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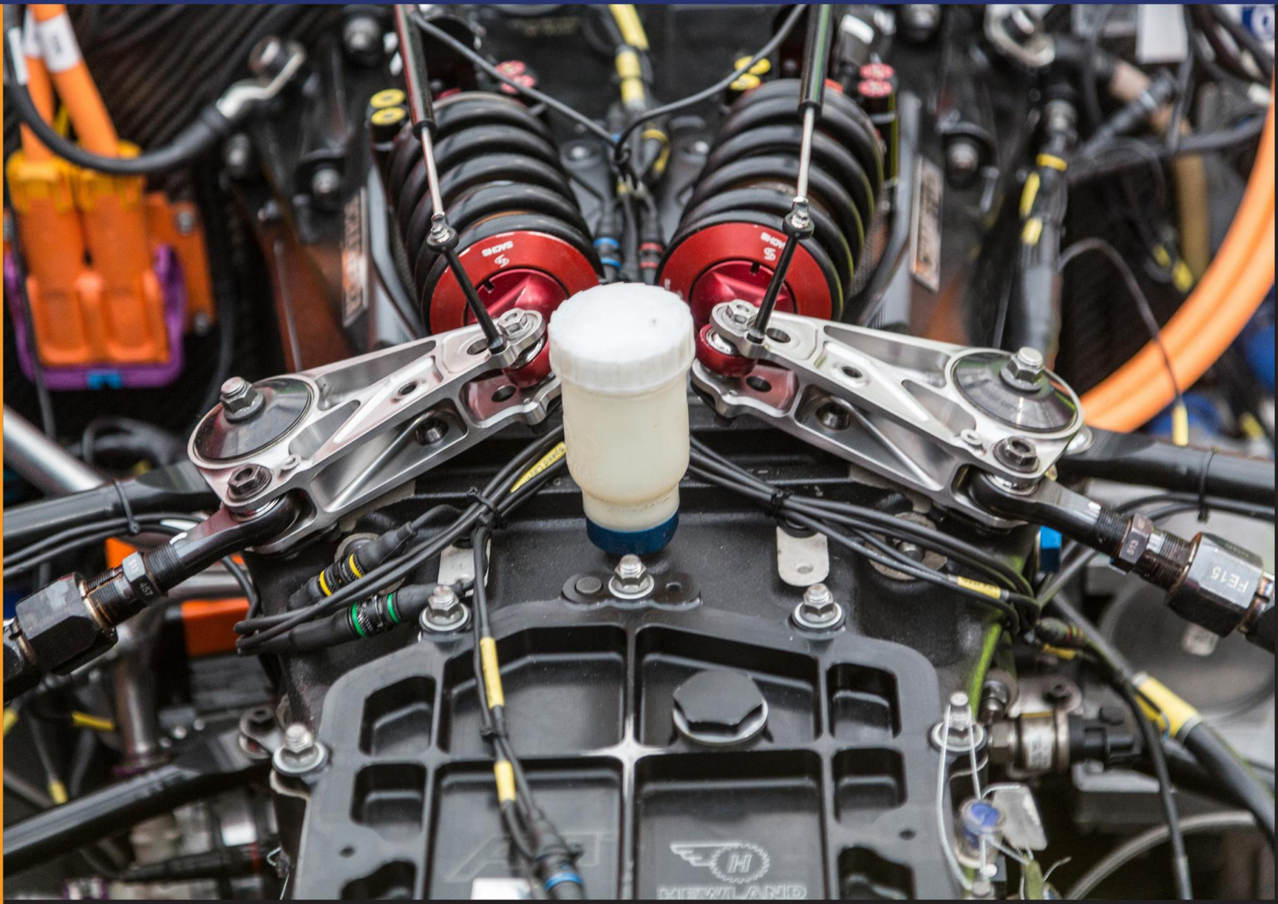

Vladimir A. Dobrushkin

APPLIED DIFFERENTIAL EQUATIONS

THE PRIMARY COURSE

SECOND EDITION

CRC Press
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A CHAPMAN & HALL BOOK

Applied Differential Equations

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Applied Differential Equations

The Primary Course

Second Edition

Vladimir A. Dobrushkin

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Contents

List of Symbols

$\|a,b\|$	any interval (closed, open, semi-open) with end points a and b.
$n!$	factorial, $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$.
$\ln x$	$= \log_e x$, natural logarithm, that is, the logarithm with base e.
$n^{\underline{k}}$	$= n(n-1)\ldots,(n-k+1)$ (falling factorial).
$\binom{n}{k}$	$= \dfrac{n^{\underline{k}}}{k!} = \dfrac{n!}{k!\,(n-x)!}$ binomial coefficient.
$\mathtt{D}$	$= \mathrm{d}/\mathrm{d}x$ or $\mathrm{d}/\mathrm{d}t$, the derivative operator.
$\mathtt{D}^n(fg)$	$= \sum_{r=0}^{n} \binom{n}{r} \left(\mathtt{D}^{n-r} f\right) \left(\mathtt{D}^r g\right)$, Leibniz formula.
$\dot{y}$	$= \mathrm{d}y/\mathrm{d}t$, derivative with respect to time variable t.
$\mathrm{H}(t)$	the Heaviside function, Definition 5.3, page 262.
$\mathrm{Si}(x)$	sine integral: $\int_0^x \frac{\sin t}{t}\,\mathrm{d}t$.
$\mathrm{Ci}(x)$	cosine integral: $-\int_x^{\infty} \frac{\cos t}{t}\,\mathrm{d}t$.
$\mathrm{sinc}(x)$	$= \dfrac{\sin(x\pi)}{x\pi}$, normalized cardinal sine function.
$\frac{1}{2} \ln \left\| \frac{1+x}{1-x} \right\|$	$= \begin{cases} \mathrm{arctanh}(x) & \text{for } \|x\| < 1, \\ \mathrm{arccoth}(x) & \text{for } \|x\| > 1. \end{cases}$
$\int \frac{v'(x)}{v(x)}\,\mathrm{d}x$	$= \ln \|v(x)\| + C = \ln C v(x), \quad v(x) \neq 0.$
$\mathbf{I}$	the identity matrix, Definition 7.6, page 411.
$\mathrm{tr}\,(\mathbf{A})$	trace of a matrix $\mathbf{A}$, Definition 7.8, page 412.
$\det\,(\mathbf{A})$	determinant of a matrix $\mathbf{A}$, §8.2.
$\mathbf{A}^{\mathrm{T}}$	transpose of a matrix $\mathbf{A}$ (also denoted as $\mathbf{A}'$), Definition 7.3, page 410.
$\mathbf{A}^{*}$	or $\mathbf{A}^{\mathrm{H}}$, adjoint of a matrix $\mathbf{A}$, Definition 7.4, page 410.
ODE	ordinary differential equation.
PDE	partial differential equation.
CAS	computer algebra system.
$\mathbf{j}$	unit pure imaginary vector on the complex plane $\mathbb{C}$: $\mathbf{j}^2 = -1$.
$x + y\mathbf{j}$	complex number, where $x = \Re\,(x + y\mathbf{j})$, $y = \Im\,(x + y\mathbf{j})$.

Preface

Applied Differential Equations is a comprehensive exposition of ordinary differential equations and an introduction to partial differential equations (due to space constraint, there is only one chapter devoted directly to PDEs) including their applications in engineering and the sciences. This text is designed for a two-semester sophomore or junior level course in differential equations and assumes previous exposure to calculus. It covers traditional material, along with novel approaches in presentation and utilization of computer capabilities, with a focus on various applications. This text intends to provide a solid background in differential equations for students majoring in a breadth of fields.

This book started as a collection of lecture notes for an undergraduate course in differential equations taught by the Division of Applied Mathematics at Brown University, Providence, RI. To some extent, it is a result of collective insights given by almost every instructor who taught such a course over the last 15 years. Therefore, the material and its presentation covered in this book were practically tested for many years.

There is no need to demonstrate the importance of ordinary and partial differential equations (ODE and PDE, for short) in science, engineering, and education—this subject has been included in the curriculum of universities around the world. Their utilization in industry and engineering is so widespread that, without a doubt, differential equations have become the most successful mathematical tool in modeling. Perhaps the most germane point for the student reader is that many curricula recommend or require a course in ordinary differential equations for graduation. The beauty and utility of differential equations and their application in mathematics, biology, chemistry, computer science, economics, engineering, geology, neuroscience, physics, the life sciences, and other fields reaffirm their inclusion in myriad curricula.

In this text, differential equations are described in the context of applications. A more comprehensive treatment of their applications is given in [9, 16]. It is important for students to grasp how to formulate a mathematical model, how to solve differential equations (analytically or numerically), how to analyze them qualitatively, and how to interpret the results. This sequence of steps is perhaps the hardest part for students to learn and appreciate, yet it is an essential skill to acquire. This book provides the common language of the subject and teaches the main techniques needed for modeling and systems analysis.

The **goals** in writing this textbook:

- To show that a course in differential equations is essential for modeling real-life phenomena. This textbook lays down a bridge between calculus, modeling, and advanced topics. It provides a basis for further serious study of differential equations and their applications. We stress the mastery of traditional solution techniques and present effective methods, including reliable numerical approximations.

- To provide qualitative analysis of ordinary differential equations. Hence, the reader should get an idea of how all solutions to the given problem behave, what are their validity intervals, whether there are oscillations, vertical or horizontal asymptotes, and what is their long term behavior. So the reader will learn various methods of solving, analysis, visualization, and approximation. This goal is hard to achieve without exploiting the capabilities of computers.

- To give an introduction to four of the most pervasive computer software[1] packages: *Maple*™, *Mathematica*®, MATLAB®, and *Maxima*—the first computer algebra system in the world. A few other such solvers are available (but will not be presented in the text). Some popular software packages have either similar syntax (such as

[1]The owner of *Maple* is Maplesoft (`http://www.maplesoft.com/`), a subsidiary of Cybernet Systems Co. Ltd. in Japan, which is the leading provider of high-performance software tools for engineering, science, and mathematics. *Mathematica* is the product of Wolfram Research company of Champaign, Illinois, USA founded by Stephen Wolfram in 1987; its URL is `http://www.wolfram.com`. MATLAB is the product of the MathWorks, Inc., 3 Apple Hill Drive, Natick, MA, 01760-2098 USA, URL: `www.mathworks.com`.

Octave) or include engines of known solvers (such as MathCad). Others should be accessible with the recent development of cloud technology such as Sage. Also, simple numerical algorithms can be handled with a calculator or a spreadsheet program.

- To give the lecturer a flexible textbook within which he or she can easily organize a curriculum matched to their specific goals. This textbook presents a large number of examples from different subjects, which facilitate the development of the student's skills to model real-world problems. Staying within a traditional context, the book contains some advanced material on differential equations.

- To give students a thorough understanding of the subject of differential equations as a whole. This book provides detailed solutions of all the basic examples, and students can learn from it without any extra help. It may be considered as a self-study text for students as well. This book recalls the basic formulas and techniques from calculus, which makes it easy to understand all derivations. It also includes advanced material in each chapter for inquisitive students who seek a deeper knowledge of this subject.

Philosophy of the Text

We share our pedagogical approach with famous mathematician Paul Halmos [23, pp. 61–62], who recommended the study of mathematics by examples. He goes on to say:

> ...it's examples, examples, examples that, for me, all mathematics is based on, and I always look for them. I look for them first, when I begin to study. I keep looking for them, and I cherish them all.

Pedagogy and Structure of the Book

Ordinary and partial differential equations is a classical subject that has been studied for about 300 years. However, education has changed with omnipresent mathematical modeling technology available to all. This textbook stresses that differential equations constitute an essential part of modeling by showing their applications, including numerical algorithms and syntax of the four most popular software packages.

It is essential to introduce information technologies early in the class. Students should be encouraged to use numerical solvers in their work because they help to illustrate and illuminate concepts and insights. It should be noted that computers cannot be used blindly because they are as smart as the programmers allow them to be—every problem requires careful examination.

This textbook stays within traditional coverage of basic topics in differential equations. It contains practical techniques for solving differential equations, some of which are not widely used in undergraduate study. Not every statement or theorem is followed by rigorous verification. Proofs are included only when they enhance the reader's understanding and challenge the student's intellectual curiosity.

Our pedagogical approach is based on the following principle: follow the author. Every section has many examples with detailed exposition focused on how to choose an appropriate technique and then how to solve the problem. There are hundreds of problems solved in detail, so a reader can master the techniques used to solve and analyze differential equations.

Notation

This text uses numbers enclosed with brackets to indicate references in the bibliography, which is located at the end of the book, starting on page 675. The text uses only standard notations and abbreviations [*et al.* (et alii from Latin) means "and others," or "and co-workers;" *i.e.* from Latin "id est" meaning that is, that is to say, or in other words; *e.g.* stands for the Latin phrase "exempli gratia," which means for example; and *etc.* means "and the others," "and other things," "and the rest"]. However, we find it convenient to type ■ at the end of proofs or at the end of a topic presented; we also use □ at the end of examples (unless a new one serves as a delimiter). We hope that the reader understands the difference between $=$ (equal) and $\equiv$ (equivalence relation). Also $\stackrel{\text{def}}{=}$ is used for short to signal that the expression follows by definition. There is no common notation for complex numbers. Since a complex number (let us denote it by z) is a vector on the plane, it is a custom to denote it by $z = x + y\mathbf{j}$ rather than $z = x\mathbf{i} + y\mathbf{j}$, where the unit vector $\mathbf{i}$ is dropped and $\mathbf{j}$ is the unit vector in the positive vertical direction. In mathematics, this vector $\mathbf{j}$ is denoted by i. For convenience, we present the list of symbols and abbreviations at the beginning of the text.

For students

This text has been written with the student in mind to make the book very friendly. There are a lot of illustrations accompanied by corresponding codes for appropriate solvers. Therefore, the reader can follow examples and learn how to use these software packages to analyze and verify obtained results, but not to replace mastering of mathematical techniques. Analytical methods constitute a crucial part of modeling with differential equations, including numerical and graphical applications. Since the text is written from the viewpoint of the applied mathematician, its presentation may sometimes be quite theoretical, sometimes intensely practical, and often somewhere in between. In addition to the examples provided in the text, students can find additional resources, including problems and tutorials on using software, at the website that accompanies this book:
http://www.cfm.brown.edu/people/dobrush/am33/computing33.html

For instructors

Universities usually offer two courses on differential equations of different levels; one is the basic first course required by curriculum, and the other covers the same material, but is more advanced and attracts students who find the basic course trivial. This text can be used for both courses, and curious students have an option to increase their understanding and obtain deeper knowledge in any topic of interest. A great number of examples and exercises make this text well suited for self-study or for traditional use by a lecturer in class. Therefore this textbook addresses the needs of two levels of audience, the beginning and the advanced.

Acknowledgments

This book would not have been written if students had not complained about the other texts unleashed on them. In addition, I have gained much from their comments and suggestions about various components of the book, and for this I would like to thank the students at Brown University.

The development of this text depended on the efforts of many people. I am very grateful to the reviewers who made many insightful suggestions that improved the text.

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Noah Donoghue, George Potter, Neil Singh, and Mark Weaver made great contributions by carefully reading the text and helping me with problems and graphs. Their suggestions improved the exposition of the material substantially.

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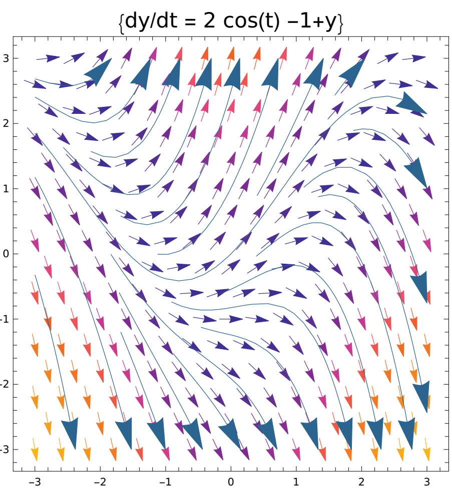
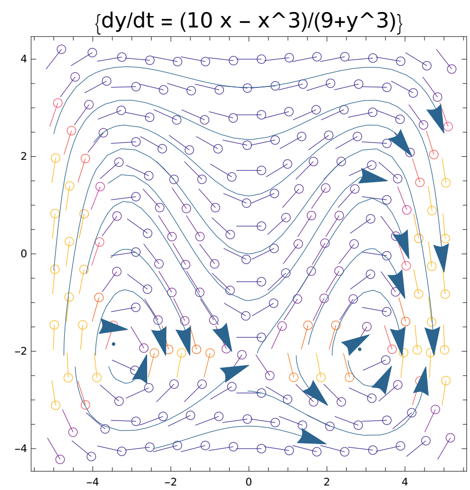
Vladimir Dobrushkin,
Providence, RI

Chapter 1





Introduction

The independent discovery of the calculus by I. Newton and G. Leibniz was immediately followed by its intensive application in mathematics, physics, and engineering. Since the late seventeenth century, differential equations have been of fundamental importance in the study, development, and application of mathematical analysis. Differential equations and their solutions play one of the central roles in the modeling of real-life phenomena.

In this chapter, we begin our study with the first order differential equations in normal form $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$, where $f(x, y)$ is a given single-valued function of two variables, called a slope or rate function. For an arbitrary function $f(x, y)$, there does not necessarily exist a function $y = \phi(x)$ that satisfies the differential equation. In fact, a differential equation usually has more than one solution. We classify first order differential equations and formulate several analytic methods that are applicable to each subclass.

One of the most intriguing things about differential equations is that for an arbitrary function f, there is no general method for finding an exact formula for the solution. For many differential equations that are encountered in real-world applications, it is impossible to express their solutions via known functions. Generally speaking, every differential equation defines its solution (if it exists) as a special function not necessarily expressible by elementary functions (such as polynomial, exponential, or trigonometric functions). Only exceptional differential equations can be explicitly or implicitly integrated. For instance, such "simple" differential equations as $y' = y^2 - x$ or $y' = e^{xy}$ cannot be solved by available methods.

1.1 Motivation

In applied mathematics, a **model** is a set of equations describing the relationships between numerical values of interest in a system. **Mathematical modeling** is the process of developing a model pertaining to physics or other sciences. Since differential equations are our main objects of interest, we consider only models that involve these equations. For example, Newton's second law, $\mathbf{F} = m\mathbf{a}$, relates the force $\mathbf{F}$ acting on a particle of mass m with the resulting acceleration $\mathbf{a} = \ddot{\mathbf{x}} \stackrel{\text{def}}{=} \mathrm{d}^2\mathbf{x}/\mathrm{d}t^2$. The transition from a physical problem to a corresponding mathematical model is not easy. It often happens that, for a particular problem, physical laws are hard or impossible to derive, though a relation between physical values can be obtained. Such a relation is usually used in the derivation of a mathematical model, which may be incomplete or somewhat inaccurate. Any such model may be subject to refining, making its predictions agree more closely with experimental results.

Many problems in the physical sciences, social sciences, biology, geology, economics, and engineering are posed mathematically in terms of an equation involving derivatives (or differentials) of the unknown function. Such an equation is called a **differential equation**, and their study was initiated by Leibniz[2] in 1676. It is customary to use his notation for derivatives: $\mathrm{d}y/\mathrm{d}x$, $\mathrm{d}^2y/\mathrm{d}x^2$, $\ldots$, or the prime (Lagrange) notation: y', y'', $\ldots$. For higher derivatives, we use the notation $y^{(n)}$ to denote the derivative of the order n. When a function depends on time, it is common to denote its first two derivatives with respect to time with dots: $\dot{y}$, $\ddot{y}$ (Newton's notation).

The next step in mathematical modeling is to determine the unknown, or unknowns, involved. Such a procedure is called solving the differential equation. The techniques used may yield solutions in analytic forms or approximations. Many software packages allow solutions to be visualized graphically. In this book, we focus on three popular commercial packages: MATLAB®, *Maple*™, *Mathematica*®, and free computer algebra systems *Maxima*, *Sage*, *SymPy*. Some attention will be given to two open-source programming languages *Python* and *R*. To motivate the reader, we begin with two well-known examples.

Example 1.1.1. (Carbon dating) The procedure for determining the age of archaeological remains was developed by the 1960 Nobel prize winner in chemistry, Willard Libby[3]. Cosmic radiation entering the Earth's atmosphere is constantly producing carbon-14 (${}_6\mathrm{C}^{14}$), an unstable radioactive isotope of ordinary carbon-12 (${}_6\mathrm{C}^{12}$). Both isotopes of carbon appear in carbon dioxide, which is incorporated into the tissues of all plants and animals, including human beings. In the atmosphere, as well as in all living organisms, the proportion of radioactive carbon-14 to ordinary (stable) carbon-12 is constant. When an organism dies, the absorption of carbon-14 by respiration and ingestion terminates. Experiments indicate that radioactive substances, such as uranium or carbon-14, decay by a certain percentage of their mass in a given unit of time. In other words, radioactive elements decay at a rate proportional to the mass present. Let $c(t)$ be the concentration of carbon-14 in dead organic material at time t, counted since the time of death. Then $c(t)$ obeys the following differential equation subject to the initial condition:

$$\frac{\mathrm{d}c(t)}{\mathrm{d}t} = -\lambda\, c(t), \qquad t > 0, \qquad c(0) = c_0, \tag{1.1.1}$$

where at the time of death $t = 0$, c_0 is the concentration of the isotope that a living organism maintains, and λ is the characteristic constant ($\lambda \approx 1.24 \times 10^{-4}$ per year for carbon-14). The technique to solve this type of differential equation will be explained later, in §2.1. We guess a solution: $c(t) = K\, e^{-\lambda t}$, with constant K, using the derivative property of the exponential function $\left(e^{kt}\right)' = k\, e^{kt}$. Since $c(0) = K$, it follows from the initial condition, $c(0) = c_0$, that

$$c(t) = c_0\, e^{-\lambda t}.$$

Suppose we know this formula to be true for $c(t)$. We determine the time of death of organic material from an examination of the concentration $c(t)$ of carbon-14 at the time t. The following relationship holds:

$$\frac{c(t)}{c_0} = e^{-\lambda t}.$$

Applying a logarithm to both sides, we obtain

$$-\lambda t = \ln[c(t)/c_0] = -\ln[c_0/c(t)],$$

from which we can find the time t of death of the organism to be

$$t = \frac{1}{\lambda}\, \ln\left(\frac{c_0}{c(t)}\right).$$

Recall that the half-life of a radioactive nucleus is defined as the time t_h during which the number of nuclei reduces to one-half of the original value. If the half-life of a radioactive element is known to be t_h, then the radioactive nuclei decay according to the law

$$N(t) = N(0)\, 2^{-t/t_h} = \frac{N(0)}{2^{t/t_h}}\,, \tag{1.1.2}$$

[2]Gottfried Wilhelm Leibniz (1646–1716) was a German scientist who first solved separable, homogeneous, and linear differential equations. He co-discovered calculus with Isaac Newton.

[3]American chemist Willard Libby (1908–1980).

where $N(t)$ is the amount of radioactive substance at time t and $t_h = (\ln 2)/\lambda$. Since the half-life of carbon-14 is approximately 5,730 years, present measurement techniques utilize this method for carbonaceous materials up to about 50,000 years old.

Example 1.1.2. **(RC-series circuit)** The most common applications of differential equations occur in the theory of electric circuits because of its importance and the pervasiveness of these equations in network theory. Figure 1.1 shows an electric circuit consisting of a resistor R and a capacitor C in series. A differential equation relating current $I(t)$ in the circuit, charge $q(t)$ on the capacitor, and voltage $V(t)$ measured at the points shown can be derived by applying Kirchhoff's voltage law, which states that the voltage $V(t)$ must equal the sum of the voltage drops across the resistor and the capacitor (see [16]). It is known that the voltage changes across the passive elements are approximately as follows.

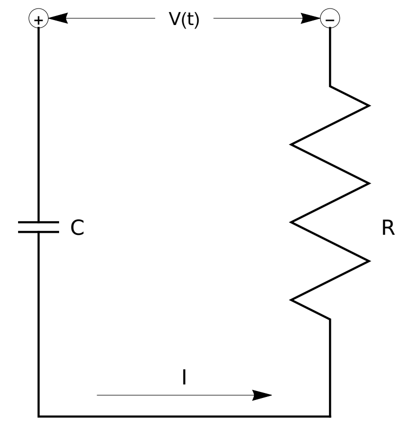


Figure 1.1: RC-Circuit.

$$\Delta V_R = RI \quad \text{for the resistor,}$$
$$\Delta V_C = q/C \quad \text{for the capacitor.}$$

Furthermore, the current is defined to be the rate of flow of charge: $I(t) = \mathrm{d}q/\mathrm{d}t$. By combining these expressions using Kirchhoff's voltage law we obtain a differential equation relating $q(t)$ and $V(t)$:

$$R\,\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{C}\,q(t) = V(t).$$

1.2 Classification of Differential Equations

To study differential equations, we need some common terminology and basic classification of equations. If an equation involves the derivative of one variable with respect to another, then the former is called a *dependent variable* and the latter is an *independent variable*. For instance, in the equation from Example 1.1.2, the charge q is a dependent variable and the time t is an independent variable.

Ordinary and Partial Differential Equations. We start classification of differential equations with the number of independent variables: whether there is a single independent variable or several independent variables. The first case is an ODE (acronym for **O**rdinary **D**ifferential **E**quation), and the second is a PDE (**P**artial **D**ifferential **E**quation).

Systems of Differential Equations. Another classification is based on the number of unknown dependent variables to be found. If two or more unknown variables are to be determined, then a system of equations is required.

Example 1.2.1. We will derive a simple model of an arms race between two countries. Let $x_i(t)$ represent the size (or cost) of the arms stocks of country i ($i = 1, 2$). Due to the cost of maintenance, we assume that an isolated country will diminish its arms stocks at a rate proportional to its size. We express this mathematically as $\dot{x}_i \stackrel{\text{def}}{=} \mathrm{d}x_i/\mathrm{d}t = -c_i x_i,\ c_i \geqslant 0$. The competition between countries, however, causes each one to increase its supply of arms at a rate proportional to the other country's arms supplies. The English meteorologist Lewis F. Richardson [47, 48] proposed a model to describe the evolution of both countries' arsenals as the solution of the following system of differential equations:

$$\dot{x}_1 = -c_1 x_1 + d_1 x_2 + g_1(x_1, x_2, t), \qquad c_1, d_1 \geqslant 0,$$
$$\dot{x}_2 = -c_2 x_2 + d_2 x_1 + g_2(x_1, x_2, t), \qquad c_2, d_2 \geqslant 0,$$

where the c's are called *cost* factors, the d's are *defense* factors, and the g's are *grievance* terms that account for other factors. □

The **order of a differential equation** is the order of the highest derivative that appears in the equation. More generally, an ordinary differential equation of the n-th order is an equation of the following form:

$$F\left(x, y(x), y'(x), \dots, y^{(n)}(x)\right) = 0. \tag{1.2.1}$$

Here $y(x)$ is an unspecified function having n derivatives and depending on $x \in (a, b)$, $a < b$; $F(x, y, p_1, \dots, p_n)$ is a given function of $n + 2$ variables. Some of the arguments, x, y, ..., $y^{(n-1)}$ (or even all of them) may not be present in Eq. (1.2.1). However, the n-th derivative, $y^{(n)}$, must be present in the ordinary differential equation, or else its order would be less than n. If this equation can be solved for $y^{(n)}(x)$, then we obtain the differential equation in the **normal** form:

$$y^{(n)}(x) = f\left(x, y, y', \dots, y^{(n-1)}\right), \quad x \in (a, b). \tag{1.2.2}$$

A *first order differential equation* is of the form

$$F(x, y, y') = 0. \tag{1.2.3}$$

If we can solve it with respect to y', then we obtain its normal form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \qquad \text{or} \qquad \mathrm{d}y = f(x, y)\,\mathrm{d}x, \tag{1.2.4}$$

where $\mathrm{d}x$ and $\mathrm{d}y$ are differentials in variables x and y, respectively.

Linear and Nonlinear Equations. The ordinary differential equation (1.2.1) is said to be *linear* if F is a linear function of the variables $y(x)$, $y'(x)$, ..., $y^{(n)}(x)$. Thus, the general linear ordinary differential equation of order n is

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = g(x). \tag{1.2.5}$$

An equation (1.2.1) is said to be *nonlinear* if it is not of this form.

For example, the van der Pol equation, $\ddot{y} - \epsilon(1 - y^2)\,\dot{y} + \delta y = 0$, is a nonlinear equation because of the presence of the term y^2. On the other hand, $y'(x) + (\sin x)\,y(x) = x^2$ is a linear differential equation of the first order because it is of the form (1.2.5). In this case, $a_1 = 1$, $a_0(x) = \sin x$, and $g(x) = x^2$.

The general forms for the first and second order linear differential equations are:

$$a_1(x)\,y'(x) + a_0(x)\,y(x) = f(x) \quad \text{and} \quad a_2(x)\,y''(x) + a_1(x)\,y'(x) + a_0(x)\,y(x) = f(x).$$

1.3 Solutions to Differential Equations

Since the unknown quantity in a differential equation is a function, it should be defined in some domain. The differential equation (1.2.1) is usually considered on some open interval $(a, b) = \{x : a < x < b\}$, where its solution along with the function $F(x, y, p_1, \dots, p_n)$ should be defined. However, it may happen that we look for a solution on a closed interval $[a, b]$ or a semi-open interval, $(a, b]$ or $[a, b)$. For instance, the bending of a plane's wing is modeled by an equation on a semi-closed interval $[0, \ell)$, where the point $x = 0$ corresponds to the connection of the wing to the body of the plane, and ℓ is the length of the wing. At $x = \ell$, the equation is not valid and its solution is not defined at this point. To embrace all possible cases, we introduce the notation $|a, b|$, which denotes an interval (a, b) possibly including the end points; hence, $|a, b|$ can denote the open interval (a, b), the closed interval $[a, b]$, or the semi-open intervals $(a, b]$ or $[a, b)$.

Definition 1.1. A **solution** or **integral** of the ordinary differential equation

$$F\left(x, y(x), y'(x), \dots, y^{(n)}(x)\right) = 0$$

on an interval $|a, b|$ $(a < b)$ is a continuous function $y(x)$ such that y, y', y'', ..., $y^{(n)}$ exist and satisfy the equation for all values of the independent variable on the interval, $x \in |a, b|$. The graphs of the solutions of a differential equation are called their **integral curves** or **streamlines**.

This means that a solution $y(x)$ has derivatives up to the order n in the interval $|a, b|$, and for every $x \in |a, b|$, the point $\left(x, y(x), y'(x), \ldots, y^{(n)}(x)\right)$ should be in the domain of F. We can show that a solution satisfies a given differential equation in various ways. The general method consists of calculating the expressions of the dependent variable and its derivatives, and substituting all of these in the given equation. The result of such a substitution should lead to the identity.

From calculus, it is known that a differential equation $y' = f(x)$ has infinitely many solutions for a smooth function $f(x)$ defined in some domain. These solutions are expressed either via an indefinite integral, $y = \int f(x)\,\mathrm{d}x + C$, or a definite integral with a variable boundary, $y = \int_{x_0}^{x} f(x)\,\mathrm{d}x - C$ or $y = -\int_{x}^{x_0} f(x)\,\mathrm{d}x + C$, where x_0 is some fixed value. The *constant of integration*, C, is assumed arbitrary in the sense that it can be given any value within a certain range. However, C actually depends on the domain where the function $y(x)$ is considered and the form in which the integral is expressed. For instance, a simple differential equation $y' = (1 + x^2)^{-1}$ has infinitely many solutions presented in three different forms:

$$\int \frac{\mathrm{d}x}{1+x^2} = \arctan x + C = \arctan\left(\frac{1+x}{1-x}\right) + C_1 = \frac{1}{2}\arccos\left(\frac{1-x^2}{1+x^2}\right) + C_2,$$

where arbitrary constants C, C_1, and C_2 can be expressed in terms of each other, but their relations depend on the domain of x. For example, $C_1 = \frac{\pi}{4} + C$ when $x < 1$, but $C_1 = C - \frac{3\pi}{4}$ for $x > 1$. Also, the antiderivative of $1/x$ (for $x \neq 0$) will usually be written as $\ln Cx$ instead of $\ln |Cx|$ because it would be assumed that $C > 0$ for a positive x and $C < 0$ for a negative x. In general, a function of an arbitrary constant is itself an arbitrary constant.

Given the observation above, one might expect that a differential equation $y' = f(x, y)$ has infinitely many solutions (if any). For instance, the function $y = x + 1$ is a solution to the differential equation

$$y' + y = x + 2.$$

To verify this, we substitute $y = x + 1$ and $y' = 1$ into the equation. Indeed, $y' + y = 1 + (x + 1) = x + 2$. It is not difficult to verify that another function $g(x) = x + 1 + e^{-x}$ is also a solution of the given differential equation, demonstrating that a differential equation may have many solutions.

To solve a differential equation means to make its solutions known (in the sense explained later). A solution in which the dependent variable is expressed in terms of the independent variable is said to be in **explicit form**. A function is *known* if it can be expressed by a formula in terms of standard and/or familiar functions (polynomial functions, exponentials, trigonometric functions, and their inverse functions). For example, we consider functions given by a convergent series as known if the terms of the series can be expressed via familiar functions. Also, quadrature (expression via integral) of a given function $f(x)$ is regarded as known.

However, we shall see in this book that functions studied in calculus are not enough to describe solutions of all differential equations. In general, a differential equation defines a function as its solution (if one exists), even if it cannot be expressed in terms of familiar functions. Such a solution is usually referred to as a special function. Thus, we use the word “solution” in a broader sense by including less convenient forms of solutions.

Any relation, free of derivatives, that involves two variables x and y and that is consistent with the differential equation (1.2.1) is said to be a solution of the equation in **implicit form**. Although we may not be able to solve the relation for y, thus obtaining a formula in x, any change in x still results in a corresponding change in y. Hence, on some interval, this could define locally a solution $y = \phi(x)$ even if we fail to find an explicit formula for it or even if the global function does not exist. In fact, we can obtain numerical values for $y = \phi(x)$ to any desired precision.

In this text, you will learn how to determine solutions explicitly or implicitly, how to approximate them numerically, how to visualize and plot solutions, and much more.

Let us consider for simplicity the first order differential equation in normal form: $y' = f(x, y)$. One can rarely find its solution in explicit form, namely, as $y = \phi(x)$. We will say that the equation

$$\Phi(x, y) = 0$$

defines a solution in *implicit form* if $\Phi(x, y)$ is a known function. How would you know that the equation $\Phi(x, y) = 0$ defines a solution $y = \phi(x)$ to the equation $y' = f(x, y)$? Assuming that the conditions of the implicit function theorem hold, we differentiate both sides of the equation $\Phi(x, y) = 0$ with respect to x:

$$\Phi_x(x, y) + \Phi_y(x, y)\, y' = 0, \qquad \text{where } \Phi_x = \partial\Phi/\partial x, \quad \Phi_y = \partial\Phi/\partial y.$$

From the equation $y' = f(x, y)$, we obtain

$$\Phi_x(x, y) + \Phi_y(x, y)\, f(x, y) = 0. \tag{1.3.1}$$

Therefore, if the function $y = \phi(x)$ is a solution to $y' = f(x, y)$, then the function $\Phi(x, y) = y - \phi(x)$ must satisfy Eq. (1.3.1). Indeed, in this case, we have $\Phi_x = -\phi'$ and $\Phi_y = 1$.

An ordinary differential equation may be given either for a restricted set of values of the independent variable or for all real values. Restrictions, if any, may be imposed arbitrarily or due to constraints relating to the equation. Such constraints can be caused by conditions imposed on the equation or by the fact that the functions involved in the equation have limited domains. Furthermore, if an ordinary differential equation is stated without explicit restrictions on the independent variable, it is assumed that all values of the independent variable are permitted with the exception of any values for which the equation is meaningless.

Example 1.3.1. The relation

$$\ln y + y^2 - \int_0^x e^{-x^2}\, \mathrm{d}x = 0 \quad (y > 0)$$

is considered to be a solution in implicit form of the differential equation

$$(1 + 2y^2)\, y' - y\, e^{-x^2} = 0 \qquad \text{or} \qquad y' = \frac{y}{1 + 2y^2}\, e^{-x^2}.$$

This can be seen by differentiating the given relationship implicitly with respect to x. This leads to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln y + y^2 - \int_0^x e^{-x^2}\, \mathrm{d}x\right] = \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\,\frac{\mathrm{d}y}{\mathrm{d}x} - e^{-x^2} = 0.$$

Therefore,

$$\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{1 + 2y^2}{y}\right) = e^{-x^2} \qquad \Longrightarrow \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{1 + 2y^2}\, e^{-x^2}.$$

Example 1.3.2. The function $y(x)$ that is defined implicitly from the equation

$$x^2 + 2y^2 = 4$$

is a solution of the differential equation $x + 2y\, y' = 0$ on the interval $(-2, 2)$ subject to $g(\pm 2) = 0$. To plot its solution in *Maple*, use the `implicitplot` command (upon invoking `plots` package):

```
implicitplot(x^2+2*y^2=4, x=-2..2, y=-1.5..1.5);
```

The same ellipse can be plotted with the aid of *Mathematica:*

```
ContourPlot[x^2 + 2 y^2 == 4, {x, -2, 2}, {y, -2, 2},
 PlotRange -> {{-2.1, 2.1}, {-1.5, 1.5}}, AspectRatio -> 1.5/2.1,
 ContourStyle -> Thickness[0.005], FrameLabel -> {"x", "y"},
 RotateLabel -> False]       (* Thickness is .5% of the figure's length *)
```

The same plot can be drawn in *Maxima* with the following commands:

```
load(draw);
draw2d(ip_grid=[100,100],  /* optional, makes a smoother plot */
      implicit(x^2 + 2*y^2 = 4, x,-2.1,2.1, y,-1.5,1.5));
```

MATLAB is capable to perform the same job:

```
[x,y] = meshgrid(-2:.1:2,-2:.1:2); contour(x.^2 + 2*y.^2)
```

The implicit relation $x^2 + 2\,y^2 = 4$ contains the *two* explicit solutions

$$y(x) = \sqrt{2 - 0.5x^2} \qquad \text{and} \qquad y(x) = -\sqrt{2 - 0.5x^2} \quad (-2 < x < 2),$$

which correspond graphically to the two semi-ellipses. Indeed, if we rewrite the given differential equation $x + 2y\, y' = 0$ in the normal form $y' = -x/(2y)$, then we should exclude $y = 0$ from consideration. Since $x = \pm 2$ correspond to $y = 0$ in both of these solutions, we must exclude these points from the domains of the explicit solutions. Note that the differential equation $x + 2y\, y' = 0$ has infinitely many solutions: $x^2 + 2y^2 = C$ $(|x| \leqslant C)$, where C is an arbitrary positive constant. □

Next, we observe that a differential equation may (and usually will) have an infinite number of solutions. A set of solutions of $y' = f(x, y)$ that depends on one arbitrary constant C deserves a special name.

Definition 1.2. A family of functions $y = \phi(x, C)$ is called the **general solution** to the differential equation $y' = f(x, y)$ in some two-dimensional domain Ω if for every point $(x, y) \in \Omega$ there exists a value of constant C such that the function $y = \phi(x, C)$ satisfies the equation $y' = f(x, y)$. A solution of this differential equation can be defined implicitly:

$$\Phi(x, y, C) = 0 \qquad \text{or} \qquad \psi(x, y) = C. \tag{1.3.2}$$

In this case, $\Phi(x, y, C)$ is called the **general integral**, and $\psi(x, y)$ is referred to as the **potential function** of the given equation $y' = f(x, y)$.

A constant C may be given any value in a suitable range. Since C can vary from problem to problem, it is often called a parameter to distinguish it from the main variables x and y. Therefore, the equation $\Phi(x, y, C) = 0$ defines a one-parameter family of curves with no intersections. Graphically, it represents a family of solution curves in the xy-plane, each element of which is associated with a particular value of C. The general solution corresponds to the entire family of curves that the equation defines.

As might be expected, the inverse statement is true: the curves of a one-parameter family are integrals of some differential equation of the first order. Indeed, let the family of curves be defined by the equation $\Phi(x, y, C) = 0$, with a smooth function Φ. Differentiating with respect to x yields a relation of the form $F(x, y, y', C) = 0$. By eliminating C from these two equations, we obtain the corresponding differential equation.

Example 1.3.3. For an arbitrary constant C, show that the function $y = C\,x + \frac{C}{\sqrt{1+C^2}}$ is the solution of the nonlinear differential equation $y - x\,y' = \dfrac{y'}{\sqrt{1+(y')^2}}$.

Solution. Taking the derivative of y shows that $y' = C$. Substitution $y = C\,x + \frac{C}{\sqrt{1+C^2}}$ and $y' = C$ into the differential equation yields

$$C\,x + \frac{C}{\sqrt{1+C^2}} - x\,C = \frac{C}{\sqrt{1+C^2}}.$$

This identity proves that the function is a solution of the given differential equation. Setting C to some value, for instance, $C = 1$, we obtain a particular solution $y = x + 1/\sqrt{2}$. A family of solutions are shown in Fig.1.2.

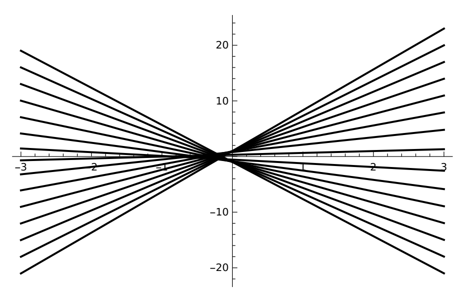


Figure 1.2: Example 1.3.3. A one-parameter family of solutions, plotted with *Mathematica*.

Example 1.3.4. Show that the function $y = \phi(x)$ in parametric form, $y(t) = t\,e^{-t}$, $x(t) = e^{t}$, is a solution to the differential equation $x^2\,y' = 1 - xy$. Note that the given differential equation has an irregular singularity at $x = 0$ (see §6.6, page 366).

Solution. The derivatives of x and y with respect to t are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = e^{t} \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = e^{-t}\,(1-t),$$

respectively. Hence,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{e^{-t}\,(1-t)}{e^{t}} = e^{-2t}\,(1-t) \\ &= e^{-2t} - t\,e^{-2t} = \frac{1}{x^2} - \frac{y}{x} = \frac{1-yx}{x^2}, \end{aligned}$$

because $x^{-2} = e^{-2t}$ and $y/x = t\,e^{-2t}$ (see Fig. 1.3).

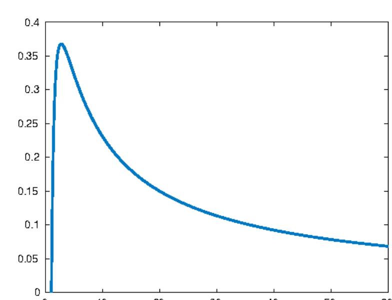


Figure 1.3: Example 1.3.4. A solution of the differential equation $x^2 y' = 1 - xy$ (plotted with MATLAB).

Example 1.3.5. Consider the one-parameter (depending on C) family of curves

$$x^2 + y^2 + Cy = 0 \qquad \text{or} \qquad C = -\frac{x^2+y^2}{y} \quad (y \neq 0).$$

On differentiating, we get $2x + 2y\,y' + Cy' = 0$. Setting $C = -(x^2+y^2)/y$ in the latter, we obtain the differential equation

$$2x + 2y\,y' - y'\left(\frac{x^2+y^2}{y}\right) = 0 \qquad \text{or} \qquad y' = 2xy/(x^2-y^2).$$

This job can be done by *Maxima* in the following steps:

```
depends(y,x);                 /* declare that y depends on x */
soln: x^2 + y^2 + C*y = 0;    /* soln is now a label for the equation */
diff(soln,x);                 /* differentiate the equation */
eliminate([%,soln], [C]);     /* eliminate C from these two equations */
solve(%, 'diff(y,x));         /* solve for y' */
```

Sometimes the integration of $y' = f(x,y)$ leads to a family of integral curves that depend on an arbitrary constant C in parametric form, namely,

$$x = \mu(t,C), \quad y = \nu(t,C).$$

This family of integral curves is called the **general solution in parametric form**.

In many cases, it is more convenient to seek a solution in parametric form, especially when the slope function is a ratio of two functions: $y' = P(x,y)/Q(x,y)$. Then introducing a new independent variable t, we can rewrite this single equation as a system of two equations:

$$\dot{x} \stackrel{\text{def}}{=} \mathrm{d}x/\mathrm{d}t = Q(x,y), \qquad \dot{y} \stackrel{\text{def}}{=} \mathrm{d}y/\mathrm{d}t = P(x,y). \tag{1.3.3}$$

1.4 Particular and Singular Solutions

A solution to a differential equation is called a **particular** (or specific) **solution** if it does not contain any arbitrary constant. By setting C to a certain value, we obtain a particular solution of the differential equation. So every specific value of C in the general solution identifies a particular solution or curve. Another way to specify a particular solution of $y' = f(x,y)$ is to impose an **initial condition**:

$$y(x_0) = y_0, \tag{1.4.1}$$

which specifies a solution curve that goes through the point (x_0, y_0) on the plane. Substituting the general solution into Eq. (1.4.1) will allow you to determine the value of the arbitrary constant. Sometimes, of course, no value of the constant will satisfy the given condition (1.4.1), which indicates that there is no particular solution with the required property among the entire family of integral curves from the general solution.

Definition 1.3. A differential equation $y' = f(x,y)$ (*or, in general,* $F(x,y,y') = 0$) subject to the initial condition $y(x_0) = y_0$, where x_0 and y_0 are specified values, is called an **initial value problem** (*IVP*). ◁

Example 1.4.1. Show that the function $y(x) = x\left[1 + \int_1^x \frac{\cos x}{x}\,\mathrm{d}x\right]$ is a solution of the following initial value problem:

$$x\,y' - y = x\,\cos x, \quad y(1) = 1.$$

Solution. The derivative of $y(x)$ is

$$y'(x) = 1 + \int_1^x \frac{\cos x}{x}\,\mathrm{d}x + x\,\frac{\cos x}{x} = 1 + \cos x + \int_1^x \frac{\cos x}{x}\,\mathrm{d}x.$$

Hence,

$$x\,y' - y = x + x\,\cos x + x\int_1^x \frac{\cos x}{x}\,\mathrm{d}x - x\left[1 + \int_1^x \frac{\cos x}{x}\,\mathrm{d}x\right] = x\,\cos x.$$

The initial condition is also satisfied since

$$y(1) = 1 \cdot \left[1 + \int_1^1 \frac{\cos x}{x}\,\mathrm{d}x\right] = 1.$$

We can verify that $y(x)$ is the solution of the given initial value problem using the following steps in *Mathematica:*

```
y[x_]=x + x*Integrate[Cos[t]/t, {t, 1, x}]
x*D[y[x], x] - y[x]
Simplify[%]
y[1]              (* to verify the initial value at x=1 *)
```

Definition 1.4. A **singular solution** of $y' = f(x,y)$ is a function that is not a special case of the general solution and for which the uniqueness of the initial value problem has failed.

Not every differential equation has a singular solution, but if it does, its singular solution cannot be determined from the general solution by setting a particular value of C, including $\pm\infty$, because integral curves of the general solution have no common points. A differential equation may have a solution that is neither singular nor a member of the family of one-parameter curves from the general solution. According to the definition, a singular solution always has a point on the plane where it meets with another solution. Such a point is usually referred to as a **branch point**. At that point, two integral curves touch because they share the same slope, $y' = f(x,y)$, but they cannot cross each other. For instance, functions $y = x^2$ and $y = x^4$ have the same slope at $x = 0$; they touch but do not cross.

A singular solution of special interest is one that consists entirely of branch points—at every point it is tangent to another integral curve. An **envelope** of the one-parameter family of integral curves is a curve in the xy-plane such that at each point it is tangent to one of the integral curves. Since there is no universally accepted definition of a singular solution, some authors define a singular solution as an envelope of the family of integral curves obtained from the general solution. Our definition of a singular solution includes not only the envelopes, but *all* solutions that have branch points. This broader definition is motivated by practical applications of differential equations in modeling real-world problems. The existence of a singular solution gives a warning signal in using the differential equation as a reliable model.

A necessary condition for the existence of an envelope is that x, y, C satisfy the equations:

$$\Phi(x,y,C) = 0 \qquad \text{and} \qquad \frac{\partial\Phi}{\partial C} = 0, \tag{1.4.2}$$

where $\Phi(x,y,C) = 0$ is the equation of the general solution. Eliminating C may introduce a function that is not a solution of the given differential equation. Therefore, any curve found from the system (1.4.2) should be checked on whether it is a solution of the given differential equation or not.

Example 1.4.2. Let us consider the equation

$$y' = 2\sqrt{y} \quad (y > 0), \tag{1.4.3}$$

where the radical takes positive sign. Suppose $y > 0$, we divide both sides of Eq. (1.4.3) by $2\sqrt{y}$, which leads to a separable equation (see §2.1 for detail)

The potential function for the given differential equation is $\psi(x,y) = \sqrt{y} - x$. Eq. (1.4.3) has also a trivial (identically zero) solution $y \equiv 0$ that consists of branch points—it is the envelope. This function is a singular solution since $y \equiv 0$ is not a member of the family of solutions $y(x) = (x+C)^2$ for **any** choice of the constant C. The envelope of the family of curves can also be found from the system (1.4.2) by solving simultaneous equations: $(x+C)^2 - y = 0$ and $\partial\Phi/\partial C = 2(x+C) = 0$, where $\Phi(x,y,C) = (x+C)^2 - y$. We can plot some solutions together with the singular solution $y = 0$ using the following *Mathematica* commands:

```
f[c_, x_] = (x + c)^2; parameters = {0, 1, 1.5, 2};
plot = Plot[Evaluate[f[#, x] & /@ parameters], {x, 0, 3},
 PlotRange -> {0, 10}, PlotTheme -> "Web",
 PlotLegends -> Table[Row[{"C=", j}], {j, parameters}]];
s = Plot[0, {x, 0, 3}, PlotStyle -> {Thickness[0.025], Blue},
   PlotRange -> {0, 10}];
Show[plot, s]
```

$$\frac{y'}{2\sqrt{y}} = 1 \qquad \text{or} \qquad \frac{\mathrm{d}\sqrt{y}}{\mathrm{d}x} = 1.$$

From chain rule, it follows that

$$\frac{\mathrm{d}\sqrt{y}}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(y)^{1/2} = \frac{1}{2}\, y^{-1/2}\, y'.$$

Hence, $\sqrt{y} = x + C$, where $x > -C$. The general solution of Eq. (1.4.3) is formed by the one-parametric family of semi-parabolas

$$y(x) = (x+C)^2, \quad \text{or} \quad C = \sqrt{y} - x, \quad x \geqslant -C.$$

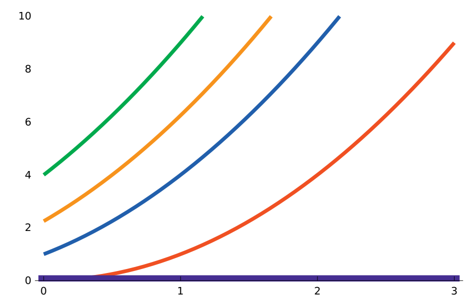


Figure 1.4: Example 1.4.2: some solutions to $y' = 2\sqrt{y}$ along with the singular solution $y \equiv 0$, plotted with *Mathematica*.

Actually, the given equation (1.4.3) has infinitely many singular solutions that could be constructed from the singular envelope $y = 0$ and the general solution by piecing together parts of solutions. An envelope does not necessarily bound the integral curves from one side. For instance, the general solution of the differential equation $y' = 3\, y^{2/3}$ consists of $y = (x+C)^3$ that fill the entire xy-plane. Its envelope is $y \equiv 0$.

Example 1.4.3. The differential equation $5y' = 2y^{-3/2}$, $y \neq 0$, has the one-parameter family of solutions $y = (x-C)^{2/5}$, which can be written in an implicit form (1.4.2) with $\Phi(x, y, C) = y^5 - (x-C)^2$. Differentiating with respect to C and equating to zero, we obtain $y \equiv 0$, which is not a solution. This example shows that conditions (1.4.2) are only necessary for the envelope's existence.

Example 1.4.4. Prove that the function $y(x)$ defined implicitly from the equation $y = \arctan(x+y) + C$, where C is a constant, is the general solution of the differential equation $(x+y)^2\, y' = 1$.

Solution. The chain rule shows that

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}(x+y)}\left[\arctan(x+y) + C\right] \frac{\mathrm{d}}{\mathrm{d}x}\left[x+y\right] \\ &= \frac{1}{1+(x+y)^2}\left[1 + \frac{\mathrm{d}y}{\mathrm{d}x}\right] \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}x}(x+y)^2 = 1 + \frac{\mathrm{d}y}{\mathrm{d}x}. \end{aligned}$$

From the latter, it follows that $y'(x+y)^2 = 1$. □

The next example demonstrates how for a function that contains an arbitrary constant as a parameter we can find the relevant differential equation for which the given function is its general solution.

Example 1.4.5. For an arbitrary constant C, show that the function $y = \frac{C-x}{1+x^2}$ is the solution of the differential equation

$$(1 + 2xy)\,\mathrm{d}x + (1+x^2)\,\mathrm{d}y = 0. \tag{1.4.4}$$

Prove that this equation has no other solutions.

Solution. The differential of this function is

$$\mathrm{d}y = y'\,\mathrm{d}x = \frac{-(1+x^2) - (C-x)2x}{(1+x^2)^2}\,\mathrm{d}x = \frac{x^2 - 1 - 2Cx}{(1+x^2)^2}\,\mathrm{d}x.$$

Multiplying both sides by $1 + x^2 > 0$, we have

$$(1+x^2)\,\mathrm{d}y = \frac{x^2-1-2Cx}{1+x^2}\,\mathrm{d}x = \frac{-x^2-1+2x^2-2Cx}{1+x^2}\,\mathrm{d}x = -\left(1 + 2x\frac{C-x}{1+x^2}\right)\,\mathrm{d}x$$

and, since $y = (C-x)/(1+x^2)$, we get

$$-(1+x^2)\,\mathrm{d}y = \frac{x^2-1-2Cx}{1+x^2}\,\mathrm{d}x = (1+2xy)\,\mathrm{d}x.$$

We are going to prove now that there is no solution other than $y = (C - x)/(1 + x^2)$. Solving for C, we find the potential function $\psi(x, y) = (1 + x^2)y + x$. Suppose the opposite, that other solutions exist; let $y = \phi(x)$ be a solution. Substituting $y = \phi(x)$ into the potential function $\psi(x, y)$, we obtain a function that we denote by $F(x)$, that is, $F(x) = (1 + x^2)\,\phi(x) + x$. Differentiation yields

$$F'(x) = 2x\phi(x) + (1 + x^2)\,\phi'(x) + 1.$$

Since $\phi'(x) = -\dfrac{1 + 2x\phi}{1 + x^2}$, we get

$$F'(x) = 2x\,\phi(x) - (1 + 2x\,\phi(x)) + 1 \equiv 0.$$

Therefore, $F(x)$ is a constant, which we denote by C. That is, $\quad \phi(x) = \dfrac{C - x}{1 + x^2}$.

1.5 Direction Fields

A geometrical viewpoint is particularly helpful for the first order equation $y' = f(x, y)$. The solutions of this equation form a family of curves in the xy-plane. At any point (x, y), the slope $\mathrm{d}y/\mathrm{d}x$ of the solution $y(x)$ at that point is given by $f(x, y)$. We can indicate this by drawing a short line segment (or arrow) through the point (x, y) with the slope $f(x, y)$. The collection of all such line segments at each point (x, y) of a rectangular grid of points is called a **direction field** or a **slope field** or **tangent field** of the differential equation $y' = f(x, y)$. A slope field with some typical trajectories is called the **phase portrait**.

By increasing the density of arrows, it would be possible, in theory at least, to approach a limiting curve, the coordinates and slope of which would satisfy the differential equation at every point. This limiting curve—or rather the relation between x and y that defines a function $y(x)$—is a solution of $y' = f(x, y)$. Therefore, the direction field gives the “flow of solutions.” Integral curves obtained from the general solution are all different: there is precisely one solution curve that passes through each point (x, y) in the domain of $f(x, y)$. They might be touched by the singular solutions (if any) forming the envelope of a family of integral curves. At each of its points, the envelope is tangent to one of integral curves because they share the same slope.

Direction fields can be plotted for differential equations even if they are not necessarily written in the normal form. If the derivative y' is determined uniquely from the equation $F(x, y, y') = 0$, the direction field can be obtained for such an equation. However, if the equation $F(x, y, y') = 0$ defines multiple values for y', then at every such point we would have at least two integral curves with distinct slopes.

Example 1.5.1. Let us consider the differential equation not in the normal form:

$$x(y')^2 - 2yy' + x = 0. \tag{1.5.1}$$

At every point (x, y) such that $y^2 \geqslant x^2 > 0$ we can assign to y' two distinct values

$$y' = \frac{y \pm \sqrt{y^2 - x^2}}{x} \qquad (y^2 - x^2 \geqslant 0, \quad x \neq 0).$$

When $y^2 \leqslant x^2$, Eq. (1.5.1) does not define y' since the root becomes imaginary. Therefore, we cannot draw a direction field for the differential equation (1.5.1) because its slope function is not a single-value function. Nevertheless, we may try to find its general solution by making a guess that it is a polynomial of the second degree: $y = Cx^2 + Bx + A$, where coefficients A, B, and C are to be determined. Substituting y and its derivative, $y' = 2Cx + B$, into Eq. (1.5.1), we get $B = 0$ and $A = 1/(4C)$. Hence, Eq. (1.5.1) has a one-parametric family of solutions

$$y = Cx^2 + \frac{1}{4C}. \tag{1.5.2}$$

For any value of C, $C \neq 0$, y^2 is greater than or equal to x^2. To check our conclusion, we use *Maple:*

```
dsolve(x*(diff(y(x),x))^2-2*y(x)*diff(y(x),x)+x=0,y(x));
phi:=(x,C)->C*x*x+0.25/C;                  # the general solution
plot({subs(C=.5,phi(x,C)),phi(x,-1),x},x=-1..1,y=-1..1,color=blue);
```

As we see from Figure 1.5, integral curves intersect each other, which would be impossible for solutions of a differential equation in the normal form. Indeed, solving Eq. (1.5.2) with respect to C, we observe that for every point (x, y), with $y^2 \geqslant x^2$, there are two distinct values of $C = \left(y \pm \sqrt{y^2 - x^2}\right)/x^2$. For instance, Eq. (1.5.1) defines two slopes at the point $(2, 1)$: $2 \pm \sqrt{3}$.
Let us find an envelope of singular solutions. According to Eq. (1.4.2), we differentiate the general solution (1.5.2) with respect to C, which gives $x^2 = -1/(4C^2)$. Eliminating C from these two equations, we obtain $x^2 - y^2 = 0$ or $y = \pm x$. Substitution into Eq. (1.5.1) yields that these two functions are its solution.
Hence, the given differential equation has two singular solutions $y = \pm x$ that form the envelope of integral curves corresponding to the general solution. □

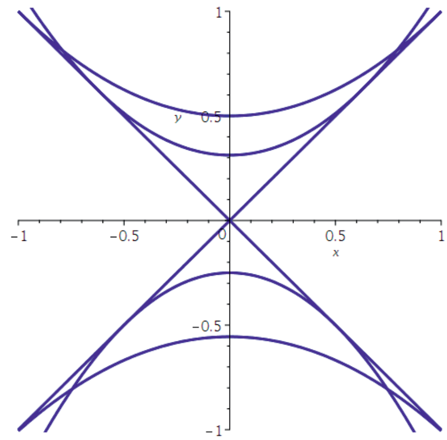


Figure 1.5: Example 1.5.1: some solutions along with two singular solutions, plotted in *Maple*.

Let us consider any region R of the xy-plane in which $f(x, y)$ is a real, single-valued, continuous function. Then the differential equation $y' = f(x, y)$ defines a direction field in the region R. A solution $y = \phi(x)$ of the given differential equation has the property that at every point its graph is tangent to the direction element at that point. The slope field provides useful qualitative information about the behavior of the solution even when you cannot solve it. Direction fields are common in physical applications, which we discuss in [16]. While slope fields prove their usefulness in qualitative analysis, they are open to several criticisms. The integral curves, being graphically obtained, are only approximations to the solutions without any knowledge of their accuracy and formulas.

If we change for a moment the notation of the independent variable x to t, for time, then we can associate the solution of the differential equation with the trajectory of a particle starting from any one of its points and then moving in the direction of the field. The path of such a particle is called a **streamline** or orbit. Thus, the function defined by a streamline is an integral of the differential equation to which the field applies. A point through which just one single integral curve passes is called an *ordinary point*.

When high precision is required, a suitably dense set of line segments on the plane region must be made. The labor involved may then be substantial. Fortunately, available software packages are very helpful for practical drawings of direction fields instead of hand sketching.

There are some free online friendly graphical programs for plotting direction fields. Try GeoGebra (`https://www.geogebra.org/m/W7dAdgqc`) or Desmos (`https://www.desmos.com/calculator/p7vd3cdmei`). Universities also provide online java applets to plot slope fields; see Monroe Community College (`https://www.monroecc.edu/faculty/paulseeburger/calcnsf/DirectionField/`) and Bluffton University (`https://homepages.bluffton.edu/~nesterd/apps/slopefields.html`). You can find some other online applications for plotting direction fields by visiting `http://www.mathscoop.com`, and `http://slopefield.nathangrigg.net`.

Maple

It is recommended to clear the memory before starting a session by invocation of either `restart` or `gc( )` for garbage collection. *Maple* is particularly useful for producing graphical output. It has two dedicated commands for plotting flow fields associated with first order differential equations—`DEplot` and `dfieldplot`. For example, the commands

```
restart;  with(DEtools):  with(plots):
dfieldplot(diff(y(x),x)=y(x)+x, y(x), x=-1..1, y=-2..2, arrows=medium);
```

allow you to plot the direction field for the differential equation $y' = y + x$. To include graphs of some solutions into the direction field, we define the initial conditions first:

```
inc:=[y(0)=0.5,y(0)=-1];
```

Then we type

```
DEplot(diff(y(x),x)=y(x)+x, y(x), x=-1..1, y=-2..2, inc, arrows=medium,
linecolor=black,color=blue,title=‘Direction field for y’=y+x‘);
```

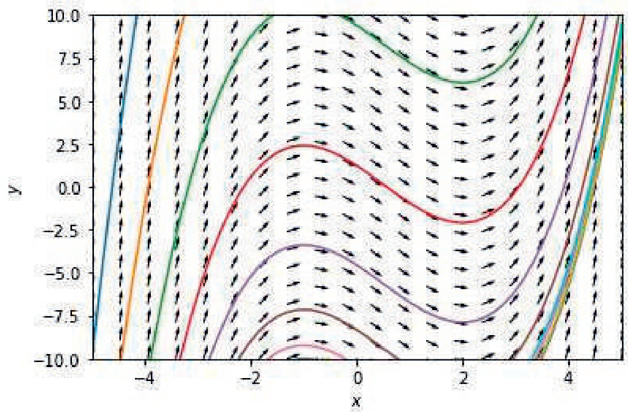


Figure 1.6: Direction field for the equation $y' = x^2 + x - 2$ along with some solutions, plotted using Python.

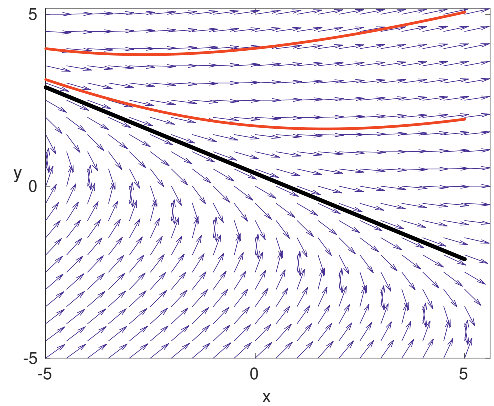


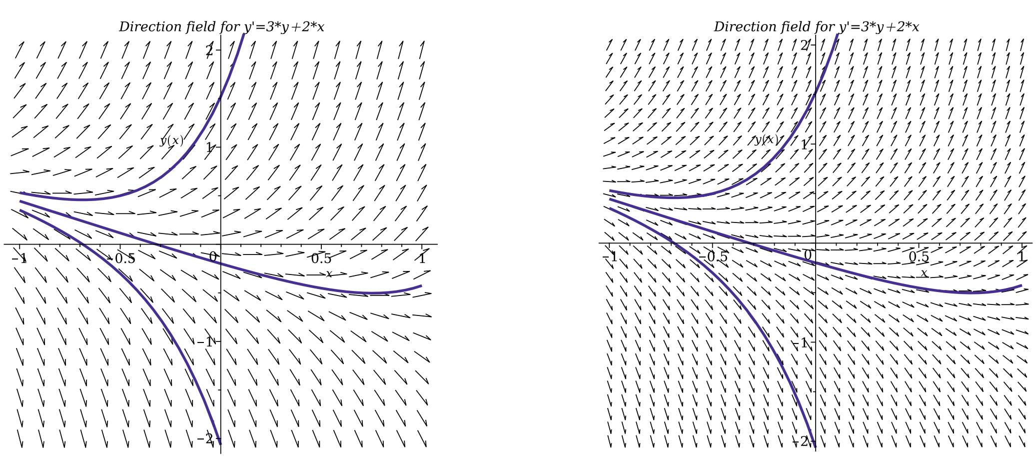
Figure 1.7: Phase portrait for $y' = (x + 2y - 5)(2x + 4y + 7)$ with the separatrix, plotted using MATLAB.

There are many options in representing a slope field, which we demonstrate in the text. A special option, `dirgrid`, specifies the number of arrows in the direction field. For instance, if we replace *Maple*'s option `arrows=medium` with `dirgrid=[16,25]`, we will get the output presented in Figure 1.8.

The computer algebra system (CAS for short) *Maple* also has an option to plot the direction fields without arrows or with comets, as can be seen in Figures 1.8, plotted with the following script:

```
dfieldplot(x*diff(y(x),x)=3*y(x)+2*x,y(x),
x=-1..1,y=-2..2,arrows=line,title=‘Direction field for xy’=3y+2x‘);
DEplot(x*diff(y(x),x)=3*y(x)+x^3,y(x),
x=-1..1,y=-2..2,arrows=comet,title=‘Direction field for xy’=3y+x*x*x‘);
```

You may draw a particular solution that goes through the point $x = \pi/2$, $y = 1$ in the same picture by typing `DEplot(equation, y(x), x-range, y-range, [y(Pi/2)=1], linecolor=blue)`.
Maple also can plot direction fields with different colors:
`dfieldplot(diff(y(x),x)=f(x,y), y(x), x-range, y-range, color=f(x,y))`.



(a) (b)

Figure 1.8: Direction field for $y' = 3y + 2x$ using (a) `arrows=medium` and (b) `dirgrid`, plotted with *Maple*.

Mathematica

It is always a good idea to start *Mathematica*'s session with clearing variables or the kernel. This CAS has several builtin commands to plot direction fields; however, we mention only two of them. With *Mathematica*, only one command is needed to draw the direction field corresponding to the differential equation $y' = f(t, y)$. By choosing, for instance, $f(t, y) = 1 - t^2 - y$, we type:

```
dfield = VectorPlot[{1,1-t^2-y}, {t, -2, 2}, {y, -2, 2}, Axes -> True,
VectorScale -> {Small,Automatic,None}, AxesLabel -> {"t", "dydt=1-t^2-y"}]
```

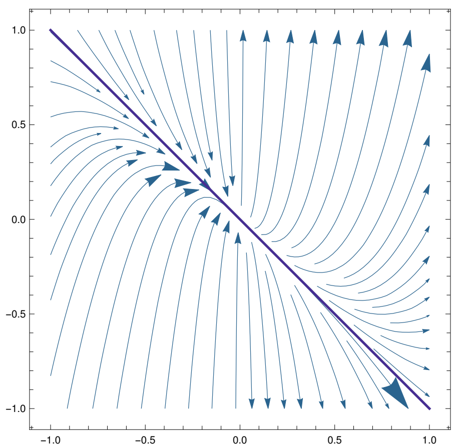


Figure 1.9: Direction field for $xy' = 3y + 2x$ with the separatrix, plotted in *Mathematica*.

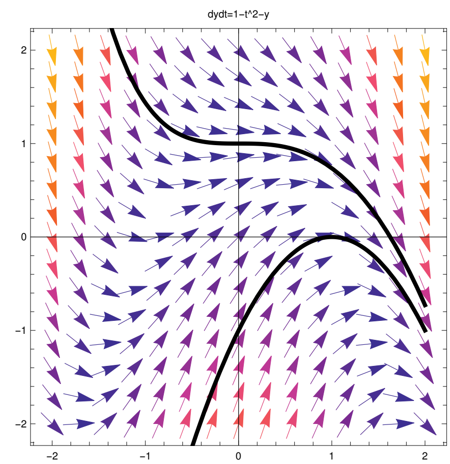


Figure 1.10: Direction field along with two solutions for the equation $y'(t) = 1 - t^2 - y(t)$, plotted with *Mathematica*.

The option `VectorScale` allows one to fix the arrows' sizes and suboption `Scaled[1]` specifies arrowhead size relative to the length of the arrow. To plot the direction field along with, for example, two solutions, we use the following commands:

```
sol1 = DSolve[{y'[t] == 1 - y[t] - t^2, y[0] == 1}, y[t], t]
sol2 = DSolve[{y'[t] == 1 - y[t] - t^2, y[0] == -1}, y[t], t]
pp1 = Plot[y[t] /. sol1, {t, -2, 2}]
pp2 = Plot[y[t] /. sol2, {t, -2, 2}]
Show[dfield, pp1, pp2]
```

For plotting streamlines/solutions, CAS *Mathematica* has a dedicated command: `StreamPlot`. If you need to plot a sequence of solutions under different initial conditions, use the following script:

```
myODE = t^2*y'[t] == (y[t])^3 - 2*t*y[t]
IC = {{0.5, 0.7}, {0.5, 4}, {0.5, 1}};
Do[ansODE[i] =
  Flatten[DSolve[{myODE, y[IC[[i, 1]]] == IC[[i, 2]]}, y[t], t]];
 myplot[i] = Plot[Evaluate[y[t] /. ansODE[i]], {t, 0.02, 5}];
 Print[myplot[i]];  , {i, 1, Length[IC]}]
```

Note that *Mathematica* uses three different notations associated with the symbol "`=`." Double equating "`==`" is used for defining an equation or for testing an equality; regular "`=`" is used to define instant assignments, while "`:=`" is used to represent the left-hand side in the form of the right-hand side afresh, that is, unevaluated.

MATLAB®

MATLAB is a numerical computing environment that also includes Live Editor. Before beginning a new session, it is recommended that you execute the `clc` command to clear the command window and the `clear` command to remove all variables from memory. In order to plot a direction field with MATLAB, you have several options. One of them includes creation of an intermediate file, say `function1.m`, yielding the slope function $f(x, y)$. Let us take a simple example $f(x, y) = xy^2$. This file will contain the following three lines (excluding comments followed after %):

```
% Function for direction field:         (function1.m)
  function F=function1(x,y);
  F=x*y*y;
F=vectorize(F); % to get a vectorized version of the function, which is optional
```

If a function $f(x, y)$ is not complicated, it could be defined directly within MATLAB code. We demonstrate it in the case of the rational slope function $f(x, y) = (x + 2y - 5)/(2x + 4y - 7)$ that is used in Example 2.2.3 on page 56:

```
der=@(x,y) (x+2*y-5)./(2*x+4*y+7); % define slope function
equ=@(x) (3-4*x)/8;                % define equilibrium
sig=@(x) -(7+2*x)/4;               % define singular
xmin=-5; xmax=5; ymin=-5; ymax=5;  % set the frame
dx=(xmax-xmin)/20;                 % set spatial steps
dy=(ymax-ymin)/20;
[X,Y]=meshgrid(xmin:dx:xmax, ymin:dy:ymax); % generate mesh
Dx=ones(size(X));       % Unit x-components of arrows
Dy=der(X,Y);            % Computed y-components
L=sqrt(Dx.^2 + Dy.^2);  % Initial lengths
Dx1=Dx./L; Dy1=Dy./L;   % Unit lengths for all arrows
quiver(X,Y, Dx1,Dy1, 'b-'); % draw the direction field
axis tight;             % sets the axis limits
% to the range of the data
xlabel('x','FontSize',16); % set labels, fontsize for them
ylabel('y','FontSize',16,'rotation',0); % and direction for letter y
set(gca, 'FontSize', 12); % set fontsize for axis
hold on
xx=xmin:dx/10:xmax;
plot(xx, equ(xx), 'k', 'LineWidth', 3);
plot(xx, sig(xx), 'k--', 'LineWidth', 3);
% below are several solutions based on different initial conditions
[x,y1]=ode45(der, [-5 5], 3.1); % IC: y(-5)=3.1
plot(x,y1,'r', 'LineWidth', 2);
[x,y2]=ode45(der, [-5 5], 4.0); % IC: y(-5)=4.0
plot(x,y2,'r', 'LineWidth', 2);
  print -deps direction_field.eps;  % or print -deps2 direction_field.eps;
  print -depsc direction_field.eps; % for color image
```

In the code above, the subroutine quiver(x, y, u, v) displays velocity vectors as arrows with components (u, v) at the points (x, y). To draw a slope field without arrows, `quiver` is not needed as the following code shows. If a graph of a function needs not to be plotted along with the slope field, comment out the last line.

```
func = @(x) 3*x.^2 - 8*x;   % slope function
ifunc = @(x) x.^3 - 4*x.^2; % integral from 0 to x
xmin = -2; xmax = 5;        % set limits for x
dx = (xmax - xmin)/100;     % step for computing ifunc
xx = xmin:dx:xmax;          % arguments for computing ifunc
sing = ifunc(xx);           % computing ifunc
dy = (max(sing)- min(sing))/100; %step along y-axis for slope field
figure; axis tight;   % set the axis limits to the range of the data
for y = min(sing):6*dy:max(sing) % loop for plotting a slope field
  for x = xmin:6*dx:xmax
    c=2/sqrt((1/dx)^2 + (func(x)/dy).^2 ); d=func(x).*c;
    tpt = [x - c, x + c]; ypt=[y - d, y + d];
    line(tpt,ypt); % plot an element of slope field
  end;
end
hold on; plot(xx, sing, 'k-', 'LineWidth', 3); hold off;
```

As you have seen from the two scripts above, although resourceful, MATLAB does not have the ability to plot direction fields naturally. However, many universities have developed some software packages that facilitate drawing direction fields for differential equations. Another option is to use a special software system, `Chebfun`, developed by L.N. Trefethen from the University of Oxford (UK). This system embraces symbolic operations with numerical calculations within MATLAB.

Maxima

Maxima and its popular graphical interface *wxMaxima*[4] are free software[5] projects. This means that you have the freedom to use them without restriction, to give copies to others, to study their internal workings and adapt them to your needs, and even to distribute modified versions. *Maxima* is a descendant of Macsyma, the first comprehensive computer algebra system, developed in the late 1960s at the Massachusetts Institute of Technology.

Maxima provides two packages for plotting direction fields, each with different strengths: `plotdf` supports interactive exploration of solutions and variation of parameters, whereas `drawdf` is non-interactive and instead emphasizes the flexible creation of high-quality graphics in a variety of formats. Let us first use `drawdf` to plot the direction field for the differential equation $y' = e^{-t} + y$:

```
load(drawdf);
drawdf(exp(-t)+y, [t,y], [t,-5,10], [y,-10,10]);
```

Note that `drawdf` normally displays graphs in a separate window. If you are using *wxMaxima* (recommended for new users) and would prefer to place your graphs within your notebook, use the `wxdrawdf` command instead. The `load` command stays the same, however.

Solution curves passing through points $y(0) = 0$, $y(0) = -0.5$, and $y(0) = -1$ can be included in the graph as follows:

```
drawdf(exp(-t)+y, [t,y], [t,-5,10], [y,-10,10],
  solns_at([0,0], [0,-0.5], [0,-1]));
```

By adding `field_degree=2`, we can draw a field of quadratic splines (similar to *Maple*'s comets) which show both slope and curvature at each grid point. Here we also specify a grid of 20 columns by 16 rows, and draw the middle solution thicker and in black.

```
drawdf(exp(-t)+y, [t,y], [t,-5,10], [y,-10,10],
  field_degree=2, field_grid=[20,16], solns_at([0,0], [0,-1]),
  color=black, line_width=2, soln_at(0,-0.5));
```

We can add arrows to the solution curves by specifying `soln_arrows=true`. This option removes arrows from the field by default and also changes the default color scheme to emphasize the solution curves.

```
drawdf(exp(-t)+y, [t,y], [t,-5,10], [y,-10,10], field_degree=2,
  soln_arrows=true, solns_at([0,0], [0,-1], [0,-0.5]),
  title="Direction field for dy/dt = exp(-t) + y",
  xlabel="t", ylabel="y");
```

Actual examples of direction fields plotted with *Maxima* are presented on the front page of Chapter 1, and scattered in the text. The following command will save the most recent plot to an encapsulated Postscript file named "plot1.eps" with dimensions 12 `cm` by 8 `cm`. Several other formats are supported as well, including PNG and PDF.

```
draw_file(terminal=eps, file_name="plot1",
  eps_width=12, eps_height=8);
```

Since `drawdf` is built upon *Maxima*'s powerful `draw` package, it accepts all of the options and graphical objects supported by `draw2d`, allowing the inclusion of additional graphics and diagrams. To investigate differential equations by varying parameters, `plotdf` is sometimes preferable. Let us explore the family of differential equations of the form $y' = y - a + b\cos t$, with a solution passing through $y(0) = 0$:

```
plotdf(y-a+b*cos(t), [t,y], [t,-5,9], [y,-5,9],
  [sliders,"a=0:2,b=0:2"], [trajectory_at,0,0]);
```

You may now adjust the values of a and b using the sliders in the plot window and immediately see how they affect the direction field and solution curves. You can also click in the field to plot new solutions through any desired point. Make sure to close the plot window before returning to your *Maxima* session.

Sage

SageMath is a free open-source mathematics software system licensed under the GPL. It builds on top of many existing open-source packages Python, *R*, Julia (computational geometry), GAP (discrete algebra), Octave, including

[4]See http://maxima.sourceforge.net/ and http://wxmaxima.sourceforge.net/

[5]See http://www.fsf.org/ for more information about free software.

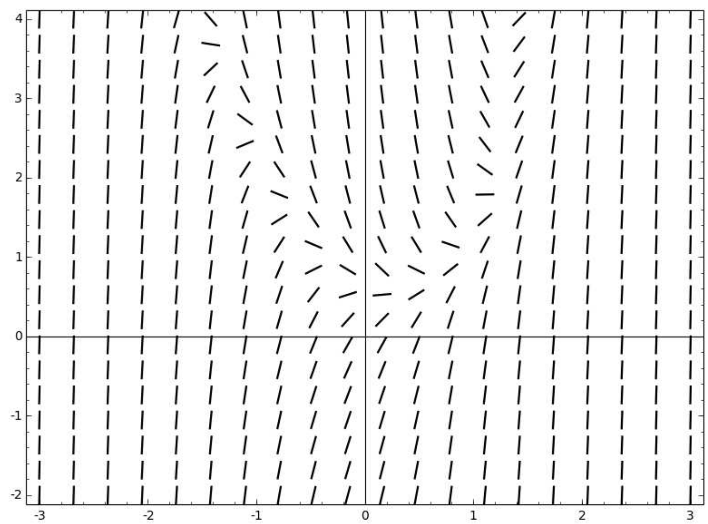


Figure 1.11: Direction field for the equation $y'(x) = 6\,x^2 - 3y + 2\,\cos(x+y)$, plotted with *Sage*.

computer algebra systems *Maxima* and SymPy, and much more. *Sage* comes with two options: one can download it[6] (for free) or use it interactively through cloud version. *SageMath* is available for every platform, and is ubiquitous throughout industry. SageMathCloud supports authoring documents written in LaTeX, Markdown, or HTML. SageMathCloud also allows you to publish documents online.

To plot a direction field for first order differential equation $y' = f(x, y)$ using *Sage* we first declare variables and then we use a standard command, which we demonstrate in the following example: $y' = 6\,x^2 - 3y + 2\,\cos(x+y)$.

```
x,y = var('x,y')
plot_slope_field(6*x^2 -3*y+2*cos(x+y), (x,-3,3), (y,-2,4), xmax=10)
```

Python

Python[7] is a high-level and general-purpose programming language (free of charge). Part of the reason that it is a popular choice for scientists and engineers is the language versatility, online community of users, and powerful analysis packages such as NumPy, SciPy, and, of course, SymPy, a CAS written completely in Python. Anaconda is a free Python distribution from Continuum Analytics that includes many useful packages for scientific computing.

The function **odeint** is available in SciPy for integrating first order vector differential equations. A higher order ordinary differential equation can always be reduced to a differential equation of this type by introducing intermediate derivatives into the vector (see §7.3). There are many optional inputs and outputs available when using odeint that can help tune the solver.

Problems

1. For each equation below, determine its order, and name the independent variable, the dependent variable, and any parameters.

   (a) $y' = y^2 + x^2$; **(b)** $\dot{P} = rP(1 - P/N)$; **(c)** $m\ddot{x} + r\dot{x} + kx = \sin t$;
   (d) $L\,\ddot{\theta} + g\,\sin\theta = 0$; **(e)** $(x\,y'(x))^2 = x^2\,y(x)$; **(f)** $2y'' + 3y' + 5y = e^{-x}$.

2. Determine a solution to the following differential equation:

$$(1 - 2x^2)\,y'' - x\,y' + 6y = 0$$

   of the form $y(x) = a + bx + cx^2$ satisfying the normalization condition $y(1) = -1$.

3. Differentiate both sides of the given equation to eliminate the arbitrary constant (denoted by C) and to obtain the associated differential equation.

[6] See http://www.sagemath.org/.
[7] https://www.python.org/

(a) $x^2 + 4xy^2 = C$; **(b)** $x^3 + 2x^2 y = C$; **(c)** $y\cos x - xy^2 = C$;
(d) $Cx^2 = y^2 + 2y$; **(e)** $e^x - \ln y = C$; **(f)** $3\tan x + \cos^2 y = C$;
(g) $Cx = \ln(xy)$; **(h)** $Cy + \ln x = 0$; **(i)** $y - 1 + \frac{x}{1+Cx} = 0$;
(j) $y = \frac{x}{x+C}$; **(k)** $y + Cx = x^4$; **(l)** $\sin^2 y - \tan x = C$.

4. Find the differential equation of the family of curves $y_n(x) = \left(1 + \frac{x}{n}\right)^n$, where $n \neq 0$ is a parameter. Show that $y = e^x$ is a solution of the equation found when $n \to \infty$.

5. Find a differential equation of fourth order having a solution $y = Ax\cos x + Bx\sin x$, where A and B are arbitrary constants.

6. Which of the following equations for the unknown function $y(x)$ are *not* ordinary differential equations? Why?
   (a) $\frac{\mathrm{d}}{\mathrm{d}x}\frac{y''(x)}{(1+(y'(x))^2)^{3/2}} = (1 + (y'(x))^2)^{1/2}$; **(b)** $y'(x) = y(x-1) - y(x)/3$;
   (c) $\int_{-\infty}^{\infty} \cos(kx) y'(x)\,\mathrm{d}x = \sqrt{2\pi}\, y'(k)$; **(d)** $y''(x) = g - y'(x)|y'(x)|$;
   (e) $\int_0^{\infty} e^{-xt} y(x)\,\mathrm{d}x = e^{-t}/t$; **(f)** $y(x) = 1 + \int_0^x y(t)\,\mathrm{d}t$.

7. Determine the order of the following ordinary differential equations:
   (a) $(y')^2 + x^2 y = 0$; **(b)** $\frac{\mathrm{d}}{\mathrm{d}x}(y\,y') = \sin x$; **(c)** $x^2 + y^2 + (y')^2 = 1$;
   (d) $x^2 y'' + \frac{\mathrm{d}}{\mathrm{d}x}(xy') + y = 0$; **(e)** $\frac{\mathrm{d}}{\mathrm{d}x}(y' \sin x) = 0$; **(f)** $t^3 u''(t) + t^2 u'(t) + t\,u(t) = 0$;
   (g) $\frac{\mathrm{d}}{\mathrm{d}t}\left[t^2 u''(t)\right] = 0$; **(h)** $(u'(t))' = t^3$; **(i)** $y'' = \sqrt{4 + (y')^2}$;
   (j) $xy'' + (y')^3 + e^x = 0$; **(k)** $xy''' + \mathrm{sign}(x) = 0$; **(l)** $(y'')^2 + y' \sin x = \cos x$.

8. Which of the following equations are linear?
   (a) $y^{(4)} + x^3 y = 0$; **(b)** $\frac{\mathrm{d}}{\mathrm{d}x}(y\,y') = \sin x$; **(c)** $\frac{\mathrm{d}}{\mathrm{d}x}[xy] = 0$;
   (d) $y' + y\sin x = 1$; **(e)** $(y')^2 - x^2 = 0$; **(f)** $y' - x^2 = 0$,
   (g) $\frac{\mathrm{d}}{\mathrm{d}x}[x^2 + y^2] = 1$; **(h)** $y'''(x) + x\,y''(x) + y^2(x) = 0$; **(i)** $y' = \sqrt{xy}$;
   (j) $y'(x) + x^2 y(x) = \cos x$; **(k)** $y' = x^2 + y^2$; **(l)** $y'' + y' \sin x = \cos x$.

9. Let $y(x)$ be a solution for the ODE $y''(x) = x\,y(x)$ that satisfies the initial conditions $y(0) = 1$, $y'(0) = 0$. (You will learn in this course that exactly one such solution exists.) Calculate $y''(0)$. The ODE is, by its nature, an equation that is meant to hold for *all* values of x. Therefore, you can take the derivative of the equation. With this in mind, calculate $y'''(0)$ and $y^{(4)}(0)$.

10. In each of the following problems, verify whether or not the given function is a solution of the given differential equation and specify the interval or intervals in which it is a solution; C always denotes a constant.
    (a) $y'' + 4y = 0$, $y = \sin 2x + C$. **(b)** $y' - y^2(x)\sin x = 0$, $y(x) = 1/\cos x$.
    (c) $2yy' = 1$, $y(x) = \sqrt{x+1}$. **(d)** $x\,y'' - y' + 4x^3 y = 0$, $y = \sin(x^2 + 1)$.
    (e) $y' = ky$, $y(x) = C\,e^{kx}$. **(f)** $y' = 1 - ky$, $k\,y(x) = 1 + C\,e^{-kx}$.
    (g) $y''' = 0$, $y(x) = C\,x^2$. **(h)** $y'' + 2y = 2\cos^2 x$, $y(x) = \sin^2 x$.
    (i) $y'' - 5y' + 6y = 0$, $y = C\,e^{3x}$. **(j)** $y'' + 2y' + y = 0$, $y(x) = C\,x\,e^{-x}$.
    (k) $y' - 2xy = 1$, $y = e^{x^2}\int_0^x e^{-t^2}\,\mathrm{d}t + C\,e^{x^2}$. **(l)** $x\,y' - y = x\sin x$, $y(x) = x\int_0^x \frac{\sin t}{t}\,\mathrm{d}t + Cx$.

11. Solutions to most differential equations cannot be expressed in finite terms using elementary functions. Some solutions of differential equations, due to their importance in applications, were given special labels (usually named after an early investigator of its properties) and therefore are referred to as *special functions*. For example, there are two known *sine integrals*: $\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,\mathrm{d}t$, $\mathrm{si}(x) = -\int_x^{\infty} \frac{\sin t}{t}\,\mathrm{d}t = \mathrm{Si}(x) - \frac{\pi}{2}$ and three cosine integrals: $\mathrm{Cin}(x) = \int_0^x \frac{1-\cos t}{t}\,\mathrm{d}t$, $\mathrm{ci}(x) = -\int_x^{\infty} \frac{\cos t}{t}\,\mathrm{d}t$, and $\mathrm{Ci}(x) = \gamma + \ln x + \int_0^x \frac{\cos t}{t}\,\mathrm{d}t$, where $\gamma \approx 0.5772$ is Euler's constant. Use definite integration to find an explicit solution to the initial value problem $x\,y' = \sin x$, subject to $y(1) = 1$.

12. Verify that the indicated function is an implicit solution of the given differential equation.
    (a) $(y+1)y' + x + 2 = 0$, $(x+2)^2 + (y+1)^2 = C^2$;
    (b) $x\,y' = (1 - y^2)/y$, $yx - \ln y = C$;
    (c) $y' = (\sqrt{(1+y)} + 1 + y)/(1 + x)$, $C\sqrt{1+x} - \sqrt{1+y} = 1$;
    (d) $y' = (y-2)^2/x^2$, $(y-2)(1 + Cx) = x$.

13. Show that the functions in parametric form satisfy the given differential equation.
    (a) $3x\,\mathrm{d}x + 2\sqrt{4 - x^2}\,\mathrm{d}y = 0$, $x = 2\cos t$, $y = 3\sin t$; **(b)** $(x + 3y)\mathrm{d}y = (5x + 3y)\mathrm{d}x$, $x = e^{-2t} + 3\,e^{6t}$, $y = -e^{-2t} + 5\,e^{6t}$;
    (c) $2x\,y' = y - 1$, $x = t^2$, $y = t + 1$; **(d)** $y\,y' = x$, $x = \tan t$, $y = \sec t$;
    (e) $2x\,y' = 3y$, $x = t^2$, $y = t^3$; **(f)** $ax\,y' = by$, $x = t^a$, $y = t^b$;
    (g) $y' = -4x$, $x = \sin t$, $y = \cos 2t$; **(h)** $9y\,y' = 4x$, $x = 3\cosh t$, $y = 2\sinh t$;
    (i) $2(y+1)\,y' = 1$, $x = t^2 + 1$, $y = t - 1$; **(j)** $y' = 2x + 10$, $x = t - 5$, $y = t^2 + 1$;
    (k) $(x+1)\,y' = 1$, $x = t - 1$, $y = \ln t$, $t > 0$; **(l)** $a^2 y\,y' = b^2 x$, $x = a\cosh t$, $y = b\sinh t$.

14. Determine the value of λ for which the given differential equation has a solution of the form $y = e^{\lambda t}$.
(a) $y' - 3y = 0$; **(b)** $y'' - 4y' + 3y = 0$; **(c)** $y'' - y' - 2y = 0$;
(d) $y'' - 2y' - 3y = 0$; **(e)** $y''' + 3y'' + 2y' = 0$; **(f)** $y''' - 3y'' + 3y' - y = 0$.

15. Determine the value of λ for which the given differential equation has a solution of the form $y = x^{\lambda}$.
(a) $x^2y'' + 2xy' - 2y = 0$; **(b)** $xy' - 2y = 0$; **(c)** $x^2y'' - 3xy' + 3y = 0$;
(d) $x^2y'' + 2xy' - 6y = 0$; **(e)** $x^2y'' - 6y = 0$; **(f)** $x^2y'' - 5xy' + 5y = 0$.

16. Show that (a) the first order differential equation $|y'| + 4 = 0$ has no solution; (b) $|y'| + y^2 + 1 = 0$ has no real solutions, but a complex one; (c) $|y'| + 4|y| = 0$ has a solution but not one involving an arbitrary constant.

17. Show that the first order differential equation $y' = 4\sqrt{y}$ has a one-parameter family of solutions of the form $y(x) = (2x + C)^2$, $2x + C \geqslant 0$, where C is an arbitrary constant, and a singular solution $y(x) \equiv 0$ which is not a member of the family $(2x + C)^2$ for any choice of C.

18. Find a differential equation for the family of lines $y = Cx - C^2$.

19. For each of the following differential equations, find a singular solution.
(a) $y' = 3x^2 - (y - x^3)^{2/3}$; **(b)** $y' = \sqrt{(x + 1)(y - 1)}$;
(c) $y' = \sqrt{x^2 - y} + 2x$; **(d)** $y' = (2y)^{-1} + (y^2 - x)^{1/3}$;
(e) $y' = (x^2 + 3y - 2)^{2/3} + 2x$; **(f)** $y' = 2(x + 1)(x^2 + 2x - 3)^{2/3}$.

20. Show that $y = \pm a$ are singular solutions to the differential equation $y\,y' = \sqrt{a^2 - y^2}$.

21. Verify that the function $y = x + 4\sqrt{x + 1}$ is a solution of the differential equation $(y - x)\,y' = y - x + 8$ on some interval.

22. The position of a particle on the x-axis at time t is $x(t) = t^{(t^t)}$ for $t > 0$. Let $v(t)$ be the velocity of the particle at time t. Find $\lim_{t\to 0} v(t)$.

23. An airplane takes off at a speed of 225 `km/hour`. A landing strip has a runway of 1.8 `km`. If the plane starts from rest and moves with a constant acceleration, what is this acceleration?

24. Let $m(t)$ be the investment resulting from a deposit m_0 after t years at the interest rate r compounded daily. Show that
$$m(t) = m_0\left[1 + \frac{r}{365}\right]^{365t}.$$
From calculus we know that $\left[1 + \frac{r}{n}\right]^{nt} \to e^{rt}$ as $n \to \infty$. Hence, $m(t) \to m_0 \exp\{rt\}$. What differential equation does the function $m(t)$ satisfy?

25. A particle moves along the abscissa so that its instantaneous acceleration is given as a function of time t by $a(t) = 2 - 3t^2$. At times $t = 1$ and $t = 4$, the particle is located at $x = 5$ and $x = -10$, respectively. Set up a differential equation and associated conditions describing the motion.

26. A particle moves along the abscissa in such a way that its instantaneous velocity is given as a function of time t by $v(t) = 6 - 3\,t^2$. At time $t = 0$, it is located at $x = 1$. Set up an initial value problem describing the motion of the particle and determine its position at any time $t > 0$.

27. A particle moves along the abscissa so that its velocity at any time $t \geqslant 0$ is given by $v(t) = 4/(t^2 + 1)$. Assuming that it is initially at π, show that it will never pass $x = 2$.

28. The slope of a family of curves at any point (x, y) of the plane is given by $1 + 2x$. Derive a differential equation of the family and solve it.

29. The graph of a nonnegative function has the property that the length of the arc between any two points on the graph is equal to the area of the region under the arc. Find a differential equation for the curve.

30. Geological dating of rocks is done using potassium-40 rather than carbon-14 because potassium has a longer half-life, 1.28×10^9 years (the half-life is the time required for the quantity to be reduced by one half). The potassium decays to argon, which remains trapped in the rocks and can be measured. Derive the differential equation that the amount of potassium obeys.

31. Prove that the equation $y' = (ay + b)/(cy + d)$ has at least one solution of the form $y = kx$ if either $b = 0$ or $ad = bc$.

32. Which straight lines through the origin are solutions of the following differential equations?
(a) $y' = \dfrac{4x + 3y}{3x + y}$; **(b)** $y' = \dfrac{x + 3y}{y - x}$; **(c)** $y' = \dfrac{x + 3y}{x - y}$;
(d) $y' = \dfrac{3y - 2x}{x - 3y}$; **(e)** $y' = \dfrac{x}{x + 2y}$; **(f)** $y' = \dfrac{2x + 3y}{2x + y}$.

33. Phosphorus (^{31}P) has multiple isotopes, two of which are used routinely in life-science laboratories dealing with DNA production. They are both beta-emitters, but differ by the energy of emissions—^{32}P has 1.71 MeV and ^{33}P has 0.25 MeV. Suppose that a sample of 32-isotope disintegrates to 71.2 `mg` in 7 days, and 33-isotope disintegrates to 82.6 `mg` during the same time period. If initially both samples were 100 `mg`, what are their half-life periods?

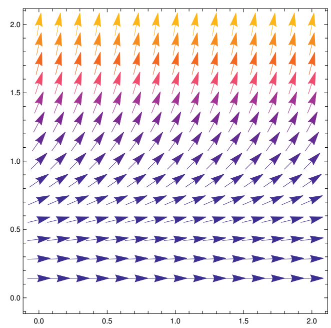


Figure 1.12: Direction field for Problem 36.

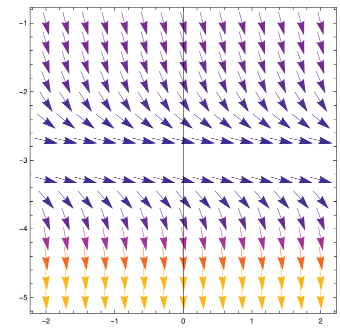


Figure 1.13: Direction field for Problem 37.

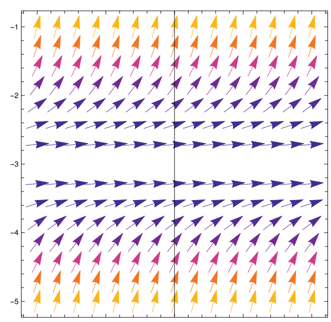


Figure 1.14: Direction field for Problem 38.

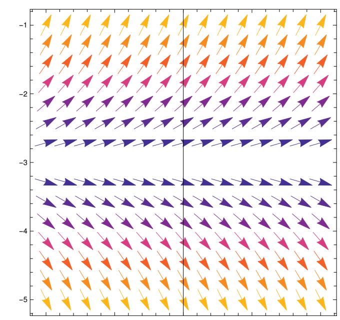


Figure 1.15: Direction field for Problem 39.

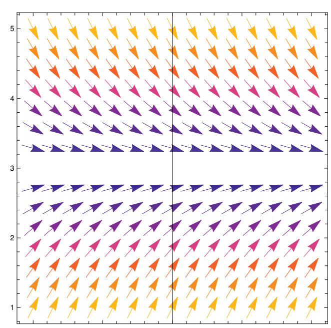


Figure 1.16: Direction field for Problem 40.

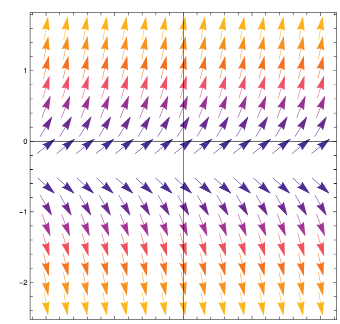


Figure 1.17: Direction field for Problem 41.

34. Show that the initial value problem $y' = 4x\sqrt{y}$, $y(0) = 0$ has infinitely many solutions.

The following problems require utilization of a computer package.

35. For the following two initial value problems

   (a) $y' = y(1-y), \quad y(0) = \frac{1}{2}$; **(b)** $y' = y(1-y), \quad y(0) = 2$;

   show that $y_a(x) = \left(1 + e^{-x}\right)^{-1}$ and $y_b = 2\left(2 - e^{-x}\right)^{-1}$ are their solutions, respectively. What are their long-term behaviors?

Consider the following list of differential equations, some of which produced the direction fields shown in Figures 1.12 through 1.17. In each of Problems 36 through 41, identify the differential equation that corresponds to the given direction field.

(a)	$\dot{y} = y^{2.5}$;	**(b)**	$\dot{y} = 3y - 1$;	**(c)**	$\dot{y} = y(y+3)^2$;	**(d)**	$\dot{y} = y^5(y+1)$;
(e)	$\dot{y} = y(y+3)^2$;	**(f)**	$\dot{y} = y^2(y+3)$;	**(g)**	$\dot{y} = (y+3)^2$;	**(h)**	$\dot{y} = y+3$;
(i)	$\dot{y} = y-3$;	**(j)**	$\dot{y} = y-3$;	**(k)**	$\dot{y} = -y+3$;	**(l)**	$\dot{y} = 3y+1$.

36. The direction field of Figure 1.12.
37. The direction field of Figure 1.13.
38. The direction field of Figure 1.14.
39. The direction field of Figure 1.15.
40. The direction field of Figure 1.16.
41. The direction field of Figure 1.17.

1.6 Existence and Uniqueness

An arbitrary differential equation of the first order $y' = f(x, y)$ does not necessarily have a solution that satisfies it. Therefore, the existence of a solution is an important problem for both the theory of differential equations and their applications.

If some phenomenon is modeled by a differential equation, then the equation should have a solution. If it does not, then presumably there is something wrong with the mathematical modeling and the simulation needs improvement. So, an engineer or a scientist would like to know whether a differential equation has a solution before investing time, effort, and computer applications in a vain attempt to solve it. An application of a software package may fail to provide a solution to a given differential equation, but this doesn't mean that the differential equation doesn't have a solution.

Whenever an initial value problem has been formulated, there are three questions that could be asked before finding a solution:

1. Does a solution of the differential equation satisfying the given conditions exist?
2. If one solution satisfying the given conditions exists, can there be a different solution that also satisfies the conditions?
3. What is the reason to determine whether an initial value problem has a unique solution if we won't be able to explicitly determine it?

A positive answer for the first question is our hunting license to go looking for a solution. In practice, one wishes to find the solution of a differential equation satisfying the given conditions to less than a finite number of decimal places. For example, if we want to draw the solution, our eyes cannot distinguish two functions which have values that differ by less than 1%. Therefore, for printing applications, the knowledge of three significant figures in the solution is admissible accuracy. This may be done, for instance, with the aid of available software packages.

In general, existence or uniqueness of an initial value problem cannot be guaranteed. For example, the initial value problem $y' = y^2$, $x < 1$, $y(1) = -1$ has a solution $y(x) = -x^{-1}$, which does not exist for $x = 0$. On the other hand, Example 1.4.2 on page 9 shows that the initial value problem may have two (or more) solutions.

For most of the differential equations in this book, there are unique solutions that satisfy certain prescribed conditions. However, let us consider the differential equation

$$xy' - 5y = 0,$$

which arose in a certain problem. Suppose a scientist has drawn an experimental curve as shown on the left side of Fig.1.18 (page 22).

The general solution of the given differential equation is $y = C\,x^5$ with an arbitrary constant C. From the initial condition $y(1) = 2$, it follows that $C = 2$ and $y = 2\,x^5$. Thus, the theoretical and experimental graphs agreed for $x > 0$, but disagree for $x < 0$.

If the scientist had erroneously assumed that a unique solution exists, s/he may decide that the mathematics was wrong. However, since the differential equation has a singular point $x = 0$, its general solution contains two arbitrary constants, A and B, one for domain $x > 0$ and another one for $x < 0$. So

$$y(x) = \begin{cases} A\,x^5, & x \geqslant 0, \\ B\,x^5, & x \leqslant 0. \end{cases}$$

Therefore, the experimental graph corresponds to the case $A = 2$ and $B = 0$.

Now suppose that for the same differential equation $xy' = 5y$ we have the initial condition at the origin: $y(0) = 0$. Then any function $y = Cx^5$ satisfies it for arbitrary C and we have infinitely many solutions to the given initial value problem (IVP). On the other hand, if we want to solve the given equation with the initial condition $y(0) = 1$, we are out of luck. There is no solution to this initial value problem!

In this section, we discuss two fundamental theorems for first order ordinary differential equations subject to initial conditions that prove the existence and the uniqueness of their solutions. These theorems provide *sufficient conditions* for the existence and uniqueness of a solution; that is, if the conditions hold, then uniqueness and/or

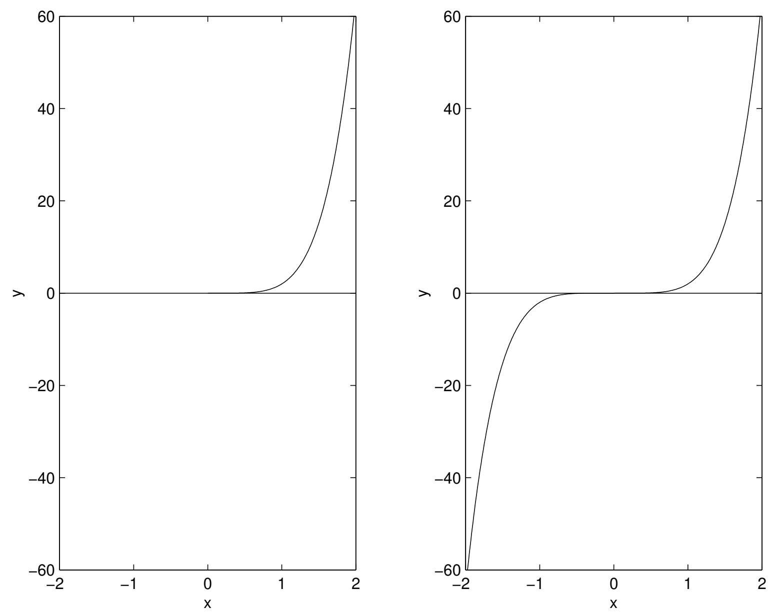


Figure 1.18: Experimental curve at the left and modeled solution at the right.

existence are guaranteed. However, the conditions are not *necessary conditions* at all; there may still be a unique solution if these conditions are not met. The following theorem guarantees the uniqueness and existence for linear differential equations.

Theorem 1.1. Let us consider the initial value problem for the linear differential equation

$$y' + q(x)y = f(x), \tag{1.6.1}$$

$$y(x_0) = y_0, \tag{1.6.2}$$

where $q(x)$ and $f(x)$ are known functions and y_0 is an arbitrary prescribed initial value. Assume that the functions $a(x)$ and $f(x)$ are continuous on an open interval $\alpha < x < \beta$ containing the point x_0. Then the initial value problem (1.6.1), (1.6.2) has a unique solution $y = \phi(x)$ on the same interval (α, β).

PROOF: In §2.5 we show that if Eq. (1.6.1) has a solution, then it must be given by the following formula:

$$y(x) = \mu^{-1}(x) \left[\int \mu(x) f(x)\, \mathrm{d}x + C \right], \quad \mu(x) = \exp\left\{ \int a(x)\, \mathrm{d}x \right\}. \tag{1.6.3}$$

When $\mu(x)$ is a nonzero differentiable function on the interval (α, β), we have from Eq. (2.5.2), page 86, that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mu(x)y(x)\right] = \mu(x) f(x).$$

Since both $\mu(x)$ and $f(x)$ are continuous functions, its product $\mu(x)f(x)$ is integrable, and formula (1.6.3) follows from the latter. Hence, from Eq. (1.6.3), the function $y(x)$ exists and is differentiable over the interval (α, β). By substituting the expression for $y(x)$ into Eq. (1.6.1), one can verify that this expression is a solution of Eq. (1.6.1). Finally, the initial condition (1.6.2) determines the constant C uniquely.

If we choose the lower limit to be x_0 in all integrals in the expression (1.6.3), then

$$y(x) = \frac{1}{\mu(x)} \left[\int_{x_0}^{x} \mu(s) f(s)\, \mathrm{d}s + y_0 \right], \qquad \mu(x) = \exp\left\{ \int_{x_0}^{x} a(s)\, \mathrm{d}s \right\}$$

is the solution of the initial value problem (1.6.1), (1.6.2). ∎

In 1886, Giuseppe Peano[8] gave sufficient conditions that only guarantee the existence of a solution for initial value problems (IVPs).

The Peano existence theorem can be viewed as a generalization of the fundamental theorem of calculus, which makes the same assertion for the first order equation $y' = f(x)$. Geometrical intuition suggests that a solution curve, if any, of the equation $y' = f(x, y)$ can be obtained by threading the segments of the direction field. We may also imagine that a solution is a trajectory or path of a particle moving under the influence of the force field. Physical intuition asserts the existence of such trajectories when that field is continuous.

Theorem 1.2. [**Peano**] Suppose that the function $f(x, y)$ is continuous in some rectangle:

$$\Omega = \{(x, y) : x_0 - a \leqslant x \leqslant x_0 + a, \quad y_0 - b \leqslant y \leqslant y_0 + b\}, \tag{1.6.4}$$

Let

$$M = \max_{(x,y)\in\Omega} |f(x, y)|, \quad h = \min\left\{a,\ \frac{b}{M}\right\}. \tag{1.6.5}$$

Then the initial value problem

$$y' = f(x, y), \qquad y(x_0) = y_0, \tag{1.6.6}$$

has a solution in the interval $[x_0 - h, x_0 + h]$.

Corollary 1.1. If the continuous function $f(x, y)$ in the domain $\Omega = \{(x, y) : \alpha < x < \beta,\ -\infty < y < \infty\}$ satisfies the inequality $|f(x, y)| \leqslant a(x)|y| + b(x)$, where $a(x)$ and $b(x)$ are positive continuous functions, then the solution to the initial value problem (1.6.1), (1.6.2) exists in the interval $\alpha < x < \beta$.

In most of today's presentations, Peano's theorem is proved with the help of either the Arzela–Ascoli compactness principle for function sequences or Banach's fix-point theorem, which are both beyond the scope of this book.

In 1890, Peano showed that the solution of the nonlinear differential equation $y' = 3y^{2/3}$ subject to the initial condition $y(0) = 0$ is not unique. He discovered, and published, a method for solving linear differential equations using successive approximations. However[9], Emile Picard[10] had independently rediscovered this method and applied it to show the existence and uniqueness of solutions to the initial value problems for ordinary differential equations. His result, known as Picard's theorem, imposes so called the Lipschitz condition in honor of the German mathematician Rudolf Lipschitz (1832–1903), who introduced it in 1876 when working out existence proofs for ordinary differential equations. Under this sufficient condition on $f(x, y)$, the corresponding initial value problem has no singular solutions.

Theorem 1.3. [**Picard**] Let $f(x, y)$ be a continuous function in a rectangular domain Ω containing the point (x_0, y_0). If $f(x, y)$ satisfies the Lipschitz condition

$$|f(x, y_1) - f(x, y_2)| \leqslant L|y_1 - y_2|$$

for some positive constant L (called the Lipschitz constant) and any x, y_1, and y_2 from Ω, then the initial value problem (1.6.6) has a unique solution in some interval $x_0 - h \leqslant x \leqslant x_0 + h$, where h is defined in Eq. (1.6.5).

PROOF: We cannot guarantee that the solution $y = \phi(x)$ of the initial value problem (1.6.6) exists in the interval $(x_0 - a, x_0 + a)$ because the integral curve $y = \phi(x)$ may extend outside of the rectangle Ω. For example, if there exists x_1 such that $x_0 - a < x_1 < x_0 + a$ and $y_0 + b = \phi(x_1)$, then for $x > x_1$ (if $x_1 > x_0$) the increasing solution $\phi(x)$ will be out of Ω.

We definitely know that the solution $y = \phi(x)$ is in the range $y_0 - b \leqslant \phi(x) \leqslant y_0 + b$ when $x_0 - h \leqslant x \leqslant x_0 + h$ with $h = \min\{a,\ b/M\}$ since the slope of the graph of the solution $y = \phi(x)$ is at least $-M$ and at most M. If the graph of the solution $y = \phi(x)$ crosses the lines $y = y_0 \pm b$, then the points of intersection with the abscissa are

[8] Giuseppe Peano (1858–1932) was a famous Italian mathematician who worked at the University of Turin. The existence theorem was published in his article [40].

[9] In 1838, Joseph Liouville first used the method of successive approximations in a special case.

[10] Charles Emile Picard (1856–1941) was one of the greatest French mathematicians of the nineteenth century. In 1899, Picard lectured at Clark University in Worcester, Massachusetts. Picard and his wife had three children, a daughter and two sons, who were all killed in World War I.

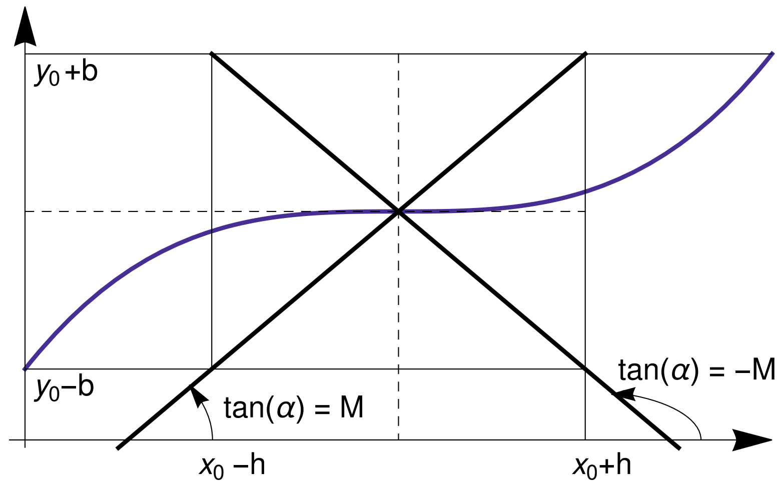


Figure 1.19: The domain of existence.

$x_0 \pm b/M$. Therefore, the abscissa at the point where the integral curve goes out of the rectangle Ω is less than or equal to $x_0 + b/M$ and is greater than or equal to $x_0 - b/M$.

Since differentiation is an unbounded operator, we transform the initial value problem (1.6.6) into an integral equation (recall that integration is a bounded operation). Upon integrating both sides of Eq. (1.6.6) from the initial point x_0 to an arbitrary value of x, we obtain

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))\,\mathrm{d}s. \tag{1.6.7}$$

The latter expression is called an **integral equation** because it contains an integral of the unknown function $y(s)$. More precisely, the equation of the form (1.6.7) is called a *Volterra integral equation of the second kind*. This integral equation is equivalent to the initial value problem (1.6.6) in the sense that any solution of one is also a solution of the other. The initial value problem (1.6.6) is not suitable for Picard's method because the differential equation contains the unbounded operator, but the integral equation does.

We prove the existence and uniqueness of Eq. (1.6.7) using **Picard's iteration method** or the **method of successive approximations**. We start by choosing an initial function ϕ_0 that satisfies the initial condition:

$$\phi_0(x) = y_0.$$

The next approximation ϕ_1 is obtained by substituting ϕ_0 for $y(s)$ in the right side of Eq. (1.6.7), namely,

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f(s, \phi_0)\,\mathrm{d}s = y_0 + \int_{x_0}^{x} f(s, y_0)\,\mathrm{d}s.$$

Let us again substitute the first order approximation in the right-hand side of Eq. (1.6.7) to obtain

$$\phi_2(x) = y_0 + \int_{x_0}^{x} f(s, \phi_1(s))\,\mathrm{d}s.$$

Each successive substitution into Eq. (1.6.7) results in a sequence of functions. In general, if the n-th approximation $\phi_n(s)$ has been obtained in this way, then the $(n+1)$-th approximation is taken to be the result of substituting ϕ_n in the right-hand side of Eq. (1.6.7). Therefore,

$$\phi_{n+1}(x) = y_0 + \int_{x_0}^{x} f(s, \phi_n(s))\,\mathrm{d}s. \tag{1.6.8}$$

When x lies in the interval $x_0 - h, x_0 + h]$, all terms of the sequence $\{\phi_n(x)\}$ exist and belong to the interval $[y_0 - b, y_0 + b]$ because

$$\left| \int_{x_0}^{x} f(s, \phi(s))\,\mathrm{d}s \right| \leqslant \max |f(x,y)| \left| \int_{x_0}^{x} \mathrm{d}s \right| = M|x - x_0| \leqslant b$$

for $|x - x_0| \leqslant b/M$, where $M = \max_{(x,y)\in\Omega} |f(x,y)|$.

The method of successive approximations gives a solution of Eq. (1.6.7) if and only if the successive approximations ϕ_{n+1} approach uniformly a certain limit as $n \to \infty$. Then the sequence $\{\phi_n(x)\}$ converges to a true solution $y = \phi(x)$ as $n \to \infty$, which in fact is the unique solution of Eq. (1.6.7):

$$y = \phi(x) = \lim_{n\to\infty} \phi_n(x). \tag{1.6.9}$$

We can identify each element $\phi_n(x)$ on the right-hand side of Eq. (1.6.9),

$$\phi_n(x) = \phi_0 + [\phi_1(x) - \phi_0] + [\phi_2(x) - \phi_1(x)] + \cdots + [\phi_n(x) - \phi_{n-1}(x)],$$

as the n-th partial sum of the telescopic series

$$\phi(x) = \phi_0 + \sum_{n=1}^{\infty} [\phi_n(x) - \phi_{n-1}(x)]. \tag{1.6.10}$$

The convergence of the sequence $\{\phi_n(x)\}$ is established by showing that the series (1.6.10) converges uniformly. To do this, we estimate the magnitude of the general term $|\phi_n(x) - \phi_{n-1}(x)|$.

We start with the first iteration:

$$|\phi_1(x) - \phi_0| = \left| \int_{x_0}^{x} f(s, y_0)\, \mathrm{d}s \right| \leqslant M \int_{x_0}^{x} \mathrm{d}s = M|x - x_0|.$$

For the second term we have

$$|\phi_2(x) - \phi_1(x)| \leqslant \int_{x_0}^{x} |f(s,\phi_1(s)) - f(s,\phi_0(s))|\, \mathrm{d}s \leqslant L \int_{x_0}^{x} M(s - x_0)\, \mathrm{d}s = \frac{ML(x-x_0)^2}{2},$$

where L is the Lipschitz constant for the function $f(x,y)$, that is, $|f(x,y_1) - f(x,y_2)| \leqslant L|y_1 - y_2|$. This allows us to make the inductive hypothesis: $|\phi_n(x) - \phi_{n-1}(x)| \leqslant \frac{ML^{n-1}h^n}{n!}$. For the n-th term, we have

$$\begin{aligned}
|\phi_n(x) - \phi_{n-1}(x)| &\leqslant \int_{x_0}^{x} |f(s,\phi_{n-1}(s)) - f(s,\phi_{n-2}(s))|\, \mathrm{d}s \\
&\leqslant L \int_{x_0}^{x} |\phi_{n-1}(s) - \phi_{n-2}(s)|\, \mathrm{d}s \leqslant L \int_{x_0}^{x} \frac{ML^{n-2}(s-x_0)^{n-1}}{(n-1)!}\, \mathrm{d}s \\
&= \frac{ML^{n-1}|x-x_0|^n}{n!} \leqslant \frac{ML^{n-1}h^n}{n!}, \qquad h = \max|x - x_0|.
\end{aligned}$$

Substituting these results into the finite sum

$$\phi_n(x) = \phi_0 + \sum_{k=1}^{n} [\phi_k(x) - \phi_{k-1}(x)],$$

we obtain

$$\begin{aligned}
|\phi_n - \phi_0| &\leqslant M|x-x_0| + \frac{ML|x-x_0|^2}{2} + \cdots + \frac{ML^{n-1}|x-x_0|^n}{n!} \\
&= \frac{M}{L} \left[L|x-x_0| + \frac{L^2|x-x_0|^2}{2} + \cdots + \frac{L^n|x-x_0|^n}{n!} \right].
\end{aligned}$$

When n approaches infinity, the sum

$$L|x-x_0| + \frac{L^2|x-x_0|^2}{2} + \cdots + \frac{L^n|x-x_0|^n}{n!}$$

approaches $e^{L|x-x_0|} - 1$. Thus, we have

$$|\phi_n(x) - y_0| \leqslant \frac{M}{L} \left[e^{L|x-x_0|} - 1 \right]$$

for any n. Therefore, by the Weierstrass M-test, the series (1.6.10) converges absolutely and uniformly on the interval $|x - x_0| \leqslant h$. It follows that the limit function (1.6.9) of the sequence (1.6.8) is a continuous function on the interval $|x - x_0| \leqslant h$. Sometimes a sequence of continuous functions converges to a limit function that is discontinuous, as Exercise 26 on page 35 shows. However, this may happen only when the sequence of functions converges pointwise, but not uniformly.

Next we will prove that $\phi(x)$ is a solution of the initial value problem (1.6.6). First of all, $\phi(x)$ satisfies the initial condition. In fact, from (1.6.8),

$$\phi_n(x_0) = y_0, \quad n = 0, 1, 2, \ldots$$

and taking limits of both sides as $n \to \infty$, we find $\phi(x_0) = y_0$. Since $\phi(x)$ is represented by a uniformly convergent series (1.6.10), it is a continuous function on the interval $x_0 - h \leqslant x \leqslant x_0 + h$.

Allowing n to approach ∞ on both sides of Eq. (1.6.8), we get

$$\phi(x) = y_0 + \lim_{n\to\infty} \int_{x_0}^{x} f(s, \phi_n(s))\,\mathrm{d}s. \tag{1.6.11}$$

Recall that the function $f(x, y)$ satisfies the Lipschitz condition for $|s - x_0| \leqslant h$:

$$|f(s, \phi_n(s)) - f(s, \phi(s))| \leqslant L|\phi_n(s) - \phi(s)|.$$

Since the sequence $\phi_n(s)$ converges uniformly to $\phi(s)$ on the interval $|s - x_0| \leqslant h$, it follows that the sequence $f(s, \phi_n(s))$ also converges uniformly to $f(s, \phi(s))$ on this interval. Therefore, we can interchange integration with the limiting operation on the right-hand side of Eq. (1.6.11) to obtain

$$\phi(x) = y_0 + \int_{x_0}^{x} \lim_{n\to\infty} f(s, \phi_n(s))\,\mathrm{d}s = y_0 + \int_{x_0}^{x} f(s, \phi(s))\,\mathrm{d}s.$$

Thus, the limit function $\phi(x)$ is a solution of the integral equation (1.6.7) and, consequently, a solution of the initial value problem (1.6.6). In general, taking the limit under the sign of integration is not permissible, as Exercise 27 (page 35) shows. However, it is true for a uniform convergent sequence. Differentiating both sides of the last equality with respect to x and noting that the right-hand side is a differentiable function of the upper limit, we find

$$\phi'(x) = f(x, \phi(x)).$$

This completes the proof that the limit function $\phi(x)$ is a solution of the initial value problem (1.6.6). ■

Uniqueness. Finally we will prove that $\phi(x)$ is the only solution of the initial value problem (1.6.6). To start, we assume the existence of another solution $y = \psi(x)$. Then

$$\phi(x) - \psi(x) = \int_{x_0}^{x} [\, f(s, \phi(s)) - f(s, \psi(s))\,]\,\mathrm{d}s$$

for $|x - x_0| \leqslant h$. Setting $U(x) = |\phi(x) - \psi(x)|$, we have

$$U(x) \leqslant \int_{x_0}^{x} |f(s, \phi(s)) - f(s, \psi(s))|\,\mathrm{d}s \leqslant L \int_{x_0}^{x} U(s)\,\mathrm{d}s.$$

By differentiating both sides with respect to x, we obtain $U'(x) - LU(x) \leqslant 0$. Multiplying by the integrating factor e^{-Lx} reduces the latter inequality to the following one:

$$\left[\, e^{-Lx}\, U(x) \,\right]' \leqslant 0.$$

The function $e^{-Lx}\, U(x)$ has a nonpositive derivative and therefore does not increase with x. After integrating from x_0 to x ($> x_0$), we obtain

$$e^{-Lx}\, U(x) \leqslant 0$$

since $U(0) = 0$. The absolute value of any number is positive; hence, $U(x) \geqslant 0$. Thus, $U(x) \equiv 0$ for $x > x_0$. The case $x < x_0$ can be treated in a similar way. Therefore $\phi(x) \equiv \psi(x)$.

Corollary 1.2. If the functions $f(x, y)$ and $\partial f/\partial y$ are continuous in a rectangle (1.6.4), then the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ has a unique solution in the interval $|x - x_0| \leqslant h$, where h is defined in Eq. (1.6.5) and the Lipschitz constant is $L = \max |\partial f(x, y)/\partial y|$.

Corollary 1.3. If the functions $f(x, y)$ and $\partial f/\partial x$ are continuous in a neighborhood of the point (x_0, y_0) and $f(x_0, y_0) \neq 0$, then the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ has a unique solution.

The statements above follow from the mean value relation

$$f(x, y_1) - f(x, y_2) = f_y(x, \xi)(y_1 - y_2),$$

where $\xi \in [y_1, y_2]$ and $f_y = \partial f/\partial y$. The proof of Corollary 1.3 is based on conversion of the original problem to its reciprocal counterpart: $\partial x/\partial y = 1/f(x, y)$. ■

The proof of Picard's theorem is an example of a constructive proof that includes an iterative procedure and an error estimate. When iteration stops at some step, it gives an approximation to the actual solution. With the availability of a computer algebra system, such an approximation can be explicitly implemented for many analytically defined slope functions. Practically, Picard's method is applicable only to polynomial slope functions. The disadvantage of Picard's method is that it only provides a solution locally—in a small neighborhood of the initial point.

Usually, the solution to the IVP exists in a wider region (see, for instance, Example 1.6.3) than Picard's theorem guarantees. Once the solution $y = \phi(x)$ of the given initial value problem is obtained, we can consider another initial condition at the point $x = x_0 + \Delta x$ and set $y_0 = \phi(x_0 + \Delta x)$. Application of Picard's theorem to this IVP may allow us to extend the solution to a larger domain. By continuing in such a way, we could extend the solution of the original problem to a bigger domain until we reach the boundary (this domain could be unbounded; in this case we define the function in the interval $x_0 - h \leqslant x < \infty$). Similarly, we can extend the solution to the left end of the initial interval $x_0 - h \leqslant x \leqslant x_0 + h$. Therefore, we may obtain some open interval $p < x < q$ (which could be unbounded) on which the given IVP has a unique solution. Such an approach is hard to call constrictive.

The ideal existence theorem would assure the existence of a solution in a longest possible interval, called the **validity interval**. It turns out that another method is known (see, for example, [28]) to extend the existence theorem that furnishes a solution in a validity interval. It was invented in 1768 by Leonhard Euler (see §3.2). However, the systematic method was developed by the famous French mathematician Augustin-Louis Cauchy (1789–1857) between the years 1820 and 1830. Later, in 1876, Rudolf Lipschitz substantially improved it. The Cauchy–Lipschitz method is based on the following fundamental inequality. This inequality can be used not only to prove the existence theorem by linear approximations, but also to find estimates produced by numerical methods that are discussed in Chapter 3.

Theorem 1.4. [Fundamental Inequality] Let $f(x, y)$ be a continuous function in the rectangle $[a, b] \times [c, d]$ and satisfying the Lipschitz condition

$$|f(x, y_1) - f(x, y_2)| \leqslant L|y_1 - y_2|$$

for some positive constant L and all pairs y_1, y_2 uniformly in x. Let $y_1(x)$ and $y_2(x)$ be two continuous piecewise differentiable functions satisfying the inequalities

$$|y_1'(x) - f(x, y_1(x))| \leqslant \epsilon_1, \qquad |y_2'(x) - f(x, y_2(x))| \leqslant \epsilon_2$$

with some positive constants $\epsilon_{1,2}$. If, in addition, these functions differ by a small amount $\delta > 0$ at some point:

$$|y_1(x_0) - y_2(x_0)| \leqslant \delta,$$

then

$$|y_1(x) - y_2(x)| \leqslant \delta\, e^{L|x-x_0|} + \frac{\epsilon_1 + \epsilon_2}{L}\left(e^{L|x-x_0|} - 1\right). \tag{1.6.12}$$

The Picard theorem can be extended for non-rectangular domains, as the following theorem[11] states.

[11]In honor of Sergey Mikhailovich Lozinskii/Lozinsky (1914–1985), a famous Russian mathematician who made an important contribution to the error estimation methods for various types of approximate solutions of ordinary differential equations.

Theorem 1.5. [**Lozinsky**] Let $f(x,y)$ be a continuous function in some domain Ω and $M(x)$ be a nonnegative continuous function on some finite interval I $(x_0 \leqslant x \leqslant x_1)$ inside Ω. Let $|f(x,y)| \leqslant M(x)$ for $x \in I$ and $(x,y) \in \Omega$. Suppose that the closed finite domain Q, defined by inequalities

$$x_0 \leqslant x \leqslant x_1, \qquad |y - y_0| \leqslant \int_{x_0}^{x} M(u)\,\mathrm{d}u,$$

is a subset of Ω and there exists a nonnegative integrable function $k(x)$, $x \in I$, such that

$$|f(x,y_2) - f(x,y_1)| \leqslant k(x)\,|y_2 - y_1|, \qquad x_0 \leqslant x \leqslant x_1, \quad (x,y_2),(x,y_1) \in Q.$$

Then formula (1.6.8) on page 24 defines the sequence of functions $\{\phi_n(x)\}$ that converges to a unique solution of the given IVP (1.6.1), (1.6.2) provided that all points $(x,\phi_n(x))$ are included in Q when $x_0 \leqslant x \leqslant x_1$. Moreover,

$$|y(x) - \phi_n(x)| \leqslant \frac{1}{n!} \int_{x_0}^{x} M(u)\,\mathrm{d}u \left[\int_{x_0}^{x} k(u)\,\mathrm{d}u\right]^n . \tag{1.6.13}$$

Actually, the Picard theorem allows us to determine the accuracy of the n-th approximation:

$$\begin{aligned} |\phi(x) - \phi_n(x)| &\leqslant \left|\int_{x_0}^{x} [f(x,\phi(x)) - f(x,\phi_{n-1}(x))]\,\mathrm{d}x\right| \\ &\leqslant \int_{x_0}^{x} |f(x,\phi(x)) - f(x,\phi_{n-1}(x))\,\mathrm{d}x| \leqslant \int_{x_0}^{x} L|\phi(x) - \phi_{n-1}(x)|\,\mathrm{d}x \\ &\leqslant \int_{x_0}^{x} L\,\mathrm{d}x \int_{x_0}^{x} L|\phi(x) - \phi_{n-2}(x)|\,\mathrm{d}x \leqslant \ldots . \end{aligned}$$

Therefore,

$$|\phi(x) - \phi_n(x)| \leqslant \frac{M}{L}\,\frac{(L|x-x_0|)^{n+1}}{n!} \leqslant \frac{ML^n}{n!}\,h^{n+1} = \frac{M}{L}\,\frac{(Lh)^{n+1}}{n!}\,, \tag{1.6.14}$$

which is in agreement with the inequality (1.6.13).

Example 1.6.1. Find the global maximum of the continuous function $f(x,y) = (x+2)^2 + (y-1)^2 + 1$ in the domain $\Omega = \{(x,y) : \ |x+2| \leqslant a, \ |y-1| \leqslant b\}$.

Solution. The function $f(x,y)$ has the global minimum at $x = -2$ and $y = 1$. This function reaches its maximum values in the domain Ω at such points that are situated furthest from the critical point $(-2,1)$. These points are vertices of the rectangle Ω, namely, $(-2 \pm a, 1 \pm b)$. Since the values of the function $f(x,y)$ at these points coincide, we have

$$\max_{(x,y)\in\Omega} f(x,y) = f(-2+a, 1+b) = a^2 + b^2 + 1.$$

Example 1.6.2. Show that the function

$$f(y) = \begin{cases} y\,\ln|y|, & \text{if } \ y \ne 0, \\ 0, & \text{if } \ y = 0 \end{cases}$$

is not a Lipschitz function on the interval $[-b,b]$, where b is a positive real number.

Solution. We prove by contradiction, assuming that $f(y)$ is a Lipschitz function. Let y_1 and y_2 be arbitrary points from this interval. Then $|f(y_1) - f(y_2)| \leqslant L|y_1 - y_2|$ for some constant L. We set $y_2 = 0$ ($\lim_{y\to 0} y\ln|y| = 0$), making $|f(y_1)| = |y_1 \ln|y_1|| \leqslant L|y_1|$ or $|\ln|y_1|| \leqslant L$ for all $y_1 \in [0,b]$, which is impossible. Indeed, for small values of an argument y, the function $\ln y$ is unbounded. Thus, the function $f(y)$ does not satisfy the Lipschitz condition.

Example 1.6.3. Let us consider the initial value problem for the Riccati equation

$$y' = x^2 + y^2, \quad y(0) = 0$$

in the rectangle $\{(x, y) : |x| \leqslant a, \quad |y| \leqslant b\}$. The maximum value of h is $\max\min\left\{a,\ \dfrac{b}{a^2+b^2}\right\} = 2^{-1/2}$, which does not exceed 1. According to Picard's theorem, the solution of the given IVP exists (and could be found by successive approximations) within the interval $|x| < h < 1$, but cannot be extended on whole line. The best we can get from the theorem is the existence of the solution on the interval $|x| \leqslant \frac{1}{\sqrt{2}} \approx .7071067810$.

This result can be improved with the aid of the Lozinsky theorem (on page 28). We consider the domain Ω defined by inequalities (containing positive numbers x_1 and A to be determined)

$$0 \leqslant x \leqslant x_1, \qquad 0 \leqslant |y| \leqslant A\,x^3\,.$$

Then $M(x) = \max_{(x,y)\in\Omega}\left(x^2+y^2\right) = x^2 + A^2x^6$, $0 \leqslant x \leqslant x_1$, and the set Q is defined by inequalities

$$0 \leqslant x \leqslant x_1, \qquad 0 \leqslant |y| \leqslant \frac{x^3}{3} + A^2\,\frac{x^7}{7}\,.$$

In order to have $Q \subset \Omega$, the following inequality must hold:

$$\frac{1}{3} + A^2\,\frac{x_1^4}{7} \leqslant A\,.$$

This is equivalent to the quadratic equation $A^2\,\frac{x_1^4}{7} - A + \frac{1}{3} = 0$, which has two roots: $A = \dfrac{1 \pm \sqrt{1 - \frac{4\,x_1^4}{21}}}{2\,x_1^4/7}$. Hence, the quadratic equation has two real roots when x_1 satisfies inequality $x_1 \leqslant \sqrt[4]{\frac{21}{4}} \approx 1.513700052$. Therefore, Lozinsky's theorem gives a larger interval of existence (more than double) than Picard's theorem.

As shown in Example 2.6.10, page 103, this Riccati equation $y' = x^2 + y^2$ has the general solution of the ratio $-u'/(2u)$, where u is a linear combination of Bessel functions (see §6.7):

$$y(x) = \frac{1}{2x} - x\,\frac{J'_{1/4}(x^2/2) + C\,Y'_{1/4}(x^2/2)}{J_{1/4}(x^2/2) + C\,Y_{1/4}(x^2/2)}\,.$$

A constant C should be chosen to satisfy the initial condition $y(0) = 0$ (see Eq. (2.7.1) on page 114 for the explicit expression). As seen from this formula, the denominator has zeroes, the smallest of them is approximately 2.003, so the solution has the asymptote $x = h$, where $h < 2.003$.

The situation changes when we consider another initial value problem (IVP):

$$y' = x^2 - y^2, \qquad y(0) = 0.$$

The slope function $x^2 - y^2$ attains its maximum in a domain containing lines $y = \pm x$:

$$\max|x^2 - y^2| = \begin{cases} x^2, & \text{if } x^2 \geqslant y^2, \\ y^2, & \text{if } y^2 \geqslant x^2. \end{cases}$$

For the function $x^2 - y^2$ in the rectangle $|x| \leqslant a$, $|y| \leqslant b$, the Picard theorem guarantees the existence of a unique solution within the interval $|x| \leqslant h$, where $h = \min\left\{a,\ \dfrac{b}{\max\{a^2, b^2\}}\right\} \leqslant 1$. To extend the interval of existence, we apply the Lozinsky theorem.

First, we consider the function $x^2 - y^2$ in the domain Ω bounded by inequalities $0 \leqslant x \leqslant x_p^*$ and $|y| \leqslant Ax^p$, where A and p are some positive constants, and x_p^* will be determined shortly. Then

$$|x^2 - y^2| \leqslant M(x) \equiv \max_{(x,y)\in\Omega}|x^2 - y^2| = \begin{cases} x^2, & \text{if } x^2 \geqslant (Ax^p)^2, \\ (Ax^p)^2 = A^2x^{2p}, & \text{if } (Ax^p)^2 \geqslant x^2. \end{cases}$$

Now we define the domain Q by inequalities $0 \leqslant x \leqslant x_p^*$, $|y| \leqslant \int_0^x M(u)\,\mathrm{d}u$, where

$$\int_0^x M(u)\,\mathrm{d}u = \int_0^{A^{1/(p-1)}} u^2\,\mathrm{d}u + \int_{A^{1/(p-1)}}^x A^2u^{2p}\,\mathrm{d}u$$
$$= \frac{1}{3}\,A^{3/(p-1)} - \frac{1}{2p+1}\,A^{4+3/(p-1)} + \frac{A^2}{2p+1}\,x^{2p+1}.$$

In order to guarantee inclusion $Q \subset \Omega$, the following inequality should hold: $\int_0^x M(u)\,\mathrm{d}u \leqslant Ax^p$. It is valid in the interval $\epsilon < x < x_p^*$, where x_p^* is the root of the equation $\int_0^x M(u)\,\mathrm{d}u = Ax^p$ and ϵ is a small number. When $A \to +0$ and $p \to 1+0$, the root, x_p^*, could be made arbitrarily large. For instance, when $A = 0.001$ and $p = 1.001$, the root is $x_p^* \approx 54.69$. Therefore, the given IVP has a solution on the whole line $-\infty < x < \infty$.

Example 1.6.4. **(Example 1.4.2 revisited)** Let us reconsider the initial value problem (page 9):

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\,y^{1/2}, \quad y(0) = 0.$$

Peano's theorem (page 23) guarantees the existence of a solution to the initial value problem since the slope function $f(x,y) = 2\,y^{1/2}$ is a continuous function. The critical point $y \equiv 0$ is obviously a solution of the initial value problem. We show that $f(x,y) = 2\,y^{1/2}$ is not a Lipschitz function by assuming the opposite. That is to say suppose there exists a positive constant L such that

$$|y_1^{1/2} - y_2^{1/2}| \leqslant L|y_1 - y_2|.$$

Setting $y_2 = 0$, we have

$$|y_1^{1/2}| \leqslant L|y_1| \quad \text{or} \quad 1 \leqslant L\,|y_1^{1/2}|.$$

The last inequality does not hold for small y_1; therefore, $f(y) = 2\,y^{1/2}$ is not a Lipschitz function. In this case, we cannot apply Picard's theorem (page 23), and the given initial value problem may have multiple solutions. According to Theorem 2.2 (page 48), since the integral

$$\int_0^y \frac{\mathrm{d}y}{2\sqrt{y}}$$

converges, the given initial value problem doesn't have a unique solution. Indeed, for arbitrary $x_0 > 0$, the function

$$y = \varphi(x) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{for } -\infty < x \leqslant x_0, \\ (x - x_0)^2, & \text{for } x > x_0, \end{cases}$$

is a singular solution of the given initial value problem. Note that $y \equiv 0$ is the envelope of the one-parameter family of curves, $y = (x - C)^2$, $x \geqslant C$, corresponding to the general solution.

Example 1.6.5. Consider the autonomous equation

$$y' = |y|.$$

The slope function $f(x,y) = |y|$ is not differentiable at $x = 0$, but it is a Lipschitz function, with $L = 1$. According to Picard's theorem, the initial value problem with the initial condition $y(0) = y_0$ has a unique solution:

$$y(x) = \begin{cases} y_0\,e^x, & \text{for } y_0 > 0, \\ 0, & \text{for } y_0 = 0, \\ y_0\,e^{-x}, & \text{for } y_0 < 0. \end{cases}$$

Since an exponential function is always positive, the integral curves never meet or cross the equilibrium solution $y \equiv 0$.

Example 1.6.6. Does the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{3}{2}\,(y - x)^{1/3}, \qquad y(0) = 0,$$

have a singular solution? Find all solutions of this differential equation.

Solution. Changing the dependent variable to $y - x = u$, we find that the differential equation with respect to u is

$$u' = 3\,u^{1/3}/2.$$

The derivative of its slope function $f(x,u) = f(u) = \frac{3}{2}\,u^{1/3}$ is $f'(u) = 1/2\,u^{-2/3}$, which is unbounded at $u = 0$. In this case, Picard's theorem (page 23) is not valid and the differential equation $u' = \frac{3}{2}\,u^{1/3}$ may have a singular solution. Since the equation for u is autonomous, we apply Theorem 2.2 (page 48). The integral

$$\int_0^u \frac{\mathrm{d}u}{f(u)} = \frac{3}{2}\int_0^u u^{-1/3}\,\mathrm{d}u = u^{2/3}|_{u=0}^{u} = u^{2/3}$$

converges. Hence, there exists another solution besides the general one, $y = \eta(x+C)^{3/2}$, where $\eta = \pm 1$ and C is an arbitrary constant. Notice that the general solution of this nonlinear differential equation depends on more than the single constant of integration; it also depends on the discrete parameter η. Returning to the variable y, we get the singular solution $y = x$ (which corresponds to $u \equiv 0$) and the general solution $y = x + \eta(x-C)^{3/2},\ x \geqslant C$.

Example 1.6.7. Let us find the solution of the problem

$$y' = y^2, \quad y(0) = 1,$$

by the method of successive approximations. We choose the first approximation $\phi_0 = 1$ according to the initial condition $y(0) = 1$. Then, from formula (1.6.8), we find

$$\phi_{n+1}(x) = 1 + \int_0^x \phi_n^2(s)\,\mathrm{d}s \qquad (n = 0, 1, 2, \ldots).$$

Hence, we have

$$\begin{aligned}
\phi_1(x) &= 1 + x;\\
\phi_2(x) &= 1 + \int_0^x (1+s)^2\,\mathrm{d}s = 1 + x + x^2 + \frac{x^3}{3};\\
\phi_3(x) &= 1 + \int_0^x \left(1 + s + s^2 + \frac{s^3}{3}\right)^2\,\mathrm{d}s\\
&= 1 + x + x^2 + x^3 + \frac{2}{3}x^4 + \frac{1}{3}\,x^5 + \frac{1}{63}\,x^7,
\end{aligned}$$

and so on. The limit function is

$$\phi(x) = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}.$$

To check our calculations, we can use either *Mathematica:*

```
Clear[phi]
T[phi_] := Function[x, 1 + Integrate[phi[t]^2, {t, 0, x}]];
f[x_] = 1;        (* specify the initial function *)
Nest[T, f, 5][x]  (* Find the result of 5th iterations *)
```

or *Maple:*

```
y0:=1:  T(phi,x):=(phi,x)->y0+eval(int(phi(t)^2,t=0..x)):
y:=array(0..n):   Y:=array(0..n):
y[0]:=x->y0:
for i from 1 to n do
    y[i]:=unapply(T(y[i-1],x),x):
    Y[i]:=plot(y[i](x),x=0..1):
od:
display([seq(Y[i],i=1..n)]);
seq(eval(y[i]),i=1..n);
```

Example 1.6.8. Using the Picard method, find the solution of the initial value problem

$$y' = x + 2y, \quad y(0) = 1.$$

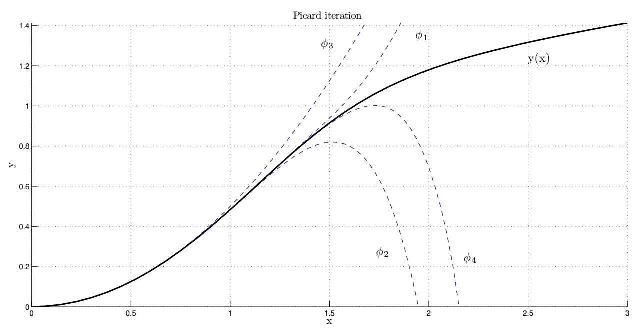


Figure 1.20: First four Picard approximations, plotted with MATLAB.

Solution. From the recursion relation

$$\phi_{n+1}(x) = 1 + \int_0^x (t + 2\,\phi_n(t))\,\mathrm{d}t = 1 + \frac{x^2}{2} + 2\int_0^x \phi_n(t)\,\mathrm{d}t, \quad n = 0, 1, 2, \ldots,$$

where $\phi_0(x) \equiv 1$ is the initial function, we obtain

$$\begin{aligned}
\phi_1(x) &= 1 + \int_0^x (t+2)\,\mathrm{d}t = 1 + 2x + \frac{x^2}{2},\\
\phi_2(x) &= 1 + \int_0^x (t + 2\,\phi_1(t))\,\mathrm{d}t = 1 + 2x + \frac{5}{2}\,x^2 + \frac{x^3}{3},\\
\phi_3(x) &= 1 + \int_0^x (t + 2\,\phi_2(t))\,\mathrm{d}t = 1 + 2x + \frac{5}{2}\,x^2 + \frac{5}{3}\,x^3 + \frac{x^4}{6},
\end{aligned}$$

and so on.

To check calculations, we compare these approximations with the exact solution

$$\begin{aligned}
y(x) &= -\frac{1}{4} - \frac{x}{2} + \frac{5}{4}\,e^{2x}\\
&= -\frac{1}{4} - \frac{x}{2} + \frac{5}{4}\left[1 + 2x + \frac{4x^2}{2} + \frac{2^3\,x^3}{3!} + \frac{2^4\,x^4}{4!} + \frac{2^5\,x^5}{5!} + \cdots\right]\\
&= -\frac{1}{4} - \frac{x}{2} + \frac{5}{4} + \frac{5}{4}\,2x + \frac{5}{4}\,4x^2 + \frac{5}{4}\,\frac{8}{6}\,x^3 + \frac{5}{4}\,\frac{16}{24}\,x^4 + \frac{5}{4}\,\frac{32}{24\cdot 5}\,x^5 + \cdots\\
&= 1 + 2x + \frac{5}{2}\,x^2 + \frac{5}{3}\,x^3 + \frac{5}{6}\,x^4 + \frac{1}{3}\,x^5 + \cdots .
\end{aligned}$$

Maple can also be used:

```
restart:  picard:=proc(f,x0,y0,n)  # n is the number of iterations
local s,y;     y:=y0;              # y0 is the initial value at x=x0
for s from 1 to n do
y:=y0+int(f(a,subs(x=a,y)),a=x0..x);
y:=sort(y,x,ascending);
od;           return(y);     end
```

Example 1.6.9. Find successive approximations to the initial value problem

$$y' = x - y^3, \quad y(0) = 0$$

in the square $|x| \leqslant 1$, $|y| \leqslant 1$. On what interval does Picard's theorem guarantee the convergence of successive approximations? Determine the error in the third approximation.

Solution. The slope function $f(x, y) = x - y^3$ has continuous partial derivatives; therefore, this function satisfies the conditions of Theorem 1.2 with the Lipschitz constant

$$L = \max_{\substack{|x|\leqslant 1\\ |y|\leqslant 1}} \left|\frac{\partial f}{\partial y}\right| = \max_{|y|\leqslant 1} |-3y^2| = 3.$$

Calculations show that

$$M = \max_{\substack{|x|\leqslant 1\\ |y|\leqslant 1}} |f(x,y)| = \max_{\substack{|x|\leqslant 1\\ |y|\leqslant 1}} |x - y^3| = 2, \quad h = \min\left\{a, \frac{b}{M}\right\} = \min\left\{1; \frac{1}{2}\right\} = \frac{1}{2}.$$

The successive approximations converge at least on the interval $[-1/2, 1/2]$. From Eq. (1.6.8), we get

$$\phi_{n+1} = \int_0^x (t - \phi_n^3(t))\,\mathrm{d}t = \frac{x^2}{2} - \int_0^x \phi_n^3(t)\,\mathrm{d}t, \quad n = 0,1,2,\ldots, \qquad \phi_0(x) \equiv 0.$$

For $n = 0, 1, 2$, we have

$$\begin{aligned}
\phi_1(x) &= \int_0^x (t-0)\,\mathrm{d}t = \frac{x^2}{2},\\
\phi_2(x) &= \int_0^x \left(t - \frac{t^6}{8}\right)\mathrm{d}t = \frac{x^2}{2} - \frac{x^7}{56},\\
\phi_3(x) &= \int_0^x \left[t - \left(\frac{t^2}{2} - \frac{t^7}{56}\right)^3\right]\mathrm{d}t = \frac{x^2}{2} - \frac{x^7}{56} + \frac{x^{12}}{896} - \frac{3x^{17}}{106,624} + \frac{x^{22}}{3,963,552}.
\end{aligned}$$

Their graphs along with the exact solution are presented in Fig. 1.20. The absolute value of the error of the third approximation can be estimated as follows:

$$|y(x) - \phi_3(x)| \leqslant \frac{ML^3}{3!}\,h^4 = \frac{2\cdot 3^3}{6}\cdot\frac{1}{2^4} = \frac{9}{16} = 0.5625.$$

Of course, the estimate could be improved if the exact solution were known. We plot the four Picard approximations along with the solution using the following MATLAB commands (which call for `quadl`, adaptive Gauss–Lobatto rules for numerical integral evaluation):

```
t = linspace(time(1),time(2),npts); % Create a discrete time grid
y = feval(@init,t,y0);              % Initialize y = y0
window = [time,space];
for n = 1:N                         % Perform N Picard iterations
    [t,y] = picard(@f,t,y,npts);    % invoke picard.m
    plot(t,y,'b--','LineWidth',1);  % Plot the nth iterant
    axis(window);  drawnow;  hold on;
end
[t,y] = ode45(@f,time,y0);          % Solve numerically the ODE
plot(t,y,'k','LineWidth',2);        % Plot the numerical solution
hold off;
axis([min(t) max(t) min(y) max(y)])
function [t,y] = picard(f,t,y,n)    % picard.m
tol = 1e-6;                         % Set tolerance
phi = y(1)*ones(size(t));           % Initialize
for j=2:n
    phi(j) = phi(j-1)+quadl(@fint,t(j-1),t(j),tol,[],f,t,y);
end        y = phi;
```

Example 1.6.10. Let

$$f(x,y) = \begin{cases} 0, & \text{if } x \leqslant 0,\ -\infty < y < \infty;\\ x, & \text{if } 0 < x \leqslant 1,\ -\infty < y < 0;\\ x - \frac{2y}{x}, & \text{if } 0 < x \leqslant 1,\ 0 \leqslant y \leqslant x^2;\\ -x, & \text{if } 0 < x \leqslant 1,\ x^2 < y < \infty. \end{cases}$$

Prove that the Picard iterations, ϕ_0, ϕ_1, ..., for the solution of the initial value problem $y' = f(x,y)$, $y(0) = 0$, do not converge on $[0, \varepsilon]$ for any $0 < \varepsilon \leqslant 1$.

Solution. It is not hard to verify that the function $f(x,y)$ is continuous and bounded in the domain $\Omega = \{\, 0 \leqslant x \leqslant 1,\ -\infty < y < \infty \,\}$. Moreover, $|f(x,y)| \leqslant 1$. Hence, the conditions of Theorem 1.2 are satisfied and the initial value problem $y' = f(x, y)$, $y(0) = 0$ has a solution. Let us find Picard's approximations for $0 \leqslant x \leqslant 1$, starting with $\phi_0 \equiv 0$:

$$\begin{aligned}
\phi_1 &= \int_0^x f(t, 0)\,\mathrm{d}t = \frac{x^2}{2};\\
\phi_2 &= \int_0^x f(t, \phi_1(t))\,\mathrm{d}t = \int_0^x f\left(t, \frac{t^2}{2}\right)\,\mathrm{d}t \equiv 0;\\
\phi_3 &= \int_0^x f(t, \phi_2(t))\,\mathrm{d}t = \int_0^x f(t, 0)\,\mathrm{d}t = \frac{x^2}{2};
\end{aligned}$$

and so on. Hence, $\phi_{2m}(x) \equiv 0$ and $\phi_{2m+1}(x) = \frac{x^2}{2}$, $m = 0, 1, \ldots$. Thus, the sequence $\phi_n(x)$ has two accumulation points (0 and $x^2/2$) for every $x \neq 0$. Therefore, Picard's approximation does not converge. This example shows that continuity of the function $f(x, y)$ is not enough to guarantee the convergence of Picard's approximations. □

There are times when the slope function $f(x, y)$ in Eq. (1.6.6), page 23, is piecewise continuous. We will see such functions in Example 2.5.4 and Problems 5 and 13 (§2.5). When abrupt changes occur in mechanical and electrical applications, the corresponding mathematical models lead to differential equations with intermittent slope functions. So, solutions to differential equations with discontinuous forcing functions may exist; however, the conditions of Peano's theorem are not valid for them. Such examples serve as reminders that the existence theorem 1.2 provides only sufficient conditions needed to guarantee a solution to the first order differential equation.

Computer-drawn pictures can sometimes make uniqueness misleading. Human eyes cannot distinguish drawings that differ within 1% to 5%. For example, solutions of the equation $y' = 3y + 2x$ in Fig. 1.8 on page 13 appear to merge; however, they are only getting very close.

Problems

1. Show that $|y(x)| = \begin{cases} C_1 x^2, & \text{for } x < 0, \\ C_2 x^2, & \text{for } x > 0. \end{cases}$ is the general solution of the differential equation $x\,y' - 2y = 0$.
2. Prove that no solution of $x^3\,y' - 2x^2\,y = 4$ can be continuous at $x = 0$.
3. Show that the functions $y \equiv 0$ and $y = x^4/16$ both satisfy the differential equation $y' = x\,y^{1/2}$ and the initial conditions $y(0) = 0$. Do the conditions of Theorem 1.2 (page 23) hold?
4. Show that the hypotheses of Theorem 1.3 (page 23) do not hold in a neighborhood of the line $y = 1$ for the differential equation $y' = |y - 1|$. Nevertheless, the initial value problem $y' = |y - 1|$, $y(1) = 1$ has a unique solution; find it.
5. Does the initial value problem $y' = \sqrt{|y|}$, $x > 0$, $y(0) = 0$ have a unique solution? Does $f(y) = \sqrt{|y|}$ satisfy the Lipschitz condition?
6. Show that the function $f(x, y) = y^2\cos x + e^{2x}$ is a Lipschitz function in the domain $\Omega = \{\,(x, y)\,:\,|y| < b\,\}$ and find the least Lipschitz constant.
7. Show that the function $f(x, y) = (2 + \sin x) \cdot y^{2/3} - \cos x$ is not a Lipschitz function in the domain $\Omega = \{\,(x, y)\,:\,|y| < b\,\}$. For what values of α and β is this function a Lipschitz function in the domain $\Omega_1 = \{\,(x, y)\,:\,0 < \alpha \leqslant y \leqslant \beta\,\}$?
8. Could the Riccati equation $y' = a(x)\,y^2 + b(x)\,y + c(x)$, where $a(x)$, $b(x)$, and $c(x)$ are continuous functions for $x \in (-\infty, \infty)$, have a singular solution?
9. Find the global maximum of the continuous function $f(x, y) = x^2 + y^2 + 2(2 - x - y)$ in the domain $\Omega = \{\,(x, y)\,:\,|x - 1| \leqslant a,\ |y - 1| \leqslant b\,\}$.
10. Consider the initial value problem $y' = -2x + 2(y + x^2)^{1/2}, \qquad y(-1) = -1$.
    (a) Find the general solution of the given differential equation. *Hint:* use the substitution $y(x) = -x^2 + u(x)$.
    (b) Derive a particular solution from the general solution that satisfies the initial value.
    (c) Show that $y = 2Cx + C^2$, where C is an arbitrary positive constant, satisfies the differential equation $y' + 2x = 2\sqrt{x^2 + y}$ in the domain $x > -C/2$.
    (d) Verify that both $y_1(x) = 2x + 1$ and $y_2(x) = -x^2$ are solutions of the given initial value problem.
    (e) Show that the function $y_2(x) = -x^2$ is a singular solution.
    (f) Explain why the existence of three solutions to the given initial value problem does not contradict the uniqueness part of Theorem 1.3 (page 23).

11. Determine a region of the xy-plane for which the given differential equation would have a unique solution whose graph passes through a given point in the region.
 (a) $(y-y^2)\,y' = x,\ y(1)=2$; **(b)** $y' = y^{2/3},\ y(0)=1$;
 (c) $(y-x)\,y' = y+x^2,\ y(1)=2$; **(d)** $y' = \sqrt{x}\,y,\ y(0)=1$;
 (e) $(x^2+y^2)\,y' = \sin y,\ y(\pi/2)=0$; **(f)** $x\,y' = y,\ y(2)=3$;
 (g) $(1+y^3)\,y' = x,\ y(1)=0$; **(h)** $y' - y = x^2, y(2)=1$.

12. Solve each of the following initial value problems and determine the domain of the solution.
 (a) $(1+\sin x)\,y' + \cot(x)\,y = \cos x$, $y(\pi/2)=1$. **(b)** $(1-x^3)\,y' - 3x^2\,y = 4x^3$, $y(0)=1$.
 (c) $(\sin x)\,y' + \cos x\,y = \cot x$, $y(3\pi/4)=2$. **(d)** $x^2\,y' + xy/\sqrt{1-x^2} = 0$, $y(\sqrt{3}/2)=1$.
 (e) $(x\sin x)\,y' + (\sin x + x\cos x)\,y = e^x$, $y(-0.5)=0$. **(f)** $y' = 2xy^2, \quad y(0) = 1/k^2$.

13. Determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist.
 (a) $(\ln t)\,y' + ty = \ln^2 t, \quad y(2)=1$; **(b)** $x(x-2)\,y' + (x-1)y + x^3y = 0,\ x(1)=2$;
 (c) $y' + (\cot x)y = x, \quad y(\pi/2)=9$; **(d)** $(t^2-9)\,y' + ty = t^4, \quad y(-1)=2$;
 (e) $(t^2-9)\,y' + ty = t^4, \quad y(4)=2$; **(f)** $(x-1)\,y' + (\sin x)y = x^3, \quad x(2)=1$.

14. Prove that two distinct solutions of a first order linear differential equation cannot intersect.

15. For each of the following differential equations, find a singular solution and the general solution.
 (a) $y' = 2x + 3(y-x^2)^{2/3}$; **(b)** $y' = \frac{1}{2y} + \frac{6x^2}{y}(y^2-x)^{3/4}$; **(c)** $y' = x^2 - \sqrt{x^3-3y}$;
 (d) $y' = \frac{\sqrt{y-1}}{x}$; **(e)** $y' = \sqrt{(y-1)(x+2)}$; **(f)** $y' = x\sqrt{2x+y} - 2$;
 (g) $y' = \sqrt{y^2-1}$; **(h)** $y' = \frac{\sqrt{y^3-x^2+2x}}{3y^2}$; **(i)** $y' = \sqrt{\frac{y-1}{y(1+x)}}$;
 (j) $y' = (y^2-1)\sqrt{y}$; **(k)** $(y')^2 + xy' = y$; **(l)** $y\,y' = \left(1-y^2\right)^{1/3}$.

16. Compute the first two Picard iterations for the following initial-value problems.
 (a) $y' = 1-(1+x)y+y^2, \quad y(0)=1$; **(b)** $y' = x - y^2, \quad y(0)=1$;
 (c) $y' = 1 + x\,\sin y, \quad y(\pi)=2\pi$; **(d)** $y' = 1+x-y^2, \quad y(0)=0$.

17. Compute the first three Picard iterations for the following initial value problems. On what interval does Picard's theorem guarantee the convergence of successive approximations? Determine the error of the third approximation.
 (a) $y' = xy, \quad y(1)=1$; **(b)** $y' = x - y^2, \quad y(0)=0$;
 (c) $x\,y' = y - y^2, \quad y(1)=1$; **(d)** $y' = 3y^2 + 4x^2, \quad y(0)=0$;
 (e) $x\,y' = y^2 - 2y, \quad y(1)=2$; **(f)** $y' = \sin x - y, \quad y(0)=0$.

18. Compute the first four Picard iterators for the differential equation $y' = x^2+y^2$ subject to the initial condition $y(0)=0$ and then another initial condition $y(1)=2$. Estimate the error of the fourth approximation for each.

19. Find the general formula for n-th Picard's approximation, $\phi_n(x)$, for the given differential equation subject to the specified initial condition.
 (a) $y' = 3\,e^{-2x} + y, \quad y(1)=0$; **(b)** $y' = e^{2x} - y, \quad y(0)=1$;
 (c) $y' = x + y, \quad y(0)=1$; **(d)** $y' = -y^2, \quad y(0)=1$.

20. Let $f(x,y)$ be a continuous function in the domain $\Omega = \{\,(x,y)\,:\, x_0 \leqslant x \leqslant x_0+\varepsilon,\ \ |y-y_0| \leqslant b\,\}$. Prove the uniqueness for the initial value problem $y' = f(x,y)$, $x_0 \leqslant x \leqslant x_0+\varepsilon$, $0<\varepsilon\leqslant a$, $y(x_0)=y_0$ if $f(x,y)$ does not increase in y for each fixed x.

21. For which nonnegative values of α does the uniqueness theorem for the differential equation $y' = |y|^\alpha$ fail?

22. For the following IVPs, show that there is no solution satisfying the given initial condition. Explain why this lack of solution does not contradict Peano's theorem.
 (a) $xy' - y = x^2$, $y(0)=1$; **(b)** $xy' = 2y - x^3$, $y(0)=1$.

23. Convert the given initial value problem into an equivalent integral equation and find the solution by Picard iteration:

$$y' = 6x - 2xy, \qquad y(0)=1.$$

24. Prove that an initial value problem for the differential equation $y' = 2\sqrt{|y|}$ has infinitely many solutions.

25. Does the equation $y' = y^{3/4}$ have a unique solution through every initial point (x_0,y_0)? Can solution curves ever intersect for this differential equation?

26. Show that the sequence of continuous functions $y_n(x) = x^n$ for $0\leqslant x\leqslant 1$ converges as $x\to 0$ pointwise to a discontinuous function.

27. Find a sequence of functions for which $\lim_{n\to\infty}\int_0^1 f_n(x)\,\mathrm{d}x \neq \int_0^1 \lim_{n\to\infty} f_n(x)\,\mathrm{d}x$.

28. Find successive approximations $\phi_0(x)$, $\phi_1(x)$, and $\phi_2(x)$ of the initial value problem for the Riccati equation $y' = 1-(1+x)y+y^2$, $y(0)=1$. Estimate the difference between $\phi_2(x)$ and the exact solution $y(x)$ on the interval $[-0.25, 0.25]$.
29. Does the initial value problem $y' = x^{-1/2}$, $y(0)=1$ have a unique solution?
30. Consider the initial value problem $y' = \sqrt{x}+\sqrt{y}$, $y(0)=0$. Using substitution $y = t^6 v^2$, $x = t^4$, derive the differential equation and the initial conditions for the function $v(t)$. Does this problem have a unique solution?

Review Questions for Chapter 1

1. Eliminate the arbitrary constant (denoted by C) to obtain a differential equation.
   (a) $y^2 = \frac{1}{2\sin x + C\cos x}$; **(b)** $\sqrt{y-x} - \ln Cx = 0$; **(c)** $x^2 + Cy^2 = 1$;
   (d) $y+1 = C\tan x$; **(e)** $C\,e^x + e^y = 1$; **(f)** $C\,e^x = \frac{1}{\sin x} + \frac{1}{\sin y}$;
   (g) $y\,e^{Cx} = 1$; **(h)** $C\,e^x = \arcsin x + \arcsin y$; **(i)** $C\,y - \ln x = 0$;
   (j) $\sqrt{1+x} = 1 + C\sqrt{1+y}$; **(k)** $y = C(x-C)^2$; **(l)** $y(x+1) + Ce^{-x} + e^x = 0$.
2. In each of the following problems, verify whether or not the given function is a solution of the given differential equation and specify the interval or intervals in which it is a solution; C always denotes a constant.
   (a) $(y')^2 - xy + y = 0$, $3\sqrt{y} = (x-1)^{3/2}$; **(b)** $y\,y' = x$, $y^2 - x^2 = C$;
   (c) $y' = 2x\,\cos(x^2)\,y$, $y = C\,e^{\sin x^2}$; **(d)** $y' + y^2 = 0$, $y(x) = 1/x$;
   (e) $y' + 2xy = 0$, $y(x) = C\,e^{-x^2}$; **(f)** $xy' = 2y$, $y = C\,x^2$;
   (g) $y' + x/y = 0$, $y(x) = \sqrt{C^2 - x^2}$; **(h)** $y' = \cos^2 y$, $\tan y = x + C$.
3. The curves of the one-parameter family $x^3 + y^3 = 3Cxy$, where C is a constant, are called **folia of Descartes**. By eliminating C, show that this family of graphs is an implicit solution to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y(y^3 - 2x^3)}{x(2y^3 - x^3)}.$$

4. Show that the functions in parametric form satisfy the given differential equation.
   (a) $x\,y' = 2y$, $x = t^2$, $y = t^4$; **(b)** $a^2 y\,y' = b^2 x$, $x = a\,\sinh t$, $y = b\,\cosh t$.
5. A particle moves along the abscissa in such a way that its instantaneous velocity is given as a function of time t by $v(t) = 4 - 3\,t^2$. At time $t = 0$, it is located at $x = 2$. Set up an initial value problem describing the motion of the particle and determine its position at any time $t > 0$.
6. Determine the values of λ for which the given differential equation has a solution of the form $y = e^{\lambda t}$.
   (a) $y'' - y' - 6y = 0$; **(b)** $y''' + 3y'' + 3y' + y = 0$.
7. Determine the values of λ for which the given differential equation has a solution of the form $y = x^{\lambda}$.
   (a) $2x^2y'' - 3xy' + 3y = 0$; **(b)** $2x^2y'' + 5xy' - 2y = 0$.
8. Find a singular solution for the given differential equation
   (a) $y' = 3x\,\sqrt{1-y^2}$; **(b)** $y' = \frac{1}{x}\,\sqrt{2x+y} - 2$;
   (c) $(x^2+1)\,y' = \sqrt{4y-1}$; **(d)** $y' = -\sqrt{y^2-4}$.
9. Verify that the function $y = \sqrt{(x^2-2x+3)/(x^2+x-2)}$ is a solution of the differential equation $2yy' = (9-3x^2)/(x^2+x-2)^2$ on some interval. Give the largest intervals of definition of this solution.
10. The sum of the x- and y-intercepts of the tangent line to a curve in the xy-plane is a constant regardless of the point of tangency. Find a differential equation for the curve.
11. Derive a differential equation of the family of circles having a tangent $y = 0$.
12. Derive a differential equation of the family of unit circles having centers on the line $y = 2x$.
13. Show that x^3 and $(x^{3/2}+5)^2$ are solutions of the nonlinear differential equation $(y')^2 - 9xy = 0$ on $(0,\infty)$. Is the sum of these functions a solution?
14. In each of the following problems, verify whether or not the given function is a solution of the given differential equation and specify the interval or intervals in which it is a solution; C always denotes a constant.

   (a) $x\,y' = y + x$, $y(x) = x\,\ln x + Cx$; **(b)** $y'\,\cos x + y\,\sin x = 1$, $y = C\,\cos x + \sin x$;

   (c) $y' + y\,e^x = 0$, $C = e^x + \ln y$; **(d)** $y' + 2xy = e^{-x^2}$, $y = (x+C)\,e^{-x^2}$;

   (e) $xy' + y = y\,\ln|xy|$, $\ln|xy| = Cx$; **(h)** $y' = \sec(y/x) + y/x$, $y = x\arcsin(\ln x)$;

(f) $\dfrac{y'}{\sqrt{1-y^2}} + \dfrac{1}{\sqrt{1-x^2}} = \arcsin x + \arcsin y, \qquad C\,e^x = \arcsin x + \arcsin y;$

(g) $t^2\, y' + 2ty - t + 1 = 0, \qquad y = \frac{1}{2} - 1/t + C/t^2.$

15. Show that $y' = y^a$, where a is a constant such that $0 < a < 1$ has the singular solution $y = 0$ and the general solution is $y = [(1-a)(x+C)]^{1/(1-a)}$. What is the limit of the general solution as $a \to 1$?

16. Which straight lines through the origin are solutions of the following differential equations?

    (a) $y' = \dfrac{x^2 - y^2}{3xy}$; **(b)** $y' = xy$; **(c)** $y' = \dfrac{5x+4y}{4x+5y}$; **(d)** $y' = \dfrac{x+3y}{3x+y}$.

17. Show that $y = Cx - C^2$, where C is an arbitrary constant, is an equation for the family of tangent lines for the parabola $y = x^2/4$.

18. Show that in general it is not possible to write every solution of $y' = f(x)$ in the form $y(x) = \int_a^x f(t)\,dt$ and compare this result with the fundamental theorem of calculus.

19. Show that the differential equation $y^2\,(y')^2 + y^2 = 1$ has the general solution family $(x+C)^2 + y^2 = 1$ and also singular solutions $y = \pm 1$.

20. The charcoal from a tree killed in the volcanic eruption that formed Crater Lake in Oregon contained 44.5% of carbon-14 found in living matter. The half-life of C^{14} is 5730 ± 40 years. About how old is Crater Lake?

21. Derive a differential equation for uranium with half-life 4.5 billion years.

22. With time measured in years, the value of λ in Eq. (1.1.1) for cobalt-60 is about 0.13. Estimate the half-life of cobalt-60.

23. The general solution of $1 - y^2 = (y\,y')^2$ is $(x-c)^2 + y^2 = 1$, where c is an arbitrary constant. Does there exist a singular solution?

24. Use a computer graphic routine to display the direction field for each of the following differential equations. Based on the slope field, determine the behavior of the solution as $x \to +\infty$. If this behavior depends on the initial value at $x = 0$, describe the dependency.

    (a) $y' = x^3 + y^3, \quad -2 \leqslant x \leqslant 2$; **(b)** $y' = \cos(x+y), \quad -4 \leqslant x \leqslant 4$;
    (c) $y' = x^2 + y^2, \quad -3 \leqslant x \leqslant 3$; **(d)** $y' = x^2y^2, \quad -2 \leqslant x \leqslant 2$.

25. Using a software solver, estimate the validity interval of each of the following initial value problems.

    (a) $y' = y^3 - xy + 1,\ y(0) = 1$; **(b)** $y' = y^2 + xy + 1,\ y(0) = 1$;
    (c) $y' = x^2y^2 - x^3,\ y(0) = 1$; **(d)** $y' = 1/(x^2+y^2),\ y(0) = 1$.

26. Using a software solver, draw a direction field for each of the given differential equations of the first order $y' = f(x, y)$. On the same graph, plot the graph of the curve defined by $f(x, y) = 0$; this curve is called the nullcline.

    (a) $y' = y^3 - x$; **(b)** $y' = y^2 + xy + x^2$; **(c)** $y' = y - x^3$;
    (d) $y' = xy/(x^2+y^2)$; **(e)** $y' = xy^{1/3}$; **(f)** $y' = \sqrt{|xy|}$.

27. A body of constant mass m is projected away from the earth in a direction perpendicular to the earth's surface. Let the positive x-axis point away from the center of the earth along the line of motion with $x = 0$ lying on the earth's surface. Suppose that there is no air resistance, but only the gravitational force acting on the body given by $w(x) = -k(x+R)^{-2}$, where k is a constant and R is the radius of the earth. Derive a differential equation for modeling body's motion.

28. Find the maximum interval for which Picard's theorem guarantees the existence and uniqueness of the initial value problem $y' = (x^2 + y^2 + 1)\,e^{1-x^2-y^2}, \quad y(0) = 0$.

29. Find all singular solutions to the differential equation $y' = y^{2/3}\,(y^2 - 1)$.

30. Under what condition on C does the solution $y(x) = y(x, C)$ of the initial value problem $y' = ky(1+y^2)$, $y(0) = C$ exist on the whole interval $[0, 1]$?

31. Determine whether Theorem 1.3 (page 23) implies that the given initial value problem has a unique solution.

    (a) $y' + yt = \sin^2 t,\ y(\pi) = 1$; **(b)** $y' = x^3 + y^3,\ y(0) = 1$;
    (c) $y' = x/y,\ y(2) = 0$; **(d)** $y' = y/x,\ y(2) = 0$;
    (e) $y' = x - \sqrt[3]{y-1},\ y(5) = 1$; **(f)** $y' = \sin y + \cos y,\ y(\pi) = 0$;
    (g) $y' = x\,\sqrt{y},\ y(0) = 0$. **(h)** $y' = x\,\ln|y|,\ y(1) = 0$.

32. Convert the given IVP, $y' = x(y-1)$, $y(0) = 1$, into an equivalent integral equation and determine the first four Picard iterations.

33. Consider the initial value problem $y' = y^2 + 4\sin^2 x$, $y(0) = 0$. According to Picard's theorem, this IVP has a unique solution in any rectangle $D = \{(x, y) : |x| < a,\ |y| < b/M\}$, where M is the maximum value of $|f(x, y)|$. Show that the unique solution exists at least on the interval $[-h, h]$, $h = \min\left\{a, \dfrac{b}{b^2+4}\right\}$.

34. Use the existence and uniqueness theorem to prove that $y = 2$ is the only solution to the IVP $y' = \frac{4x}{x^2+9}\,(y^2 - 4), \quad y(0) = 2$.

35. Prove that the differential equation $y' = x - 1/y$ has a unique solution on $(0, \infty)$.

36. Determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist.
 (a) $y' + \frac{x}{1-x^2}\,y = x,\ y(0) = 0;$ **(b)** $(1+x^3)\,y' + xy = x^2,\ y(0) = 1;$
 (c) $y' + (\tan t)y = t^2,\ y(\pi/2) = 9;$ **(d)** $(x^2+1)\,y' + xy = x^2,\ y(0) = 0;$
 (e) $t\,y' + t^2\,y = t^4,\ y(\pi) = 1;$ **(f)** $(\sin^2 t)\,y' + y = \cos t,\ y(1) = 1.$

37. Compute the first two Picard iterations for the following initial-value problems.
 (a) $y' = x^2 + y^2, \quad y(0) = 0;$ **(b)** $y' = (x^2 + y^2)^2, \quad y(0) = 1;$
 (c) $y' = \sin(xy), \quad y(\pi) = 2;$ **(d)** $y' = e^{xy}, \quad y(0) = 1.$

38. Compute the first three Picard iterations for the following initial value problems. On what interval does Picard's theorem guarantee the convergence of successive approximations? Determine the error of the third approximation when the given function is defined in the rectangle $|x - x_0| < a,\ |y - y_0| < b$.
 (a) $y' = x^2 + xy^2,\ y(0) = 1;$ **(b)** $x\,y' = y^2 - 1,\ y(1) = 2;$
 (c) $y' = xy^2,\ y(1) = 1;$ **(d)** $y' = x^2 - y,\ y(0) = 0;$
 (e) $xy' = y^2,\ y(1) = 1;$ **(f)** $y' = 2y^2 + 3x^2,\ y(0) = 0;$
 (g) $xy' = 2y^2 - 3x^2,\ y(0) = 0;$ **(h)** $y' = y + \cos x,\ y(\pi/2) = 0.$

39. Compute the first four Picard iterations for the given initial value problems. Estimate the error of the fourth approximation when the given function is defined in the rectangle $|x - x_0| < a,\ |y - y_0| < b$.
 (a) $y' = 3x^2 + xy^2 + y,\ y(0) = 0;$ **(b)** $y' = y - e^x + x,\ y(0) = 0;$
 (c) $y' + y = 2\sin x,\ y(0) = 1;$ **(d)** $y' = -y - 2t,\ y(0) = 1.$

40. Find the general formula for n-th Picard's approximation, $\phi_n(x)$, for the given differential equation subject to the specified initial condition.
 (a) $y' = y - e^{2x},\ y(0) = 1;$ **(b)** $y' = 2y + x,\ y(1) = 0;$
 (c) $y' = y - x^2,\ y(0) = 1;$ **(d)** $y' = y^2,\ y(1) = 1.$

41. Find the formula for the eighth Picard's approximation, $\phi_8(x)$, for the given differential equation subject to the specified initial condition. Also find the integral of the given initial value problem and compare its Taylor's series with $\phi_8(x)$.
 (a) $y' = 2y + \cos x,\ y(0) = 0;$ **(b)** $y' = \sin x - y,\ y(0) = 1;$
 (c) $y' = x^2 y + 1,\ y(0) = 0;$ **(d)** $y' + 2y = 3x^2,\ y(1) = 1.$

42. Sometimes the quadratures that are required to carry further the process of successive approximations are difficult or impossible. Nevertheless, even the first few approximations are often quite good. For each of the following initial value problems, find the second Picard's approximation and compare it with the exact solution at points $x_k = k/4$, $k = -1, 0, 1, 2, 3, 4, 5, 6, 7, 8$.
 (a) $y' = 2\sqrt{y}, \quad y(0) = 1;$ **(b)** $y' = \frac{2-e^{-y}}{1+2x}, \quad y(0) = 0.$

43. The accuracy of Picard's approximations depends on the choice of the initial approximation, $\phi_0(x)$. For the following problems, calculate the second Picard's approximation for two given initial approximations, $\phi_0(x) = 1$ and $y_0(x) = x$, and compare it with the exact solution at points $x_k = k/4$, $k = -1, 0, 1, 3, 4, 5, 6, 7, 8$.
 (a) $y' = y^2, \quad y(1) = 1;$ **(b)** $y' = y^{-2}, \quad y(1) = 1.$

44. **The Grönwall[12] inequality:** Let x, g, and h be real-valued continuous functions on a real t-interval I: $a \leqslant t \leqslant b$. Let $h(t) \geqslant 0$ on I, $g(t)$ be differentiable, and suppose that for $t \in I$,

$$x(t) \leqslant g(t) + \int_a^t h(\tau)x(\tau)\,\mathrm{d}\tau.$$

Prove that on I

$$x(t) \leqslant g(t) + \int_a^t h(\tau)g(\tau)\,\exp\left\{\int_\tau^t h(s)\,\mathrm{d}s\right\}\,\mathrm{d}\tau.$$

Hint: Differentiate the given inequality and use an integrating factor $\mu(t) = \exp\left\{-\int_a^t h(\tau)\,\mathrm{d}\tau\right\}$.

45. For a positive constant $k > 0$, find the general solution of the differential equation $\dot{y} = \sqrt{|y|} + k$. Show that while the slope function $\sqrt{|y|} + k$ does not satisfy a Lipschitz condition in any region containing $y = 0$, the initial value problem for this equation has a unique solution.

[12] In honor of the Swedish mathematician Thomas Hakon Grönwall (1877–1932), who proved this inequality in 1919.