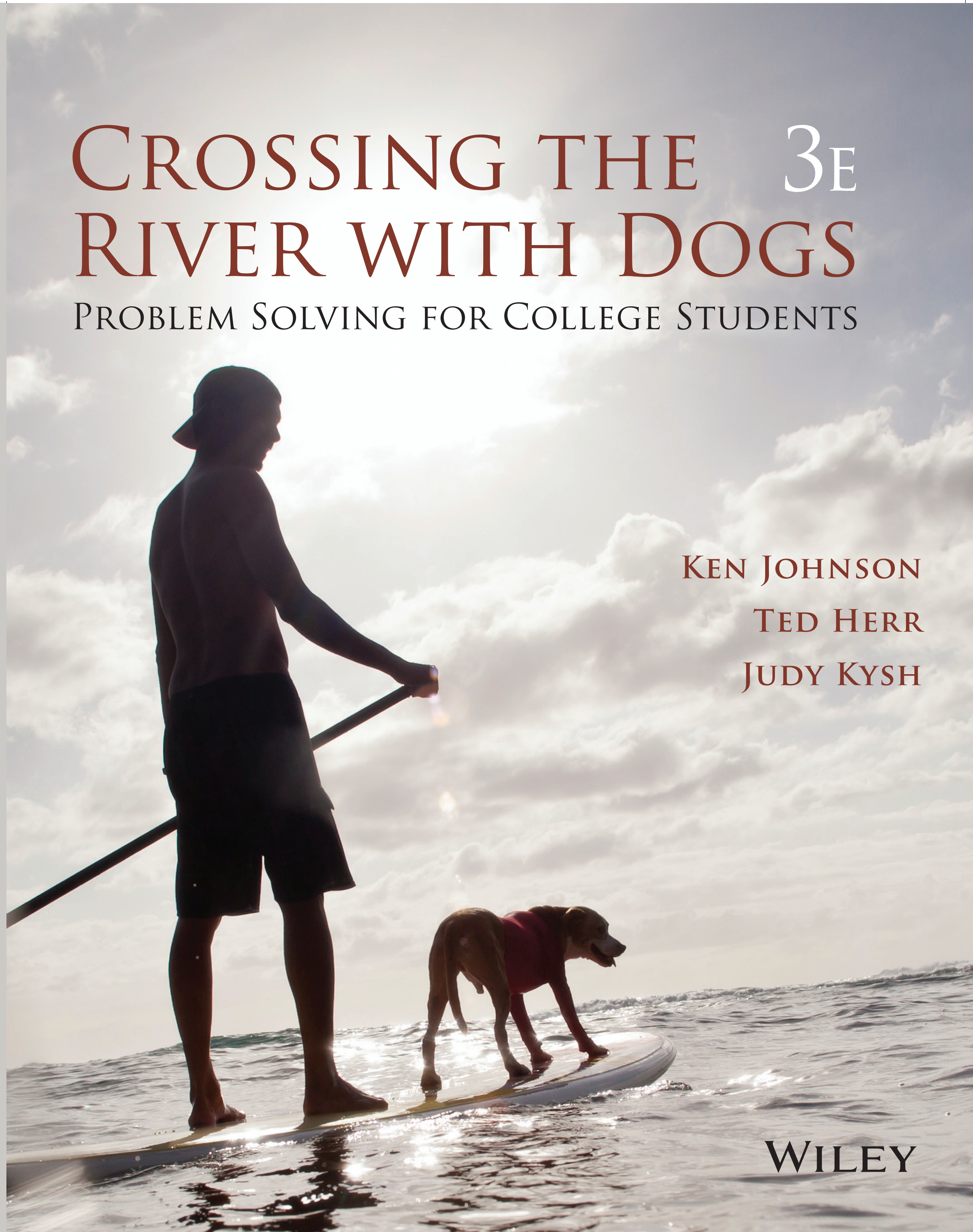
Crossing the River with Dogs 3E

Problem Solving for College Students

Ken Johnson
Ted Herr
Judy Kysh



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Cover Design: Wiley
Cover Image: © Noel Hendrickson/Getty Images

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ISBN 978-1-119-27509-1
90000
9 781119 275091

Crossing the River with Dogs

Problem Solving for College Students

Third Edition

Ken Johnson
SIERRA COLLEGE

Ted Herr
ROSEVILLE HIGH SCHOOL

Judy Kysh
SAN FRANCISCO STATE UNIVERSITY

WILEY

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This book was set in 10.5/14 StempelSchneidler by SPi Global and printed and bound by Strategic Content Imaging.

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ISBN: 978-1-119-27509-1 (PBK)
ISBN: 978-1-119-29869-4 (EVALC)

Library of Congress Cataloging in Publication Data:

Names: Johnson, Ken, 1956- author. | Herr, Ted, 1955- author. | Kysh, Judy, 1940- author.
Title: Crossing the river with dogs : problem solving for college students / by Ken Johnson, Ted Herr, Judy Kysh.
Description: Third edition. | Hoboken, NJ : John Wiley & Sons, 2018. | Includes bibliographical references and index. |
Identifiers: LCCN 2018000525 (print) | LCCN 2018004092 (ebook) | ISBN 9781119298755 (pdf) | ISBN 9781119298731 (epub) | ISBN 9781119275091 (pbk.)
Subjects: LCSH: Problem solving. | Mathematics.
Classification: LCC QA63 (ebook) | LCC QA63 .J64 2018 (print) | DDC 510—dc23 LC record available at https://lccn.loc.gov/2018000525

The inside back cover will contain printing identification and country of origin if omitted from this page. In addition, if the ISBN on the back cover differs from the ISBN on this page, the one on the back cover is correct.

To Janie, Allyson, and Armand

Contents

Preface

What Is Problem Solving?

Problem solving has been defined as knowing what to do when you don't know what to do. As a core human activity, it covers many daily functions such as going to the store, buying ingredients, and cooking a nutritionally sound meal. But beyond just grocery shopping, problem solving is used in every profession from the arts to computer science to politics. Having the skill to solve problems is essential in every work place, as well as having the skills to work collaboratively and communicate your reasoning and solutions.

Problem solving involves struggle: grappling with something that may not, at first, make sense, using what you know, figuring out what you don't know, and trying something to work toward a solution. Problem solving involves failure! Sometimes we have to try out ideas that don't work, learn from the results and decide to try something else. An article in the Harvard Business Review states it: "A tolerance for 'getting it wrong the first time' can be the better strategy, as long as people iterate rapidly and frequently, and learn quickly from their failures.*

Unfortunately many students think that math is something you "get" or you don't "get." They believe the math myth that either you have the ability and can just do problems quickly, or you don't have it and you never will be able to solve math problems. This belief turns out to be false, but for many it has become a mindset! Everybody can solve problems, and there are problem solving strategies for dealing with math and other problems that everyone can learn. This book focuses on strategies for dealing with the hard parts of problems, things to try when you are not sure what to do. Just as importantly, it focuses on building collaborative and communication skills to enable you to convince your peers of your reasoning and solutions with precision and with confidence.

The Genesis of *Crossing the River with Dogs*

This book has had its genesis in many places: George Pólya's *How to Solve It;* Carol Meyer and Tom Sallee's *Make It Simpler; Topics of Problem Solving* by Randall Charles and colleagues; and the Lane County Problem Solving series; as well as several other diverse works.

* "Six Myths of Product Development." *Harvard Business Review. https://hbr.org/2012/05/six-myths-of-product-development*

These earlier works inspired us to develop a coherent course in mathematical problem solving. The topic was a natural for collaborative learning, and it also provided for extensive communication as a unifying theme.

The earlier editions of this book anticipated the publication of the Common Core Standards for Mathematical Practice and the need for a course where students can learn how to *make sense of problems and persevere in solving them* (Practice 1). From the first edition the goal for every problem has been to engage students in *reasoning abstractly and quantitatively, constructing viable arguments, and critiquing the reasoning of others* (Practices 2 & 3). The problems are chosen to engage students in *modeling with mathematics* and *using appropriate tools strategically* (Practices 4 & 5) and they learn specific strategies such as drawing a diagram, making an organized list, or creating a manipulative, an equation, or a graph. Other strategies such as looking for patterns, identifying subproblems, and starting with a simpler problem develop their skills in *looking for and making use of structure* and *looking for and expressing regularity* (Practices 7 & 8). And because of the emphasis on the communication of their reasoning, students work on their skills in problem presentation, which means that they must learn to *attend to precision* (Practice 6).

Incorporating Problem Solving into College-Level Courses

All students can benefit from this book, whether majoring in math or not. This book is appropriate for a problem-solving course with an intermediate algebra prerequisite that could be taken as a general education math class. It is also appropriate for a liberal arts mathematics course. In addition, it works well in teacher credential programs for future elementary or secondary math teachers. Because much of the course is taught using groups, the course can provide teacher candidates a particularly valuable experience with this mode of learning and encourage them to incorporate cooperative learning in their own classes.

What's New in the Third Edition?

For this edition, we did the following:

- Rewrote the language in several problems to make the problems easier to understand, or to make the problem situation more modern.

- Added new problems to Problem Set A in several chapters.
- Added problems to the More Practice section in several chapters.
- In Problem Set B throughout the text, made slight changes to many problems, leading to new answers.
- Replaced many Problem Set B problems with new problems.

Expected Student Outcomes

From our experiences in our own high school and college classrooms, we find that this book fits best in courses that emphasize reasoning, communication, collaboration, and problem-solving strategies. The curriculum, the course content, and the delivery of instruction are uniquely consistent: Students learn best by working in groups, and the skills required for real workplace problem solving are those skills of collaboration.

By the end of the course, students will see a marked improvement in their writing, oral communication, and collaboration skills. They will acquire and internalize mathematical problem-solving strategies and increase their repertoire of learning strategies.

The students' own reflections on this course speak to its power. Students believe they are better equipped to continue their educational programs and courses of study at the end of a problem-solving course and report that it is the most useful course they have ever taken. Furthermore, students have increased their ability to think about and evaluate their thinking process (meta-cognition) while solving problems.

We believe this kind of learning is rare and special. We proudly present to you the new edition of *Crossing the River with Dogs: Problem Solving for College Students*.

Ken Johnson, Ted Herr, and Judy Kysh

Instructor Resources

Instructor resources are available to qualified adopters. These resources provide instructors with the following:

- Teaching suggestions, such as organizing a class structure, incorporating collaborative learning, and dealing with unique assessment issues.
- Chapter summaries and frequently asked questions.
- Notes on the chapter problems.
- Alternate Problem Sets A and B, which allow instructors to assign different problems each semester.
- Answers to the problem sets that occur in the book as well as the alternate versions that appear in the *Instructor Resources.*

For more information on these teaching resources, please visit www.wiley.com and search "Johnson Problem Solving" to find the Instructor Companion Site.

Acknowledgments

The support for this book has come in many forms. We wish to publicly thank the people who supported and inspired us and those who contributed problems and solutions:

Jen Adorjan, Reina Alvarez, Kevin Alviso, Allyson Angus Stewart, Sally Appert, Alan Barson, Michelle Bautista, Andreas Bazos, Dave Beauchamp, Susan Behl, Dan Bennett, Sarah Bianchet, John Bingham, Paul Bohanna, Monica Borchard, Brianne Brickner, Josh Bridges, Chuck Buchwald, Judi Caler, Raul Carbajal, Barbara Carmichael, Gary Cerar, Randall Charles, Jeremy Chew, Evelyn Click, Cory Craig, Bob Daniel, Lisa DaValle, James Davis, Ginny Day, Carolyn Donohoe, Tab Dorando, Brandon Duckett, Don Elias, Halee Epling, Jessica Erickson, Carina Euyen, Katie Evenson, Katie Fitzpatrick, Russell Fredy, Ryan Gaffney, Greg Garretson, Nancy Gatzman, Don Gernes, Ed Gieszelmann, Michelle Goal, Joanne Gold, Aletta Gonzales, Suzanne Goodell, Gary Haas, Leah Hall, Rachel Hamilton, Kerry Harrigan, Nancy Harrigan, Tim Harrison, Alyssa Haugen, Haley Haus, Kyle Heckey, Holly Herr, Joe Herr, Ryan Hietpas, Alicia Hupp, Kelly Iames, Steve Jackson, Erica Johnson, Gary Johnson, Jennifer Johnson, Lucille Johnson, Rick Johnson, Heather 'Fonzie' Jordan, Erick Kale, Jay Kesler, Nick Landauer, David Lee, Steve Legé, Ashley Lien, Eric Lofholm, Chris Lowman, Irving Lubliner, Brad Marietti, Alyse Mason-Herr, Melissa McConnell, Jonathan Merzel, Amanda Mizell, Jacque Moncrief, Peter Murdock, Matthew Nelson, Ed Patriquin, Patrick Pergamit, Grover Phillips, Jill Rafael, Alex Ramm, Steve Rasmussen, Alyssa Reindl, Jason Reynolds, Travis Roberts, Kristen Rogers, Tom Sallee, Shari Seigworth, Sherrie Serros, Dave Seymour, Alan Shanahan, Robin Shortt, Tanja Slomski, Cheyenne Smedley-Hanson, Alyssa Spagnola, Gary Steiger, Danny Stevenson, Max Stockton, Brian Strand, Kristen Sullivan, Matt Swearingen, Tony Tani, Eric Theer, Andrea Trihub, Joanne Weatherly, Steve Weatherly, Stephanie Wells, Brandy Whittle, Amy Wilcox, Russell Wilcox, Joe Yates, Barbara Ybarra, Byron Young, the Northern California Mathematics Project, and the students enrolled in the problem-solving courses at Luther Burbank High School, Sacramento, California, and Sierra College, Rocklin, California.

We also owe thanks to our colleagues who were the sources for some of these problem ideas. You'll recognize in this book some classic problems (many with new twists) and familiar themes from the lore of the problem-solving community at large. It would be impossible to even guess at, much less acknowledge, a source for each of those problems—but we're grateful to be in a profession that values sharing ideas.

In particular, we would like to thank the authors of the Classic Problems that appear in this book. These people were some of the forefathers of problem solving who were inspirational to us: Henry Dudeney, James Fixx, Martin Gardner, Boris Kordemsky, Sam Loyd, Julio Mira, George Pólya, Raymond Smullyan, Ian Stewart, George Summers, and David Rock and Doug Brumbaugh and the other contributors to the Problem of The Week Website at Ole Miss.

We are very appreciative of the comments and criticisms of the reviewers of this edition of the text:

- Carol J. Bell, Northern Michigan University
- Rebecca J. Fox, Tiffin University
- Thomas Fox, University of Houston Clear Lake
- Brandon Huff, Lewis and Clark Community College
- Victoria Ingalls, Tiffin University
- Dr. William A. Jasper, Sam Houston State University
- Jay Kesler, Sierra College
- Winifred A. Mallam, Texas Woman's University
- Pamela Omer, Western New England University
- Steven Post, Edgewood College
- Jill Rafael, Sierra College
- Mary B. Swarthout, Sam Houston State University
- Dr. Aaron T. Wilson, University of Texas Rio Grande Valley

We'd also like to acknowledge the reviewers of the previous editions:

- Richard Borie, University of Alabama
- Maria Brunett, Montgomery College
- Phyllis Chinn, Humboldt State University

- Michael Coco, Lynchburg College
- Joyce Cutler, Framingham State University
- Sandra Bowen Franz, University of Cincinnati
- Stuart Moskowitz, Humboldt State University
- Kathy Nickell, College of DuPage
- Laurie Pines, San Jose State University
- Christopher Reisch, Jamestown Community College
- Carla Tayeh, Eastern Michigan University
- Joan Weinstein, Pine Manor College
- John Wilkins, California State University, Dominguez Hills

And thank you to Jim Stenson, the math checker of the first edition.

We would like to thank Joanna Dingle, Giana Milazzo, Annie Sophia Thapasumony, Judy Howarth, Gladys Soto, and Laurie Rosatone, as well as Jennifer Brady, Jonathan Cottrell, Sheena Goldstein, Pauline Tan, David Dietz, Michael O'Neal and the rest of the current and former editorial staff at John Wiley and Sons that worked with us on this textbook. We'd also like to thank Dan Bennett, Greer Lleuad, Allyndreth Cassidy, and Steve Rasmussen, formerly of Key Curriculum Press, for believing in this project 25 years ago and helping us become better writers. And thanks to Dan Piraro, who took a chance on illustrating a math book, and contributed to the whimsical style of the text.

A special acknowledgment is reserved for Tom Sallee. For each of us he played similar, overlapping roles. As a professor of mathematics at the University of California, Davis, he was an instrumental part of our development in mathematics, education, and mathematical problem solving and as mathematics teachers. For all of this, we are grateful.

And finally, thanks to Ken and Ted's children: Daniel, Gary, and Will Johnson and Alyse, Jeremy, and Kevin Mason-Herr.

Introduction

Letter to the Student

A mathematics course based on this book will be different from other math classes you've encountered. You may solve some equations, but mostly you'll be asked to think about problems, solve them, then write about your solutions. You may know students who have survived math classes without understanding the material, but that won't happen in this class. Our goal for you is that when you finish this course, you will be able to understand the mathematics you are doing and explain your reasoning in writing and to other students.

This course is based on the idea that you'll learn the strategies people in the real world use to solve problems. You will develop specific problem-solving strategies, communication skills, and attitudes. Learning problem-solving strategies will help you to do well on standardized tests used by colleges, graduate schools, and some employers. These tests often assess problem-solving skills more than the ability to solve equations. You will also learn to have fun doing mathematics. For centuries, people have considered challenging problems, often called puzzles or brain teasers, to be a source of entertainment.

In this course, you will be asked to solve some tough problems. You will be able to solve most of them by being persistent and by talking with other students. When you come across an especially difficult problem, don't give up. Use the techniques you've already learned in the course, such as drawing diagrams, asking other people for help, looking at your notes, and trying other approaches.

You will be expected to talk to your classmates! Your instructor will ask you to get help from one another. Thus, not all of the learning you'll do will be "book learning." You'll also learn how to work with others. Research has shown that even the best students learn from working with their peers. The communication skills you learn in this class will help you throughout your lifetime.

What you get out of this course will depend on how much you are willing to invest. You have a chance to take an active role in your education.

Enjoy the journey!

Answers to Questions That Students Usually Ask

Some people have said that America is not ready for a math book with the word *dogs* in the title, but we think the country can handle it. This book is different from many other mathematics books, from the title page to the last page. For one thing, this book is meant to be enjoyable to read. It is also meant to teach problem-solving **strategies,** and it incorporates research on how students best learn mathematics.

This book was written to take advantage of the strength of cooperative learning and the benefits of communicating your math work to others. You've probably attended classes in which your instructor encouraged you to work with others. Your instructor used this approach because research shows that students learn more when they work together. In addition to working with your fellow students, reading the book, and learning from your instructor, you will be expected to communicate about your work and your mathematical thinking. You will do this by presenting your **solutions** to the entire class and by writing up complete solutions to problems. You will do presentations and write-ups because talking and writing allow you to develop your thinking.

What is problem solving?

Problem solving has been defined as figuring out what to do when you don't know what to do. In some of your math courses, you probably learned about mathematical ideas by first working on an example and then practicing with exercises. An **exercise** asks you to repeat a method you learned from a similar example. A **problem** is usually more complex than an exercise. It is harder to solve because you don't have a preconceived notion about how to solve it. In this course you will learn general, wide-ranging strategies for solving problems. These strategies, many of them popularized by George (György) Pólya's classic book *How to Solve It* (Princeton University Press, 1988), apply to many different types of problems.[1]

Many of the problems you do in this course will be new to you. That is, you won't have seen a similar example in class or have a "recipe" to follow. To solve the problems, you will use the broad **heuristic,** or discovery-based, strategies you'll learn in this book. You may sometimes find that your first approach to a problem doesn't work. When this happens, don't be afraid to abandon your first approach and try something else. Be persistent. If you get frustrated with a problem, put it aside and come back to it later. Let your subconscious work on it—you may find yourself solving problems in your sleep! But don't give up on any problem.

Why should we work together? Can't we learn just as well on our own?

You will have many opportunities to work with other students in class. You should also try to get together with other students outside of class. When you work with other students, you are free to make conjectures, ask questions, make mistakes, and express your ideas and opinions.

[1]First published in 1945, *How to Solve It* was evidence that Pólya was far ahead of his time in his approach to mathematical problem solving.

You don't have to worry about being criticized for your thoughts or your wrong answers. You won't always proceed down the correct path. Support one another, question one another, ask another person to explain what you don't understand, and make sure the other members of your group understand, too.

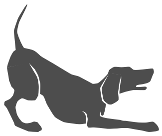
Are there any limits to working together? What about searching for solutions on the Internet?

The point of this class is for you to learn new problem-solving strategies, and use your skills to solve a variety of problems. While we encourage you to work together, you have to be reasonable about the amount of help you accept. It is fine to collaborate to reach a solution, with each member of the group having input to the final solution. It is not OK to copy someone else's work, with none of your own thinking. It is also not OK to search for solutions on the Internet. This is a class about becoming a powerful problem solver and discovering the power of collaboration to solve difficult problems. This is not a class about getting an answer by any means necessary.

How should we study? How should we read the book?

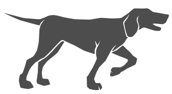
The book is organized into chapters. Each chapter introduces a new problem-solving strategy and presents several problems within the text, which are identified by an icon of an attentive dog. You are asked to first solve each problem, then read its sample solution. To get the maximum benefit from this book, work the problems before reading the solutions. Even if you successfully solved a problem, read the solution anyway because it may differ from your own or bring up some points you hadn't thought about. Remember, your purpose is not just to get answers, but to learn more solution strategies. Although you'll learn the most by trying the problems yourself, if you don't have time to solve all of them, you will still get a lot out of at least reading the problems and their solutions. You'll learn the least if you don't read the book at all. Be willing to read the text slowly and carefully.

Sometimes you will see a problem that you saw earlier in another chapter. You'll solve this same problem again, but you'll use a different strategy. Solving the same problem in many different ways will help you become a better problem solver. You'll also often see sample solutions that actual students came up with as they worked through the problems in this book. Their work shows that there are many valid ways to tackle a problem, even if one particular approach doesn't result in a correct answer.



Some issues that cause confusion occur occasionally in the problem discussions. When these points of confusion come up, they are considered further and clarified in the discussion following the problem analysis. These points of possible confusion are marked by an icon of a confused dog.

Margin notes will clarify important steps in solutions to problems. An oar icon marks these spots.



Every chapter concludes with a summary of the main points of the strategy. In addition, important problem-solving advice is given in the body of the text. A pointing dog icon marks these spots.

There are many useful references in the back of the book. In the text, there are many words in **bold** print. You will find the definition for these words in the glossary. There is a page of geometry formulas on area, volume, and triangle properties. You will find information on metric conversions, as well as common abbreviations for different units of measure. There is also a reminder on how to work with fractions. There is a section on divisibility rules and how to determine if a number is prime. There is an answer section for the More Practice problems. Finally, there is an index of problem titles and a general index.

What other problems will we do besides those discussed in the text?

Each chapter ends with a Problem Set A. The problems in these sets can always be solved with the strategy presented in that particular chapter. Some of the problems could also be solved by using a strategy you learned in an earlier chapter. However, because you're learning a new strategy with each new chapter, solve the Set A problems with their chapter-specific strategy. (To get more benefit from this class, you may also want to try solving many of the problems using other strategies.)

In most chapters, at the end of Problem Set A, you will find one or two classic problems. There have been many famous puzzle writers through the years, beginning with Henry Dudeney in England and Sam Loyd in the United States at the end of the nineteenth century. In the twentieth century, Martin Gardner, Raymond Smullyan, George

Summers, Boris Kordemsky, and George Pólya joined them. These individuals, along with several others, created problems that are now considered classics by the mathematics community. In this book, we give you the opportunity to solve some of these famous problems. In most cases, you can solve them with the strategy you learned in the chapter where you find them.

After the Classics section, there is a section called More Practice. The problems that appear here are relatively straightforward and you should use them to study from. Do the problems in this section after finishing the chapter to solidify your grasp of the strategy. Answers for these problems appear in the back of the book as one way to check your solution.

Beginning in Chapter 3, a Problem Set B ends the chapter. The five Set B problems in each chapter can be solved with any strategy you've already learned, and it's up to you to pick an appropriate strategy. The Set B problems are more difficult than the Set A problems. In fact, toward the end of the book, many of the Set B problems are extremely difficult. Many instructors use Problem Set B as a week-long (or longer) assignment. They use the sets as take-home tests and allow students to work together, with each student turning in his or her own work. Each student is expected to provide answers and to explain their reasoning for each solution.

What is the role of the instructor in this course?

In many courses, the instructor is the final authority who determines whether the student is right or wrong. In this course your instructor will play that role at times, such as when she or he grades your work. But there will probably be times when the instructor will not play that role. For example, during student presentations several people may have different answers to the same problem. When this happens, it's natural to ask your instructor who is right. In this course, your instructor may let you make up your own minds. Your in-class groups can discuss which answer they think is correct and why. Explaining why is a very important part of this course. Not relying on the instructor to verify your work will help you become a better problem solver. You will learn to carefully evaluate your own work as well as the work of others.

Why aren't there any other answers in the back of the book?

Answers appear in the back of the book for the More Practice problems. There are no other answers in the back of the book. When you are learning a strategy for the first time, an answer given too early will cut off the thinking processes that you need to develop. When working in a group of students in class, the group needs to come to a conclusion about approaches and answers. If there were answers in the back of the book, much of this process would be inhibited.

Will the skills I learn in this class help me to get a job?

What do employers want in their employees? The Internet is full of top-ten lists that explore this question. There are many common themes to such lists. Employers want their employees to exhibit:

a. Problem-solving skills

b. Creativity in finding solutions to complex problems

c. The ability to analyze solutions to determine their workability

d. The ability to work well with other people, especially people different from yourself

e. The ability to apply previous knowledge to new situations

f. Good communication skills—both oral and written

g. Determination and persistence

h. A strong work ethic

i. Honesty and integrity

Every skill on this list will be greatly developed by studying and actively working on the problems in this book. Look at this course as a challenge—as an opportunity to learn new skills and show them off. You will use your creativity and problem-solving skills to solve a variety of different problems. We are sure that those who really buy into this will emerge from the course with an enriched toolbox of problem-solving skills.

Some Comments on Answers

When you turn in written work, you should write your answer to each problem in the form of a sentence, including any appropriate units. Be sure that you answer the question that is being asked, and make your answer entirely clear. Don't expect the reader to dig through your work to find your answer.

Think carefully about what your answer means, and make sure the form of the answer makes sense and is reasonable given the circumstances of the problem. For example, if the answer to a question is a certain number of people and your answer is a fraction or a decimal, think about what the question's answer should be. Does it make sense to round your answer up or down, or to leave it the way it is? Consider the following situation.

The Vans

There are 25 people going on a trip. They are traveling by van, and each van has a capacity of 7 people.

Some people might think the answer being sought is $3\frac{4}{7}$ vans, but the answer depends on what question is posed. Here are some possible questions:

a. How many vans will be needed to transport all 25 people?

b. How many vans can be filled to capacity?

c. How many vans will have to be filled to capacity?

d. What is the average number of people in a van?

e. Must any van have 7 people in it?

f. How many more people could fit into the vans that will be required?

The answer to each of these questions is different, even though the situation is the same. The difference is in the question asked. For example, the van problem looks like it could be solved by dividing 25 by 7, but only one of the questions above looks like a division problem. In the van problem, we're working with units that are generally considered to be indivisible (vans or people) as opposed to units that are clearly divisible (pizzas). The answer to our division problem is reasonable only if our answer's units are also reasonable. That is, no matter what arithmetic is done with the numbers in this problem, the answer must still apply to human beings going somewhere in vans. Keep these issues in mind when you work problems.

Some Introductory Problems

During this course you will learn many problem-solving strategies and use them to solve many different problems. Solve the problems in this introduction with whatever strategy you wish. You will have an opportunity to share your solutions to some or all of these problems with a small group or the whole class. You can solve these problems with a variety of different strategies. In fact, you may want to solve each problem several times, using a different strategy each time. (The solutions to the example problems in this book are shown following those problems, but the solutions to these introductory problems are not shown.)

1. SOCCER GAME

At the conclusion of a soccer game whose two teams each included 11 players, each player on the winning team "gave five" to (slapped hands with) each player on the losing team. Each player on the winning team also gave five to each *other* player on the winning team. How many fives were given?

2. ELEVATOR

The capacity of an elevator is either 20 children or 15 adults. If 12 children are currently in the elevator, how many adults can still get in?

3. THEATER GROUP

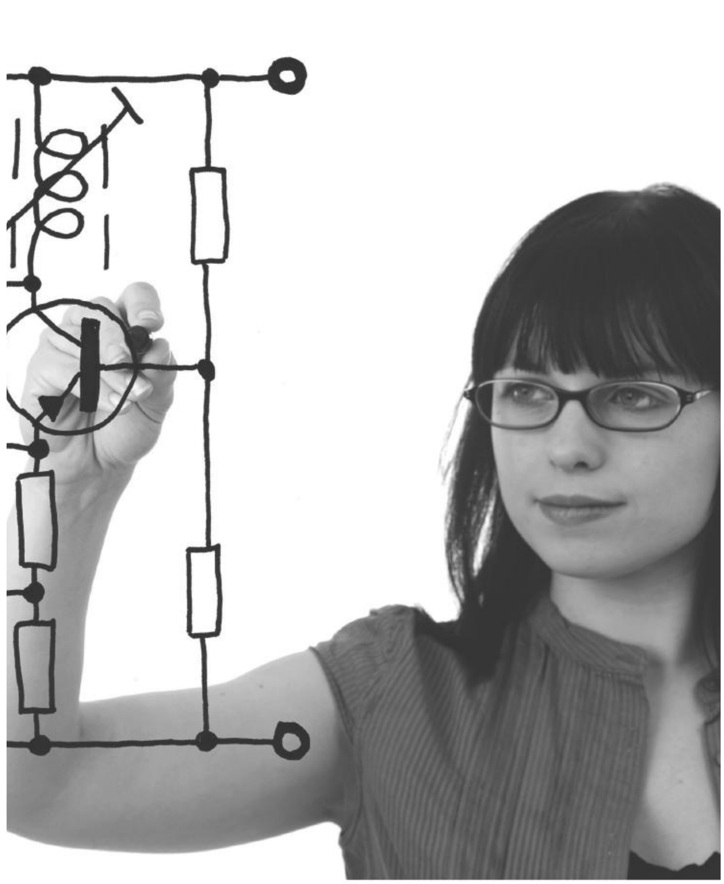
There are eight more women than men in a theater group. The group has a total of 44 members. How many men and how many women are in the group?

4. DUCKS AND COWS

Farmer Brown had ducks and cows. One day she noticed that the animals had a total of 12 heads and 32 feet. How many of the animals were ducks and how many were cows?

5. STRANGE NUMBER

If you take a particular two-digit number, reverse its **digits** to make a second two-digit number, and add these two numbers together, their sum will be 121. What is the original number?



Draw a Diagram

Diagrams are often the key to getting started on a problem. They can clarify relationships that appear complicated when written. Electrical engineers draw diagrams of circuit boards to help them visualize the relationships among a computer's electrical components.

You've probably heard the old saying "One picture is worth a thousand words." Most people nod in agreement when this statement is made, without realizing just how powerful a picture, or a **diagram,** can be. (Note that words in **bold** type are terms that are defined in this book's glossary.) A diagram has many advantages over verbal communication. For example, a diagram can show positional relationships far more easily and clearly than a verbal description can. To attempt to clarify ideas in their own minds, some people talk to themselves or to others about those ideas. Similarly, a diagram can help clarify ideas and solve problems that lend themselves to visual representations.

One of the best examples of a diagram in the professional world is a blueprint. An architect's blueprint expresses ideas concisely in a visual form that leaves little to interpretation. Words are added only to indicate details that are not visually evident. A blueprint illustrates one of the strengths of diagrams: the ability to present the "whole picture" immediately.

Problem solving often revolves around how information is organized. When you draw a diagram, you organize information spatially, which then allows the visual part of your brain to become more involved in the problem-solving process. In this chapter, you will learn how you can use diagrams to clarify ideas and solve a variety of problems. You'll improve your diagramming abilities, and you'll discover that a diagram can help you understand and correctly interpret the information contained in a problem. You'll also see the value of using diagrams as a problem-solving strategy.

Solve this problem by drawing a diagram.

VIRTUAL BASKETBALL LEAGUE

Andrew and his friends have formed a fantasy basketball league in which each team will play three games against each of the other teams. There are seven teams: the (Texas A&M) Aggies, the (Purdue) Boilermakers, the (Alabama) Crimson Tide, the (Oregon) Ducks, the (Boston College) Eagles, the (Air Force) Falcons, and the (Florida) Gators. How many games will be played in all? Do this problem before reading on.

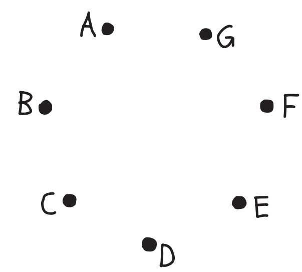
As you read in the Introduction, you'll see many different problems as you work through this book. The problems are indicated by an icon of an attentive dog. To get the maximum benefit from the book, solve each of the problems before reading on. You gain a lot by solving problems, even if your answers are incorrect. The *process* you use to solve each problem is what you should concentrate on.

You could use many different diagrams to solve the Virtual Basketball League problem, but you could also solve this problem in ways that do not involve diagrams. As you also read in the Introduction, throughout this book you will see some of the same problems in different chapters and solve them with different strategies. You will become a better problem solver in two ways: by solving many different problems and by solving the same problem in many different ways. In this chapter, the solutions involve diagrams. If you solved the Virtual Basketball League problem without using a diagram, try solving it again with a diagram before reading on.

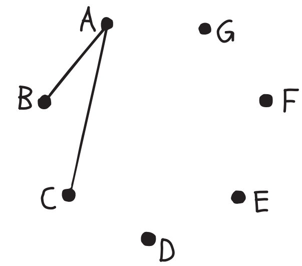
What comes next is a solution process that is attributed to a student. The people mentioned in this book are real students who took a problem-solving class at either Sierra College in Rocklin, California, or at Luther Burbank High School in Sacramento, California. In those classes, the students presented their solutions on the board to their classmates. Ted Herr and Ken Johnson, two of the authors of this book, taught these classes. Our students presented their solutions because we felt that the other students in class would benefit greatly from seeing many different approaches to the same problem. We didn't judge each student's solution in any way. Rather, we asked each member of the class to examine each solution that was presented and decide which approach or approaches were valid or, perhaps, better. The purpose behind shifting this responsibility from the instructor to the students is to give the students practice in evaluating problem solving.

We have tried to re-create the same learning atmosphere in this book. Sometimes you'll see several different approaches to a problem in this book, but for the most part those approaches and the resulting solutions won't be judged. You are encouraged to evaluate the quality of the approaches. You may have been led to believe that there is always one right way—and many wrong ways—to solve problems. This notion couldn't be further from the truth. There are many right ways to solve problems, and you are encouraged to solve the problems in this book more than once, using different methods.

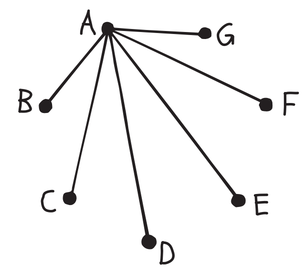
Here's how Rita solved the Virtual Basketball League problem: She drew a diagram that showed the letters representing each team arranged in a circle.



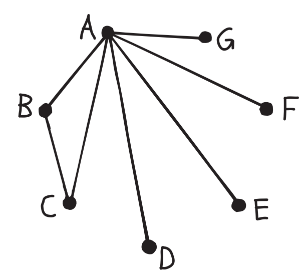
She then drew a line from A to B to represent the games played between the Aggies and the Boilermakers. Then she drew a line from A to C to represent the games played between the Aggies and the Crimson Tide.



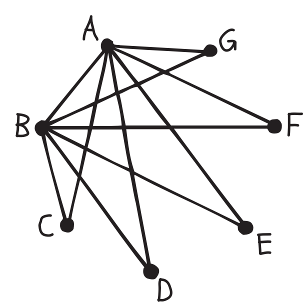
She finished representing the Aggies' games by drawing lines from A to D, E, F, and G.



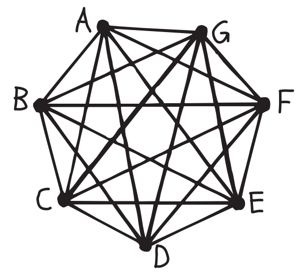
Next she drew the lines for the Boilermakers. She'd already drawn a line from A to B to represent the games the Boilermakers played against the Aggies, so the first line she drew for the Boilermakers was from B to C.



She continued drawing lines to represent the games that the Boilermakers played against each other team.



From C she drew lines only to D, E, F, and G because the lines from C to A and from C to B had already been drawn. She continued in this way, completing her diagram by drawing the lines needed to represent the games played by the rest of the teams in the league. Note that when she finally got to the Gators, she did not need to draw any more lines because the games the Gators played against each other team had already been represented with a line.



She then counted the lines she'd drawn. There were 21. She multiplied 21 by 3 (remember that each line represented three games) and came up with an answer of 63 games. Finally, Rita made sure that she'd answered the question asked. The question was "How many games will be played in all?" Her answer, "Sixty-three games will be played," accurately answers the question.

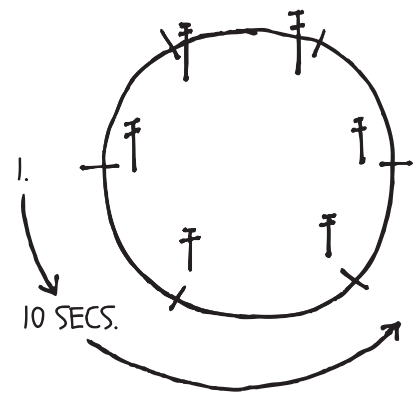
Mirka solved this problem with the diagram below. She also used the letters A, B, C, D, E, F, and G to represent the teams. She arranged the letters in a row and, as Rita did, she drew lines from team to team to represent games played. She started by drawing lines from A to the other letters, then from B to the other letters, and so on. She drew 21 lines, multiplied 21 by 3, and got an answer of 63 games.

MODEL TRAIN

Esther's model train is set up on a circular track. Six telephone poles are spaced evenly around the track. The engine of Esther's train takes 10 seconds to go from the first pole to the third pole. How long would it take the engine to go all the way around the track? Solve the problem before reading on.

If you read the problem quickly and solved it in your head, you might think the answer is 20 seconds. After all, the problem states that the engine can go from the first pole to the third pole in 10 seconds, which is three poles out of six and apparently halfway around the track. So it would take the engine 2 times 10, or 20 seconds, to go all the way around the track. But this answer is wrong. The correct answer becomes apparent when you look at a diagram.

Rena's diagram is shown at right. Rena explained that the train goes one-third of the way around the track in 10 seconds, not halfway around the track. So the train goes around the entire track in 3 times 10 seconds, or 30 seconds.



Phong drew the same diagram, but he interpreted it differently. He explained that if it takes 10 seconds to go from the first pole to the third pole, then it takes 5 seconds to go from the first pole to the second pole. So it takes 5 seconds to go from pole to pole. There are six poles, so it takes the train 30 seconds to go all the way around the track.

Pete interpreted the problem as Phong did, but he didn't draw a diagram. Thus, he neglected the fact that the train must return from the sixth pole to the first pole in order to travel all the way around the track. Therefore, he got the incorrect answer 25 seconds.

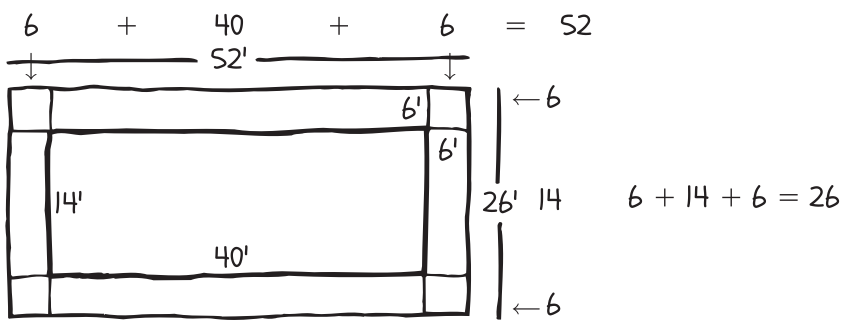
The diagram helped Rena and Phong solve the Model Train problem. If you used a diagram to solve the problem, you probably got the correct solution. If you were able to get the correct solution without drawing a diagram, think back on your process. You probably visualized the train track in your mind, so even though you didn't actually draw a diagram, you could "see" a picture.

Do you get the picture? Do you see why diagrams are important? Research shows that most good problem solvers draw diagrams for almost every problem they solve. Don't resist drawing a diagram because you think that you can't draw, or that smart people use only equations to solve problems, or whatever. Just draw it!

THE POOL DECK

Curly used a shovel to dig his own swimming pool. He figured he needed a pool because digging it was hard work and he could use it to cool off after working on it all day. He also planned to build a rectangular concrete deck around the pool that would be 6 feet wide at all points. The pool is rectangular and measures 14 feet by 40 feet. What is the **area** of the deck? As usual, solve this problem before continuing.

Jeff drew the diagram below to show the correct dimensions of the deck and pool, which together are 12 feet longer and 12 feet wider than the pool alone.

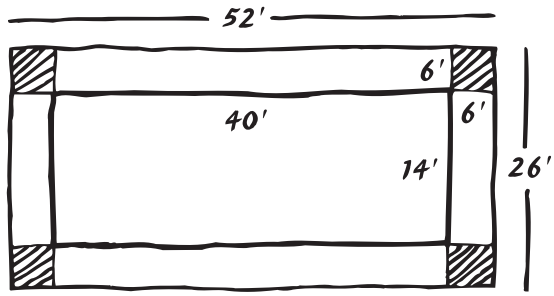


The diagram helps show the difficult parts of the problem. However, Jeff solved the problem incorrectly by finding the outside **perimeter** of the pool and the deck together, then multiplying the perimeter by the width of the deck.

52 feet + 26 feet + 52 feet + 26 feet = 156 feet
156 feet x 6 feet = 936 square feet

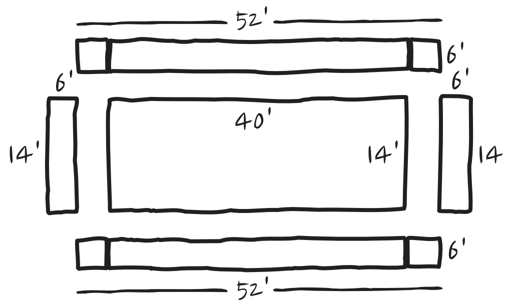
His approach is incorrect because it counts each corner twice.

Rajesh used the same diagram, but he solved the problem by first computing the area of the deck along the sides of the pool, then adding in the corners of the deck.



Two lengths: **40 ft x 6 ft x 2 = 480 sq ft**
Two widths: **14 ft x 6 ft x 2 = 168 sq ft**
Four corners: **6 ft x 6 ft x 4 = 144 sq ft**
Total **792 sq ft**

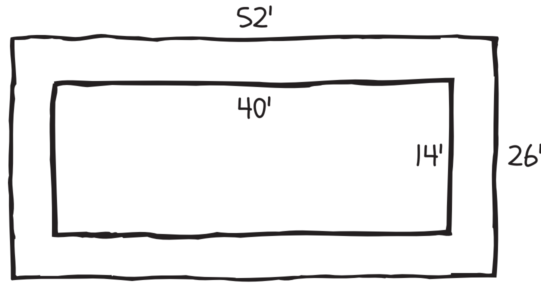
May's diagram shows the corners attached to the length of the deck.



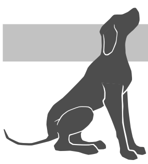
She calculated the area as follows:

52 ft x 6 ft = 312 sq ft
312 sq ft x 2 = 624 sq ft for extended lengths
14 ft x 6 ft = 84 sq ft
84 sq ft x 2 = 168 sq ft for widths
Total = 624 sq ft + 168 sq ft = 792 sq ft

Herb solved this problem by first computing the area of the pool and the deck together, then subtracting the area of the pool, leaving the area of the deck.



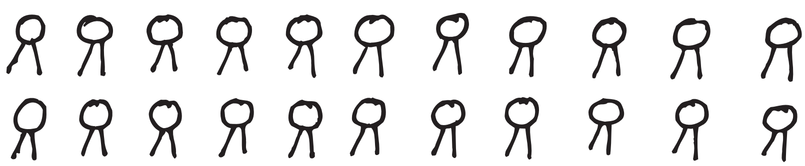
Area of entire figure = 52 ft x 26 ft = 1,352 sq ft
Area of pool alone = 40 ft x 14 ft = 560 sq ft
Area of deck = 1352 ft – 560 ft = 792 sq ft



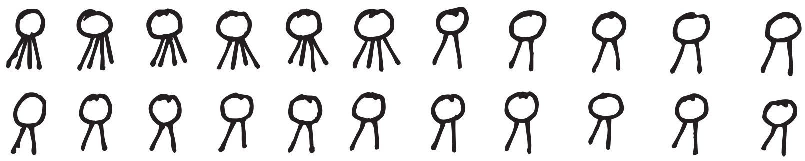
FARMER BEN

Farmer Ben has only ducks and cows. He can't remember how many of each he has, but he doesn't need to remember because he knows he has 22 animals and that 22 is also his age. He also knows that the animals have a total of 56 legs, because 56 is also his father's age. Assuming that each animal has all legs intact and no extra limbs, how many of each animal does Farmer Ben have? Do this problem, and then read on.

Trent drew the following diagram and explained his thinking: "These 22 circles represent the 22 animals. First, I made all of the animals into ducks." (Trent is not much of an artist, so you just have to believe that these are ducks.) "I gave each animal two legs because ducks have two legs."

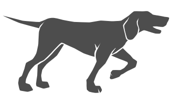


"Then I converted the ducks into cows by drawing extra legs. The ducks alone had 44 of the 56 legs initially, so I drew 12 more legs, or six pairs, on 6 ducks to turn them into cows. So there are 6 cows and 16 ducks."



Of course, Farmer Ben might have a problem when his father turns 57 next year.

Draw a Diagram



Any idea that can be represented with a picture can be communicated more effectively with that picture. By making visible what a person is thinking, a diagram becomes a problem-solving strategy. A diagram clarifies ideas and communicates those ideas to anyone who looks at it. Diagrams are used in many jobs, especially those that require a planning stage. Occupational diagrams include blueprints, project flow charts, and concept maps, to name a few. Diagrams are often necessary to show position, directions, or complicated multidimensional relationships, because pictures communicate these ideas more easily and more clearly than words.

If you can visualize it, draw a diagram.

Problem Set A

You must draw a diagram to solve each problem.

1. WORM JOURNEY

A worm is at the bottom of a 12-foot wall. Every day the worm crawls up 3 feet, but at night it slips down 2 feet. How many days does it take the worm to get to the top of the wall?

2. UPS AND DOWNS OF SHOPPING

Roberto is shopping in a large department store with many floors. He enters the store on the middle floor from a skyway and immediately goes to the credit department. After making sure his credit is good, he goes up three floors to the housewares department. Then he goes down five floors to the children's department. Then he goes up six floors to the TV department. Finally, he goes down ten floors to the main entrance of the store, which is on the first floor, and leaves to go to another store down the street. How many floors does the department store have?

3. FOLLOW THE BOUNCING BALL

A ball rebounds one-half the height from which it is dropped. The ball is dropped from a height of 160 feet and keeps on bouncing. What is the total vertical distance the ball will travel from the moment it is dropped to the moment it hits the floor for the fifth time?

4. FLOOR TILES

How many 9-inch-square floor tiles are needed to cover a rectangular floor that measures 12 feet by 15 feet?

5. STONE NECKLACE

Arvilla laid out the stones for a necklace in a big circle, with each stone spaced an equal distance from its neighbors. She then counted the stones in order around the circle. Unfortunately, before she finished counting she lost track of where she had started, but she realized that she could figure out how many stones were in the circle after she noticed that the sixth stone was directly opposite the seventeenth stone. How many stones are in the necklace?

6. DANGEROUS MANEUVERS

Somewhere in the Mojave Desert, the army set up training camps named Arachnid, Feline, Canine, Lupine, Bovine, and Thirty-Nine. Several camps are connected by roads:

Arachnid is 15 miles from Canine, Bovine is 12 miles from Lupine, Feline is 6 miles from Thirty-Nine, Lupine is 3 miles from Canine, Bovine is 9 miles from Thirty-Nine, Bovine is 7 miles from Canine, Thirty-Nine is 1 mile from Arachnid, and Feline is 11 miles from Lupine. No other pairs of training camps are connected by roads.

Note: This problem continues on the next page.

Answer each of the following questions (in each answer, indicate both the mileage and the route): What is the shortest route from

Feline to Bovine?	Canine to Thirty-Nine?
Lupine to Thirty-Nine?	Lupine to Bovine?
Canine to Feline?	Arachnid to Feline?
Arachnid to Lupine?	

7. RACE

Becky, Ruby, Isabel, Lani, Alma, and Sabrina ran an 800-meter race. Alma beat Isabel by 7 meters. Sabrina beat Becky by 12 meters. Alma finished 5 meters ahead of Lani but 3 meters behind Sabrina. Ruby finished halfway between the first and last women. In what order did the women finish? What were the distances between them?

8. A WHOLE LOTTA SHAKIN' GOIN' ON!

If six people met at a party and all shook hands with one another, how many handshakes would be exchanged?

9. HAYWIRE

A telephone system in a major manufacturing company has gone haywire. The system will complete certain calls only over certain sets of wires. So, to get a message to someone, an employee of the company first has to call another employee to start a message on a route to the person the call is for. As far as the company can determine, these are the connections:

Cherlondia can call Al and Shirley (this means that Cherlondia can call them, but neither Al nor Shirley can call Cherlondia). Al can call Max. Wolfgang can call Darlene, and Darlene can call Wolfgang back. Sylvia can call Dalamatia and Henry. Max can get calls only from Al. Carla can call Sylvia and Cherlondia. Shirley can call Darlene. Max can call Henry. Darlene can call Sylvia. Henry can call Carla. Cherlondia can call Dalamatia.

How would you route a message from

Cherlondia to Darlene?	Shirley to Henry?
Carla to Max?	Max to Dalamatia?
Sylvia to Wolfgang?	Cherlondia to Sylvia?
Henry to Wolfgang?	Dalamatia to Henry?

10. ROCK CLIMBING

Amy is just learning how to rock climb. Her instructor takes her to a 26-foot climbing wall for her first time. She climbs 5 feet in 2 minutes but then slips back 2 feet in 10 seconds. This pattern (up 5 feet, down 2 feet) continues until she reaches the top. How long will it take her to reach the very top of the wall?

This problem was written by Jen Adorjan, a student at Sierra College in Rocklin, California.

11. CIRCULAR TABLE

In Amanda and Emily's apartment, a round table is shoved into the corner of the room. The table touches the two walls at points that are 17 inches apart. How far is the center of the table from the corner?

12. THE HUNGRY BOOKWORM

Following is an expansion of a well-known problem:

The four volumes of *The World of Mathematics* by James R. Newman are sitting side by side on a bookshelf, in order, with volume 1 on the left. A bookworm tunnels through the front cover of volume 1 all the way through the back cover of volume 4. Each book has a front cover and a back cover that each measure $1/16$ inch. The pages of each book measure $1\frac{1}{8}$ inches. How far does the bookworm tunnel?

13. BUSING TABLES

Brian buses tables at a local café. To bus a table, he must clear the dirty dishes and reset the table for the next set of customers. One night he noticed that for every three-fifths of a table that he bused, another table of customers would get up and leave. He also noticed that right after he finished busing a table, a new table of customers would come into the restaurant. However, once every table was empty (no diners were left in the restaurant), nobody else came into the restaurant. Suppose there were six tables with customers and one unbused table. How many new tables of customers would come in before the restaurant was empty? After the last table of customers had left, how many tables were unbused?

This problem was written by Brian Strand, a student at Sierra College in Rocklin, California.

14. WRITE YOUR OWN PROBLEM

In each chapter, you'll be given the opportunity to write your own problem that can be solved by using the strategy you studied in that chapter. The book will give you suggestions for how to go about writing these problems. Each time you write your own problem, solve it yourself to be sure that it's solvable. Then give it to another student to solve and, as needed, to help you with the problem's wording.

Create your own Draw a Diagram problem. Model it after either this chapter's Worm Journey problem or Ups and Downs of Shopping problem.

Classic Problems

15. THE WEIGHT OF A BRICK

If a brick balances with three-quarters of a brick and three-quarters of a pound, then how much does the brick weigh?

Adapted from *Mathematical Puzzles of Sam Loyd,* Vol. 2, edited by Martin Gardner.

16. THE MOTORCYCLIST AND THE HORSEMAN

A motorcyclist was sent by the post office to meet a plane at the airport. The plane landed ahead of schedule, and its mail was taken toward the post office by horse. After half an hour, the horseman met the motorcyclist on the road and gave him the mail. The motorcyclist returned to the post office 20 minutes before he was expected. How many minutes early did the plane land?

Adapted from *The Moscow Puzzles* by Boris Kordemsky.

More Practice

1. APARTMENT BUILDING

Joden just moved into a 12-story apartment building, and he is still having trouble finding which floor he lives on. He knows that he lives in the first apartment on the floor, but doesn't know which floor. He starts by going to the first floor and knocks on the door. Mrs. Smith answers and tells him to go up 8 floors and ask Mr. Jones. Joden does that and asks Mr. Jones where he lives. Mr. Jones doesn't know, but he says, "Go down 2 floors to Bryn's apartment and ask him." Bryn didn't know either. He told Joden to go up 5 floors to see his friend Trudie, because she knows where everyone lives. Trudie had no clue who Joden was. She said that the only one who might know where he lived was the new guy 7 floors below her. Joden goes down 7 floors and knocks on the new guy's apartment. No one answers. He stands there thinking for a while and finally realizes that he is the new guy. He opens the door and walks into his apartment. Which floor does Joden live on?

This problem was written by Jeremy Chew, a student at Sierra College in Rocklin, California.

2. MOVIE LINE

A bunch of people were standing in line for a movie. Averi got there late and realized that she knew every person in line. She decided not to get in line until she figured out who to cut in with. She first stopped and talked to Jake, who was at the back of the line. Then she moved forward by passing 3 people and talked to Alexandra. She then moved forward again by passing 9 people and talked to Walter. Then she moved backward by passing 4 people and talked to Annie. Then she moved forward by passing 12 people and talked to Katie. Finally she moved backward by passing 2 people and joined Carli in line. Carli was originally the person in the exact middle of the line. Including Averi (who was now in line) how many people are in the line? *Note:* Moving forward refers to moving toward the front of the line and moving backward refers to moving toward the back of the line.

3. BACKBOARD

Ei liked to play tennis. One day she didn't have anyone to play with, so she took her racket and tennis ball and began to hit the ball against the backboard. She hit it at the backboard, the ball bounced off the backboard and came back to her, and she hit it again, and so on. She started out 40 feet from the backboard. But she didn't hit the ball hard enough—the bounce came back only 90% as far, so she had to run up to hit it again. Again she didn't hit it hard enough, and it again only came back 90% as far as she had hit it. This continued for two more hits, each time the ball coming back 90% as far. Finally on the fifth hit she hit it really hard and it came back five times as far as she had hit it, going way over her head and hitting the fence. She got frustrated and picked up her ball and went home. How far was the distance from the backboard to the fence? What was the total horizontal distance that the tennis ball traveled?

4. WORKING OUT

At the gym, there are 12 weight machines that Gina liked to use: 6 upper body machines and 6 lower body machines. All 12 machines were in one row, with 8 feet between each upper body machine and 8 feet between each lower body machine. The upper body machines were separated from the lower body machines by 16 feet. The lower body machines were numbered 1 through 6, and the upper body machines were numbered 7 through 12. Gina started at machine 6 (lower body) and then walked to machine 7 (upper body), then to machine 5 (lower body), then to machine 8 (upper body), and so on, alternating between lower body and upper body. After each 3 lower body machines, she would walk to the mat that was 20 feet past machine 1 and do some stretches. After each 3 upper body machines, she would walk to the wall that was 30 feet past machine 12 and do some stretching. When she finally finished lifting and stretching (the last thing she did was upper body stretching on the wall), how many feet had she walked in all?

5. ANYA AND SOPHIA AND MASHA AND MIKE

Anya, Sophia, Masha, and Mike went to New York City to see a few Broadway shows. They stayed on a floor in a small hotel which had four rooms along one long hallway. Anya was staying in the first room, Sophia in the second room, Masha in the third room, and Mike in the fourth room, at the opposite end of the hall from Anya. The elevator was along the hallway, exactly halfway between Sophia's door and Masha's door. One afternoon, Sophia and Masha got out of the elevator and walked 90 feet to Mike's door to say hello. Then they walked back the other direction all the way down the hall to Anya's door to leave a note. The total distance from Mike's door to Anya's door was 160 feet. Then they turned around and walked 30 feet to Sophia's door. How far did Masha have to walk to return from Sophia's door to her own room's door?

This problem was written by Sierra College professor Jill Rafael.

2

Make a Systematic List

When you make a systematic list, you reveal the structure of a problem. Sometimes the list is all you need to solve it. Train schedules are systematic lists that help travelers find information easily and quickly.

LOOSE CHANGE

Leslie has 25¢ in her pocket but does not have a quarter. If you can tell her all possible combinations of coins she could have that add up to 25¢, she will give you the 25¢. Solve this problem before continuing.

Many people start solving this problem as follows: "Let's see, we could have 5 nickels, or 2 dimes and 1 nickel. We might have 25 pennies. Oh yeah, we could have 10 pennies, 1 dime, and 1 nickel. Perhaps we could have . . . ." Solving the problem this way is extremely inefficient. It could take a long time to figure out all the ways to make 25¢, and you still might not be sure that you'd thought of all the ways.

A better way to solve the problem is to make a **systematic list**. A systematic list is just what its name says it is: a list generated through some kind of system. A **system** is any procedure that allows you to do something (like organize information) in a methodical way. The system used in generating a systematic list should be understandable and clear so that the person making the list can verify its completeness quickly. Additionally, another person should be able to understand the system and verify the solution without too much effort.

Many systematic lists are in the form of a table whose columns are labeled with the information given in a problem. The rows of the table are used to indicate possible combinations. As you read the following solutions for the Loose Change problem, make your own systematic list. Label the columns of the list Dimes, Nickels, and Pennies, and then fill in the rows with combinations of coins that add up to 25¢.

Dimes	Nickels	Pennies
2	1	0
2	0	5
1	3	0
1	2	5
1	1	10
1	0	15

and so on

Brooke started her list in the first row of the Dimes column by showing the maximum number of dimes Leslie could have: two. In the Nickels column, she showed the maximum number of nickels possible with two dimes: one. In the second row, she decreased the number of nickels by one because it's possible to make 25¢ without using nickels. She then filled in the Pennies column by showing how many pennies she had to add to her dimes and nickels to make 25¢. After finding all the ways to make 25¢ with two dimes, Brooke continued filling in her list with combinations that include only one dime. In the third row, she showed the maximum number of nickels possible in one-dime combinations: three. As she did for the two-dime combinations, she decreased the number of nickels by one in each row until she ran out of nickels.

Dimes	Nickels	Pennies
2	1	0
2	0	5
1	3	0
1	2	5
1	1	10
1	0	15
0	5	0
0	4	5
0	3	10
0	2	15
0	1	20
0	0	25

Brooke's completed list is shown at right. It includes all the possible zero-dime combinations. Finish your own list before reading on.

Brooke's systematic list is not the only one that will solve this problem. Heather used a different system. Before you look at her entire solution, which follows, cover all but the first three rows of the table at bottom right with a piece of paper. Look at the uncovered rows to figure out her system, and then complete the list yourself.

Pennies	Nickels	Dimes
25	0	0
20	1	0
15	2	0
15	0	1
10	3	0
10	1	1
5	4	0
5	2	1
5	0	2
0	5	0
0	3	1
0	1	2

Heather explained her system like this: "I started with the largest number of pennies, which was 25. Then I let the pennies go down by fives and filled in the nickels and dimes to make up the difference."

Many people find Heather's system to be more difficult than Brooke's system. What do you think?

There are even more ways to do this problem as well. Kaitlyn made her list as shown on the right.

10¢	5¢	1¢
20	5	0
20	0	5
10	15	0
10	10	5
10	5	10
10	0	15
0	25	0
0	20	5
0	15	10
0	10	15
0	5	20
0	0	25

Kaitlyn explained, “I wrote 20 in the 10 cent column because two dimes is 20 cents. Then I wrote 5 in the 5 cent column because one nickel is 5 cents. In this way, each row adds up to 25 cents. I also made sure that I only used 20, 10, or 0 in the 10 cent column and 25, 20, 15, 10, 5, or 0 in the 5 cent column, because obviously you can’t have half a nickel or something like that. I also froze the number in the first column and played with the other two columns figuring out all possible ways before I changed the number in the first column. So there were 2 ways to put 20 in the first column, 4 ways to put 10 in the first column, and 6 ways to put 0 in the first column. That’s a nice pattern. There are a total of 12 ways.”

Mo made her list in another way (shown on the next page). “I saw what the others had done, but I didn’t like the chart set up with the three columns. So I looked at the problem differently. I simply listed each coin, using 10 for a dime, 5 for a nickel, and 1 for a penny. I started off with two dimes and one nickel so I wrote 10-10-5. Then I traded the nickel in for five pennies so I wrote 10-10-1-1-1-1-1. Then I changed it to one dime and used three nickels. Continuing, I froze the 10, and changed one of the 5’s to five 1’s. Then I changed another 5 to five 1’s, and finally the third 5 to five 1’s. Finally I got rid of the 10, and started with five 5’s. Then four 5’s and five 1’s, then three 5’s, etc.”

Mo went on, “The triangular pattern of the 1’s was really cool. It helped me see that I hadn’t missed any. I also noticed that my list was in exactly the same order as Brooke’s list and Kaitlyn’s list, but we thought about the problem in totally different ways. I liked Kaitlyn’s explanation of freezing a column, and then unfreezing it and freezing it

```
10      10      5
10      10      |||||
10      5       5       5
10      5       5       |||||
10      5       |||||   |||||
10      |||||   |||||   |||||
5       5       5       5       5
5       5       5       5       |||||
5       5       5       |||||   |||||
5       5       |||||   |||||   |||||
5       |||||   |||||   |||||   |||||
|||||   |||||   |||||   |||||   |||||
```

again. I did the same thing, but without the columns. First I froze two 10's, then I froze one 10, and finally I froze zero 10's. I got the same answer of 12 ways."

Making systematic lists is a way to solve problems by organizing information. In this chapter, you'll make systematic lists to organize information in tables and charts. You will also learn a little about using a special type of diagram called a **tree diagram**. Many of the strategies you'll explore later in this book involve organizing information in some sort of table or chart, and you'll learn other strategies that involve organizing information spatially.

There is often more than one correct approach to solving a problem.

Remember that there is often more than one useful approach to solving a problem. This is often true with devising systematic lists. When you solved the Loose Change problem, you may have used a different list from what was shown. The four students here—Brooke, Heather, Kaitlyn, and Mo—all used systematic lists to solve the problem, but each list was different. They all incorporated the idea of **freezing** an entry in the list, and working with the other entries until that possibility was exhausted. Then they unfroze that entry, changed it, and froze it again. This idea of freezing and unfreezing is often a key element of systematic lists. Any list is fine so long as you have a system that you understand and can use effectively. If you find that your original system is too confusing, scrap it and start over with a different system.

Just as you can use the *same* strategy, such as making a list, to solve a problem in different ways, you will also often find that you can use *more than one strategy* to solve a given problem. In Chapter 1 you solved the Virtual Basketball League problem with a diagram. Solve the problem again, but this time use a systematic list. Don't refer back to the diagram solution!

VIRTUAL BASKETBALL LEAGUE

Andrew and his friends have formed a fantasy basketball league in which each team will play three games against each of the other teams. There are seven teams: the (Texas A&M) Aggies, the (Purdue) Boilermakers, the (Alabama) Crimson Tide, the (Oregon) Ducks, the (Boston College) Eagles, the (Air Force) Falcons, and the (Florida) Gators. How many games will be played in all? Do this problem before reading on.

Michael is a basketball player, and he's always interested in the matchups. In this problem there are seven teams, which Michael quickly assembled into pairs of teams for games:

Aggies vs Crimson Tide	Crimson Tide vs Ducks
Boilermakers vs Gators	Gators vs Aggies
Falcons vs Aggies	Crimson Tide vs Gators
Ducks vs Eagles	Boilermakers vs Aggies
Crimson Tide vs Gators	Falcons vs Eagles
Eagles vs Boilermakers	Ducks vs Gators
Crimson Tide vs Gators	Crimson Tide vs Aggies
Eagles vs Ducks	Ducks vs Boilermakers
Boilermakers vs Eagles	Gators vs Eagles

Don't list the same combination twice.

Is Michael's list systematic? Are all possible matchups represented? Does the list contain omissions or duplications?

Instead of trying to verify the accuracy of Michael's nonsystematic list, look at the first two columns of Monica's systematic list at right.

AB	BC
AC	BD
AD	BE
AE	BF
AF	BG
AG	

Monica represented each of the teams by the first letter of its name. For example, AB represents a matchup between the Aggies and the Boilermakers. She started her list by showing the matchups between the Aggies and the other six teams. In the second column of her list, she showed the matchups between the Boilermakers and the other teams. Note that she didn't include the matchup between the Aggies and the Boilermakers because she'd already shown it in the first column.

Look for patterns within your list.

She continued by listing, in order, the opposing teams for each remaining matchup. The complete list is shown below.

AB	BC	CD	DE	EF	FG
AC	BD	CE	DF	EG	
AD	BE	CF	DG		
AE	BF	CG			
AF	BG				
AG					

There are 21 different pairs of teams, and each pair played 3 games against each other. So to answer the question "How many games will be played in all?" multiply 21 by 3. The answer is 63 games.

Now compare Monica's solution to this problem's diagram solutions in Chapter 1. You can see that the diagram lines, which represent games, were drawn systematically so that they'd be easy to understand and follow. Diagrams are often systematic. Notice also that the diagram lines correspond exactly to the pairs in Monica's list.

PENNY'S DIMES, PART I

Nick's daughter Penny has 25 dimes. She likes to arrange them into three piles, putting an odd number of dimes into each pile. In how many ways could she do this? Solve this problem before continuing.

Randy solved this problem by making a systematic list of the possible combinations. He made three columns for his list and called them

Pile 1, Pile 2, and Pile 3. In the first row of the list, he indicated the first combination of dimes. He put 1 dime in the first pile and 1 dime in the second pile. This left 23 dimes for the third pile. In the second row, he started again with 1 dime in the first pile, then increased the second pile by 2 and decreased the third pile by 2. (Remember that each pile contains an *odd* number of dimes.) He continued in this way for a while, as shown above.

Pile 1	Pile 2	Pile 3
1	1	23
1	3	21
1	5	19
1	7	17
1	9	15
1	11	13
1	13	11

At this point in his list, Randy needed to decide whether or not 1, 13, 11 is a repeat of 1, 11, 13. In other words, is 13 in one pile and 11 in the other the same as 11 in one pile and 13 in the other? Randy decided that the piles were indistinguishable and therefore that these two combinations were indeed the same. He realized that crossing out repeats would save him a lot of work and make his list a lot shorter. So he crossed out the row with 1, 13, and 11. The next combination would be 1, 15, 9, which is a repeat of 1, 9, 15. So he concluded that he'd exhausted the combinations for 1 dime in the first pile.

Pile 1	Pile 2	Pile 3
1	1	23
1	3	21
1	5	19
1	7	17
1	9	15
1	11	13
~~1~~	~~13~~	~~11~~
~~3~~	~~1~~	~~21~~
3	3	19

Next he began finding combinations that started with 3 dimes in the first pile. The first combination he wrote down was 3, 1, 21. He quickly crossed out this combination because he realized that 3, 1, 21 was a repeat of the second combination in the list, 1, 3, 21. So he started with 3, 3, 19. He continued listing combinations with 3 dimes in the first pile until he reached 3, 11, 11. He stopped at this combination because he knew the next combination would be 3, 13, 9, which again would be a repeat.

Randy then moved on to listing combinations with 5 dimes in the first pile. To avoid repeating 5, 1, 19 and 5, 3, 17, he started his combinations with 5, 5, 15. He realized that when he changed the number in the first pile, he had to use that *same number* in the second pile to avoid repeating an earlier arrangement. He also noticed that he began to get repetitious combinations after the number in the second pile became as high as the number in the third pile. For example, when he reached 1, 13, 11, he had a repeat of 1, 11, 13. So here is the primary pattern present in this list: When moving from the first pile to the second pile to the third pile, the numbers cannot decrease. The second pile must be equal to or greater than the first pile, and the third pile must be equal to or greater than the second pile. This type of pattern can appear in many systematic lists.

Finding a primary pattern helps to shorten the list.

Randy continued with his list, using the pattern he'd discovered. When he began listing combinations with 9 dimes in the first pile, his first combination was 9, 9, 7. At that point his list was complete, because 9, 9, 7 is a repeat of 7, 9, 9.

Randy's complete list is shown at right. There are 16 ways to form three piles of dimes.

Pile 1	Pile 2	Pile 3
1	1	23
1	3	21
1	5	19
1	7	17
1	9	15
1	11	13
3	3	19
3	5	17
3	7	15
3	9	13
3	11	11
5	5	15
5	7	13
5	9	11
7	7	19
7	9	9

You can solve this problem differently by experimenting with other systems—we encourage you to do so. One possible system would begin with 23 dimes in the first pile. You might also decide to solve the problem again, but this time assume that the three piles are distinguishable, which leads to a much longer list that has 78 possibilities. You should make this list, too. You'll have to modify the system that Randy used, because it will no longer be true that 1, 3, 21 is the same as 3, 1, 21.

FRISBIN

On a famous episode of *Star Trek*, Captain Kirk and the gang played a card game called Phisbin. This problem is about another game, called Frisbin. The object of Frisbin is to throw three Frisbees at three different-sized bins that are set up on the ground about 20 feet away from the player. If a Frisbee lands in the largest bin, the player scores 1 point. If a Frisbee lands in the medium-sized bin, the player scores 5 points. If a Frisbee lands in the smallest bin, the player scores 10 points. Kirk McCoy is playing the game. If all three of his Frisbees land in bins, how many different total scores can he make? Make a systematic list for this problem before reading on.

You can make two different types of systematic lists for this problem. An example of each follows.

Derrick set up a list with columns titled 10 Points, 5 Points, 1 Point, and Total. He began by indicating the maximum number of 10-point throws: 3. He continued by indicating the other possible 10-point throws: 2, 1, and 0. In each row he adjusted the 5-point and 1-point throws so that three throws were always accounted for. After calculating all the point totals, Derrick concluded that Kirk McCoy can make ten different total scores.

10 POINTS	5 POINTS	1 POINT	TOTAL
3	0	0	30
2	1	0	25
2	0	1	21
1	2	0	20
1	1	1	16
1	0	2	12
0	3	0	15
0	2	1	11
0	1	2	7
0	0	3	3

Notice the system in the list. The 10 Points column starts with the highest possible number of throws, then decreases by 1. The column entry stays at each particular possible number of throws (3, then 2, then 1, and finally 0) as long as it can. The 5 Points column follows a similar process: It starts with the highest possible number of 5-point throws for each particular score and decreases by 1 each time. The 1 Point column makes up the difference in the scores.

Derrick made this list very quickly, and anyone seeing the list for the first time should immediately be able to follow the system. To help ensure that the system is evident, we have provided an explanation of it. In this course, when you write solutions to problems that you'll turn in to your instructor, you'll be asked to also provide a written explanation of your work. By explaining your work, you'll not only become a better problem solver, but you'll also become proficient at explaining your reasoning, which is a very valuable skill.

Julian used a different method, shown next. He labeled each column with the number of the three possible throws. Then he wrote down the points for each throw. Describe Julian's system.

THROW 1	THROW 2	THROW 3	TOTAL
10	10	10	30
10	10	5	25
10	10	1	21
10	5	5	20
10	5	1	16
10	1	1	12
5	5	5	15
5	5	1	11
5	1	1	7
1	1	1	3

Julian started by freezing the first two throws at 10 points. He then adjusted the third throw to include each possibility. Then he unfroze the second throw, changed it to 5 points, and froze it again while he adjusted the third throw to include each remaining possibility. (Note that he did not list 10, 5, 10 as a possibility, because that would be a repeat of the second entry in the list, 10, 10, 5.) Then he changed the second throw to 1, and this finished the possibilities where the first throw was 10. He then changed the first throw to 5, froze it again, and adjusted the other two columns. (Again, note that he did not list 5, 10, 10 as a possibility here, because that was included earlier as 10, 10, 5. A rearrangement of the same three numbers is not considered a new possibility, unless the order of the throws makes a difference.) Finally, he finished the list with 1, 1, 1. The point totals came out exactly the same as those in Derrick's list, but Julian's approach made it easier to add up the total scores.

Take another look at each entry in Julian's list. Do you notice anything? Study the list before reading on. You should notice a special property about each entry in the list. Reading left to right: the numbers in each row either remain the same or decrease. The numbers never get bigger going from left to right. So, for example, you will never see an entry like 10, 1, 5 or 5, 5, 10 in your list. Why? Think about it before reading on. Julian designed the system to prevent repeated entries. He froze the larger numbers at the beginning of each row and adjusted the other columns downward. Starting with 10, 10 in the first two columns, he adjusted column three going from 10 to 5 to 1. Then he continued to freeze column one at 10, while he changed column two to 5, but he could not begin column three with 10 because he would repeat an earlier entry. So he used only 5 and then 1 for column three.

This idea of a row never increasing is a property of many systematic lists of this type. You may remember that Randy found a similar pattern in his systematic list for Penny's Dimes Part 1 on page 35, but his list never decreased. For Frisbin you could also have created a list where you never decreased going from left to right. For example, try making a systematic list in a similar style to Julian's list, but begin with the entry 1, 1, 1. Then freeze the first two 1's, and change the third column to 5 and then to 10. Finish the list in this way, and you will notice that reading each row from left to right, you will never decrease.

Finally, let's look at this problem one more time, with a different system. Cali's list does not employ the idea described above of never decreasing or never increasing in each row. What system did Cali use in the list below? Before reading further, study her list to figure out her system.

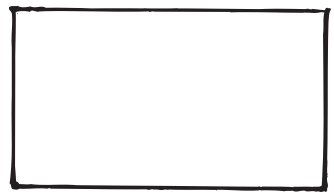
Throw 1	Throw 2	Throw 3	Total
10	10	10	30
5	5	5	15
1	1	1	3
10	10	5	25
10	10	1	21
5	5	10	20
5	5	1	11
1	1	10	12
1	1	5	7
1	5	10	16

Cali started by listing those situations where all three throws landed in the same bin. Then she listed the situations where two throws landed in the same bin. Finally, she listed the one possibility where all three landed in different bins.

AREA AND PERIMETER

A rectangle has an area measuring 120 square centimeters. Its length and width are whole numbers of centimeters. What are the possible combinations of length and width? Which possibility gives the smallest perimeter? Work this problem before continuing.

Tuan explained his solution for this problem: "I read that the area of the rectangle was 120 square centimeters. The first thing I did was to draw a picture of a rectangle.



"I had no idea whether this rectangle was long and skinny, or shaped like a square. But I did know that the area was supposed to be 120 square centimeters. So I made a list of whole-number pairs that could be multiplied to get 120.

WIDTH	LENGTH	AREA
1 cm	120 cm	120 cm^2
2 cm	60 cm	120 cm^2
3 cm	40 cm	120 cm^2
4 cm	30 cm	120 cm^2
5 cm	24 cm	120 cm^2
6 cm	20 cm	120 cm^2
8 cm	15 cm	120 cm^2
10 cm	12 cm	120 cm^2

"I knew I was done at this point because the next pair of factors of 120 is 12 and 10, which I'd already used. A 12-by-10 rectangle is the same as a 10-by-12 rectangle turned on its side, and I saw no need to list it twice. I also realized that neither 7 nor 9 would work for the width, because they don't divide evenly into 120.

"Now I had to find which possibility gives the smallest perimeter. I knew that the perimeter of a rectangle is the distance around the rectangle, so I needed to add up the length and width. But this would only give me half of the perimeter, so I would have to double the sum of the length and width. I added the Perimeter column to my chart."

WIDTH	LENGTH	AREA	PERIMETER
1 cm	120 cm	120 cm^2	242 cm
2 cm	60 cm	120 cm^2	124 cm
3 cm	40 cm	120 cm^2	86 cm
4 cm	30 cm	120 cm^2	68 cm
5 cm	24 cm	120 cm^2	58 cm
6 cm	20 cm	120 cm^2	52 cm
8 cm	15 cm	120 cm^2	46 cm
10 cm	12 cm	120 cm^2	44 cm

"Now I can see from my chart that the rectangle measuring 10 centimeters by 12 centimeters (which does have an area of 120 cm^2) gives the smallest perimeter of 44 centimeters."

WHICH PAPERS SHOULD KRISTEN WRITE?

For her Shakespeare course, Kristen is to read all five of the following plays and choose three of them to write papers about: Richard III, The Tempest, Macbeth, A Midsummer Night's Dream, and Othello. How many different sets of three books can Kristen write papers about? Do the problem before continuing.

Li explained her systematic list, shown at right: "I decided to abbreviate the names of the books so I wouldn't have to write out the whole names each time. I used R3, TT, Mac, AMND, and Oth. Then I just made a list. I made my list by letting R3 stay in front as long as it could and rearranged the other four books into the remaining two spots. Once I had all the combinations that include R3, I dropped it from the list. Then I used TT in the first spot and listed all the combinations that included it. Then I dropped TT, and finally I used Mac in the first spot. I listed the

R3	TT	Mac
R3	TT	Oth
R3	TT	AMND
R3	Mac	Oth
R3	Mac	AMND
R3	Oth	AMND
TT	Mac	Oth
TT	Mac	AMND
TT	Oth	AMND
Mac	Oth	AMND