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C. T. Sun, PhD, is Neil A. Armstrong Distinguished Professor Emeritus of Aeronautics and Astronautics at Purdue University. Dr. Sun was the inaugural recipient of the AIAA-ASC James H. Starnes Award and the 2007 ASME Warner T. Koiter Medal.

Ashfaq Adnan, PhD, is Professor in the Mechanical and Aerospace Engineering Department at the University of Texas at Arlington and a Fellow of ASME. His research focus is on deformation, damage, and failure of biological, bioinspired, and engineered materials at multiple length scales.

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THIRD EDITION

C. T. Sun and Ashfaq Adnan

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To my wife, Iris, and my children, Edna, Clifford, and Leslie – C.T. Sun

To my loving parents Afroza Nasreen and Dr. Md. Golbar Hussain, my beautiful wife, Most, and my sons, Aayan and Aayat – A. Adnan

Contents

Preface to the Third Edition

The purpose of the third edition is to correct some typographical errors in the second edition, add 3D elasticity equations, describe methods for structural idealization, and add a number of worked out and exercise problems. The Chapter 1 in the second edition is broken to two chapters. The Chapter 1 in the new edition includes discussions on the role of structural analysis in aircraft component design process, an overview on (i) fixed wing (ii) rotor craft (iii) lighter than air vehicles, and (iv) drones, a show the road map for developing simplified geometry for structural analysis, basic structural elements and control surfaces and materials. Example problems are added. Chapter 2 provides a brief overview of various types of mechanical loads such as axial, shear, torsion, and bending. Solved example problems include simple design problems including designing a pressurized thin walled cylinder (as simplified fuselage), thin cantilever beam (as simplified wing), designing against joint failure, etc. In Chapter 3, the concept of elasticity is elaborately discussed. A new road map is added to show how the concepts of statics, solid mechanics and elasticity are connected. The role of elasticity in aircraft structure design and its limitation are added. In Chapter 4, a discussion is added on the limitation of solid mechanics in describing torsion problems for noncircular section. Additional problems are added. In Chapter 5, a new discussion is added to describe structural idealizations. New example problems are added. The expansions in the remaining chapters are concentrated on new examples and exercise problems.

The authors are indebted to many students and colleagues for some corrections and valuable suggestions. In particular, Ashfaq Adnan is indebted to his former colleague late Dr. Wen Chan. Ashfaq Adnan is thankful to Aayan Adnan for his assistance in making many new drawings. Ms. Rajni Chahal and Dr. Wei-Tsen (Eric) Lu are acknowledged for their contributions in the worked-out problems and instruction materials.

Preface to the Second Edition

The purpose of the second edition is to correct a number of typographical errors in the first edition, add more examples and problems for the student, and introduce a few new topics, including primary warping, effects of boundary constraints, Saint-Venant's principle, the concept of shear lag, the Timoshenko beam theory, and a brief introduction to the effect of plasticity on fracture. All these additions are direct extensions of the existing contents in the first edition. Consequently, the background-building chapters, Chapters 1 and 2, need no modification. The expansions are concentrated in Chapters 3, 4, and 6 and amount to about a 25% increase in the number of pages.

The author is indebted to many students and colleagues for numerous corrections and valuable suggestions. He is indebted also to Dr. G. Huang for his assistance in making many new drawings.

Preface to the First Edition

This book is intended for junior or senior level aeronautical engineering students with a background in the first course of mechanics of solids. The contents can be covered in a semester at a normal pace.

The selection and presentation of materials in the course of writing this book were greatly influenced by the following developments. First, commercial finite element codes have been used extensively for structural analyses in recent years. As a result, many simplified ad hoc techniques that were important in the past have lost their useful roles in structural analyses. This development leads to the shift of emphasis from the problem-solving drill to better understanding of mechanics, developing the student's ability in formulating the problem, and judging the correctness of numerical results. Second, fracture mechanics has become the most important tool in the study of aircraft structure damage tolerance and durability in the past thirty years. It seems highly desirable for undergraduate students to get some exposure to this important subject, which has traditionally been regarded as a subject for graduate students. Third, advanced composite materials have gained wide acceptance for use in aircraft structures. This new class of materials is substantially different from traditional metallic materials. An introduction to the characteristic properties of these new materials seems imperative even for undergraduate students.

In response to the advent of the finite element method, consistent elasticity approach is employed. Multidimensional stresses, strains, and stress–strain relations are emphasized. Displacement, rather than strain or stress, is used in deriving the governing equations for torsion and bending problems. This approach will help the student understand the relation between simplified structural theories and 3-D elasticity equations.

The concept of fracture mechanics is brought in via the original Griffith's concept of strain energy release rate. Taking advantage of its global nature and its relation to the change of the total strain energies stored in the structure before and after crack extension, the strain energy release rate can be calculated for simple structures without difficulty for junior and senior level students.

The coverage of composite materials consists of a brief discussion of their mechanical properties in Chapter 1, the stress–strain relations for anisotropic solids in Chapter 2, and a chapter (Chapter 8) on analysis of symmetric laminates of composite materials. This should be enough to give the student a background to deal correctly with composites and to avoid regarding a composite as an aluminum alloy with the Young's modulus taken equal to the longitudinal modulus of the composite. Such a brief introduction to composite materials and laminates is by no means sufficient to be used as a substitute for a course (or courses) dedicated to composites.

A classical treatment of elastic buckling is presented in Chapter 7. Besides buckling of slender bars, the postbuckling concept and buckling of structures composed of thin sheets are also briefly covered without invoking an advanced background in solid mechanics. Postbuckling strengths of bars or panels are often utilized in aircraft structures. Exposure, even very brief, to this concept seems justified, especially in view of the mathematics employed, which should be quite manageable for student readers of this book.

The author expresses his appreciation to Mrs. Marilyn Engel for typing the manuscript and to James Chou and R. Sergio Hasebe for making the drawings.

C.T. Sun

About the Companion Website

This book is accompanied by a companion website:

www.wiley.com/go/Sun/aircraftstructures3

This website includes:

- Lecture slides
- A solutions manual

Scan this QR code to visit the companion website.

1 Characteristics of Aircraft Structures and Materials

1.1 INTRODUCTION

An aircraft is a vehicle that is used for flight in the air. A vehicle like this is typically built by assembling many component structures such as wing, fuselage, landing gears, stabilizers, etc. Each component structure is typically built by assembling many substructures. Each substructure can be made out of different materials. The main difference between aircraft structures and materials and civil engineering structures and materials lies in their weight. The main driving force in aircraft structural design and aerospace material development is to reduce weight. In general, materials with high stiffness, high strength, and light weight are most suitable for aircraft applications.

Aircraft structures must be designed to ensure that every part of the material is used to its full capability. A typical aircraft design cycle involves three major steps – (i) conceptual design, (ii) preliminary design, and (iii) detail design. In any of these design stages, different factors such as aerodynamics, avionics, propulsion, and structural integrity are simultaneously taken into account. As such, aircraft structures are not designed for structural safety and integrity only; many nonstructural requirements impose additional restrictions in designing aircraft structural components. For instance, an airfoil is chosen according to aerodynamic lift and drag characteristics. As such, the size and shape of an aircraft structural component are usually predetermined. Such restrictions significantly limit the number of solutions for structural problems in terms of global configurations. Often, the solutions resort to the use of special materials developed for applications in aerospace vehicles.

The nonstructural and weight-saving design requirements generally lead to the use of shell-like structures (monocoque constructions) and stiffened shell structures (semimonocoque constructions). The geometrical details of aircraft structures are much more complicated than those of civil engineering structures. They usually require the assemblage of thousands of parts. Technologies for joining the parts are especially important for aircraft construction.

Because of their high stiffness/weight and strength/weight ratios, aluminum and titanium alloys have been the dominant aircraft structural materials for many decades. However, the recent advent of advanced fiber-reinforced composites has changed the outlook. Composites may now achieve weight savings of 30–40% over aluminum or titanium counterparts. As a result, composites have been used increasingly in aircraft structures.

1.2 TYPES OF AIRCRAFT STRUCTURES

Most aircraft are built as fixed-wing vehicles and are commonly known as airplanes. Other categories include rotorcrafts, glider, lighter-than-air vehicles, etc. Presence of air is essential for generating lift on these vehicles. As such, structural design of such vehicles depends on how airload is transmitted to the structural elements.

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1.2.1 Fixed-Wing Aircraft

A fixed-wing aircraft is a kind of air vehicle that is heavier-than-air but can fly in the air by generating lift using the wings. An aircraft with a powered engine is generally called an airplane (Figure 1.1a). The unpowered version of fixed-wing aircraft is called gliders (Figure 1.1b).

1.2.2 Rotorcraft

A rotorcraft (Figure 1.1c) or rotary-wing aircraft is a heavier-than-air vehicle that generates lift using rotary wings or rotor blades, which revolve around a rotor. Depending on how rotor blades function, rotorcrafts are categorized as helicopters, autogyros, or gyrodynes. Recently, small-scale multirotor rotorcrafts are widely used for surveillance or video-capturing purposes. Designing blades for the rotorcraft is far more complex than designing a fixed-wing aircraft because of the complex aerodynamic forces.

1.2.3 Lighter-than-Air Vehicles

Aircraft such as balloons, nonrigid blimps, and airships (also known as dirigibles) are designed to contain sufficient amount of lighter-than-air gases (typically helium) so that lift can be generated from the lifting gas (Figure 1.2).

1.2.4 Drones

Drones (Figure 1.3) are small-scale air vehicles that can be fixed-wing type or rotary-wing type. The size of a drone is significantly smaller than a typical airplane or rotorcraft. As such, most drones are powered by electrical sources. Other than their size, the lifting mechanism of a drone is similar to the conventional fixed-wing or rotary-wing vehicles.

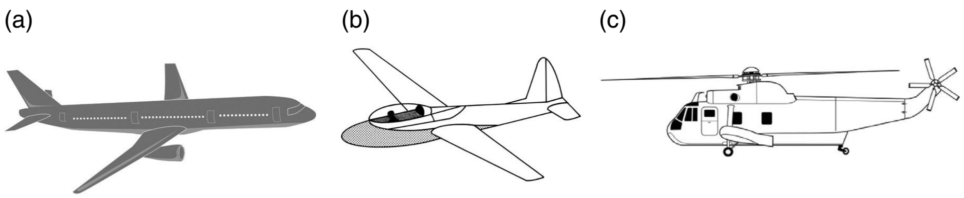

Fig. 1.1. (a) Powered fixed-wing aircraft, (b) glider, and (c) rotorcraft.

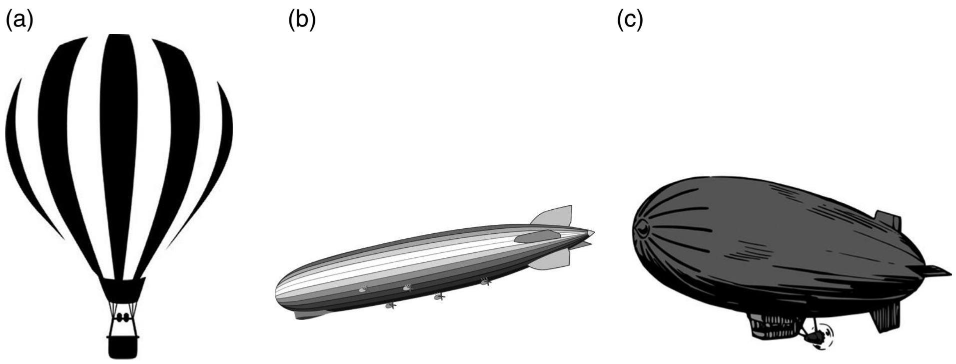

Fig. 1.2. Various lighter-than-air vehicles: (a) hot-air balloon, (b) blimp, and (c) dirigible.

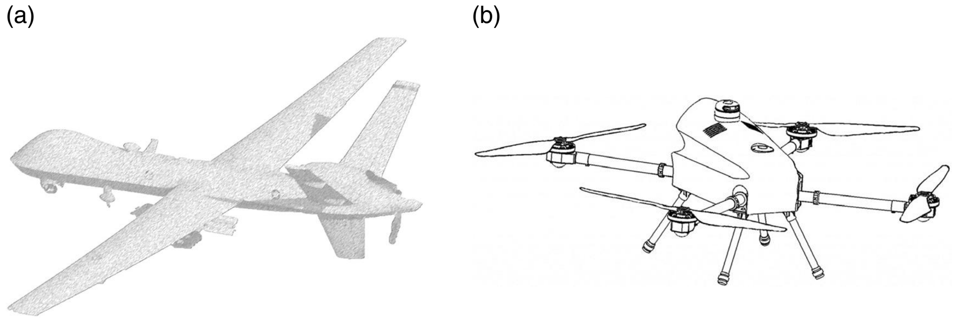

Fig. 1.3. (a) Fixed-wing drone; (b) multirotor rotary wing drone.

1.3 BASIC STRUCTURAL ELEMENTS IN AIRCRAFT STRUCTURE

An aircraft has many integrated parts, as shown in Figure 1.4. In general, these parts can be categorized into basic structural elements such as wing, fuselage, landing gears, tail units (horizontal and vertical stabilizers), and control surfaces such as aileron, rudder, and elevator.

1.3.1 Fuselage

The fuselage is the main structural element of a fixed-wing aircraft. It provides space for cargo, control system and pilots, passengers and cabin crews, and other accessories and equipment. In single-engine aircraft, the fuselage also carries the power plant. As shown in Figure 1.5, a fuselage can be constructed in various configurations such as truss, semimonocoque, and monocoque.

1.3.2 Wing

The main function of the wing is to pick up the air and power plant loads and transmit them to the fuselage. The wing cross-section takes the shape of an airfoil, which is designed based on aerodynamic considerations. In general, wings are constructed based on monospar, multispar, or box beam

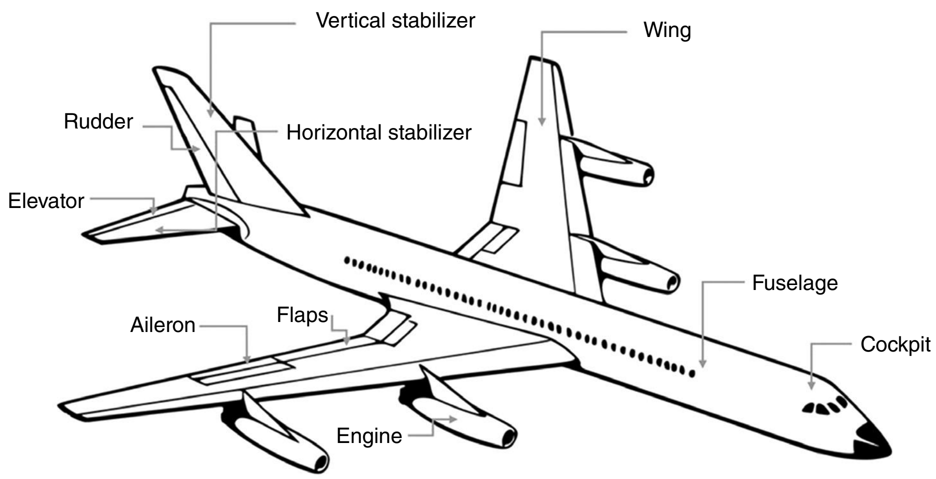

Fig. 1.4. Fixed-wing aircraft parts.

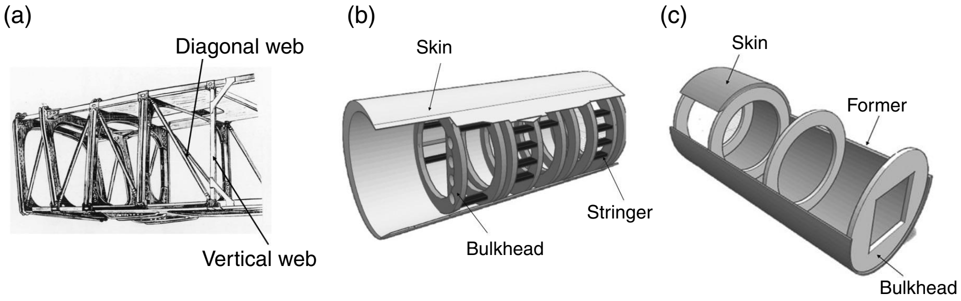

Fig. 1.5. Various fuselage configurations: (a) truss type, (b) semimonocoque type, and (c) monocoque type.

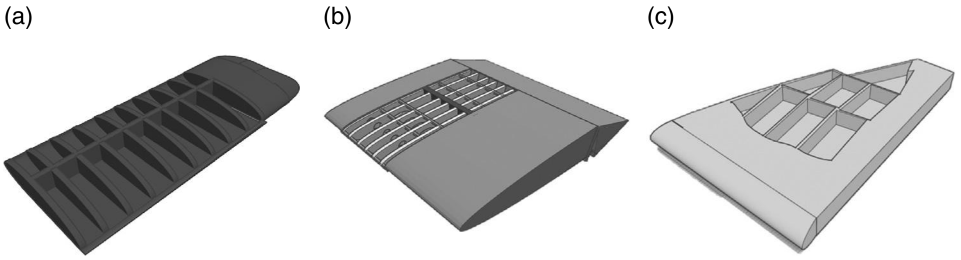

Fig. 1.6. Various wing configurations: (a) monospar (b) multispar type and (c) box beam type.

configurations, as shown in Figure 1.6. These three design configurations are considered as the basic designs, and aircraft manufacturers may adopt a modified configuration. In the monospar wing configuration, only one main spanwise member is present. Ribs or bulkheads are used to provide the necessary aerodynamic contour or shape to the airfoil. The multispar wing configuration has more than one main longitudinal member in its construction. To attain the desired aerodynamic shape, ribs or bulkheads are often included. The box beam wing configuration has two main longitudinal members and connecting bulkheads to attain the required airfoil contour.

1.3.3 Landing Gear

The landing gear is used to support an aircraft during landing and while it is on the ground. Small aircraft flying at low speeds generally have fixed gear. On the other hand, faster and more complex aircraft have retractable landing gear. To avoid parasite drag forces, the landing gear is retracted into the fuselage or wings after take-off.

1.3.4 Control Surfaces

Since an aircraft is free to rotate around three mutually perpendicular axes (longitudinal, transverse, and vertical) intersecting at its center of gravity (CG), a pilot must be able to control rotation about each of these axes to control overall position and direction of the aircraft. Aircraft flight control surfaces are aerodynamic devices that allow a pilot to maneuver and control the aircraft's flight in midair. As shown in Figure 1.4, there are three basic control surfaces, namely aileron, rudder, and elevator. Rotation about the transverse axis, defined by the line that passes through an aircraft from wingtip to wingtip, is called *pitch*. The elevators are the major control surfaces for pitch. Ailerons control the rotation about the longitudinal axis, called roll. This axis passes through the aircraft from nose to tail. The rotation about the vertical axis is called yaw, and the primary control of yaw is done with the rudder.

1.4 AIRCRAFT MATERIALS

Traditional metallic materials used in aircraft structures are aluminum, titanium, and steel alloys. In the past three decades, applications of advanced fiber composites have rapidly gained momentum. To date, some new commercial jets, such as the Boeing 787, already contain composite materials up to 50% of their structural weight.

Selection of aircraft materials depends on many considerations that can, in general, be categorized as cost and structural performance. Cost includes initial material cost, manufacturing cost, and maintenance cost. The key material properties that are pertinent to maintenance cost and structural performance are as follows:

- Density (weight)
- Stiffness (Young's modulus)
- Strength (ultimate and yield strengths)
- Durability (fatigue)
- Damage tolerance (fracture toughness and crack growth)
- Corrosion.

Seldom is a single material able to deliver all desired properties in all components of the aircraft structure. A combination of various materials is often necessary. Table 1.1 lists the basic mechanical properties of some metallic aircraft structural materials.

1.4.1 Steel Alloys

Among the three metallic materials, steel alloys have highest densities, and are used only where high strength and high yield stress are critical. Examples include landing gear units and highly loaded fittings. The high strength steel alloy 300 M is commonly used for landing gear components. This steel alloy has a strength of 1.9 GPa (270 ksi) and a yield stress of 1.5 GPa (220 ksi).

Besides being heavy, steel alloys are generally poor in corrosion resistance. Components made of these alloys must be plated for corrosion protection.

Table 1.1. Mechanical properties of metals at room temperature in aircraft structures.

	Property[a]				
	E		σ_u	σ_Y	ρ
Material	GPa (msi)	ν	MPa (ksi)	MPa (ksi)	g/cm^3 (lb/in^3)
Aluminum					
2024-T3	72 (10.5)	0.33	449 (65)	324 (47)	2.78 (0.10)
7075-T6	71 (10.3)	0.33	538 (78)	490 (71)	2.78 (0.10)
Titanium					
Ti-6Al-4V	110 (16.0)	0.31	925 (134)	869 (126)	4.46 (0.16)
Steel					
AISI4340	200 (29.0)	0.32	1790 (260)	1483 (212)	7.8 (0.28)
300 M	200 (29.0)	0.32	1860 (270)	1520 (220)	7.8 (0.28)

[a] σ_u, tensile ultimate stress; σ_Y, tensile yield stress.

1.4.2 Aluminum Alloys

Aluminum alloys have played a dominant role in aircraft structures for many decades. They offer good mechanical properties with low weight. Among the aluminum alloys, the 2024 and 7075 alloys are perhaps the most used. The 2024 alloys (2024-T3, T42) have excellent fracture toughness and slow crack growth rate as well as good fatigue life. The code number following T for each aluminum alloy indicates the heat treatment process. The 7075 alloys (7075-T6, T651) have higher strength than the 2024 but lower fracture toughness. The 2024-T3 is used in the fuselage and lower wing skins, which are prone to fatigue due to applications of cyclic tensile stresses. For the upper wing skins, which are subjected to compressive stresses, fatigue is less of a problem, and 7075-T6 is used.

The recently developed aluminum lithium alloys offer improved properties over conventional aluminum alloys. They are about 10% stiffer and 10% lighter and have superior fatigue performance.

1.4.3 Titanium Alloys

Titanium such as Ti–6Al–4V (the number indicates the weight percentage of the alloying element) with a density of 4.5 g/cm^3 is lighter than steel (7.8 g/cm^3) but heavier than aluminum (2.7 g/cm^3). See Table 1.1. Its ultimate and yield stresses are almost double those of aluminum 7075-T6. Its corrosion resistance in general is superior to both steel and aluminum alloys. While aluminum is usually not for applications above 350 °F, titanium, on the other hand, can be used continuously up to 1000 °F.

Titanium is difficult to machine, and thus the cost of machining titanium parts is high. Near net shape forming is an economic way to manufacture titanium parts. Despite its high cost, titanium has found increasing use in military aircraft. For instance, the F-15 contained 26% (structural weight) titanium.

1.4.4 Fiber-Reinforced Composites

Materials made into fiber forms can achieve significantly better mechanical properties than their bulk counterparts. A notable example is glass fiber versus bulk glass. The tensile strength of glass fiber can be two orders of magnitude higher than that of bulk glass. In this century, fiber science has made gigantic strides, and many high-performance fibers have been introduced. Listed in Table 1.2 are the mechanical properties of some high-performance manufactured fibers.

Fibers alone are not suitable for structural applications. To utilize the superior properties of fibers, they are embedded in a matrix material that holds the fibers together to form a solid body capable of carrying complex loads.

Table 1.2. Mechanical properties of fibers.

	Property		
	E	σ_u	ρ
Material	GPa (msi)	GPa (ksi)	g/cm^3
E-glass	77.0 (11)	2.50 (350)	2.54
S-glass	85.0 (12)	3.50 (500)	2.48
Silicon carbide (Nicalon)	190.0 (27)	2.80 (400)	2.55
Carbon (Hercules AS4)	240.0 (35)	3.60 (510)	1.80
Carbon (Hercules HMS)	360.0 (51)	2.20 (310)	1.80
Carbon (Toray T300)	240.0 (35)	3.50 (500)	1.80
Boron	385.0 (55)	3.50 (500)	2.65
Kevlar-49 (Aramid)	130.0 (18)	2.80 (400)	1.45
Kevlar-29	65.0 (9.5)	2.80 (400)	1.45

Table 1.3. Longitudinal mechanical properties of fiber composites.

Material	Type	Property E GPa (msi)	σ_u GPa (ksi)	ρ g/cm^3
Carbon/epoxy	T300/5208	140.0 (20)	1.50 (210)	1.55
	IM6/3501-6	177.0 (25.7)	2.86 (414)	1.55
	AS4/3501-6	140.0 (20)	2.10 (300)	1.55
Boron/aluminum	B/Al 2024	210.0 (30)	1.50 (210)	2.65
Glass/epoxy	S2 Glass/epoxy	43.0 (6.2)	1.70 (245)	1.80
Aramid/epoxy	Kev 49/epoxy	70.0 (10)	1.40 (200)	1.40

Matrix materials that are currently used for forming composites include three major categories: polymers, metals, and ceramics. The resulting composites are usually referred to as polymer matrix composites (PMCs), metal matrix composites (MMCs), and ceramic matrix composites (CMCs). Table 1.3 presents properties of a list of composites. The range of service temperature of a composite is often determined by its matrix material. PMCs are usually for lower temperature (less than 300 °F) applications, and CMCs are intended for applications in hot (higher than 1500 °F) environments, such as jet engines.

Fiber composites are stiff, strong, and light and are thus most suitable for aircraft structures. They are often used in the form of laminates that consist of a number of unidirectional laminae with different fiber orientations to provide multidirectional load capability. Composite laminates have excellent fatigue life, damage tolerance, and corrosion resistance. Laminate constructions offer the possibility of tailoring fiber orientations to achieve optimal structural performance of the composite structure.

PROBLEMS

1.1 Specific modulus of a material is defined by Young's modulus divided by density. Specific strength is also defined the same manner. Compare the specific moduli and specific strengths of carbon composites, aluminum, titanium, and steel. Which one performs better?

1.2 Consider a commercial jetliner that weighs about 200 000 kgs. It is found that 80% of the structure is made of 2024-T3 aluminum and 15% is made of carbon/epoxy composites. The remaining 5% part of the airplane is equally distributed by Ti-6Al-4V and AISI4340. If the jetliner is now redesigned with the following material distributions, what will be the new weight of the airplane? How much weight is saved, percentage-wise?

Carbon composite = 65%

Aluminum = 30%

Titanium = 2.5%

Steel = 2.5%.

2 Loads on Aircraft Structures

2.1 INTRODUCTION

Aircraft structures are required to support mainly two classes of structural loads – in-ground loads and in-flight loads. The in-ground loads are imposed on the aircraft during various ground activities such as taxiing, cargo/passenger loading/unloading, towing, and hoisting. The in-flight loads are imposed on the aircraft structures due to aerodynamic loads. To maintain desired cabin pressure at a higher altitude, certain aircraft structure is also pressurized. In general, loads on aircraft structures can be felt in the form of axial tension and compression, shear, bending, and torsional loads. Each aircraft structural element is designed to take specific types of these loads. Collectively, these elements can efficiently provide the capability for sustaining loads on an airplane.

Designing the structural members against specific type of loading and deformation requires stress analysis. In general, a structural analysis roadmap incorporates the concept of statics, mechanics of solids, and advanced concepts such as elasticity. Employment of suitable failure, fracture, and fatigue theories will allow a stress analyst design a structural member. As such, it is essential to understand how stress–strain relation can be established for the basic structural elements. The governing equations for these basic structural elements are introduced in the elementary courses such as statics and mechanics of solids (also known as solid mechanics or mechanics of materials). In the following subsections, the governing equations are reviewed briefly and their behavior discussed.

2.2 BASIC STRUCTURAL ELEMENTS

2.2.1 Axial Member

Axial members are used to carry extensional or compressive loads applied in the direction of the axial direction of the member. The resulting stress is uniaxial:

$$\sigma = E\varepsilon \tag{2.1}$$

where E and ε are the Young's modulus and normal strain, respectively, in the loading direction. The total axial force F provided by the member is

$$F = A\sigma = EA\varepsilon = EA\frac{\delta}{L} \tag{2.2}$$

where A is the cross-sectional area of the member, δ is the axial displacement, and L is the length of the axial member. The quantity EA is termed the axial stiffness of the member, which depends on the modulus of the material and the cross-sectional area of the member. It is obvious that the axial stiffness of axial members cannot be increased (or decreased) by changing the shape of the cross-section. In other words, a circular rod and a channel (see Figure 2.1a,b) will carry the same axial load as long as they have the same cross-sectional area.

Axial members are usually slender and are susceptible to buckling failure when subjected to compression. It is known that the buckling load is directly proportional to the bending stiffness and

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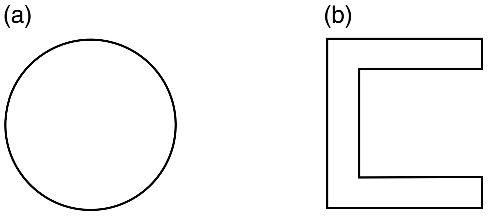


Fig. 2.1. (a) Circular rod; (b) channel.

inversely proportional to the square of the effective length. Buckling strength can be increased by increasing the bending stiffness and by shortening the length of the buckle mode. For buckling, the channel section is better since it has higher bending stiffness than the circular section. However, because of the slenderness of most axial members used in aircraft (such as stringers), the bending stiffness of these members is usually very small and is not sufficient to achieve the necessary buckling strength. In practice, the buckling strength of axial members is enhanced by providing lateral supports along the length of the member with more rigid ribs (in wings) and frames (in fuselage).

EXAMPLE 2.1

The AISI4340 steel rod is subjected to the loading shown in Figure 2.2. If the cross-sectional area of the rod is 50 mm^2 determine the displacement of its end D.

The normal forces developed in sections AB, BC, and CD are in the free body diagrams. The modulus of elasticity E for AISI4340 steel is 200 MPa.

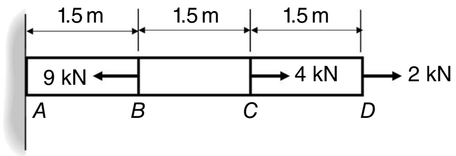


Fig. 2.2. Axially loaded steel rod.

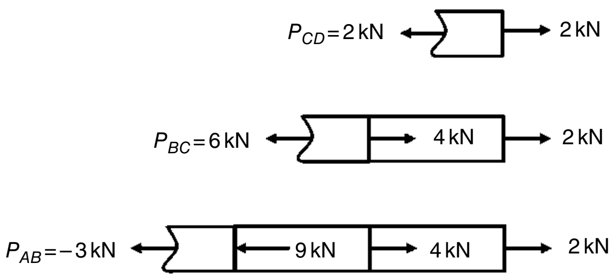
The cross-sectional area of all the sections are as follows:

$$A = 50\,\text{mm}^2 = 50 \times 10^{-6}\,\text{m}^2$$

$$\delta_D = \sum \frac{P_i L_i}{E_i A_i} = \frac{L}{AE}\left(P_{AB} + P_{BC} + P_{CD}\right)$$

$$= \frac{1.5}{\left(50 \times 10^{-6}\right)\left(200 \times 10^{6}\right)}\left(-3 + 6 + 2\right) = 0.75\,\text{mm}$$

Free body diagram:



2.2.2 Shear Panel

A shear panel is a thin sheet of material used to carry in-plane shear load. Consider a shear panel of uniform thickness t under uniform shear stress τ as shown in Figure 2.3. The total shear force in the x-direction provided by the panel is given by

$$V_x = \tau ta = G\gamma ta \tag{2.3}$$

where G is the shear modulus and γ is the shear strain. Thus, for a flat panel, the shear force V_x is proportional to its thickness and the lateral dimension a.

For a curved panel under a state of constant shear stress τ (see Figure 2.4), the resulting shear force of the shear stress on the thin-walled section may be decomposed into a horizontal component V_x and a vertical component V_y as

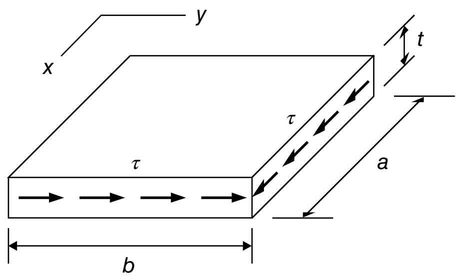


Fig. 2.3. Shear panel under uniform shear stress.

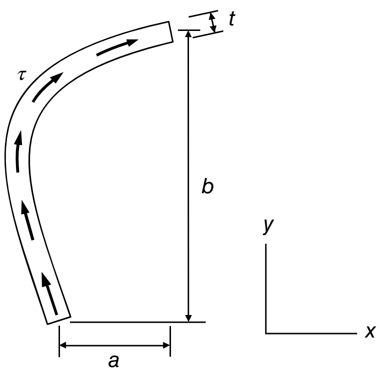


Fig. 2.4. Curved panel under a state of constant shear stress.

$$V_x = \tau ta \tag{2.4}$$

$$V_y = \tau tb \tag{2.5}$$

Thus, the components of the resultant force of the shear stress τ have the relation

$$\frac{V_x}{V_y} = \frac{a}{b}$$

Since this relation does not depend on the contour shape of the section of the panel, a flat panel would be the most efficient (in material usage) in providing a shear force for given values of a and b.

2.2.3 Bending Member (Beam)

A structural member that can carry bending moments is called a **beam**. A beam can also act as an axial member carrying longitudinal tension and compression. According to simple beam theory, bending moment M is related to beam deflection w as

$$M = -EI\frac{d^2w}{dx^2} \tag{2.6}$$

where EI is the bending stiffness of the beam. The area moment of inertia I depends on the geometry of the cross-section.

Except for pure moment loading, a beam is designed to carry both bending moments and transverse shear forces as the latter usually produce the former. For a beam of a large span/depth ratio, the bending stress is usually more critical than the transverse shear stress. This is illustrated by the example of a cantilever beam shown in Figure 2.5.

It can be found that the maximum bending moment and bending stress occur at the fixed root of the cantilever beam. We have

$$\sigma_{\max} = \frac{M_{\max}(h/2)}{I} = \frac{VL(h/2)}{bh^3/12} = \frac{6VL}{bh^2} \tag{2.7}$$

The transverse shear stress distribution is parabolic over the beam depth with maximum value occurring at the neutral plane, i.e.

$$\tau_{\max} = \frac{3}{2}\frac{V}{bh} \tag{2.8}$$

From the ratio

$$\frac{\sigma_{\max}}{\tau_{\max}} = \frac{4L}{h} \tag{2.9}$$

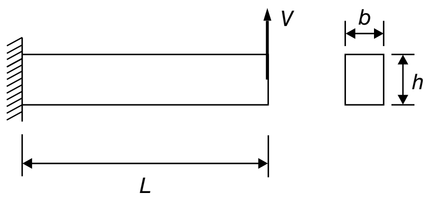


Fig. 2.5. Cantilever beam.

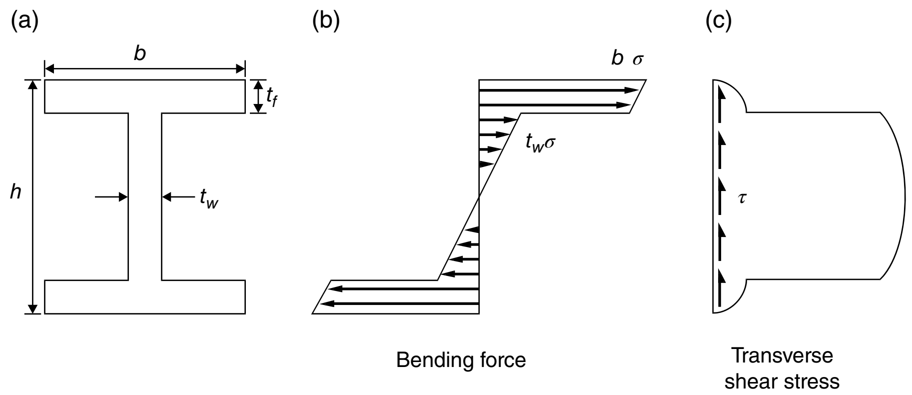


Fig. 2.6. (a) Wide-flange beam; (b) bending force distribution; (c) shear stress distribution.

it is evident that bending stress plays a more dominant role than transverse shear stress if the span-to-depth ratio is large (as in wing structure). For such beams, attention is focused on optimizing the cross-section to increase bending stiffness.

In the elastic range, bending stress distribution over depth is linear with maximum values at the farthest positions from the neutral axis. The material near the neutral axis is underutilized. Thus, the beam with a rectangular cross-section is not an efficient bending member.

In order to utilize the material to its full capacity, material in a beam must be located as far as possible from the neutral axis. An example is the wide-flange beam shown in Figure 2.6a. Although the bending stress distribution is still linear over the depth, the bending line force (bending stress times the width) distribution is concentrated at the two flanges as shown in Figure 2.6b because $b \gg t_w$. For simplicity, the small contribution of the vertical web to bending can be neglected.

The transverse shear stress distribution in the wide-flange beam is shown in Figure 2.6c. The vertical web is seen to carry essentially all the transverse shear load; its variation over the web is small and can be practically assumed to be constant. For all practical purposes, the wide-flange beam can be regarded as two axial members (flanges) connected by a flat shear panel.

2.2.4 Torsion Member

Torque is an important form of load to aircraft structures. In a structural member, torque is formed by shear stresses acting in the plane of the cross-section. Consider a hollow cylinder subjected to a torque T as shown in Figure 2.7. The torque-induced shear stress τ is linearly distributed along the radial direction. The torque is related to the twist angle θ per unit length as

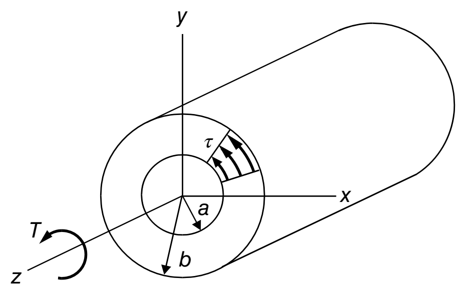


Fig. 2.7. Hollow cylinder subjected to a torque.

$$T = GJ\theta \tag{2.10}$$

where J is the torsional constant. For hollow cylinders, J is equal to the polar moment of inertia of the cross-section, i.e.

$$J = I_p = \frac{1}{2}\pi\left(b^4 - a^4\right) = \frac{1}{2}\pi(b-a)(b+a)\left(b^2 + a^2\right) \tag{2.11}$$

The term GJ is usually referred to as torsional stiffness.

If the wall thickness $t = b - a$ is small compared with the inner radius, then an approximate expression of J is given by

$$J = 2t\pi\bar{r}^3 \tag{2.12}$$

where $\bar{r} = (a+b)/2$ is the average value of the outer and inner radii. Thus, for a thin-walled cylinder, the torsional stiffness is proportional to the 3/2 power of the area $\left(\pi\bar{r}^2\right)$ enclosed by the wall.

EXAMPLE 2.2

Compare the torsional stiffness of the solid circular section and the thin hollow sections shown in Figure 2.8. Both sections are made of 2024-T3 aluminum.

The shear modulus of 2024-T3 aluminum is 27 GPa.

The polar moment of inertia J for the circulation section is

$$J_{\text{solid}} = \frac{\pi(1)^4}{2} = 0.5\pi\ \text{cm}^4$$

The polar moment of inertia J for the thin hollow section is

$$J_{\text{hollow}} = 2(0.1)(\pi)\left(5^3\right) = 25\pi\ \text{cm}^4$$

Obviously, the thin hollow section will have 50 times higher torsional stiffness (GJ) than the solid section.

The area ratio between these two sections are

$$\frac{A_{\text{hollow}}}{A_{\text{solid}}} = \frac{2\pi\bar{r}t}{\pi r^2} = \frac{2\bar{r}t}{r^2} = \frac{1\ \text{cm}^2}{4\ \text{cm}^2} = 0.25$$

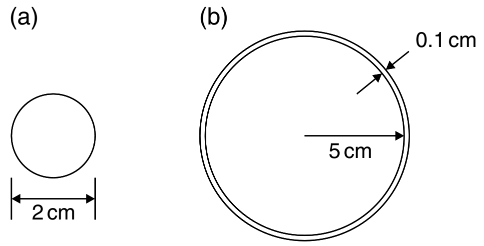


Fig. 2.8. Torsion member with (a) a solid section and (b) hollow section.

It means the hollow section is not only four times lighter than the solid section, it has 50 times higher resistance to torsional deformation. As such, this example illustrates that a thin-walled structure can be made into a very efficient torsion member.

2.3 WING AND FUSELAGE

The wing and fuselage are the two major airframe components of an airplane. The horizontal and vertical tails bear close resemblance to the wing. Hence, these two components are taken for discussion to exemplify the principles of structural mechanics employed in aircraft structures.

2.3.1 Load Transfer

Wing and fuselage structures consist of a collection of basic structural elements. Each component, as a whole, acts like a beam and a torsion member. For illustrative purposes, let us consider the box beam shown in Figure 2.9. The box beam consists of stringers (axial members) that are located at the maximum allowable distance from the neutral axis to achieve the most bending capability, and the thin skin (shear panel), which encloses a large area to provide a large torque capability. The design of Figure 2.10 would be fine if the load is directly applied in the form of global torque T and bending moment M_x. In reality, aircraft loads are in the form of air pressure (or suction) on the skin, concentrated loads from the landing gear, power plants, passenger seats, etc. These loads are to be "collected" locally and transferred to the major load-carrying members. Without proper care, these loads may produce excessive local deflections that are not permissible from aerodynamic considerations.

Using the box beam of Figure 2.9 as an example, we assume that a distributed air pressure is applied on the top and bottom surfaces of the beam. The skin (shear panel) is thin and has little bending stiffness to resist the air pressure. To avoid incurring large deflections in the skin, longitudinal stringers (stiffeners) can be added, as shown in Figure 2.10, to pick up the air loads. These stiffeners are usually slender axial members with a moderate amount of bending stiffness. Therefore, the transverse loads picked up by the stiffeners must be transferred "quickly" to more rigid ribs or frames at sections A and B (see Figure 2.9) to avoid excessive deflections. The ribs collect all transverse loads from the stiffeners and transfer them to the two wide-flange beams (spars) that are designed to take transverse shear loads. The local-to-global load transfer is thus complete. Note that besides serving as a local load distributor, the stiffeners also contribute to the total bending capability of the box beam.

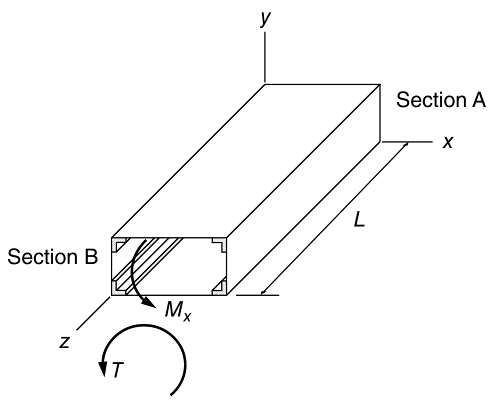


Fig. 2.9. Box beam.

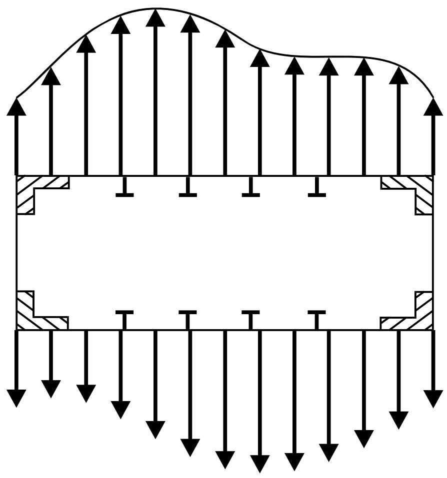


Fig. 2.10. Longitudinal stringers in a box beam.

2.3.2 Wing Structure

The wing as a whole performs the combined function of a beam and a torsion member. It consists of axial members in stringers, bending members in spars, and shear panels in the cover skin and webs of spars. The spar is a heavy beam running spanwise to take transverse shear loads and spanwise bending. It is usually composed of a thin shear panel (the web) with a heavy cap or flange at the top and bottom to take bending. A typical spar construction is depicted in Figure 2.11. A multiple-spar wing construction is shown in Figure 2.12.

Wing ribs are planar structures capable of carrying in-plane loads. They are placed chordwise along the wing span. Besides serving as load redistributers, ribs also hold the skin stringer to the designed contour shape. Ribs reduce the effective buckling length of the stringers (or the stringer-skin system) and thus increase their compressive load capability. Figure 2.12 shows a typical rib construction. Note that the rib is supported by spanwise spars.

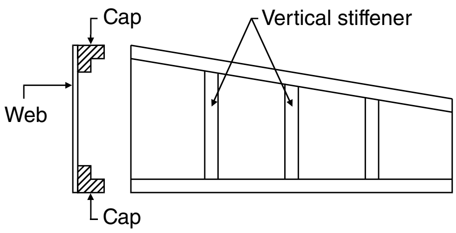


Fig. 2.11. Typical spar construction.

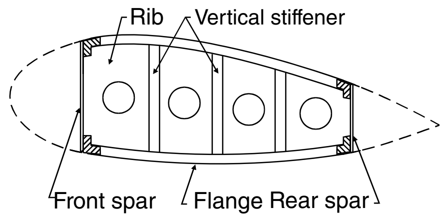


Fig. 2.12. Typical rib construction.

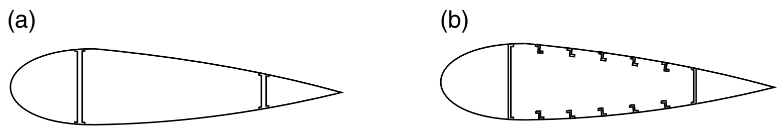


Fig. 2.13. Typical two-spar wing cross-sections for subsonic aircraft: (a) spars only; (b) spars and stringers.

The cover skin of the wing together with the spar webs form an efficient torsion member. For subsonic airplanes, the skin is relatively thin and may be designed to undergo postbuckling. Thus, the thin skin can be assumed to make no contribution to bending of the wing box, and the bending moment is taken by spars and stringers. Figure 2.13 presents two typical wing cross-sections for two-spar subsonic aircraft. One type (Figure 2.13) consists only of spars (the concentrated flange type) to take bending. The other type (the distributed flange type, Figure 2.13) uses both spars and stringers to take bending.

Supersonic airfoils are relatively thin compared with subsonic airfoils. To withstand high surface air loads and to provide additional bending capability of the wing box structure, thicker skins are often necessary. In addition, to increase structural efficiency, stiffeners can be manufactured (either by forging or machining) as integral parts of the skin.

2.3.3 Fuselage

Unlike the wing, which is subjected to large distributed air loads, the fuselage is subjected to relatively small air loads. The primary loads on the fuselage include large concentrated forces from wing reactions, landing gear reactions, and pay loads. For airplanes carrying passengers, the fuselage must also withstand internal pressures. Under normal operating conditions, most commercial airliners fly between 31 000 and 38 000 feet. Air density gradually reduces as altitude increases. As such, the partial pressure of oxygen also reduces with altitude. Prolonged exposure to such reduced oxygen environment is dangerous to human health and can be lethal. As such, pressurization inside the cabin (i.e. fuselage) is necessary at altitude above 10 000 feet. To specifically determine how much air needs to be pumped into the cabin, the functional relation between the barometric pressure with altitude is required:

$$P(H) = P_0 \exp\left(-\frac{Mg}{RT}H\right) \tag{2.13}$$

here, $P(H)$ is the barometric pressure, $P_0 = P(H = 0)$ is the average sea level atmospheric pressure = 101.325 kPa, M is the molar mass of air = ~0.029 kg/mol, R is the universal gas constant = 8.314 J/K mol, and T is temperature in Kelvin, which also depends on altitude and generally reduces by about 6.5 °C for every 1000 feet altitude hike. Using Eq. (2.13), it is possible to find the required pressure, $P_{req} = P(H) - P_0$ to bring the cabin pressure at a desired pressure level. For example:

$$P_{req}(H = 5000\ \text{ft}) = 47.25\ \text{kPa}$$

$$P_{req}(H = 15\,000\ \text{ft}) = 89.2\ \text{kPa}$$

$$P_{req}(H = 38\,000\ \text{ft}) = 101\ \text{kPa}$$

These estimations are made assuming desired cabin pressure is same as in-ground atmospheric pressure (101.3 kPa). In reality, most commercial airliners maintain lower than atmospheric pressure, something in the range of 60 kPa when they fly at maximum altitude allowed.

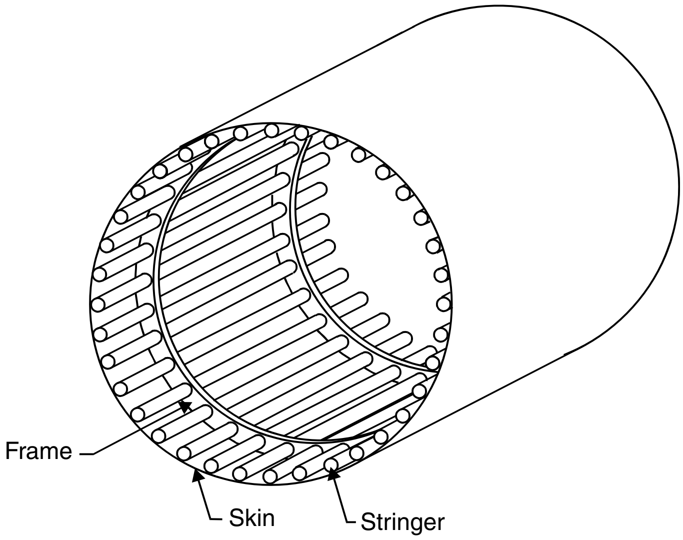


Fig. 2.14. Fuselage structure.

Because of internal pressure, the fuselage undergoes hydrostatic expansion. For a very long fuselage flying at a high altitude, the expansion could be unachievable unless low cabin pressure is maintained or aircraft fuselage is built with composite materials that offer low thermal expansion. In general, the fuselage has an efficient circular cross-section with a semimonocoque construction consisting of a thin shell stiffened by longitudinal axial elements (stringers and longerons) supported by many transverse frames or rings along its length; see Figure 2.14. The fuselage skin carries the shear stresses produced by torques and transverse forces. It also bears the hoop stresses produced by internal pressures. The stringers carry bending moments and axial forces. They also stabilize the thin fuselage skin.

EXAMPLE 2.3

Consider a sealed thin-walled cylindrical shell of radius = 2 m, thickness = 10 mm, and length = 50 m. If the cylinder is 11 500 m above the ground, determine how much stress will be developed on the cylinder if it needs to maintain atmospheric pressure inside. If the cylinder is made of aluminum, what will be size of the cylinder at 11 500 m. Consider air pressure drops by 101 kPa below the atmospheric ground pressure when measured 11 500 m above the ground.

Since the radius (2 m) to thickness (10 mm) ratio of this cylinder is very large, the cylinder can be treated as a thin-walled cylinder (if the ratio is more than 10, the cylinder can be considered thin-walled).

As such, the developed longitudinal and circumferential stresses would be

$$\sigma_{\text{longitudinal}} = \sigma_l = \frac{pr}{2t} = \frac{101 \times 10^3 \times 2}{2 \times 10 \times 10^{-3}} = 10.1\ \text{MPa}$$

$$\sigma_{\text{circumferential}} = \sigma_t = \frac{pr}{t} = \frac{101 \times 10^3 \times 2}{10 \times 10^{-3}} = 20.2\ \text{MPa}$$

The direction of the these stresses are shown in Figure 2.15. Since both stresses are tensile, pressurizing the cylinder with 101 kPa pressure will cause hydrostatic expansion of the cylinder. Total lengthwise expansion will be

$$\delta_l = \frac{\sigma_l}{E} L = \frac{10.1 \times 10^6 \times 50}{70 \times 10^9} = 7\ \text{mm}$$

Fuselage frames often take the form of a ring. They are used to maintain the shape of the fuselage and to shorten the span of the stringers between supports in order to increase the buckling strength of the stringer. The loads on the frames are usually small and self-equilibrated. Consequently, their constructions are light. To distribute large concentrated forces such as those from the wing structure, heavy bulkheads are needed. Figure 2.16 shows the fuselage of a Boeing 777 under construction.

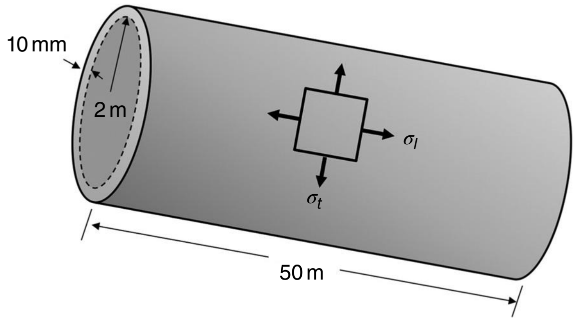


Fig. 2.15. Thin-walled pressurized cylinder.

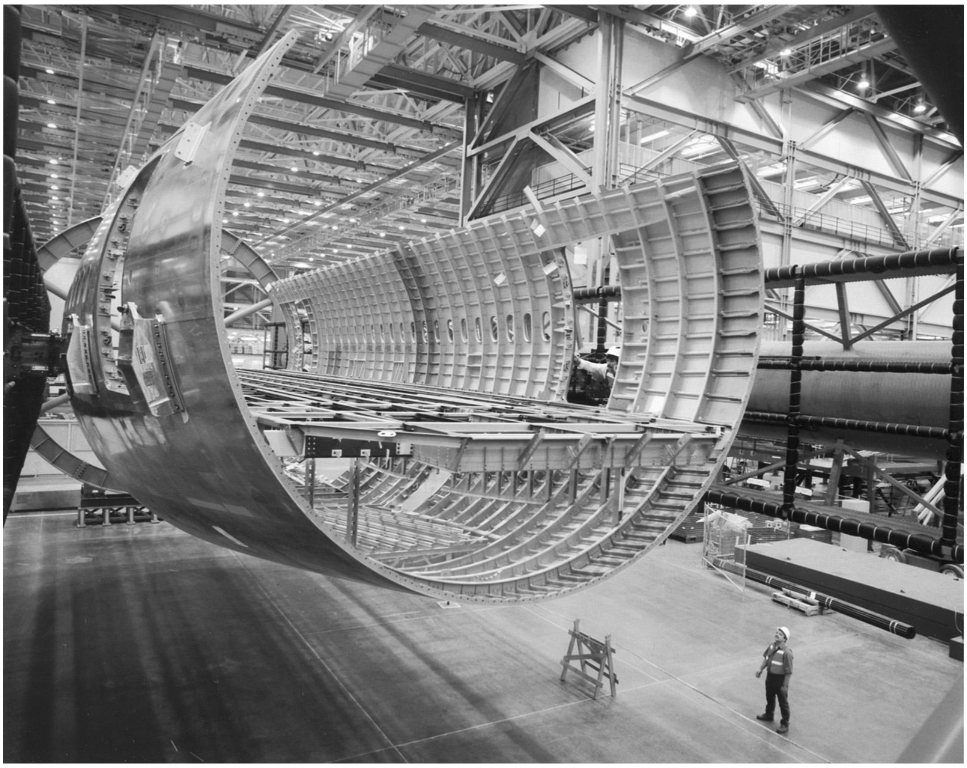


Fig. 2.16. Fuselage of a Boeing 777 under construction. *Source:* Courtesy of the Boeing Company.

PROBLEMS

2.1 The beam of a rectangular thin-walled section (i.e. t is very small) is designed to carry both bending moment M and torque T. If the total wall contour length $L = 2(a + b)$ (see Figure 2.17) is fixed, find the optimum b/a ratio to achieve the most efficient section if $M = T$ and $\sigma_{\text{allowable}} = 2\tau_{\text{allowable}}$. Note that for closed thin-walled sections such as the one in Figure 2.17, the shear stress due to torsion is

$$\tau = \frac{T}{2abt}$$

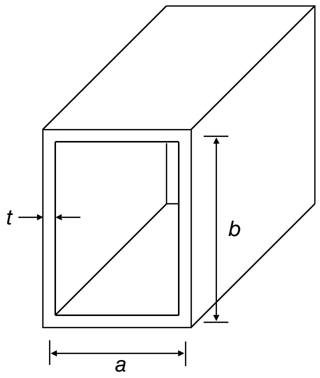


Fig. 2.17. Closed thin-walled section.

2.2 Do Problem 2.1 with $M = \alpha T$, where $\alpha = 0$ to ∞.

2.3 The dimensions of a steel (300 M) I-beam are $b = 50$ mm, $t = 5$ mm, and $h = 200$ mm (Figure 2.18). Assume that t and h are to be fixed for an aluminum (7075-T6) I-beam. Find the width b for the aluminum beam so that its bending stiffness EI is equal to that of the steel beam. Compare the weights-per-unit length of these two beams. Which is more efficient weightwise?

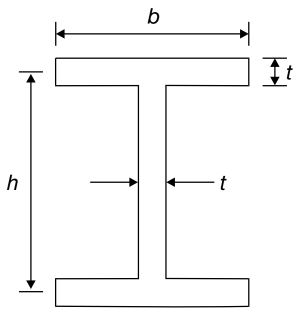


Fig. 2.18. Dimensions of the cross-section of an I-beam.

2.4 Use AS4/3501-6 carbon/epoxy composite to make the I-beam as stated in Problem 2.3. Compare its weight with that of the aluminum beam.

2.5 Derive the relations given by (2.4) and (2.5).

2.6 The sign convention (positive directions of resultants) used in the beam theory depends on the coordinate system chosen. Consider the moment–curvature relation

$$M = -EI\frac{d^2w}{dx^2}$$

in reference to the coordinate system shown in Figure 2.19. If w is regarded as a positive displacement (or deflection) in the positive y-direction, find the positive direction of the bending moment. State the reason.

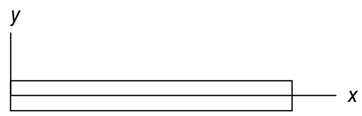


Fig. 2.19. Coordinate system for a beam.

2.7 Compare the load-carrying capabilities of two beams having the respective cross-sections shown in Figure 2.20. Use bending rigidity as the criterion for comparison. It is given that $a = 4$ cm, $t = 0.2$ cm, and the two cross-sections have the same area.

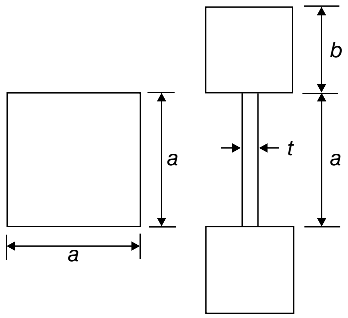


Fig. 2.20. Cross-sections of two beams.

2.8 The assembly shown in Figure 2.21 is fastened together by two bolts made of 7075-T6 aluminum. Determine the required diameter of the bolts if the assembly is subjected to the loading as shown. Use a factor of safety = 1.4. The shear strength of 7075-T6 aluminum is 330 MPa.

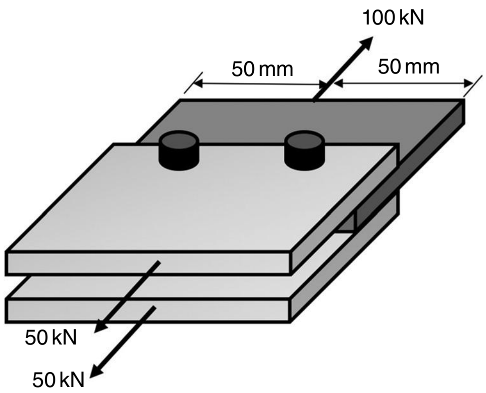


Fig. 2.21. A joint fastened together using two bolts.

2.9 A thin-walled cylinder (radius = 1 m) is constructed of 10-mm thick aluminum plates (2024-T3) that are fastened together at their ends with two 5-mm steel cover plates (AISI4340), as shown in Figure 2.22. The assembly is joined using rivets having a diameter of 10 mm and spaced 50 mm apart. If internal pressure of the cylinder is 10 MPa, determine (a) the circumferential stress in the cylinder and (b) the shear stress in the rivets.

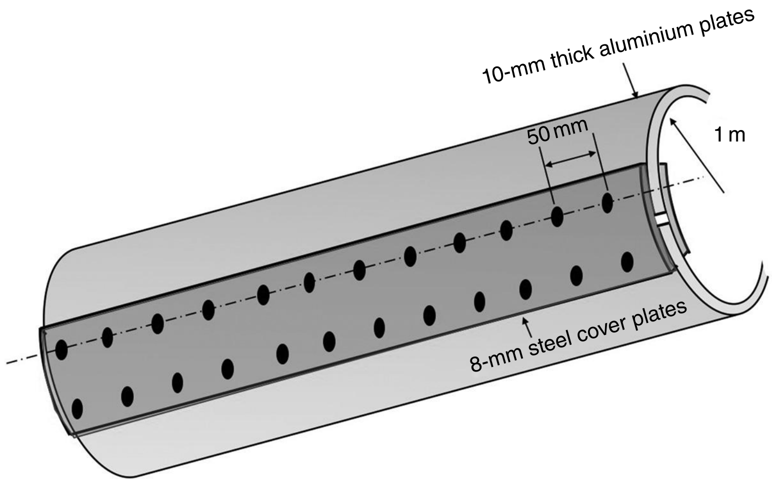


Fig. 2.22. A thin-walled cylinder fastened together with cover plates.

2.10 Consider the rivets in the Problem 2.9. Determine the maximum spacing between the rivets so that internal pressure is still safe.

2.11 Consider the rivets in the Problem 2.9. If the rivet spacing is fixed, determine the minimum rivet diameter so that the internal pressure is safe.

2.12 Two beams, *AC* and *CD*, are hinged at *C* through a frictionless pin. Beam *CD* is cantilevered at *D* and Beam *AC* is simply supported at *A*. Draw shear force and bending moment diagram for the connected beam and identify the location of maximum shear force and bending moment.

3 Introduction to Elasticity

3.1 INTRODUCTION

The primary design requirement of aircraft structural components is to ensure that the designed structures do not fail or excessively deform due to various ground and air loads. From mechanics point of view, a basic structural design process starts by establishing the relation between external forces with the internal resultants. The external loads are typically defined in the form of prescribed displacements, traction/stress vector, point forces, moments, or distributed, normal, or shear forces. For structural analysis, the resultant forces and moments are expressed in terms of stress components. In a deformable body, the internal resultant forces and moments deform the body. The states of deformation are expressed in terms of strain components. Depending on the external loading type and geometry of the structure, the applied loading can lead to uniform or nonuniform internal stress distributions. Various branches of mechanics are simultaneously utilized to analytically perform a basic structural design process.

Theory of elasticity is a branch of mechanics that describes the elastic relation between stress and strain of a deformable body in equilibrium. The mathematical formulations allow solving problems including uniform and nonuniform distributions of stresses. Nonuniform distribution of stresses in a deformable body generally comes from material nonhomogeneity (e.g. presence of holes, flaws, sharp changes in geometry, etc.) or loading types. In principle, theory of elasticity requires basic understanding of displacement, forces, stress, and strains in terms of mathematical equations. As such, fundamental understanding in statics and solid mechanics is essential to analyze the stress–strain relation in deformable bodies using the theory of elasticity.

Statics is the branch of mechanics concerned with the analysis of loads (force, moments) on physical systems in static equilibrium. The foundation of statics is essentially an extension of Newton's first law. Using statics, it is possible to obtain internal forces and moments on a deformable body subjected to external forces and moments. However, it is not possible to relate forces with deformation using statics. To obtain such relation, the concepts from mechanics of solid is utilized.

Solid mechanics is the branch of mechanics that describes the relation between stress (obtained from the internal forces/moments of a body) and strain (obtained from the associated deformation in the body) of a body in static equilibrium. The relation between stress and strain is established using Hooke's law where properties of materials such as Young's modulus (E) and Poisson's ratio (v) appear as the "bridge" between stress and strain. Therefore, properties of materials must be known if one wants to employ the concepts of solid mechanics.

The governing equations in the theory of elasticity are built upon displacement-based formulations or stress-based formulations. In the displacement formulation, the displacement vectors are obtained first. Then, strain–displacement relations are established in terms of differential equations. Using the generalized Hooke's law, the stress components are obtained. In the **stress formulation**, stresses are obtained first, and then strains are computed using the stress–strain relations. Finally, displacements are obtained using the strain–displacement relations.

The major steps in the displacement and stress formulations are shown in Figure 3.1.

The theory of elasticity plays an important role for stress analysis in aircraft structural components. However, such theory is only applicable to simple structural components, and hence, not suitable for

Mechanics of Aircraft Structures, Third Edition. C. T. Sun and Ashfaq Adnan.

Companion website: www.wiley.com/go/Sun/aircraftstructures3

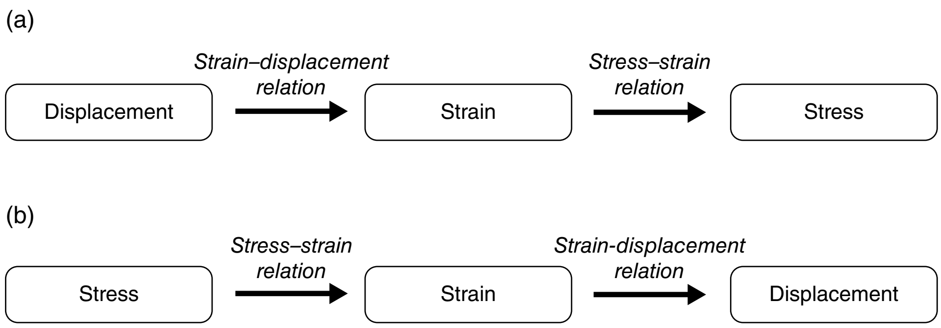


Fig. 3.1. (a) Displacement formulation and (b) stress formulation approaches in theory of elasticity.

analyzing real aerospace structures. Advanced, experimentally validated computational methods such as finite element analysis are generally used for this purpose. The theory of elasticity is often employed as a guideline for advanced computational methods. In this chapter, the governing equations of elasticity are developed using the displacement-based formulations.

3.2 CONCEPT OF DISPLACEMENT

Consider a material point P at the position $\mathbf{x}(x, y, z)$ before deformation (see Figure 3.2). After deformation, P moves to a new position $P'(x', y', z')$. The change of position during deformation, which is measured in terms of the displacement vector $\mathbf{u}$, has three components: u, v, and w in the x, y, and z directions, respectively. The new location of the point (x, y, z) after deformation is given by

$$\begin{aligned} x' &= x + u \quad \text{or} \quad & u &= x' - x \\ y' &= y + u & v &= y' - y \\ z' &= z + w & w &= z' - z \end{aligned} \tag{3.1}$$

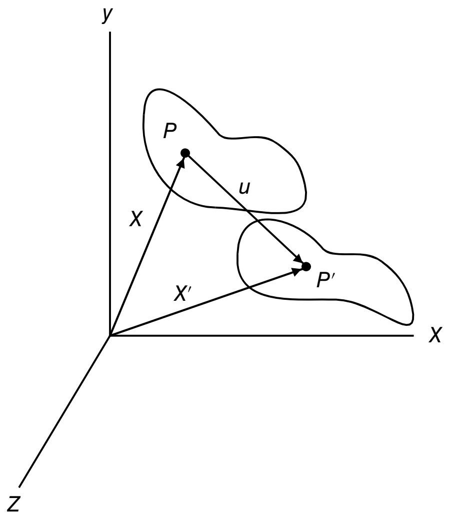


Fig. 3.2. Displacement of material point P after deformation.

Thus, the deformed configuration is uniquely defined if the displacement components u, v, and w are given everywhere in the body of interest.

Consider an axial member (i.e. a one-dimensional, 1-D, body) of original length L_0. Assume the axial strain to be uniform in the member. Then the axial strain everywhere in the member is calculated by

$$\varepsilon = \frac{\Delta L}{L_0} \tag{3.2}$$

where ΔL is the total elongation of the member. The elongation ΔL can be regarded as the difference in displacement $u_1 = u(x_1)$ at the right end and $u_0 = u(x_0)$ at the left end (see Figure 3.3), i.e.

$$\Delta L = u_1 - u_0$$

The function $u(x) = u_0 + \varepsilon_0(x - x_0)$ gives the axial displacement at any point x in the axial member.

If the strain is not uniform, then (Eq. (3.2)) gives an average strain. To determine the strain at a point, a small segment $L_0 = \Delta x$ must be considered. Consider two points x_0 and $x_0 + \Delta x$ that are separated by a small distance Δx. Let the displacements at these two points be

$$u_0 = u(x_0)$$

and

$$u_1 = u(x_0 + \Delta x)$$

respectively. The difference in displacement between these two points is

$$\Delta u = u_1 - u_0 = u(x_0 + \Delta x) - u(x_0) \tag{3.3}$$

which can also be regarded as the elongation of the material between these two points. The axial strain in this segment (or at point x_0) is defined as

$$\varepsilon = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{du}{dx} \tag{3.4}$$

Thus, axial strain can be obtained from the derivative of the displacement function.

If a rod is subjected to a uniform tension and $\varepsilon = \varepsilon_0 =$ constant, then

$$\frac{du}{dx} = \varepsilon_0, \qquad x_0 \le x \le x_0 + L_0$$

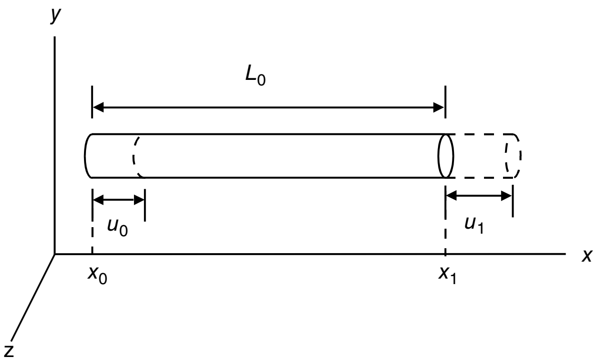


Fig. 3.3. Elongation.

Integrate the equation above to obtain

$$u = \varepsilon_0 x + C$$

Let $u(x_0) = u_0$; then, from the equation above, $C = u_0 - \varepsilon_0 x_0$, and the displacement function is given by

$$u = \varepsilon_0 (x - x_0) + u_0 \tag{3.5}$$

3.3 STRAIN

Consider two points P and Q in a solid body. The coordinates of P and Q are (x, y, z) and $(x + \Delta x, y, z)$, respectively. The distance between the two points before deformation is Δx (see Figure 3.4).

After deformation, let the displacement of P in the x-direction be $u = u(x, y, z)$ and of Q be $u' = u(x + \Delta x, y, z)$. The new distance between these two points (P' and Q') in the x-direction after deformation is

$$(x + \Delta x + u') - (x + u) = \Delta x + \Delta u \tag{3.6}$$

where $\Delta u \equiv u' - u$ is the *change of length* in the x-direction for material connecting P and Q after deformation. The strain is defined just as in an axial member:

$$\varepsilon_{xx} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x} \tag{3.7}$$

This is the x-component of the normal strain, which measures the *deformation in the x-direction* at a point (x, y, z).

Similarly, the y and z components of the normal strain at the point are given by

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \tag{3.8}$$

and

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \tag{3.9}$$

respectively. Comparing the strain component ε_{xx} with the strain in the 1-D case (or in an axial member), we may interpret ε_{xx} as the elongation per unit length of an "infinitesimal" axial element of the material at a point (x, y, z) in the x-direction. Similar interpretations can be given to ε_{yy} and ε_{zz}.

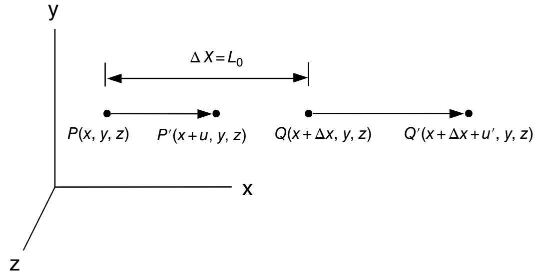


Fig. 3.4. Neighboring points P and Q in a solid body.

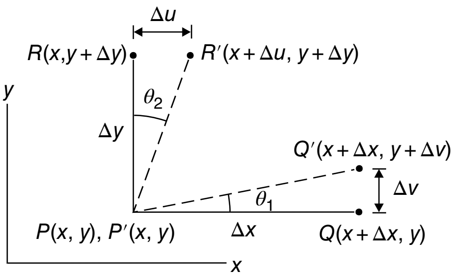


Fig. 3.5. Rotations of material line elements in the *x–y* plane.

The three normal strain components are not sufficient to describe a general state of deformation in a 3-D body. Additional shear strain components are needed to describe the distortional deformation.

For simplicity, let us consider a 2-D case. Let P, Q, R be three neighboring points all lying on the $x-y$ plane as shown in Figure 3.5. Let P', Q', and R' be the corresponding positions after deformation. For no loss of generality, we assume that P does not move. In addition, we assume that the infinitesimal elements $\overline{PQ}$ and $\overline{PR}$ do not experience any elongation. Thus, the positions of P' and Q' (see Figure 3.5) are given by

$$Q': \quad (x + \Delta x, y + \Delta v)$$
$$R': \quad (x + \Delta u, y + \Delta y)$$

For Q', the displacement increment Δv is

$$\Delta v = \underbrace{v(x + \Delta x, y)}_{\text{displacement at } Q} - \underbrace{v(x, y)}_{\text{displacement at } P} \tag{3.10}$$

Similarly for R', the displacement increment Δu can be written as

$$\Delta u = u(x, y + \Delta y) - u(x, y) \tag{3.11}$$

The rotations θ_1 and θ_2 of elements $\overline{PQ}$ and $\overline{PR}$ are assumed to be small and are given by

$$\theta_1 = \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = \frac{\partial v}{\partial x}$$

and

$$\theta_2 = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta y} = \frac{\partial u}{\partial y}$$

respectively. The total change of angle between $\overline{PQ}$ and $\overline{PR}$ after deformation is defined as the shear strain component in the *x–y* plane:

$$\gamma_{xy} = \gamma_{yx} \equiv \theta_1 + \theta_2 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{3.12}$$

Similar shear strain components in the *y–z* plane and *x–z* plane are defined as

$$\gamma_{zy} = \gamma_{yz} \equiv \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \tag{3.13}$$

$$\gamma_{zx} = \gamma_{xz} \equiv \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \tag{3.14}$$

Thus, a general state of deformation at a point in a solid is described by three normal strain components ε_{xx}, ε_{yy}, ε_{zz} and three shear strain components γ_{xy}, γ_{yz}, γ_{xz}.

3.3.1 Rigid Body Motion

If a body undergoes a displacement without inducing strains in the body, then the motion is a rigid body motion. For instance, the displacements

$$\begin{aligned} u &= u_0 = \text{constant} \\ v &= v_0 = \text{constant} \\ w &= w_0 = \text{constant} \end{aligned}$$

represent a rigid body translational motion and do not yield any strains.

Another rigid body motion is the rigid body rotation. The following displacements represent a rigid body rotation in the x–y plane.

$$\begin{aligned} u &= -\alpha y \\ v &= \alpha x \\ w &= 0 \end{aligned} \tag{3.15}$$

It is easy to verify that no strains are associated with the displacement field above.

EXAMPLE 3.1

Consider a 2-D body (a unit square ABCD) in the x–y plane as shown in Figure 3.6. After deformation, the four corner points move to A', B', C', and D', respectively, due to the displacement field given by

$$\begin{aligned} u &= 0.01y \\ v &= 0.015x \end{aligned} \tag{3.16}$$

Describe the nature of the deformation.

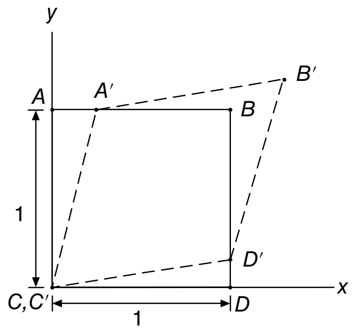


Fig. 3.6. Shear deformation in the x–y plane.

Using (Eq. (3.1)), the new position of point A after deformation is given by

$$x' = 0 + u|_{x=0,y=1} = 0.01$$
$$y' = 1 + v|_{x=0,y=1} = 1 + 0 = 1$$
$$\text{New coordinates of } A' : (0.01, 1)$$

Similarly, we obtain the new positions of B, C, and D.

$$B' : (1.01, 1.015)$$
$$C' : (0, 0)$$
$$D' : (1, 0.015)$$

Since the deformed configuration is linear in x and y, it can be determined from the new positions A', B', C', and D' as shown by the dashed lines in Figure 3.6.

The strains corresponding to the displacements given by Eq. (3.16) are

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = 0, \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} = 0$$
$$\gamma_{xy} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0.015 + 0.01 = 0.025$$

This is a simple shear deformation.

For comparison, consider the following displacement field,

$$u = 0.2y$$

$$v = -0.2x$$

Using (Eq. (3.1)), the new positions A', B', C', and D' can be obtained as shown by the dashed lines in Figure 3.7.

It can be easily shown that the displacement field does not yield any strain component. As such, this is a rigid body rotation.

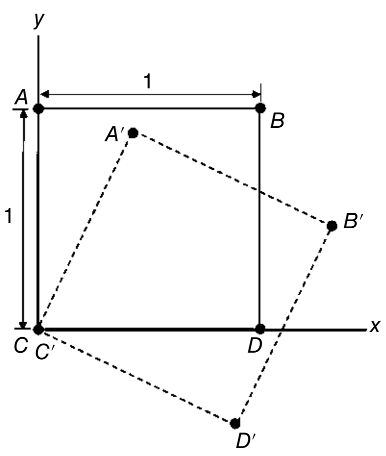


Fig. 3.7. Rigid body rotation.

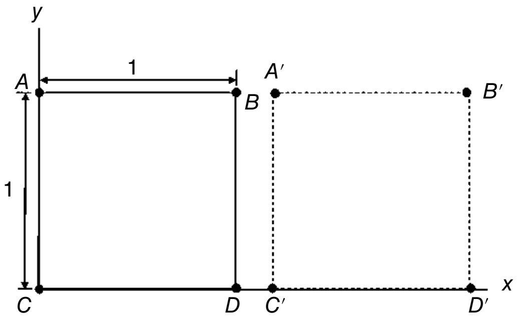


Fig. 3.8. Rigid body translation.

The following displacement field

$$u = 1.2$$

$$v = 0$$

which does not yield any strain as well, is a rigid body translation (Figure 3.8).

3.4 STRESS

For an axial member, the force is always parallel to the member, and the stress is defined as

$$\sigma = \frac{P}{A} \tag{3.17}$$

where A is the cross-sectional area. If $A = 1$ unit area, then $\sigma = P$.

The concept of stress can easily be extended to 3-D bodies subjected to loads applied in arbitrary directions. Consider an infinitesimal plane surface of area ΔA with a unit normal vector **n**. The total resultant force acting on this area is **ΔF** (force is a vector; see Figure 3.9). The **stress vector t** is defined as

$$\mathbf{t} = \lim_{\Delta A \to 0} \frac{\Delta \mathbf{F}}{\Delta A} \tag{3.18}$$

Thus, **t** can be considered as the force per unit area acting on the given plane surface.

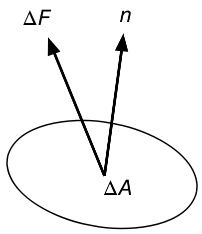


Fig. 3.9. Total resultant force on an area.

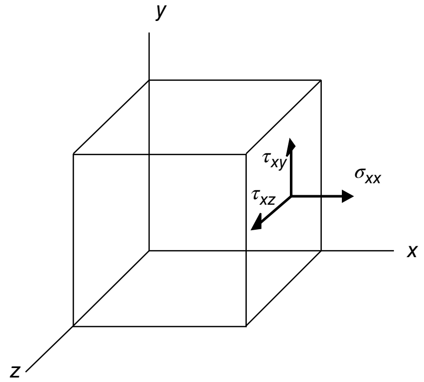


Fig. 3.10. Stress vector with three components on the x-face.

Consider the special plane surface with the unit normal vector parallel to the x-axis. On this face, the stress vector **t** has three components, which are denoted by σ_{xx}, τ_{xy}, τ_{xz} as shown in Figure 3.10. Similarly, on the y and z faces the force intensities are given by the components of the respective stress vectors as

$$\sigma_{yy}, \tau_{yx}, \tau_{yz} \text{ and } \sigma_{zz}, \tau_{zx}, \tau_{zy}.$$

The two subscripts in each stress component have special meaning. The first subscript refers to the plane where the stresses originate, and the second subscript denotes the direction along which the stresses act. For example, the symbol τ_{xy} refers to a stress component that originates from the plane normal to the x-axis and acts along the y-axis direction. In a cubic stress element, there are six planes, and three stress components can be originated from each plane. Of the six planes, any plane for which the normal is directed toward the positive x-, y-, or z-axis is considered as the positive plane. Otherwise, the plane is a negative plane. If the direction of a stress component is along the positive x-, y-, or z-axis, then the direction is a positive direction, otherwise it is a negative direction. Collectively, any stress component will be considered positive if it originates from a positive plane and acts along a positive direction or originates from a negative plane and acts toward a negative direction. Otherwise, it will be a negative stress component.

It is evident that for arbitrary applied loadings, a cubic stress element can have at most eighteen stress components. Depending on the geometry or applied loading types, the stresses inside a body could be uniform or nonuniform.

Consider a thin rectangular plate of width b and thickness t is subjected to a tensile force P (uniformly distributed) as shown in Figure 3.11a. The internal stress $\sigma = \frac{P}{bt}$ is uniform everywhere. Now consider the same plate with a circular hole of radius a at the center, as shown in Figure 3.11b. When this plate is subjected to the tensile force P, the stresses near the hole is not uniform. In both scenarios, these stresses must be in static equilibrium.

3.5 EQUATIONS OF EQUILIBRIUM IN A UNIFORM STRESS FIELD

Consider an infinitesimal solid element under a state of uniform stress. The stress element shown in Figure 3.11a is an example of such element. The stress components on the six faces of this element are shown in Figure 3.12. Since the body is in equilibrium, the six equations of equilibrium must be satisfied, i.e.

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum F_z = 0 \tag{3.19}$$

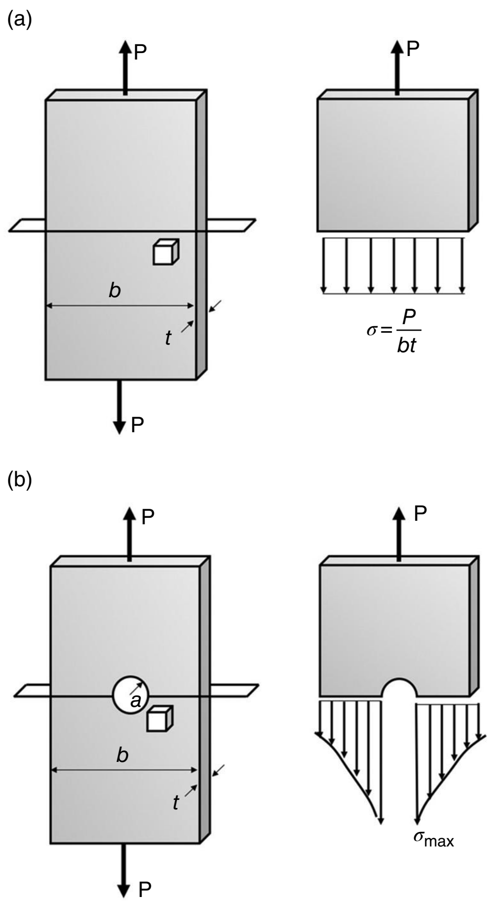


Fig. 3.11. Example of how (a) uniform and (b) nonuniform stress fields are developed within a structure subjected to same loading.

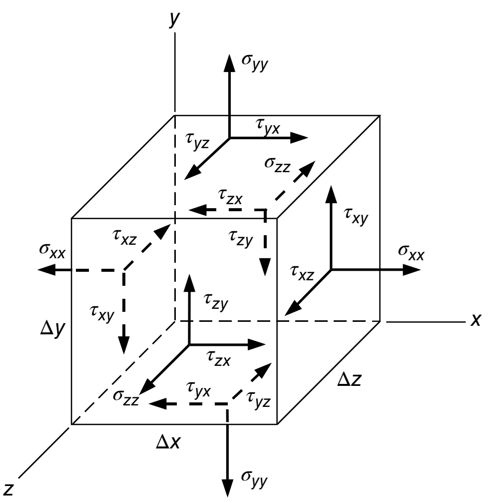


Fig. 3.12. Infinitesimal solid element under uniform stress.

$$\sum M_x = 0, \quad \sum M_y = 0, \quad \sum M_z = 0 \tag{3.20}$$

The force Eq. (3.19) can be expanded as:

$$\sum F_x = (\sigma_{xx} - \sigma_{xx})\Delta y \Delta z + (\tau_{yx} - \tau_{yx})\Delta z \Delta x + (\tau_{zx} - \tau_{zx})\Delta x \Delta y = 0$$

$$\sum F_y = (\sigma_{yy} - \sigma_{yy})\Delta z \Delta x + (\tau_{zy} - \tau_{zy})\Delta x \Delta y + (\tau_{xy} - \tau_{xy})\Delta y \Delta z = 0$$

$$\sum F_z = (\sigma_{zz} - \sigma_{zz})\Delta x \Delta y + (\tau_{yz} - \tau_{yz})\Delta z \Delta x + (\tau_{xz} - \tau_{xz})\Delta y \Delta z = 0$$

It is obvious that the force Eq. (3.19) are satisfied automatically.

The moment equation M_x can be expressed by taking the moment (with respect to the center of the solid element)

$$\sum M_x = \tau_{zy}\Delta x \Delta y.\frac{\Delta z}{2} + \tau_{zy}\Delta x \Delta y.\frac{\Delta z}{2} - \tau_{yz}\Delta x \Delta z.\frac{\Delta y}{2} - \tau_{yz}\Delta x \Delta z.\frac{\Delta y}{2} = (\tau_{zy} - \tau_{yz})\Delta x \Delta y \Delta z$$

Similarly, the moment equations M_y and M_z are

$$\sum M_y = (\tau_{zx} - \tau_{xz})\Delta x \Delta y \Delta z$$

$$\sum M_z = (\tau_{yx} - \tau_{xy})\Delta x \Delta y \Delta z$$

To satisfy the moment Eq. (3.20), the following relations among the shear stress components are necessary:

$$\tau_{xy} = \tau_{yx}, \quad \tau_{yz} = \tau_{zy}, \quad \tau_{xz} = \tau_{zx} \tag{3.21}$$

Thus, only six stress components are independent, including three normal stress components σ_{xx}, σ_{yy}, σ_{zz} and three shear stress components, say, τ_{yz}, τ_{xz}, τ_{xy}.

3.6 EQUATIONS OF EQUILIBRIUM IN A NONUNIFORM STRESS FIELD

Consider an infinitesimal element $\Delta x \times \Delta y \times \Delta z$ in which the stress field is not uniform. Figure 3.11b shows how nonuniform stress field can be developed inside a material. The stress components acting on the faces of the element are shown in Figure 3.13.

If the element is in equilibrium, then the six equations of equilibrium, (Eqs. (3.19) and (3.20)), must be satisfied. Consider one of the equations of equilibrium, say, $\Sigma F_x = 0$. We have

$$\begin{aligned}(\sigma_{xx} + \Delta\sigma_{xx})\Delta y\, \Delta z - (\sigma_{xx})\Delta y\, \Delta z \\ + (\tau_{yx} + \Delta\tau_{yx})\Delta x\, \Delta \mathrm{z} - (\tau_{yx})\Delta x\, \Delta z \\ + (\tau_{zx} + \Delta\tau_{zx})\Delta x\, \Delta \mathrm{y} - (\tau_{zx})\Delta x\, \Delta \mathrm{y} \\ = 0\end{aligned}$$

Dividing the equation above by Δx, Δy, Δz, we obtain

$$\frac{\Delta\sigma_{xx}}{\Delta x} + \frac{\Delta\tau_{yx}}{\Delta y} + \frac{\Delta\tau_{zx}}{\Delta z} = 0$$

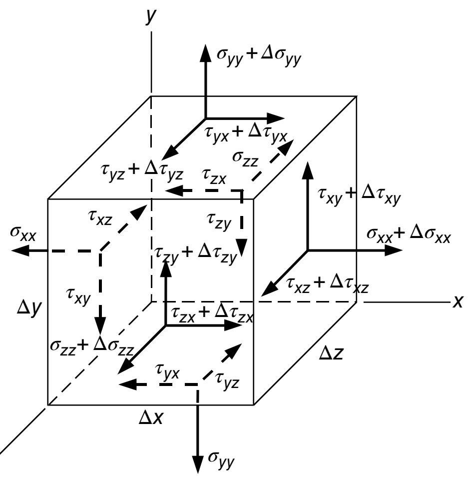


Fig. 3.13. Stress components acting on the faces of the element under a nonuniform state of stress.

Taking the limit Δx, Δy, $\Delta z \to 0$, the equilibrium equation above becomes

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0 \tag{3.22}$$

Similarly, equations $\Sigma F_y = 0$ and $\Sigma F_z = 0$ lead to

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0 \tag{3.23}$$

and

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0 \tag{3.24}$$

respectively.

It can easily be verified that the moment equations $\Sigma M_x = \Sigma M_y = \Sigma M_z = 0$ lead to

$$\tau_{xy} = \tau_{yx}, \quad \tau_{yz} = \tau_{zy}, \quad \tau_{xz} = \tau_{zx} \tag{3.25}$$

which are identical to the relation given by (Eq. (3.21)).

Equations (3.22)–(3.25) are the equilibrium equations of a point in a body. If a body is in equilibrium, the stress field must satisfy these equations everywhere in the body.

EXAMPLE 3.2

(a) Determine whether the following stress field is in static equilibrium.

$$[\sigma] = \begin{bmatrix} 4x & 4x + 5y & 0 \\ 4x + 5y & 4y & 0 \\ 0 & 0 & 0 \end{bmatrix}$$