), Hankel, and many others.

Problems

N **1.** Solve each of the following initial value problems and plot the solutions for several values of y_0. Then describe in a few words how the solutions resemble, and differ from, each other.

a. $dy/dt = -y + 5, \quad y(0) = y_0$
b. $dy/dt = -2y + 5, \quad y(0) = y_0$
c. $dy/dt = -2y + 10, \quad y(0) = y_0$

G **2.** Follow the instructions for Problem 1 for the following initial-value problems:

a. $dy/dt = y - 5, \quad y(0) = y_0$
G **b.** $dy/dt = 2y - 5, \quad y(0) = y_0$
c. $dy/dt = 2y - 10, \quad y(0) = y_0$

3. Consider the differential equation

$$dy/dt = -ay + b,$$

where both a and b are positive numbers.

a. Find the general solution of the differential equation.
G **b.** Sketch the solution for several different initial conditions.
c. Describe how the solutions change under each of the following conditions:

i. a increases.
ii. b increases.
iii. Both a and b increase, but the ratio b/a remains the same.

4. Consider the differential equation $dy/dt = ay - b$.

a. Find the equilibrium solution y_e.
b. Let $Y(t) = y - y_e$; thus $Y(t)$ is the deviation from the equilibrium solution. Find the differential equation satisfied by $Y(t)$.

5. Undetermined Coefficients. Here is an alternative way to solve the equation

$$\frac{dy}{dt} = ay - b. \tag{31}$$

a. Solve the simpler equation

$$\frac{dy}{dt} = ay. \tag{32}$$

Call the solution $y_1(t)$.
b. Observe that the only difference between equations (31) and (32) is the constant $-b$ in equation (31). Therefore, it may seem reasonable to assume that the solutions of these two equations also differ only by a constant. Test this assumption by trying to find a constant k such that $y = y_1(t) + k$ is a solution of equation (31).
c. Compare your solution from part b with the solution given in the text in equation (17).

Note: This method can also be used in some cases in which the constant b is replaced by a function $g(t)$. It depends on whether you can guess the general form that the solution is likely to take. This method is described in detail in Section 3.5 in connection with second-order equations.

6. Use the method of Problem 5 to solve the equation

$$\frac{dy}{dt} = -ay + b.$$

7. The field mouse population in Example 1 satisfies the differential equation

$$\frac{dy}{dt} = \frac{p}{2} - 450.$$

a. Find the time at which the population becomes extinct if $p(0) = 850$.
b. Find the time of extinction if $p(0) = p_0$, where $0 < p_0 < 900$.
N **c.** Find the initial population p_0 if the population is to become extinct in 1 year.

8. The falling object in Example 2 satisfies the initial value problem

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}, \quad v(0) = 0.$$

a. Find the time that must elapse for the object to reach 98% of its limiting velocity.
b. How far does the object fall in the time found in part a?

9. Consider the falling object of mass 10 kg in Example 2, but assume now that the drag force is proportional to the square of the velocity.

a. If the limiting velocity is 49 m/s (the same as in Example 2), show that the equation of motion can be written as

$$\frac{dv}{dt} = \frac{1}{245}\left(49^2 - v^2\right).$$

Also see Problem 21 of Section 1.1.

b. If $v(0) = 0$, find an expression for $v(t)$ at any time.

G **c.** Plot your solution from part b and the solution (26) from Example 2 on the same axes.

d. Based on your plots in part c, compare the effect of a quadratic drag force with that of a linear drag force.

e. Find the distance $x(t)$ that the object falls in time t.

N **f.** Find the time T it takes the object to fall 300 m.

10. A radioactive material, such as the isotope thorium-234, disintegrates at a rate proportional to the amount currently present. If $Q(t)$ is the amount present at time t, then $dQ/dt = -rQ$, where $r > 0$ is the decay rate.

a. If 100 mg of thorium-234 decays to 82.04 mg in 1 week, determine the decay rate r.

b. Find an expression for the amount of thorium-234 present at any time t.

c. Find the time required for the thorium-234 to decay to one-half its original amount.

11. The **half-life** of a radioactive material is the time required for an amount of this material to decay to one-half its original value. Show that for any radioactive material that decays according to the equation $Q' = -rQ$, the half-life τ and the decay rate r satisfy the equation $r\tau = \ln 2$.

12. According to Newton's law of cooling (see Problem 19 of Section 1.1), the temperature $u(t)$ of an object satisfies the differential equation

$$\frac{du}{dt} = -k(u - T),$$

where T is the constant ambient temperature and k is a positive constant. Suppose that the initial temperature of the object is $u(0) = u_0$.

a. Find the temperature of the object at any time.

b. Let τ be the time at which the initial temperature difference $u_0 - T$ has been reduced by half. Find the relation between k and τ.

13. Consider an electric circuit containing a capacitor, resistor, and battery; see Figure 1.2.3. The charge $Q(t)$ on the capacitor satisfies the equation[5]

$$R\frac{dQ}{dt} + \frac{Q}{C} = V,$$

where R is the resistance, C is the capacitance, and V is the constant voltage supplied by the battery.

G **a.** If $Q(0) = 0$, find $Q(t)$ at any time t, and sketch the graph of Q versus t.

b. Find the limiting value Q_L that $Q(t)$ approaches after a long time.

G **c.** Suppose that $Q(t_1) = Q_L$ and that at time $t = t_1$ the battery is removed and the circuit is closed again. Find $Q(t)$ for $t > t_1$ and sketch its graph.

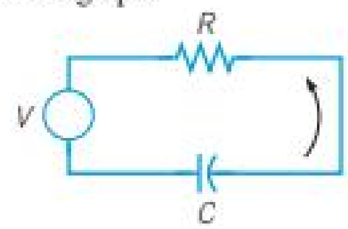

FIGURE 1.2.3 The electric circuit of Problem 13.

N **14.** A pond containing 1,000,000 gal of water is initially free of a certain undesirable chemical (see Problem 17 of Section 1.1). Water containing 0.01 g/gal of the chemical flows into the pond at a rate of 300 gal/h, and water also flows out of the pond at the same rate. Assume that the chemical is uniformly distributed throughout the pond.

a. Let $Q(t)$ be the amount of the chemical in the pond at time t. Write down an initial value problem for $Q(t)$.

b. Solve the problem in part a for $Q(t)$. How much chemical is in the pond after 1 year?

c. At the end of 1 year the source of the chemical in the pond is removed; thereafter pure water flows into the pond, and the mixture flows out at the same rate as before. Write down the initial value problem that describes this new situation.

d. Solve the initial value problem in part c. How much chemical remains in the pond after 1 additional year (2 years from the beginning of the problem)?

e. How long does it take for $Q(t)$ to be reduced to 10 g?

G **f.** Plot $Q(t)$ versus t for 3 years.

[5]This equation results from Kirchhoff's laws, which are discussed in Section 3.7.

1.3 Classification of Differential Equations

The main purposes of this book are to discuss some of the properties of solutions of differential equations and to present some of the methods that have proved effective in finding solutions or, in some cases, in approximating them. To provide a framework for our presentation, we describe here several useful ways of classifying differential equations. Mastery of this vocabulary is essential to selecting appropriate solution methods and to describing properties of solutions of differential equations that you encounter later in this book—and in the real world.

Ordinary and Partial Differential Equations. One important classification is based on whether the unknown function depends on a single independent variable or on several independent variables. In the first case, only ordinary derivatives appear in the differential equation, and it is said to be an **ordinary differential equation**. In the second case, the derivatives are partial derivatives, and the equation is called a **partial differential equation**.

All the differential equations discussed in the preceding two sections are ordinary differential equations. Another example of an ordinary differential equation is

$$L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{1}{C}Q(t) = E(t), \tag{1}$$

for the charge $Q(t)$ on a capacitor in a circuit with capacitance C, resistance R, and inductance L; this equation is derived in Section 3.7. Typical examples of partial differential equations are the heat conduction equation

$$\alpha^2\frac{\partial^2u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \tag{2}$$

and the wave equation

$$a^2\frac{\partial^2u(x,t)}{\partial x^2} = \frac{\partial^2u(x,t)}{\partial t^2}. \tag{3}$$

Here, α^2 and a^2 are certain physical constants. Note that in both equations (2) and (3) the dependent variable u depends on the two independent variables x and t. The heat conduction equation describes the conduction of heat in a solid body, and the wave equation arises in a variety of problems involving wave motion in solids or fluids.

Systems of Differential Equations. Another classification of differential equations depends on the number of unknown functions that are involved. If there is a single function to be determined, then one differential equation is sufficient. However, if there are two or more unknown functions, then a system of differential equations is required. For example, the Lotka-Volterra, or predator-prey, equations are important in ecological modeling. They have the form

$$\begin{aligned} \frac{dx}{dt} &= ax - \alpha xy \\ \frac{dy}{dt} &= -cy + \gamma xy, \end{aligned} \tag{4}$$

where $x(t)$ and $y(t)$ are the respective populations of the prey and predator species. The positive constants $a, \alpha, c,$ and γ are based on empirical observations and depend on the particular species being studied. Systems of equations are discussed in Chapters 7 and 9; in particular, the Lotka-Volterra equations are examined in Section 9.5. In some areas of application it is not unusual to encounter very large systems containing hundreds, or even many thousands, of differential equations.

Order. The **order** of a differential equation is the order of the highest derivative that appears in the equation. The equations in the preceding sections are all first-order equations, whereas equation (1) is a second-order equation. Equations (2) and (3) are also second-order partial differential equations. More generally, the equation

$$F\left(t, u(t), u'(t), \ldots, u^{(n)}(t)\right) = 0 \tag{5}$$

is an ordinary differential equation of the n^{th} order. Equation (5) expresses a relation between the independent variable t and the values of the function u and its first n derivatives $u', u'', \ldots, u^{(n)}$. It is convenient and customary in differential equations to write y for $u(t)$, with $y', y'', \ldots, y^{(n)}$ standing for $u'(t), u''(t), \ldots, u^{(n)}(t)$. Thus equation (5) is written as

$$F\left(t, y, y', \ldots, y^{(n)}\right) = 0. \tag{6}$$

For example,

$$y''' + 2e^t y'' + yy' = t^4 \tag{7}$$

is a third-order differential equation for $y = u(t)$. Occasionally, other letters will be used instead of t and y for the independent and dependent variables; the meaning should be clear from the context.

We assume that it is always possible to solve a given ordinary differential equation for the highest derivative, obtaining

$$y^{(n)} = f\left(t, y, y', y'', \ldots, y^{(n-1)}\right). \tag{8}$$

This is mainly to avoid the ambiguity that may arise because a single equation of the form (6) may correspond to several equations of the form (8). For example, the equation

$$(y')^2 + ty' + 4y = 0 \tag{9}$$

leads to the two equations

$$y' = \frac{-t + \sqrt{t^2 - 16y}}{2} \quad \text{or} \quad y' = \frac{-t - \sqrt{t^2 - 16y}}{2}. \tag{10}$$

Linear and Nonlinear Equations. A crucial classification of differential equations is whether they are linear or nonlinear. The ordinary differential equation

$$F\left(t, y, y', \ldots, y^{(n)}\right) = 0$$

is said to be **linear** if F is a linear function of the variables $y, y', \ldots, y^{(n)}$; a similar definition applies to partial differential equations. Thus the general linear ordinary differential equation of order n is

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t). \tag{11}$$

Most of the equations you have seen thus far in this book are linear; examples are the equations in Sections 1.1 and 1.2 describing the falling object and the field mouse population. Similarly, in this section, equation (1) is a linear ordinary differential equation and equations (2) and (3) are linear partial differential equations. An equation that is not of the form (11) is a **nonlinear** equation. Equation (7) is nonlinear because of the term yy'. Similarly, each equation in the system (4) is nonlinear because of the terms that involve the product of the two unknown functions xy.

A simple physical problem that leads to a nonlinear differential equation is the oscillating pendulum. The angle $\theta = \theta(t)$ that an oscillating pendulum of length L makes with the vertical direction (see Figure 1.3.1) satisfies the equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0, \tag{12}$$

whose derivation is outlined in Problems 22 through 24. The presence of the term involving $\sin\theta$ makes equation (12) nonlinear.

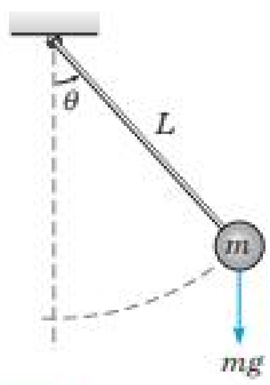

FIGURE 1.3.1 An oscillating pendulum.

The mathematical theory and methods for solving linear equations are highly developed. In contrast, for nonlinear equations the theory is more complicated, and methods of solution are less satisfactory. In view of this, it is fortunate that many significant problems lead to linear ordinary differential equations or can be approximated by linear equations. For example, for the pendulum, if the angle θ is small, then $\sin\theta \cong \theta$ and equation (12) can be approximated by the linear equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0. \tag{13}$$

This process of approximating a nonlinear equation by a linear one is called **linearization**; it is an extremely valuable way to deal with nonlinear equations. Nevertheless, there are many physical phenomena that simply cannot be represented adequately by linear equations. To study these phenomena, it is essential to deal with nonlinear equations.

In an elementary text it is natural to emphasize the simpler and more straightforward parts of the subject. Therefore, the greater part of this book is devoted to linear equations and various methods for solving them. However, Chapters 8 and 9, as well as parts of Chapter 2, are concerned with nonlinear equations. Whenever it is appropriate, we point out why nonlinear equations are, in general, more difficult and why many of the techniques that are useful in solving linear equations cannot be applied to nonlinear equations.

Solutions. A **solution** of the n^{th} order ordinary differential equation (8) on the interval $\alpha < t < \beta$ is a function ϕ such that $\phi', \phi'', \ldots, \phi^{(n)}$ exist and satisfy

$$\phi^{(n)}(t) = f\left(t, \phi(t), \phi'(t), \ldots, \phi^{(n-1)}(t)\right) \tag{14}$$

for every t in $\alpha < t < \beta$. Unless stated otherwise, we assume that the function f of equation (8) is a real-valued function, and we are interested in obtaining real-valued solutions $y = \phi(t)$.

Recall that in Section 1.2 we found solutions of certain equations by a process of direct integration. For instance, we found that the equation

$$\frac{dp}{dt} = \frac{p}{2} - 450 \tag{15}$$

has the solution

$$p(t) = 900 + ce^{t/2}, \tag{16}$$

where c is an arbitrary constant.

It is often not so easy to find solutions of differential equations. However, if you find a function that you think may be a solution of a given equation, it is usually relatively easy to determine whether the function is actually a solution: just substitute the function into the equation.

For example, in this way it is easy to show that the function $y_1(t) = \cos t$ is a solution of

$$y'' + y = 0 \tag{17}$$

for all t. To confirm this, observe that $y_1'(t) = -\sin t$ and $y_1''(t) = -\cos t$; then it follows that $y_1''(t) + y_1(t) = 0$. In the same way you can easily show that $y_2(t) = \sin t$ is also a solution of equation (17).

Of course, this does not constitute a satisfactory way to solve most differential equations, because there are far too many possible functions for you to have a good chance of finding the correct one by a random choice. Nevertheless, you should realize that you can verify whether any proposed solution is correct by substituting it into the differential equation. This can be a very useful check; it is one that you should make a habit of considering.

Some Important Questions. Although for the differential equations (15) and (17) we are able to verify that certain simple functions are solutions, in general we do not have such solutions readily available. Thus a fundamental question is the following: Does an equation of the form (8) always have a solution? The answer is "No." Merely writing down an equation of the form (8) does not necessarily mean that there is a function $y = \phi(t)$ that satisfies it. So, how can we tell whether some particular equation has a solution? This is the question of *existence* of a solution, and it is answered by theorems stating that under certain restrictions on the function f in equation (8), the equation always has solutions. This is not a purely theoretical concern for at least two reasons. If a problem has no solution, we would prefer to know that fact before investing time and effort in a vain attempt to solve the problem. Further, if a sensible physical problem is modeled mathematically as a differential equation, then the equation should have a solution. If it does not, then presumably there is something wrong with the formulation. In this sense an engineer or scientist has some check on the validity of the mathematical model.

If we assume that a given differential equation has at least one solution, then we may need to consider how many solutions it has, and what additional conditions must be specified to single out a particular solution. This is the question of *uniqueness*. In general, solutions of differential equations contain one or more arbitrary constants of integration, as does the solution (16) of equation (15). Equation (16) represents an infinity of functions corresponding to the infinity of possible choices of the constant c. As we saw in Section 1.2, if p is specified at some time t, this condition will determine a specific value for c; even so, we have not yet ruled out the possibility that there may be other solutions of equation (15) that also have the prescribed value of p at the prescribed time t. As in the question of existence of solutions, the issue of uniqueness has practical as well as theoretical implications. If we are fortunate enough to find a solution of a given problem, and if we know that the problem has a unique solution, then we can be sure that we have completely solved the problem. If there may be other solutions, then perhaps we should continue to search for them.

A third important question is: Given a differential equation of the form (8), can we actually determine a solution, and if so, how? Note that if we find a solution of the given equation, we have at the same time answered the question of the existence of a solution. However, without knowledge of existence theory we might, for example, use a computer to find a numerical approximation to a "solution" that does not exist. On the other hand, even though we may know that a solution exists, it may be that the solution is not expressible in terms of the usual elementary functions—polynomial, trigonometric, exponential, logarithmic, and hyperbolic functions. Unfortunately, this is the situation for most differential equations. Thus, we discuss both elementary methods that can be used to obtain exact solutions of certain relatively simple problems, and also methods of a more general nature that can be used to find approximations to solutions of more difficult problems.

Technology Use in Differential Equations. Technology provides many extremely valuable tools for the study of differential equations. For many years computers have been used to execute numerical algorithms, such as those described in Chapter 8, to construct numerical approximations to solutions of differential equations. These algorithms have been refined to an extremely high level of generality and efficiency. A few lines of computer code, written in a high-level programming language and executed (often within a few seconds) on a relatively inexpensive computer, suffice to approximate to a high degree of accuracy the solutions of a wide range of differential equations. More sophisticated routines are also readily available. These routines combine the ability to handle very large and complicated systems with numerous diagnostic features that alert the user to possible problems as they are encountered.

The usual output from a numerical algorithm is a table of numbers, listing selected values of the independent variable and the corresponding values of the dependent variable. With appropriate software it is easy to display the solution of a differential equation graphically, whether the solution has been obtained numerically or as the result of an analytical procedure of some kind. Such a graphical display is often much more illuminating and helpful in understanding and interpreting the solution of a differential equation than a table of numbers or a complicated analytical formula. There are on the market several well-crafted and relatively inexpensive special-purpose software packages for the graphical investigation of differential equations. The increased power and sophistication of modern smartphones, tablets, and other mobile devices have brought powerful computational and graphical capability within the reach of individual students. You should consider, in the light of your own circumstances, how best to take advantage of the available computing resources. You will surely find it enlightening to do so.

Another aspect of computer use that is very relevant to the study of differential equations is the availability of extremely powerful and general software packages that can perform a wide variety of mathematical operations. Among these are Maple, Mathematica, and MATLAB, each of which can be used on various kinds of personal computers or workstations. All three of these packages can execute extensive numerical computations and have versatile graphical facilities. Maple and Mathematica also have very extensive analytical capabilities. For example, they can perform the analytical steps involved in solving many differential equations, often in response to a single command. Anyone who expects to deal with differential equations in more than a superficial way should become familiar with at least one of these products and explore the ways in which it can be used.

For you, the student, these computing resources have an effect on how you should study differential equations. To become confident in using differential equations, it is essential to understand how the solution methods work, and this understanding is achieved, in part, by working out a sufficient number of examples in detail. However, eventually you should plan to utilize appropriate computational tools to complete as many as possible of the routine (often repetitive) details, while you focus on the proper formulation of the problem and on the interpretation of the solution. Our viewpoint is that you should always try to use the best methods and tools available for each task. In particular, you should strive to combine numerical, graphical, and analytical methods so as to attain maximum understanding of the behavior of the solution and of the underlying process that the problem models. You should also remember that some tasks can best be done with pencil and paper, while others require the use of some sort of computational technology. Good judgment is often needed in selecting an effective combination.

Historical Background, Part III: Recent and Ongoing Advances. The numerous differential equations that resisted solution by analytical means led to the investigation of methods of numerical approximation (see Chapter 8). By 1900 fairly effective numerical integration methods had been devised, but their implementation was severely restricted by the need to execute the computations by hand or with very primitive computing equipment. Since World War II the development of increasingly powerful and versatile computers has vastly enlarged the range of problems that can be investigated effectively by numerical methods. Extremely refined and robust numerical integrators were developed during the same period and now are readily available, even on smartphones and other mobile devices. These technological advances have brought the ability to solve a great many significant problems within the reach of individual students.

Another characteristic of modern differential equations is the creation of geometric or topological methods, especially for nonlinear equations. The goal is to understand at least the qualitative behavior of solutions from a geometrical, as well as from an analytical, point of view. If more detailed information is needed, it can usually be obtained by using numerical approximations. An introduction to geometric methods appears in Chapter 9. We conclude this brief historical review with two examples that illustrate how computational and real-world experiences have motivated important analytical and theoretical discoveries.

In 1834 John Scott Russell (1808–1882), a Scottish civil engineer, was conducting experiments to determine the most efficient design for canal boats when he noticed that "when the boat suddenly stopped" the water being pushed by the boat "accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it [the boat] behind, [the water] rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water."[6] Many mathematicians did not believe that the solitary traveling waves reported by Russell existed. These objections were silenced when the doctoral dissertation of Dutch mathematician Gustav de Vries (1866–1934) included a nonlinear partial differential equation model for water waves in a shallow canal. Today these equations are known as the Korteweg-de Vries (KdV) equations. Diederik Johannes Korteweg (1848–1941) was de Vries's thesis advisor. Unknown to either Korteweg or de Vries, their Korteweg-de Vries model appeared as a footnote ten years earlier in French mathematician Joseph Valentin Boussinesq's (1842–1929) 680-page treatise *Essai sur la théorie des eaux courantes*. The work of Boussinesq and of Korteweg and deVries remained largely unnoticed until two Americans, physicist Norman J. Zabusky (1929–) and mathematician Martin David Kruskal (1925–2006), used computer simulations to discover, in 1965, that all solutions of the KdV equations eventually consist of a finite set of localized traveling waves. Today, nearly 200 years after Russell's observations and 50 years after the computational experiments of Zabusky and Kruskal, the study of "solitons" remains an active area of differential equations research. Other early contributors to nonlinear wave propagation include David Hilbert (German, 1862–1943), Richard Courant (German-American, 1888–1972), and John von Neumann (Hungarian-American, 1903–1957); we will encounter some of these ideas again in Chapter 9.

Computational results were also an essential element in the discovery of "chaos theory." In 1961, Edward Lorenz (1917–2008), an American mathematician and meteorologist at the Massachusetts Institute of Technology, was developing weather prediction models when he observed different results upon restarting a simulation in the middle of the time period using

[6] "Report on Waves," in *Proceedings of the Fourteenth Meeting of the British Association for the Advancement of Science*, 1845, pp. 311–390, plus plates 47–57, http://www.macs.hw.ac.uk/~chris/Scott-Russell/SR44.pdf.

previously computed results. (Lorenz restarted the computation with three-digit approximate solutions, not the six-digit approximations that were stored in the computer.) In 1976 the Australian mathematician Sir Robert M. May (1938–) introduced and analyzed the logistic map, showing that there are special values of the problem's parameter where the solutions undergo drastic changes. The common trait that small changes in the problem produce large changes in the solution is one of the defining characteristics of chaos. May's logistic map is discussed in more detail in Section 2.9. Other classical examples of what we now recognize as "chaos" include the work by French mathematician Henri Poincaré (1854–1912) on planetary motion and the studies of turbulent fluid flow by Soviet mathematician Andrey Nikolaevich Kolmogorov (1903–1987), American mathematician Mitchell Feigenbaum (1944–), and many others. In addition to these and other classical examples of chaos, new examples continue to be found.

Solitons and chaos are just two of many examples where computers, and especially computer graphics, have given a new impetus to the study of systems of nonlinear differential equations. Other unexpected phenomena (Section 9.8), such as strange attractors (David Ruelle, Belgium, 1935–) and fractals (Benoit Mandelbrot, Poland, 1924–2010), have been discovered, are being intensively studied, and are leading to important new insights in a variety of applications. Although it is an old subject about which much is known, the study of differential equations in the twenty-first century remains a fertile source of fascinating and important unsolved problems.

Problems

In each of Problems 1 through 4, determine the order of the given differential equation; also state whether the equation is linear or nonlinear.

1. $t^2\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + 2y = \sin t$

2. $(1+y^2)\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + y = e^t$

3. $\dfrac{d^4y}{dt^4} + \dfrac{d^3y}{dt^3} + \dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = 1$

4. $\dfrac{d^2y}{dt^2} + \sin(t+y) = \sin t$

In each of Problems 5 through 10, verify that each given function is a solution of the differential equation.

5. $y'' - y = 0;\quad y_1(t) = e^t,\ y_2(t) = \cosh t$

6. $y'' + 2y' - 3y = 0;\quad y_1(t) = e^{-3t},\ y_2(t) = e^t$

7. $ty' - y = t^2;\quad y = 3t + t^2$

8. $y'''' + 4y''' + 3y = t;\quad y_1(t) = t/3,\ y_2(t) = e^{-t} + t/3$

9. $t^2y'' + 5ty' + 4y = 0,\ t > 0;\quad y_1(t) = t^{-2},\ y_2(t) = t^{-2}\ln t$

10. $y' - 2ty = 1;\quad y = e^{t^2}\displaystyle\int_0^t e^{-s^2}\,ds + e^{t^2}$

In each of Problems 11 through 13, determine the values of r for which the given differential equation has solutions of the form $y = e^{rt}$.

11. $y' + 2y = 0$

12. $y'' + y' - 6y = 0$

13. $y''' - 3y'' + 2y' = 0$

In each of Problems 14 and 15, determine the values of r for which the given differential equation has solutions of the form $y = t^r$ for $t > 0$.

14. $t^2y'' + 4ty' + 2y = 0$

15. $t^2y'' - 4ty' + 4y = 0$

In each of Problems 16 through 18, determine the order of the given partial differential equation; also state whether the equation is linear or nonlinear. Partial derivatives are denoted by subscripts.

16. $u_{xx} + u_{yy} + u_{zz} = 0$

17. $u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$

18. $u_t + uu_x = 1 + u_{xx}$

In each of Problems 19 through 21, verify that each given function is a solution of the given partial differential equation.

19. $u_{xx} + u_{yy} = 0;\quad u_1(x,y) = \cos x \cosh y,$
$u_2(x,y) = \ln(x^2 + y^2)$

20. $\alpha^2 u_{xx} = u_t;\quad u_1(x,t) = e^{-\alpha^2 t}\sin x,$
$u_2(x,t) = e^{-\alpha^2\lambda^2 t}\sin\lambda x,\ \lambda$ a real constant

21. $a^2u_{xx} = u_{tt};\quad u_1(x,t) = \sin(\lambda x)\sin(\lambda a t),$
$u_2(x,t) = \sin(x - at),\ \lambda$ a real constant

22. Follow the steps indicated here to derive the equation of motion of a pendulum, equation (12) in the text. Assume that the rod is rigid and weightless, that the mass is a point mass, and that there is no friction or drag anywhere in the system.

a. Assume that the mass is in an arbitrary displaced position, indicated by the angle θ. Draw a free-body diagram showing the forces acting on the mass.

b. Apply Newton's law of motion in the direction tangential to the circular arc on which the mass moves. Then the tensile force in the rod does not enter the equation. Observe that you need to find the component of the gravitational force in the tangential direction. Observe also that the linear acceleration, as opposed to the angular acceleration, is $Ld^2\theta/dt^2$, where L is the length of the rod.

c. Simplify the result from part b to obtain equation (12) in the text.

23. Another way to derive the pendulum equation (12) is based on the principle of conservation of energy.

a. Show that the kinetic energy T of the pendulum in motion is

$$T = \frac{1}{2} m L^2 \left(\frac{d\theta}{dt} \right)^2.$$

b. Show that the potential energy V of the pendulum, relative to its rest position, is

$$V = mgL(1 - \cos\theta).$$

c. By the principle of conservation of energy, the total energy $E = T + V$ is constant. Calculate dE/dt, set it equal to zero, and show that the resulting equation reduces to equation (12).

24. A third derivation of the pendulum equation depends on the principle of angular momentum: The rate of change of angular momentum about any point is equal to the net external moment about the same point.

a. Show that the angular momentum M, or moment of momentum, about the point of support is given by $M = mL^2 d\theta/dt$.

b. Set dM/dt equal to the moment of the gravitational force, and show that the resulting equation reduces to equation (12). Note that positive moments are counterclockwise.

References

Computer software for differential equations changes too fast for particulars to be given in a book such as this. A Google search for Maple, Mathematica, Sage, or MATLAB is a good way to begin if you need information about one of these computer algebra and numerical systems.

There are many instructional books on computer algebra systems, such as the following:

Cheung, C.-K., Keough, G. E., Gross, R. H., and Landraitis, C., *Getting Started with Mathematica* (3rd ed.) (New York: Wiley, 2009).

Meade, D. B., May, M., Cheung, C.-K., and Keough, G. E., *Getting Started with Maple* (3rd ed.) (New York: Wiley, 2009).

For further reading in the history of mathematics, see books such as those listed below:

Boyer, C. B., and Merzbach, U. C., *A History of Mathematics* (2nd ed.) (New York: Wiley, 1989).

Kline, M., *Mathematical Thought from Ancient to Modern Times* (3 vols.) (New York: Oxford University Press, 1990).

A useful historical appendix on the early development of differential equations appears in

Ince, E. L., *Ordinary Differential Equations* (London: Longmans, Green, 1927; New York: Dover, 1956).

Encyclopedic sources of information about the lives and achievements of mathematicians of the past are

Gillespie, C. C., ed., *Dictionary of Scientific Biography* (15 vols.) (New York: Scribner's, 1971).

Koertge, N., ed., *New Dictionary of Scientific Biography* (8 vols.) (New York: Scribner's, 2007).

Koertge, N., ed., *Complete Dictionary of Scientific Biography* (New York: Scribner's, 2007 [e-book]).

Much historical information can be found on the Internet. One excellent site is the MacTutor History of Mathematics archive

http://www-history.mcs.st-and.ac.uk/history/

created by O'Connor, J. J., and Robertson, E. F., Department of Mathematics and Statistics, University of St. Andrews, Scotland.

CHAPTER 2

First-Order Differential Equations

This chapter deals with differential equations of first order

$$\frac{dy}{dt} = f(t, y), \tag{1}$$

where f is a given function of two variables. Any differentiable function $y = \phi(t)$ that satisfies this equation for all t in some interval is called a solution, and our objective is to determine whether such functions exist and, if so, to develop methods for finding them. Unfortunately, for an arbitrary function f, there is no general method for solving the equation in terms of elementary functions. Instead, we will describe several methods, each of which is applicable to a certain subclass of first-order equations.

The most important of these are linear equations (Section 2.1), separable equations (Section 2.2), and exact equations (Section 2.6). Other sections of this chapter describe some of the important applications of first-order differential equations, introduce the idea of approximating a solution by numerical computation, and discuss some theoretical questions related to the existence and uniqueness of solutions. The final section includes an example of chaotic solutions in the context of first-order difference equations, which have some important points of similarity with differential equations and are simpler to investigate.

2.1 Linear Differential Equations; Method of Integrating Factors

If the function f in equation (1) depends linearly on the dependent variable y, then equation (1) is a first-order linear differential equation. In Sections 1.1 and 1.2 we discussed a restricted type of first-order linear differential equation in which the coefficients are constants. A typical example is

$$\frac{dy}{dt} = -ay + b, \tag{2}$$

where a and b are given constants. Recall that an equation of this form describes the motion of an object falling in the atmosphere.

Now we want to consider the most general first-order linear differential equation, which is obtained by replacing the coefficients a and b in equation (2) by arbitrary functions of t. We will usually write the general **first-order linear differential equation** in the standard form

$$\frac{dy}{dt} + p(t)y = g(t), \tag{3}$$

where p and g are given functions of the independent variable t. Sometimes it is more convenient to write the equation in the form

$$P(t)\frac{dy}{dt} + Q(t)y = G(t), \tag{4}$$

where P, Q, and G are given. Of course, as long as $P(t) \neq 0$, you can convert equation (4) to equation (3) by dividing both sides of equation (4) by $P(t)$.

In some cases it is possible to solve a first-order linear differential equation immediately by integrating the equation, as in the next example.

EXAMPLE 1

Solve the differential equation

$$(4+t^2)\frac{dy}{dt}+2ty=4t. \tag{5}$$

Solution:

The left-hand side of equation (5) is a linear combination of dy/dt and y, a combination that also appears in the rule from calculus for differentiating a product. In fact,

$$(4+t^2)\frac{dy}{dt}+2ty=\frac{d}{dt}\left((4+t^2)y\right);$$

it follows that equation (5) can be rewritten as

$$\frac{d}{dt}\left((4+t^2)y\right)=4t. \tag{6}$$

Thus, even though y is unknown, we can integrate both sides of equation (6) with respect to t, thereby obtaining

$$(4+t^2)y=2t^2+c, \tag{7}$$

where c is an arbitrary constant of integration. Solving for y, we find that

$$y=\frac{2t^2}{4+t^2}+\frac{c}{4+t^2}. \tag{8}$$

This is the general solution of equation (5).

Unfortunately, most first-order linear differential equations cannot be solved as illustrated in Example 1 because their left-hand sides are not the derivative of the product of y and some other function. However, Leibniz discovered that if the differential equation is multiplied by a certain function $\mu(t)$, then the equation is converted into one that is immediately integrable by using the product rule for derivatives, just as in Example 1. The function $\mu(t)$ is called an **integrating factor** and our main task in this section is to determine how to find it for a given equation. We will show how this method works first for an example and then for the general first-order linear differential equation in the standard form (3).

EXAMPLE 2

Find the general solution of the differential equation

$$\frac{dy}{dt}+\frac{1}{2}y=\frac{1}{2}e^{t/3}. \tag{9}$$

Draw some representative integral curves; that is, plot solutions corresponding to several values of the arbitrary constant c. Also find the particular solution whose graph contains the point $(0, 1)$.

Solution:

The first step is to multiply equation (9) by a function $\mu(t)$, as yet undetermined; thus

$$\mu(t)\frac{dy}{dt}+\frac{1}{2}\mu(t)y=\frac{1}{2}\mu(t)e^{t/3}. \tag{10}$$

The question now is whether we can choose $\mu(t)$ so that the left-hand side of equation (10) is the derivative of the product $\mu(t)y$. For any differentiable function $\mu(t)$ we have

$$\frac{d}{dt}(\mu(t)y)=\mu(t)\frac{dy}{dt}+\frac{d\mu(t)}{dt}y. \tag{11}$$

Thus the left-hand side of equation (10) and the right-hand side of equation (11) are identical, provided that we choose $\mu(t)$ to satisfy

$$\frac{d\mu(t)}{dt}=\frac{1}{2}\mu(t). \tag{12}$$

Our search for an integrating factor will be successful if we can find a solution of equation (12). Perhaps you can readily identify a function that satisfies equation (12): What well-known function from calculus has a derivative that is equal to one-half times the original function? More systematically, rewrite equation (12) as

$$\frac{1}{\mu(t)}\frac{d\mu(t)}{dt} = \frac{1}{2},$$

which is equivalent to

$$\frac{d}{dt}\ln|\mu(t)| = \frac{1}{2}. \tag{13}$$

Then it follows that

$$\ln|\mu(t)| = \frac{1}{2}t + C,$$

or

$$\mu(t) = ce^{t/2}. \tag{14}$$

The function $\mu(t)$ given by equation (14) is an integrating factor for equation (9). Since we do not need the most general integrating factor, we will choose c to be 1 in equation (14) and use $\mu(t) = e^{t/2}$.

Now we return to equation (9), multiply it by the integrating factor $e^{t/2}$, and obtain

$$e^{t/2}\frac{dy}{dt} + \frac{1}{2}e^{t/2}y = \frac{1}{2}e^{5t/6}. \tag{15}$$

By the choice we have made of the integrating factor, the left-hand side of equation (15) is the derivative of $e^{t/2}y$, so that equation (15) becomes

$$\frac{d}{dt}(e^{t/2}y) = \frac{1}{2}e^{5t/6}. \tag{16}$$

By integrating both sides of equation (16), we obtain

$$e^{t/2}y = \frac{3}{5}e^{5t/6} + c, \tag{17}$$

where c is an arbitrary constant. Finally, on solving equation (17) for y, we have the general solution of equation (9), namely,

$$y = \frac{3}{5}e^{t/3} + ce^{-t/2}. \tag{18}$$

To find the solution passing through the point $(0, 1)$, we set $t = 0$ and $y = 1$ in equation (18), obtaining $1 = 3/5 + c$. Thus $c = 2/5$, and the desired solution is

$$y = \frac{3}{5}e^{t/3} + \frac{2}{5}e^{-t/2}. \tag{19}$$

Figure 2.1.1 includes the graphs of equation (18) for several values of c with a direction field in the background. The solution satisfying $y(0) = 1$ is shown by the green curve.

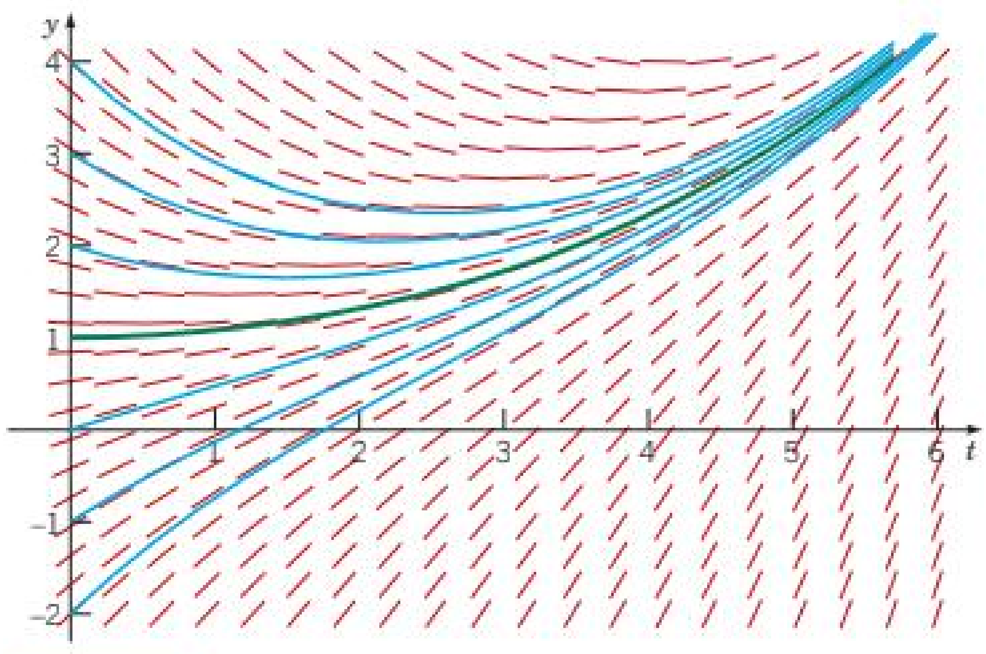

FIGURE 2.1.1 Direction field and integral curves of $y' + \frac{1}{2}y = \frac{1}{2}e^{t/3}$; the green curve passes through the point $(0, 1)$.

Let us now extend the method of integrating factors to equations of the form

$$\frac{dy}{dt} + ay = g(t), \tag{20}$$

where a is a given constant and $g(t)$ is a given function. Proceeding as in Example 2, we find that the integrating factor $\mu(t)$ must satisfy

$$\frac{d\mu}{dt} = a\mu, \tag{21}$$

rather than equation (12). Thus the integrating factor is $\mu(t) = e^{at}$. Multiplying equation (20) by $\mu(t)$, we obtain

$$e^{at}\frac{dy}{dt} + ae^{at}y = e^{at}g(t),$$

or

$$\frac{d}{dt}(e^{at}y) = e^{at}g(t). \tag{22}$$

By integrating both sides of equation (22), we find that

$$e^{at}y = \int e^{at}g(t)\,dt + c, \tag{23}$$

where c is an arbitrary constant. For many simple functions $g(t)$, we can evaluate the integral in equation (23) and express the solution y in terms of elementary functions, as in Example 2. However, for more complicated functions $g(t)$, it is necessary to leave the solution in integral form. In this case

$$y = e^{-at}\int_{t_0}^{t} e^{as}g(s)\,ds + ce^{-at}. \tag{24}$$

Note that in equation (24) we have used s to denote the integration variable to distinguish it from the independent variable t, and we have chosen some convenient value t_0 as the lower limit of integration. (See Theorem 2.4.1.) The choice of t_0 determines the specific value of the constant c but does not change the solution. For example, plugging $t = t_0$ into the solution formula (24) shows that $c = y(t_0)e^{at_0}$.

EXAMPLE 3

Find the general solution of the differential equation

$$\frac{dy}{dt} - 2y = 4 - t \tag{25}$$

and plot the graphs of several solutions. Discuss the behavior of solutions as $t \to \infty$.

Solution:

Equation (25) is of the form (20) with $a = -2$; therefore, the integrating factor is $\mu(t) = e^{-2t}$. Multiplying the differential equation (25) by $\mu(t)$, we obtain

$$e^{-2t}\frac{dy}{dt} - 2e^{-2t}y = 4e^{-2t} - te^{-2t},$$

or

$$\frac{d}{dt}\left(e^{-2t}y\right) = 4e^{-2t} - te^{-2t}. \tag{26}$$

Then, by integrating both sides of this equation, we have

$$e^{-2t}y = -2e^{-2t} + \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t} + c,$$

where we have used integration by parts on the last term in equation (26). Thus the general solution of equation (25) is

$$y = -\frac{7}{4} + \frac{1}{2}t + ce^{2t}. \tag{27}$$

Figure 2.1.2 shows the direction field and graphs of the solution (27) for several values of c. The behavior of the solution for large values of t is determined by the term ce^{2t}. If $c \neq 0$, then the solution grows exponentially large in magnitude, with the same sign as c itself. Thus the solutions diverge as t becomes large. The boundary between solutions that ultimately grow positively and those that ultimately grow negatively occurs when $c = 0$. If we substitute $c = 0$ into equation (27) and then set $t = 0$, we find that $y = -7/4$ is the separation point on the y-axis. Note that for this initial value, the solution is $y = -\frac{7}{4} + \frac{1}{2}t$; it grows positively, but linearly rather than exponentially.

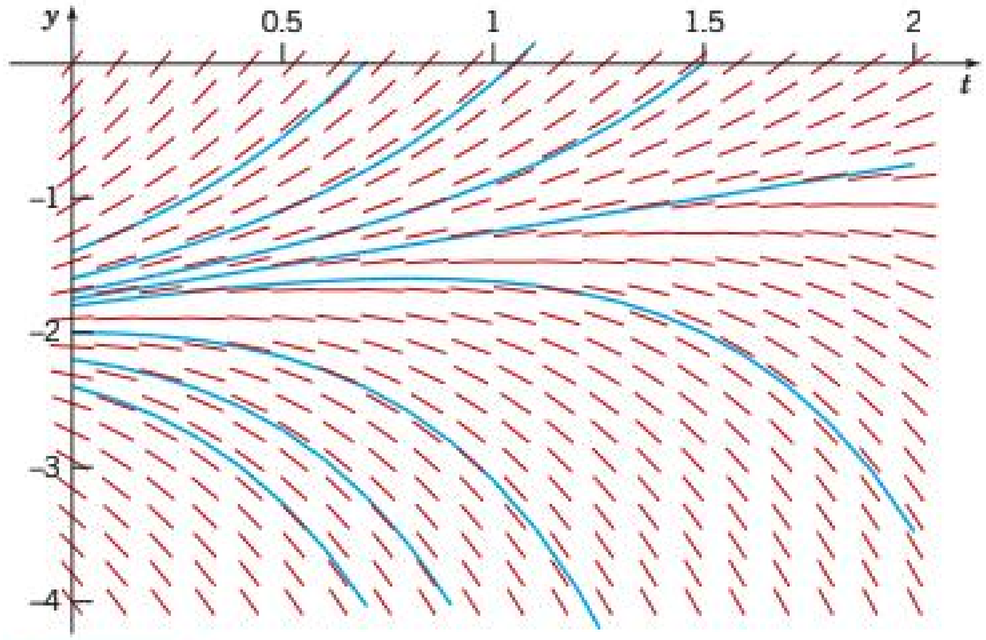

FIGURE 2.1.2 Direction field and integral curves of $y' - 2y = 4 - t$.

Now we return to the general first-order linear differential equation (3)

$$\frac{dy}{dt} + p(t)y = g(t),$$

where p and g are given functions. To determine an appropriate integrating factor, we multiply equation (3) by an as yet undetermined function $\mu(t)$, obtaining

$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t). \tag{28}$$

Following the same line of development as in Example 2, we see that the left-hand side of equation (28) is the derivative of the product $\mu(t)y$, provided that $\mu(t)$ satisfies the equation

$$\frac{d\mu(t)}{dt} = p(t)\mu(t). \tag{29}$$

If we assume temporarily that $\mu(t)$ is positive, then we have

$$\frac{1}{\mu(t)}\frac{d\mu(t)}{dt} = p(t),$$

and consequently

$$\ln|\mu(t)| = \int p(t)\,dt + k.$$

By choosing the arbitrary constant k to be zero, we obtain the simplest possible function for μ, namely,

$$\mu(t) = \exp\int p(t)\,dt. \tag{30}$$

Note that $\mu(t)$ is positive for all t, as we assumed. Returning to equation (28), we have

$$\frac{d}{dt}(\mu(t)y) = \mu(t)g(t). \tag{31}$$

Hence

$$\mu(t)y = \int \mu(t)g(t)\,dt + c, \tag{32}$$

where c is an arbitrary constant. Sometimes the integral in equation (32) can be evaluated in terms of elementary functions. However, in general this is not possible, so the general solution of equation (3) is

$$y = \frac{1}{\mu(t)}\left(\int_{t_0}^{t} \mu(s)g(s)\,ds + c\right), \tag{33}$$

where again t_0 is some convenient lower limit of integration. Observe that equation (33) involves two integrations, one to obtain $\mu(t)$ from equation (30) and the other to determine y from equation (33).

EXAMPLE 4

Solve the initial value problem

$$ty' + 2y = 4t^2, \tag{34}$$

$$y(1) = 2. \tag{35}$$

Solution:

In order to determine $p(t)$ and $g(t)$ correctly, we must first rewrite equation (34) in the standard form (3). Thus we have

$$y' + \frac{2}{t}y = 4t, \tag{36}$$

so $p(t) = 2/t$ and $g(t) = 4t$. To solve equation (36), we first compute the integrating factor $\mu(t)$:

$$\mu(t) = \exp\left(\int \frac{2}{t}\,dt\right) = e^{2\ln|t|} = t^2.$$

On multiplying equation (36) by $\mu(t) = t^2$, we obtain

$$t^2y' + 2ty = (t^2y)' = 4t^3,$$

and therefore

$$t^2y = \int 4t^3\,dt = t^4 + c,$$

where c is an arbitrary constant. It follows that, for $t > 0$,

$$y = t^2 + \frac{c}{t^2} \tag{37}$$

is the general solution of equation (34). Integral curves of equation (34) for several values of c are shown in Figure 2.1.3.

To satisfy initial condition (35), set $t = 1$ and $y = 2$ in equation (37): $2 = 1 + c$, so $c = 1$; thus

$$y = t^2 + \frac{1}{t^2}, \quad t > 0 \tag{38}$$

is the solution of the initial value problem (24), (25). This solution is shown by the green curve in Figure 2.1.3. Note that it becomes unbounded and is asymptotic to the positive y-axis as $t \to 0$ from the right. This is the effect of the infinite discontinuity in the coefficient $p(t)$ at the origin. It is important to note that while the function $y = t^2 + 1/t^2$ for $t < 0$ is part of the general solution of equation (34), it is not part of the solution of this initial value problem.

This is the first example in which the solution fails to exist for some values of t. Again, this is due to the infinite discontinuity in $p(t)$ at $t = 0$, which restricts the solution to the interval $0 < t < \infty$.

Looking again at Figure 2.1.3, we see that some solutions (those for which $c > 0$) are asymptotic to the positive y-axis as $t \to 0$ from the right, while other solutions (for which $c < 0$) are asymptotic to the negative y-axis. If we generalize the initial condition (35) to

$$y(1) = y_0, \tag{39}$$

then $c = y_0 - 1$ and the solution (38) becomes

$$y = t^2 + \frac{y_0 - 1}{t^2}, \quad t > 0 \tag{40}$$

Note that when $y_0 = 1$, so $c = 0$, the solution is $y = t^2$, which remains bounded and differentiable even at $t = 0$. (This is the red curve in Figure 2.1.3.)

As in Example 3, this is another instance where there is a critical initial value, namely, $y_0 = 1$, that separates solutions that behave in one way from others that behave quite differently.

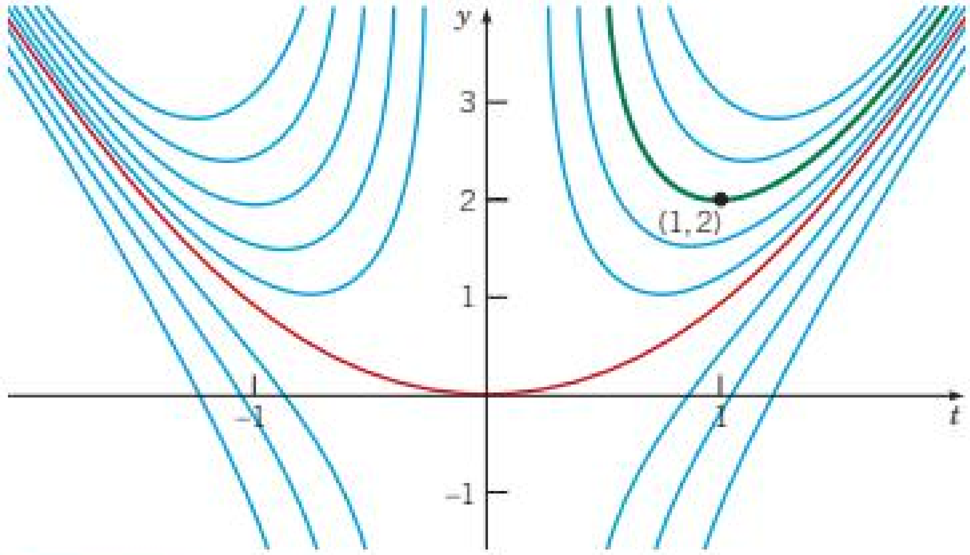

FIGURE 2.1.3 Integral curves of the differential equation $ty' + 2y = 4t^2$; the green curve is the particular solution with $y(1) = 2$. The red curve is the particular solution with $y(1) = 1$.

EXAMPLE 5

Solve the initial value problem

$$2y' + ty = 2, \tag{41}$$

$$y(0) = 1. \tag{42}$$

Solution:

To convert the differential equation (41) to the standard form (3), we must divide equation (41) by 2, obtaining

$$y' + \frac{t}{2}y = 1. \tag{43}$$

Thus $p(t) = t/2$, and the integrating factor is $\mu(t) = \exp(t^2/4)$. Then multiply equation (43) by $\mu(t)$, so that

$$e^{t^2/4}y' + \frac{t}{2}e^{t^2/4}y = e^{t^2/4}. \tag{44}$$

The left-hand side of equation (44) is the derivative of $e^{t^2/4}y$, so by integrating both sides of equation (44), we obtain

$$e^{t^2/4}y = \int e^{t^2/4}\,dt + c. \tag{45}$$

The integral on the right-hand side of equation (45) cannot be evaluated in terms of the usual elementary functions, so we leave the integral unevaluated. By choosing the lower limit of integration as the initial point $t = 0$, we can replace equation (45) by

$$e^{t^2/4}y = \int_0^t e^{s^2/4}\,ds + c, \tag{46}$$

where c is an arbitrary constant. It then follows that the general solution y of equation (41) is given by

$$y = e^{-t^2/4}\int_0^t e^{s^2/4}\,ds + ce^{-t^2/4}. \tag{47}$$

To determine the particular solution that satisfies the initial condition (42), set $t = 0$ and $y = 1$ in equation (47):

$$\begin{aligned} 1 &= e^0 \int_0^0 e^{-s^2/4}\,ds + ce^0 \\ &= 0 + c, \end{aligned}$$

so $c = 1$.

The main purpose of this example is to illustrate that sometimes the solution must be left in terms of an integral. This is usually at most a slight inconvenience, rather than a serious obstacle. For a given value of t, the integral in equation (47) is a definite integral and can be approximated to any desired degree of accuracy by using readily available numerical integrators. By repeating this process for many values of t and plotting the results, you can obtain a graph of a solution. Alternatively, you can use a numerical approximation method, such as those discussed in Chapter 8, that proceed directly from the differential equation and need no expression for the solution. Software packages such as Maple, Mathematica, MATLAB and Sage readily execute such procedures and produce graphs of solutions of differential equations.

Figure 2.1.4 displays graphs of the solution (47) for several values of c. The particular solution satisfying the initial condition $y(0) = 1$ is shown in black. From the figure it may be plausible to conjecture that all solutions approach a limit as $t \to \infty$. The limit can also be found analytically (see Problem 22).

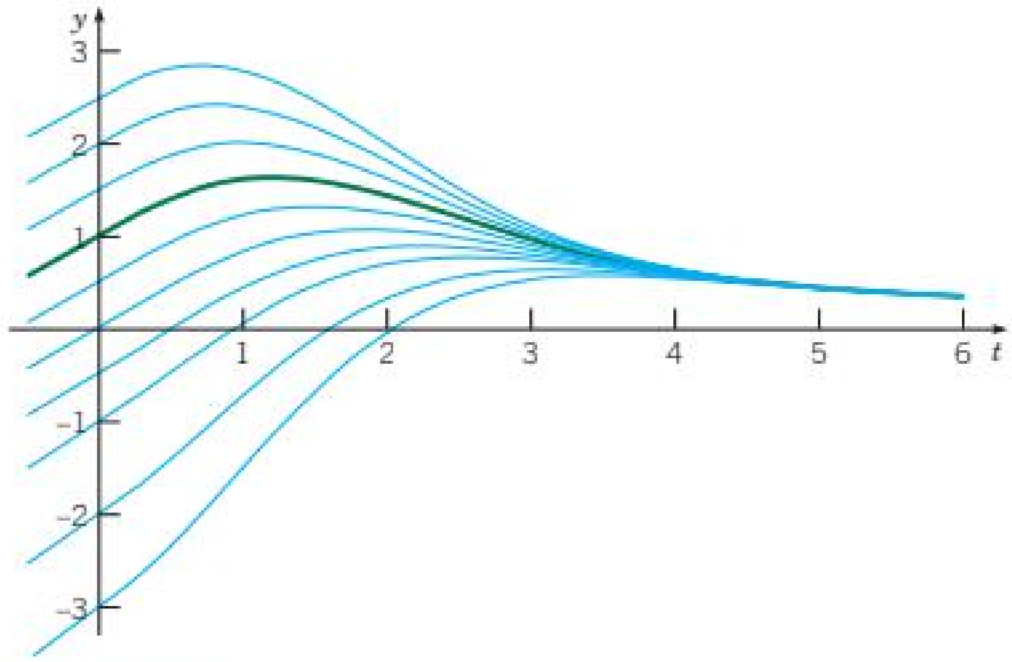

FIGURE 2.1.4 Integral curves of $2y' + ty = 2$; the green curve is the particular solution satisfying the initial condition $y(0) = 1$.

Problems

In each of Problems 1 through 8:

a. Draw a direction field for the given differential equation.

b. Based on an inspection of the direction field, describe how solutions behave for large t.

c. Find the general solution of the given differential equation, and use it to determine how solutions behave as $t \to \infty$.

1. $y' + 3y = t + e^{-2t}$

2. $y' - 2y = t^2 e^{2t}$

3. $y' + y = te^{-t} + 1$

4. $y' + \frac{1}{t}y = 3\cos(2t), \quad t > 0$

5. $y' - 2y = 3e^t$

6. $ty' - y = t^2 e^{-t}, \quad t > 0$

7. $y' + y = 5\sin(2t)$

8. $2y' + y = 3t^2$

In each of Problems 9 through 12, find the solution of the given initial value problem.

9. $y' - y = 2te^{2t}, \quad y(0) = 1$

10. $y' + 2y = te^{-2t}, \quad y(1) = 0$

11. $y' + \frac{2}{t}y = \frac{\cos t}{t^2}, \quad y(\pi) = 0, \quad t > 0$

12. $ty' + (t+1)y = t, \quad y(\ln 2) = 1, \quad t > 0$

In each of Problems 13 and 14:

a. Draw a direction field for the given differential equation. How do solutions appear to behave as t becomes large? Does the behavior depend on the choice of the initial value a? Let a_0 be the value of a for which the transition from one type of behavior to another occurs. Estimate the value of a_0.

b. Solve the initial value problem and find the critical value a_0 exactly.

c. Describe the behavior of the solution corresponding to the initial value a_0.

13. $y' - \frac{1}{2}y = 2\cos t, \quad y(0) = a$

14. $3y' - 2y = e^{-\pi t/2}, \quad y(0) = a$

In each of Problems 15 and 16:

G **a.** Draw a direction field for the given differential equation. How do solutions appear to behave as $t \to 0$? Does the behavior depend on the choice of the initial value a? Let a_0 be the critical value of a, that is, the initial value such that the solutions for $a < a_0$ and the solutions for $a > a_0$ have different behaviors as $t \to \infty$. Estimate the value of a_0.

b. Solve the initial value problem and find the critical value a_0 exactly.

c. Describe the behavior of the solution corresponding to the initial value a_0.

15. $ty' + (t+1)y = 2te^{-t}, \quad y(1) = a, \quad t > 0$

16. $(\sin t)y' + (\cos t)y = e^t, \quad y(1) = a, \quad 0 < t < \pi$

G **17.** Consider the initial value problem

$$y' + \frac{1}{2}y = 2\cos t, \quad y(0) = -1.$$

Find the coordinates of the first local maximum point of the solution for $t > 0$.

N **18.** Consider the initial value problem

$$y' + \frac{2}{3}y = 1 - \frac{1}{2}t, \quad y(0) = y_0.$$

Find the value of y_0 for which the solution touches, but does not cross, the t-axis.

19. Consider the initial value problem

$$y' + \frac{1}{4}y = 3 + 2\cos(2t), \quad y(0) = 0.$$

a. Find the solution of this initial value problem and describe its behavior for large t.

N **b.** Determine the value of t for which the solution first intersects the line $y = 12$.

20. Find the value of y_0 for which the solution of the initial value problem

$$y' - y = 1 + 3\sin t, \quad y(0) = y_0$$

remains finite as $t \to \infty$.

21. Consider the initial value problem

$$y' - \frac{3}{2}y = 3t + 2e^t, \quad y(0) = y_0.$$

Find the value of y_0 that separates solutions that grow positively as $t \to \infty$ from those that grow negatively. How does the solution that corresponds to this critical value of y_0 behave as $t \to \infty$?

22. Show that all solutions of $2y' + ty = 2$ [equation (41) of the text] approach a limit as $t \to \infty$, and find the limiting value.
Hint: Consider the general solution, equation (47). Show that the first term in the solution (47) is indeterminate with form $0 \cdot \infty$. Then, use l'Hôpital's rule to compute the limit as $t \to \infty$.

23. Show that if a and λ are positive constants, and b is any real number, then every solution of the equation

$$y' + ay = be^{-\lambda t}$$

has the property that $y \to 0$ as $t \to \infty$.
Hint: Consider the cases $a = \lambda$ and $a \neq \lambda$ separately.

In each of Problems 24 through 27, construct a first-order linear differential equation whose solutions have the required behavior as $t \to \infty$. Then solve your equation and confirm that the solutions do indeed have the specified property.

24. All solutions have the limit 3 as $t \to \infty$.

25. All solutions are asymptotic to the line $y = 3 - t$ as $t \to \infty$.

26. All solutions are asymptotic to the line $y = 2t - 5$ as $t \to \infty$.

27. All solutions approach the curve $y = 4 - t^2$ as $t \to \infty$.

28. **Variation of Parameters.** Consider the following method of solving the general linear equation of first order:

$$y' + p(t)y = g(t). \tag{48}$$

a. If $g(t) = 0$ for all t, show that the solution is

$$y = A\exp\left(-\int p(t)\,dt\right), \tag{49}$$

where A is a constant.

b. If $g(t)$ is not everywhere zero, assume that the solution of equation (48) is of the form

$$y = A(t)\exp\left(-\int p(t)\,dt\right), \tag{50}$$

where A is now a function of t. By substituting for y in the given differential equation, show that $A(t)$ must satisfy the condition

$$A'(t) = g(t)\exp\left(\int p(t)\,dt\right). \tag{51}$$

c. Find $A(t)$ from equation (51). Then substitute for $A(t)$ in equation (50) and determine y. Verify that the solution obtained in this manner agrees with that of equation (33) in the text. This technique is known as the method of **variation of parameters**; it is discussed in detail in Section 3.6 in connection with second-order linear equations.

In each of Problems 29 and 30, use the method of Problem 28 to solve the given differential equation.

29. $y' - 2y = t^2e^{2t}$

30. $y' + \frac{1}{t}y = \cos(2t), \quad t > 0$

2.2 Separable Differential Equations

In Section 1.2 we used a process of direct integration to solve first-order linear differential equations of the form

$$\frac{dy}{dt} = ay + b, \tag{1}$$

where a and b are constants. We will now show that this process is actually applicable to a much larger class of nonlinear differential equations.

We will use x, rather than t, to denote the independent variable in this section for two reasons. In the first place, different letters are frequently used for the variables in a differential equation, and you should not become too accustomed to using a single pair. In particular, x often occurs as the independent variable. Further, we want to reserve t for another purpose later in the section.

The general first-order differential equation is

$$\frac{dy}{dx} = f(x, y). \tag{2}$$

Linear differential equations were considered in the preceding section, but if equation (2) is nonlinear, then there is no universally applicable method for solving the equation. Here, we consider a subclass of first-order equations that can be solved by direct integration.

To identify this class of equations, we first rewrite equation (2) in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0. \tag{3}$$

It is always possible to do this by setting $M(x, y) = -f(x, y)$ and $N(x, y) = 1$, but there may be other ways as well. When M is a function of x only and N is a function of y only, then equation (3) becomes

$$M(x) + N(y)\frac{dy}{dx} = 0. \tag{4}$$

Such an equation is said to be **separable**, because if it is written in the **differential form**

$$M(x)\,dx + N(y)\,dy = 0, \tag{5}$$

then, if you wish, terms involving each variable may be placed on opposite sides of the equation. The differential form (5) is also more symmetric and tends to suppress the distinction between independent and dependent variables.

A separable equation can be solved by integrating the functions M and N. We illustrate the process by an example and then discuss it in general for equation (4).

EXAMPLE 1

Show that the equation

$$\frac{dy}{dx} = \frac{x^2}{1 - y^2} \tag{6}$$

is separable, and then find an equation for its integral curves.

Solution:

If we write equation (6) as

$$-x^2 + (1 - y^2)\frac{dy}{dx} = 0, \tag{7}$$

then it has the form (4) and is therefore separable. Recall from calculus that if y is a function of x, then by the chain rule,

$$\frac{d}{dx}f(y) = \frac{d}{dy}f(y)\,\frac{dy}{dx} = f'(y)\frac{dy}{dx}.$$

For example, if $f(y) = y - y^3/3$, then

$$\frac{d}{dx}\left(y - \frac{y^3}{3}\right) = (1 - y^2)\frac{dy}{dx}.$$

Thus the second term in equation (7) is the derivative with respect to x of $y - y^3/3$, and the first term is the derivative of $-x^3/3$. Thus equation (7) can be written as

$$\frac{d}{dx}\left(-\frac{x^3}{3}\right) + \frac{d}{dx}\left(y - \frac{y^3}{3}\right) = 0,$$

or

$$\frac{d}{dx}\left(-\frac{x^3}{3} + y - \frac{y^3}{3}\right) = 0.$$

Therefore, by integrating (and multiplying the result by 3), we obtain

$$-x^3 + 3y - y^3 = c, \tag{8}$$

where c is an arbitrary constant.

Equation (8) is an equation for the integral curves of equation (6). A direction field and several integral curves are shown in Figure 2.2.1. Any differentiable function $y = \phi(x)$ that satisfies equation (8) is a solution of equation (6). An equation of the integral curve passing through a particular point (x_0, y_0) can be found by substituting x_0 and y_0 for x and y, respectively, in equation (8) and determining the corresponding value of c.

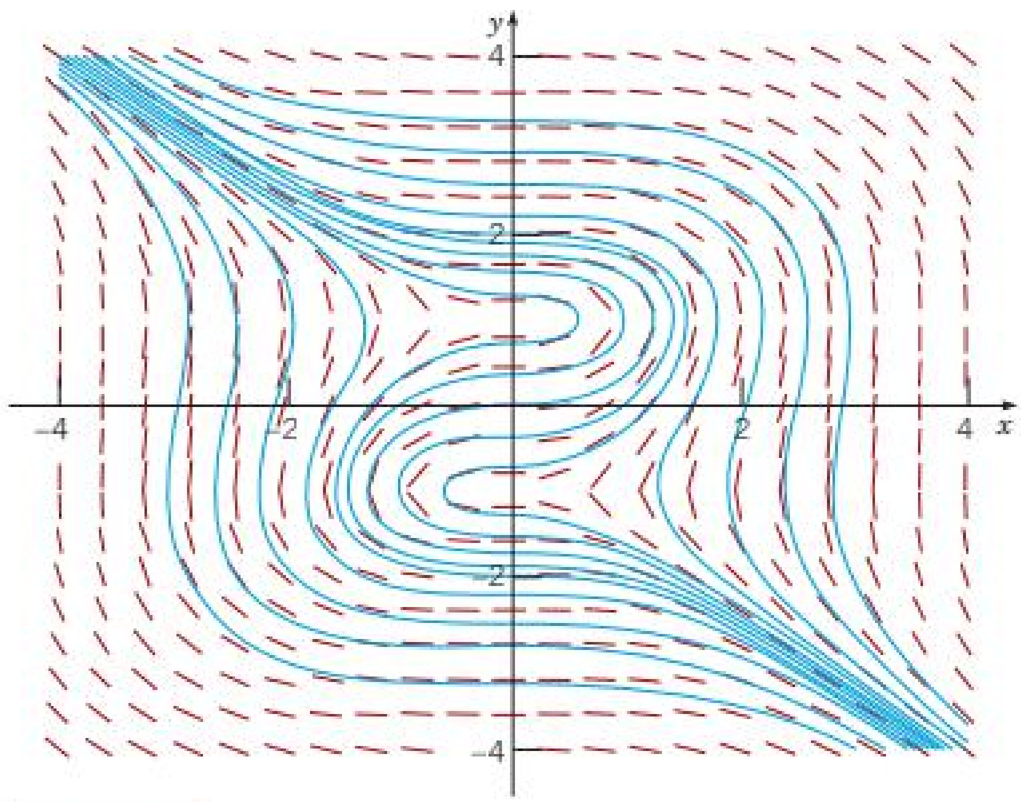

FIGURE 2.2.1 Direction field and integral curves of $y' = x^2/(1 - y^2)$.

Essentially the same procedure can be followed for any separable equation. Returning to equation (4), let H_1 and H_2 be any antiderivatives of M and N, respectively. Thus

$$H_1'(x) = M(x), \quad H_2'(y) = N(y), \tag{9}$$

and equation (4) becomes

$$H_1'(x) + H_2'(y)\frac{dy}{dx} = 0. \tag{10}$$

If y is regarded as a function of x, then according to the chain rule,

$$H_2'(y)\frac{dy}{dx} = \frac{d}{dy}H_2(y)\frac{dy}{dx} = \frac{d}{dx}H_2(y). \tag{11}$$

Consequently, we can write equation (10) as

$$\frac{d}{dx}\big(H_1(x) + H_2(y)\big) = 0. \tag{12}$$

By integrating equation (12) with respect to x, we obtain

$$H_1(x) + H_2(y) = c, \tag{13}$$

where c is an arbitrary constant. Any differentiable function $y = \phi(x)$ that satisfies equation (13) is a solution of equation (4); in other words, equation (13) defines the solution implicitly rather than explicitly. In practice, equation (13) is usually obtained from equation (5) by integrating the first term with respect to x and the second term with respect to y. The justification for this is the argument that we have just given.

The differential equation (4), together with an initial condition

$$y(x_0) = y_0, \tag{14}$$

forms an initial value problem. To solve this initial value problem, we must determine the appropriate value for the constant c in equation (13). We do this by setting $x = x_0$ and $y = y_0$ in equation (13) with the result that

$$c = H_1(x_0) + H_2(y_0). \tag{15}$$

Substituting this value of c in equation (13) and noting that

$$H_1(x) - H_1(x_0) = \int_{x_0}^{x} M(s)ds, \quad H_2(y) - H_2(y_0) = \int_{y_0}^{y} N(s)ds,$$

we obtain

$$\int_{x_0}^{x} M(s)ds + \int_{y_0}^{y} N(s)ds = 0. \tag{16}$$

Equation (16) is an implicit representation of the solution of the differential equation (4) that also satisfies the initial condition (14). Bear in mind that to determine an explicit formula for the solution, you need to solve equation (16) for y as a function of x. Unfortunately, it is often impossible to do this analytically; in such cases you can resort to numerical methods to find approximate values of y for given values of x.

EXAMPLE 2

Solve the initial value problem

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1, \tag{17}$$

and determine the interval in which the solution exists.

Solution:

The differential equation can be written as

$$2(y-1)dy = (3x^2 + 4x + 2)dx.$$

Integrating the left-hand side with respect to y and the right-hand side with respect to x gives

$$y^2 - 2y = x^3 + 2x^2 + 2x + c, \tag{18}$$

where c is an arbitrary constant. To determine the solution satisfying the prescribed initial condition, we substitute $x = 0$ and $y = -1$ in equation (18), obtaining $c = 3$. Hence the solution of the initial value problem is given implicitly by

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3. \tag{19}$$

To obtain the solution explicitly, we must solve equation (19) for y in terms of x. That is a simple matter in this case, since equation (19) is quadratic in y, and we obtain

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}. \tag{20}$$

Equation (20) gives two solutions of the differential equation, only one of which, however, satisfies the given initial condition. This is the solution corresponding to the minus sign in equation (20), so we finally obtain

$$y = \phi(x) = 1 - \sqrt{x^3 + 2x^2 + 2x + 4} \tag{21}$$

as the solution of the initial value problem (15). Note that if we choose the plus sign by mistake in equation (20), then we obtain the solution of the same differential equation that satisfies the initial condition $y(0) = 3$. Finally, to determine the interval in which the solution (21) is valid, we must find the interval in which the quantity under the radical is positive. The only real zero of this expression is $x = -2$, so the desired interval is $x > -2$. Some integral curves of the differential equation are shown in Figure 2.2.2. The green curve passes through the point $(0, -1)$ and thus is the solution of the initial value problem (15). Observe that the boundary of the interval of validity of the solution (21) is determined by the point $(-2, 1)$ at which the tangent line is vertical.

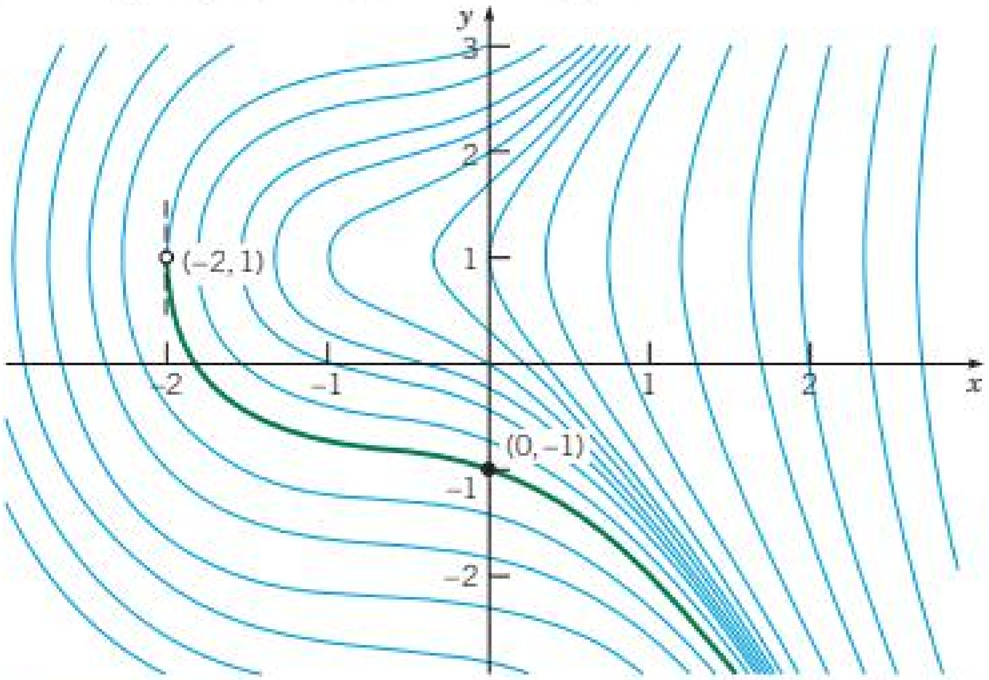

FIGURE 2.2.2 Integral curves of $y' = (3x^2 + 4x + 2)/2(y - 1)$; the solution satisfying $y(0) = -1$ is shown in green and is valid for $x > -2$.

EXAMPLE 3

Solve the separable differential equation

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3} \tag{22}$$

and draw graphs of several integral curves. Also find the solution passing through the point $(0, 1)$ and determine its interval of validity.

Solution:

Rewriting equation (22) as

$$(4 + y^3)dy = (4x - x^3)dx,$$

integrating each side, multiplying by 4, and rearranging the terms, we obtain

$$y^4 + 16y + x^4 - 8x^2 = c, \tag{23}$$

where c is an arbitrary constant. Any differentiable function $y = \phi(x)$ that satisfies equation (23) is a solution of the differential equation (22). Graphs of equation (23) for several values of c are shown in Figure 2.2.3.

To find the particular solution passing through $(0, 1)$, we set $x = 0$ and $y = 1$ in equation (23) with the result that $c = 17$. Thus the solution in question is given implicitly by

$$y^4 + 16y + x^4 - 8x^2 = 17. \tag{24}$$

It is shown by the green curve in Figure 2.2.3. The interval of validity of this solution extends on either side of the initial point as long as the function remains differentiable. From the figure we see that the interval ends when we reach points where the tangent line is vertical. It follows from the differential equation (22) that these are points where $4 + y^3 = 0$, or $y = (-4)^{1/3} \cong -1.5874$. From equation (24) the corresponding values of x are $x \cong \pm 3.3488$. These points are marked on the graph in Figure 2.2.3.

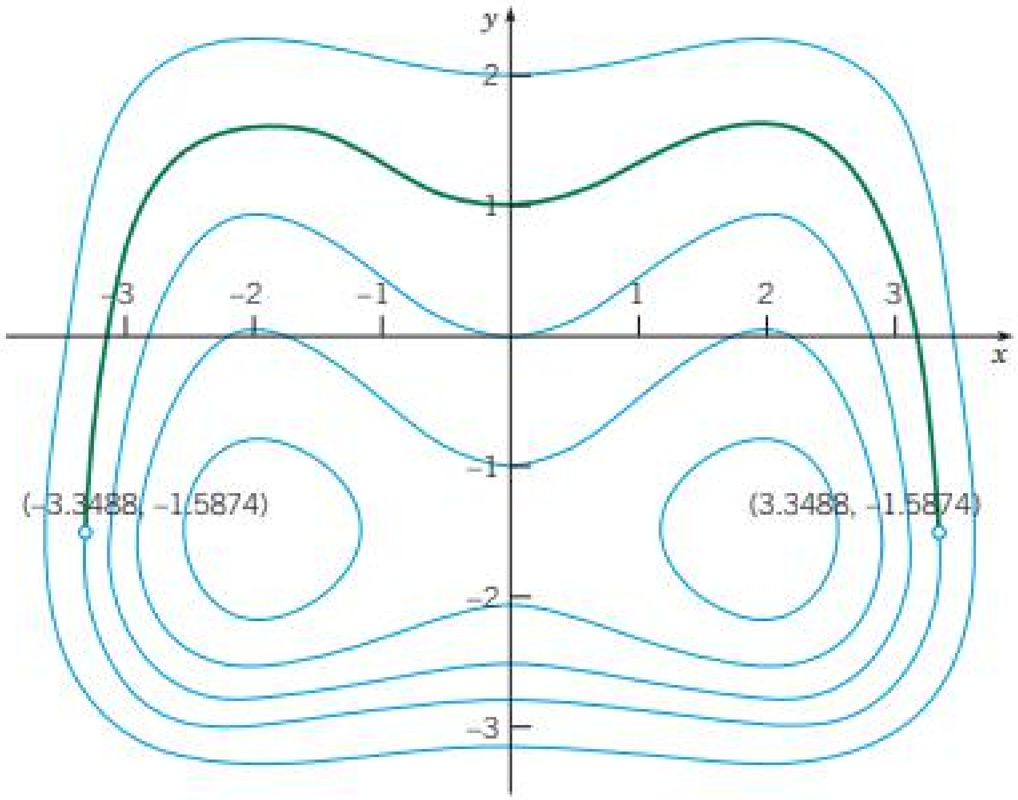

FIGURE 2.2.3 Integral curves of $y' = (4x - x^3)/(4 + y^3)$. The solution passing through $(0, 1)$ is shown by the green curve.

Note 1: Sometimes a differential equation of the form (2):

$$\frac{dy}{dx} = f(x, y)$$

has a constant solution $y = y_0$. Such a solution is usually easy to find because if $f(x, y_0) = 0$ for some value y_0 and for all x, then the constant function $y = y_0$ is a solution of the differential equation (2). For example, the equation

$$\frac{dy}{dx} = \frac{(y-3)\cos x}{1 + 2y^2} \tag{25}$$

has the constant solution $y = 3$. Other solutions of this equation can be found by separating the variables and integrating.

Note 2: The investigation of a first-order nonlinear differential equation can sometimes be facilitated by regarding both x and y as functions of a third variable t. Then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \tag{26}$$

If the differential equation is

$$\frac{dy}{dx} = \frac{F(x, y)}{G(x, y)}, \tag{27}$$

then, by comparing numerators and denominators in equations (26) and (27), we obtain the system

$$\frac{dx}{dt} = G(x, y), \quad \frac{dy}{dt} = F(x, y). \tag{28}$$

At first sight it may seem unlikely that a problem will be simplified by replacing a single equation by a pair of equations, but in fact, the system (28) may well be more amenable to investigation than the single equation (27). Chapter 9 is devoted to nonlinear systems of the form (28).

Note 3: In Example 2 it was not difficult to solve explicitly for y as a function of x. However, this situation is exceptional, and often it will be better to leave the solution in implicit form, as in Examples 1 and 3. Thus, in the problems below and in other sections where nonlinear equations appear, the words "solve the following differential equation" mean to find the solution explicitly if it is convenient to do so, but otherwise to find an equation defining the solution implicitly.

Problems

In each of Problems 1 through 8, solve the given differential equation.

1. $y' = \dfrac{x^2}{y}$
2. $y' + y^2 \sin x = 0$
3. $y' = \cos^2(x)\cos^2(2y)$
4. $xy' = (1 - y^2)^{1/2}$
5. $\dfrac{dy}{dx} = \dfrac{x - e^{-x}}{y + e^y}$
6. $\dfrac{dy}{dx} = \dfrac{x^2}{1 + y^2}$
7. $\dfrac{dy}{dx} = \dfrac{y}{x}$
8. $\dfrac{dy}{dx} = \dfrac{-x}{y}$

In each of Problems 9 through 16:

a. Find the solution of the given initial value problem in explicit form.

G b. Plot the graph of the solution.

c. Determine (at least approximately) the interval in which the solution is defined.

9. $y' = (1 - 2x)y^2, \quad y(0) = -1/6$
10. $y' = (1 - 2x)/y, \quad y(1) = -2$
11. $x\,dx + ye^{-x}dy = 0, \quad y(0) = 1$
12. $dr/d\theta = r^2/\theta, \quad r(1) = 2$
13. $y' = xy^3(1 + x^2)^{-1/2}, \quad y(0) = 1$
14. $y' = 2x/(1 + 2y), \quad y(2) = 0$
15. $y' = (3x^2 - e^x)/(2y - 5), \quad y(0) = 1$
16. $\sin(2x)\,dx + \cos(3y)\,dy = 0, \quad y(\pi/2) = \pi/3$

Some of the results requested in Problems 17 through 22 can be obtained either by solving the given equations analytically or by plotting numerically generated approximations to the solutions. Try to form an opinion about the advantages and disadvantages of each approach.

G 17. Solve the initial value problem

$$y' = \frac{1 + 3x^2}{3y^2 - 6y}, \quad y(0) = 1$$

and determine the interval in which the solution is valid.
Hint: To find the interval of definition, look for points where the integral curve has a vertical tangent.

G 18. Solve the initial value problem

$$y' = \frac{3x^2}{3y^2 - 4}, \quad y(1) = 0$$

and determine the interval in which the solution is valid.
Hint: To find the interval of definition, look for points where the integral curve has a vertical tangent.

G 19. Solve the initial value problem

$$y' = 2y^2 + xy^2, \quad y(0) = 1$$

and determine where the solution attains its minimum value.

G 20. Solve the initial value problem

$$y' = \frac{2 - e^x}{3 + 2y}, \quad y(0) = 0$$

and determine where the solution attains its maximum value.

G 21. Consider the initial value problem

$$y' = \frac{ty(4 - y)}{3}, \quad y(0) = y_0.$$

a. Determine how the behavior of the solution as t increases depends on the initial value y_0.

b. Suppose that $y_0 = 0.5$. Find the time T at which the solution first reaches the value 3.98.

G 22. Consider the initial value problem

$$y' = \frac{ty(4 - y)}{1 + t}, \quad y(0) = y_0 > 0.$$

a. Determine how the solution behaves as $t \to \infty$.

b. If $y_0 = 2$, find the time T at which the solution first reaches the value 3.99.

c. Find the range of initial values for which the solution lies in the interval $3.99 < y < 4.01$ by the time $t = 2$.

23. Solve the equation

$$\frac{dy}{dx} = \frac{ay + b}{cy + d},$$

where a, b, c, and d are constants.

24. Use separation of variables to solve the differential equation

$$\frac{dQ}{dt} = r(a + bQ), \quad Q(0) = Q_0,$$

where a, b, r, and Q_0 are constants. Determine how the solution behaves as $t \to \infty$

Homogeneous Equations. If the right-hand side of the equation $dy/dx = f(x, y)$ can be expressed as a function of the ratio y/x only, then the equation is said to be homogeneous.[1] Such equations can always be transformed into separable equations by a change of the dependent variable. Problem 25 illustrates how to solve first-order homogeneous equations.

[1]The word "homogeneous" has different meanings in different mathematical contexts. The homogeneous equations considered here have nothing to do with the homogeneous equations that will occur in Chapter 3 and elsewhere.

N **25.** Consider the equation

$$\frac{dy}{dx} = \frac{y - 4x}{x - y}. \quad (29)$$

a. Show that equation (29) can be rewritten as

$$\frac{dy}{dx} = \frac{(y/x) - 4}{1 - (y/x)}; \quad (30)$$

thus equation (29) is homogeneous.

b. Introduce a new dependent variable v so that $v = y/x$, or $y = xv(x)$. Express dy/dx in terms of x, v, and dv/dx.

c. Replace y and dy/dx in equation (30) by the expressions from part b that involve v and dv/dx. Show that the resulting differential equation is

$$v + x\frac{dv}{dx} = \frac{v - 4}{1 - v},$$

or

$$x\frac{dv}{dx} = \frac{v^2 - 4}{1 - v}. \quad (31)$$

Observe that equation (31) is separable.

d. Solve equation (31), obtaining v implicitly in terms of x.

e. Find the solution of equation (29) by replacing v by y/x in the solution in part d.

f. Draw a direction field and some integral curves for equation (29). Recall that the right-hand side of equation (29) actually depends only on the ratio y/x. This means that integral curves have the same slope at all points on any given straight line through the origin, although the slope changes from one line to another. Therefore, the direction field and the integral curves are symmetric with respect to the origin. Is this symmetry property evident from your plot?

The method outlined in Problem 25 can be used for any homogeneous equation. That is, the substitution $y = xv(x)$ transforms a homogeneous equation into a separable equation. The latter equation can be solved by direct integration, and then replacing v by y/x gives the solution to the original equation. In each of Problems 26 through 31:

a. Show that the given equation is homogeneous.

b. Solve the differential equation.

G **c.** Draw a direction field and some integral curves. Are they symmetric with respect to the origin?

26. $\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}$

27. $\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}$

28. $\frac{dy}{dx} = \frac{4y - 3x}{2x - y}$

29. $\frac{dy}{dx} = -\frac{4x + 3y}{2x + y}$

30. $\frac{dy}{dx} = \frac{x^2 - 3y^2}{2xy}$

31. $\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy}$

2.3 Modeling with First-Order Differential Equations

Differential equations are of interest to nonmathematicians primarily because of the possibility of using them to investigate a wide variety of problems in the physical, biological, and social sciences. One reason for this is that mathematical models and their solutions lead to equations relating the variables and parameters in the problem. These equations often enable you to make predictions about how the natural process will behave in various circumstances. It is often easy to vary parameters in the mathematical model over wide ranges, whereas this may be very time-consuming or expensive, if not impossible, in an experimental setting. Nevertheless, mathematical modeling and experiment or observation are both critically important and have somewhat complementary roles in scientific investigations. Mathematical models are validated by comparison of their predictions with experimental results. On the other hand, mathematical analyses may suggest the most promising directions to explore experimentally, and they may indicate fairly precisely what experimental data will be most helpful.

In Sections 1.1 and 1.2 we formulated and investigated a few simple mathematical models. We begin by recapitulating and expanding on some of the conclusions reached in those sections. Regardless of the specific field of application, there are three identifiable steps that are always present in the process of mathematical modeling.

Step 1: Construction of the Model. In this step the physical situation is translated into mathematical terms, often using the steps listed at the end of Section 1.1. Perhaps most critical at this stage is to state clearly the physical principle(s) that are believed to govern the process. For example, it has been observed that in some circumstances heat passes from a warmer to a cooler body at a rate proportional to the temperature difference, that objects move about in accordance with Newton's laws of motion, and that isolated insect populations grow at a rate proportional to the current population. Each of these statements involves a rate of change (derivative) and consequently, when expressed mathematically, leads to a differential equation. The differential equation is a mathematical model of the process.

It is important to realize that the mathematical equations are almost always only an approximate description of the actual process. For example, bodies moving at speeds comparable to the speed of light are not governed by Newton's laws, insect populations do not grow indefinitely as stated because of eventual lack of food or space, and heat transfer is affected by factors other than the temperature difference. Thus you should always be aware of the limitations of the model so that you will use it only when it is reasonable to believe that it is accurate. Alternatively, you can adopt the point of view that the mathematical equations exactly describe the operation of a simplified physical model, which has been constructed (or conceived of) so as to embody the most important features of the actual process. Sometimes, the process of mathematical modeling involves the conceptual replacement of a discrete process by a continuous one. For instance, the number of members in an insect population changes by discrete amounts; however, if the population is large, it seems reasonable to consider it as a continuous variable and even to speak of its derivative.

Step 2: Analysis of the Model. Once the problem has been formulated mathematically, you are often faced with the problem of solving one or more differential equations or, failing that, of finding out as much as possible about the properties of the solution. It may happen that this mathematical problem is quite difficult, and if so, further approximations may be indicated at this stage to make the problem mathematically tractable. For example, a nonlinear equation may be approximated by a linear one, or a slowly varying coefficient may be replaced by a constant. Naturally, any such approximations must also be examined from the physical point of view to make sure that the simplified mathematical problem still reflects the essential features of the physical process under investigation. At the same time, an intimate knowledge of the physics of the problem may suggest reasonable mathematical approximations that will make the mathematical problem more amenable to analysis. This interplay of understanding of physical phenomena and knowledge of mathematical techniques and their limitations is characteristic of applied mathematics at its best, and it is indispensable in successfully constructing useful mathematical models of intricate physical processes.

Step 3: Comparison with Experiment or Observation. Finally, having obtained the solution (or at least some information about it), you must interpret this information in the context in which the problem arose. In particular, you should always check that the mathematical solution appears physically reasonable. If possible, calculate the values of the solution at selected points and compare them with experimentally observed values. Or ask whether the behavior of the solution after a long time is consistent with observations. Or examine the solutions corresponding to certain special values of parameters in the problem. Of course, the fact that the mathematical solution appears to be reasonable does not guarantee that it is correct. However, if the predictions of the mathematical model are seriously inconsistent with observations of the physical system it purports to describe, this suggests that errors have been made in solving the mathematical problem, that the mathematical model itself needs refinement, or that observations must be made with greater care.

The examples in this section are typical of applications in which first-order differential equations arise.

EXAMPLE 1 | Mixing

At time $t = 0$ a tank contains Q_0 lb of salt dissolved in 100 gal of water; see Figure 2.3.1. Assume that water containing $\frac{1}{4}$ lb of salt per gallon is entering the tank at a rate of r gal/min and that the well-stirred mixture is draining from the tank at the same rate. Set up the initial value problem that describes this flow process. Find the amount of salt $Q(t)$ in the tank at any time, and also find the limiting amount Q_L that is present after a very long time. If $r = 3$ and $Q_0 = 2Q_L$, find the time T after which the salt level is within 2% of Q_L. Also find the flow rate that is required if the value of T is not to exceed 45 min.

Solution:

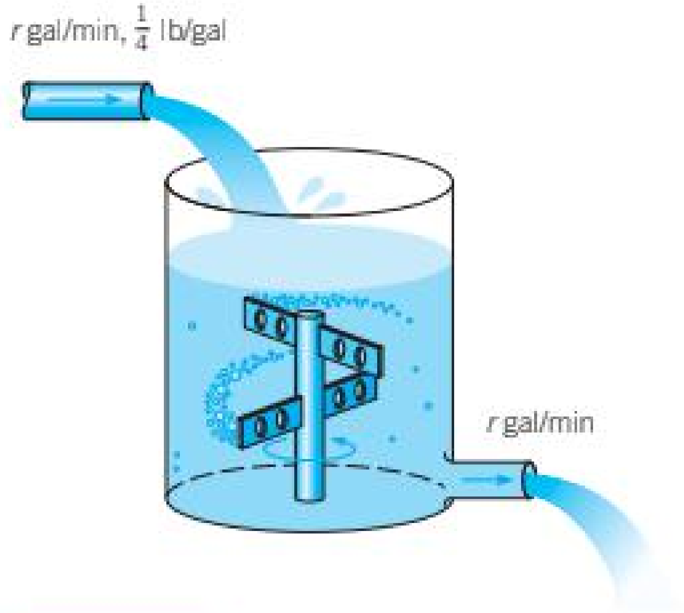

FIGURE 2.3.1 The water tank in Example 1.

We assume that salt is neither created nor destroyed in the tank. Therefore, variations in the amount of salt are due solely to the flows in and out of the tank. More precisely, the rate of change of salt in the tank, dQ/dt, is equal to the rate at which salt is flowing in minus the rate at which it is flowing out. In symbols,

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}. \tag{1}$$

The rate at which salt enters the tank is the concentration $\frac{1}{4}$ lb/gal times the flow rate r gal/min, or $r/4$ lb/min. To find the rate at which salt leaves the tank, we need to multiply the concentration of salt in the tank by the rate of outflow, r gal/min. Since the rates of flow in and out are equal, the volume of water in the tank remains constant at 100 gal, and since the mixture is "well-stirred," the concentration throughout the tank is the same, namely, $Q(t)/100$ lb/gal. Therefore, the rate at which salt leaves the tank is $rQ(t)/100$ lb/min. Thus the differential equation governing this process is

$$\frac{dQ}{dt} = \frac{r}{4} - \frac{rQ}{100}. \tag{2}$$

The initial condition is

$$Q(0) = Q_0. \tag{3}$$

Upon thinking about the problem physically, we might anticipate that eventually the mixture originally in the tank will be essentially replaced by the mixture flowing in, whose concentration is $\frac{1}{4}$ lb/gal. Consequently, we might expect that ultimately the amount of salt in the tank would be very close to 25 lb. We can also find the limiting amount $Q_L = 25$ by setting dQ/dt equal to zero in equation (2) and solving the resulting algebraic equation for Q.

To solve the initial value problem (2), (3) analytically, note that equation (2) is linear. (It is also separable, see Problem 24 in Section 2.2.) Rewriting the differential equation (2) in the standard form for a linear differential equation, we have

$$\frac{dQ}{dt} + \frac{rQ}{100} = \frac{r}{4}. \tag{4}$$

Thus the integrating factor is $e^{rt/100}$ and the general solution is

$$Q(t) = 25 + ce^{-rt/100}, \tag{5}$$

where c is an arbitrary constant. To satisfy the initial condition (3), we must choose $c = Q_0 - 25$. Therefore, the solution of the initial value problem (2), (3) is

$$Q(t) = 25 + (Q_0 - 25)e^{-rt/100}, \tag{6}$$

or

$$Q(t) = 25(1 - e^{-rt/100}) + Q_0 e^{-rt/100}. \tag{7}$$

From either form of the solution, (6) or (7), you can see that $Q(t) \to 25$ (lb) as $t \to \infty$, so the limiting value Q_L is 25, confirming our physical intuition.