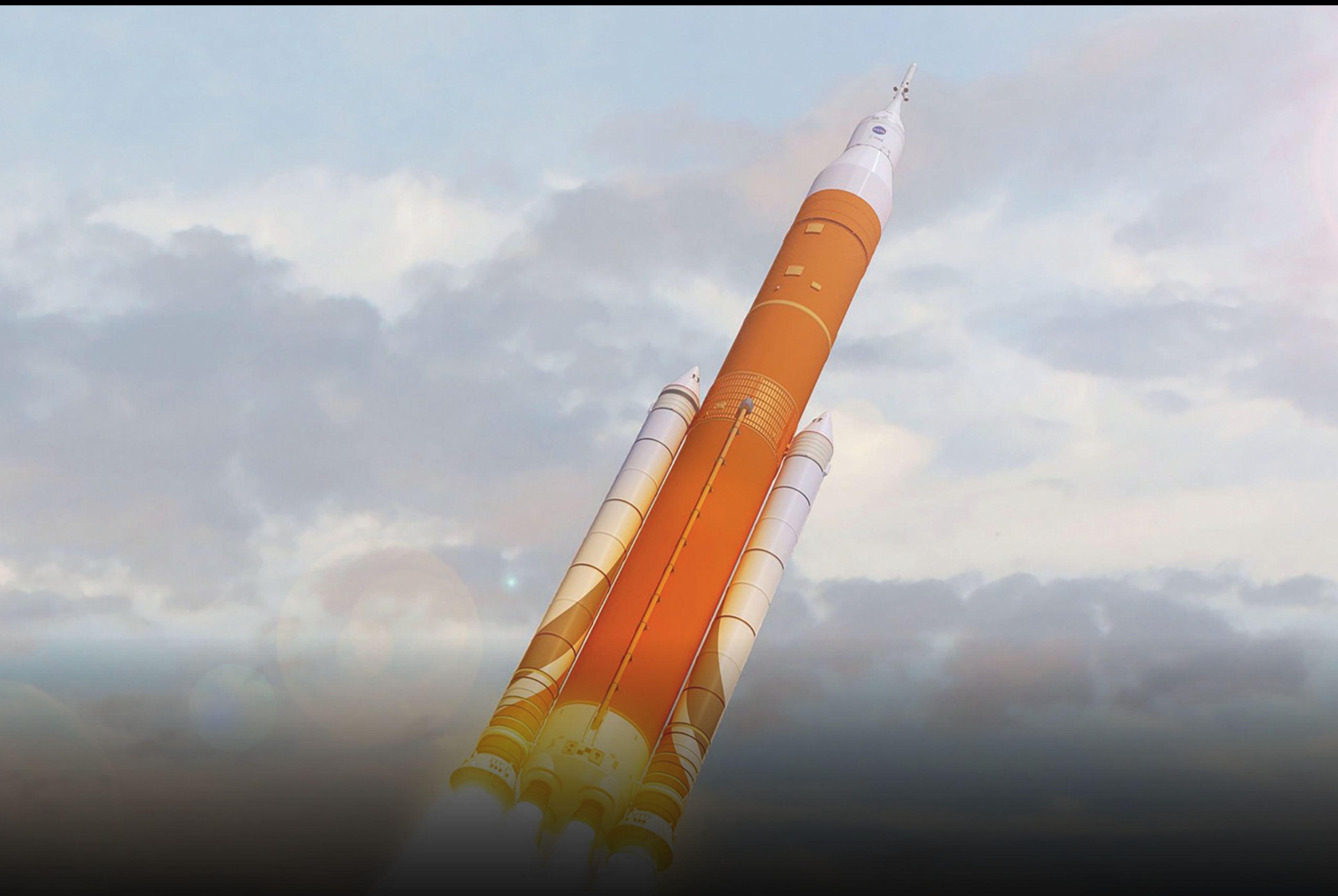
Engineering Mechanics:

Dynamics

Third Edition



McGraw Hill

Gary L. Gray | Francesco Costanzo
Robert J. Witt | Michael E. Plesha

Engineering Mechanics

DYNAMICS

Engineering Mechanics
DYNAMICS

THIRD EDITION

Gary L. Gray
Department of Engineering Science and Mechanics
Penn State University

Francesco Costanzo
Department of Engineering Science and Mechanics
Penn State University

Robert J. Witt
Department of Engineering Physics
University of Wisconsin–Madison

Michael E. Plesha
Department of Engineering Physics
University of Wisconsin–Madison

Mc Graw Hill

ENGINEERING MECHANICS: DYNAMICS, THIRD EDITION

This book is printed on acid-free paper.

1 2 3 4 5 6 7 8 9 LWI 26 25 24 23 22

ISBN 978-1-264-97974-5 (bound edition)
MHID 1-264-97974-6 (bound edition)
ISBN 978-1-264-98212-7 (loose-leaf edition)
MHID 1-264-98212-7 (loose-leaf edition)

Portfolio Manager: *Beth Bettcher*
Product Developers: *Heather Ervolino and Joan Weber*
Marketing Manager: *Lisa Granger*
Content Project Managers: *Laura Bies and Rachael Hillebrand*
Buyer: *Susan K. Culbertson*
Designer: *David W. Hash*
Content Licensing Specialist: *Lorraine Buczek*
Cover Image: *NASA/MSFC; NASA*
Compositor: *Aptara®, Inc.*

Library of Congress Cataloging-in-Publication Data available upon request

mheducation.com/highered

ABOUT THE AUTHORS

Gary L. Gray is an Associate Professor of Engineering Science and Mechanics in the Department of Engineering Science and Mechanics at Penn State in University Park, PA. He received a B.S. in mechanical engineering (cum laude) from Washington University in St. Louis, MO, an S.M. in engineering science from Harvard University, and M.S. and Ph.D. degrees in engineering mechanics from the University of Wisconsin–Madison. His primary research interests are in dynamical systems, dynamics of mechanical systems, and mechanics education. For his contributions to mechanics education, he has been awarded the Outstanding and Premier Teaching Awards from the Penn State Engineering Society, the Outstanding New Mechanics Educator Award from the American Society for Engineering Education, the Learning Excellence Award from General Electric, and the Collaborative and Curricular Innovations Special Recognition Award from the Provost of Penn State. In addition to dynamics, he also teaches mechanics of materials, mechanical vibrations, numerical methods, advanced dynamics, and engineering mathematics.

Robert J. Witt retired from the University of Wisconsin–Madison, Department of Engineering Physics, in 2020 after a 33-year teaching career. He received his B.S. in mechanical engineering from the University of California–Davis and his M.S. and Ph.D. in nuclear engineering from MIT. His research interests were in computational methods in fluid and solid mechanics, with particular applications to nuclear systems. He taught 20 different courses over the span of his career, including statics, dynamics, mechanics of materials, and a variety of other classes in applied mechanics, computational methods, and nuclear engineering. He is coauthor of the book *Concepts and Applications of Finite Element Analysis* (with R. D. Cook, D. S. Malkus, and M. E. Plesha).

Michael E. Plesha is a Professor Emeritus of Engineering Mechanics in the Department of Engineering Physics at the University of Wisconsin–Madison. Professor Plesha received his B.S. from the University of Illinois–Chicago in structural engineering and materials, and his M.S. and Ph.D. from Northwestern University in structural engineering and applied mechanics. His primary research areas are computational mechanics, focusing on the development of finite element and discrete element methods for solving static and dynamic nonlinear problems, and the development of constitutive models for characterizing the behavior of materials. Much of his work focuses on problems featuring contact, friction, and material interfaces. Applications include nanotribology, high-temperature rheology of ceramic composite materials, modeling geomaterials including rock and soil, penetration mechanics, and modeling crack growth in structures. He is coauthor of the book *Concepts and Applications of Finite Element Analysis* (with R. D. Cook, D. S. Malkus, and R. J. Witt). In addition to teaching statics and dynamics, he also has extensive experience teaching courses in basic and advanced mechanics of materials, mechanical vibrations, and finite element methods.

Francesco Costanzo is a Professor of Engineering Science and Mechanics in the Engineering Science and Mechanics Department at Penn State. He received the Laurea in Ingegneria Aeronautica from the Politecnico di Milano, Milan, Italy. After coming to the United States as a Fulbright scholar, he received his Ph.D. in aerospace engineering from Texas A&M University. His primary research interest is the mathematical and numerical modeling of material behavior. His specific research interests include the theoretical and numerical characterization of dynamic fracture in materials subject to thermo-mechanical loading, the development of multi-scale methods for predicting continuum-level material properties from molecular calculations, and modeling and computational problems in bio-medical applications. In addition to scientific research, he has contributed to various projects for the advancement of mechanics education under the sponsorship of several organizations, including the National Science Foundation. For his contributions, he has received various awards, including the 1998 and the 2003 GE Learning Excellence Awards and the 1999 ASEE Outstanding New Mechanics Educator Award.

The authors thank their families for their patience, understanding, and, most importantly, encouragement during the long years it took to bring these books to completion. Without their support, none of this would have been possible.

BRIEF CONTENTS

Dynamics

501-1111

TABLE OF CONTENTS

Welcome to dynamics! Dynamics is the science that relates motion to the forces that cause and are caused by that motion. Dynamics is at the heart of the design and analysis of mechanical systems whose operating principles rely on motion or are meant to control motion. The engineering applications of dynamics are many and varied. Traditional applications include the design of mechanisms, engines, turbines, and airplanes. Other (perhaps less known) applications include the kinesiology of the human body, the analysis of cell motion, and the design of some micro- and nano-size devices, including both sensors and actuators. All of these applications stem from the combination of kinematics, which describes the geometry of motion, with a few basic principles anchored in Newton's laws of motion, such as the work-energy and the impulse-momentum principles.

With this book we hope to provide a teaching and learning experience that is not only effective but also motivates the study and application of dynamics. We have structured the book to achieve four main objectives. First, we provide a rigorous introduction to the fundamental principles of particle and rigid body dynamics. In a constantly changing technological landscape, it is by relying on fundamentals that we can find new ways of applying what we know. Second, we incorporate those pedagogical principles that recent research in math, science, and engineering education has identified as essential for improving student learning. While it is commonly accepted that a good conceptual understanding is important to improve problem-solving skills, it has been discovered that problem-solving skills and concepts need to be taught in different ways. Third, we have made modeling the underlying theme of our approach to problem solving. We believe that modeling, understood as the making of sensible assumptions to reduce a real complex problem to a simpler but solvable problem, is also something that must be taught and discussed alongside the basic principles. Fourth, we emphasize a systematic approach to solving every problem, an integral part of which is creating the aforementioned model. The four objectives that animate this textbook have been incorporated in a series of clearly identifiable features that are used consistently throughout the book. We believe these features make the book new and unique, and we hope that they will improve both the teaching and the learning experience.

This book is the second volume of a Statics and Dynamics series. Let's see in detail what makes these books different.

Why Another Statics and Dynamics Series?

These books provide thorough coverage of all the pertinent topics traditionally associated with statics and dynamics. Indeed, many of the currently available texts also provide this. However, these texts offer several major innovations that enhance the learning objectives and outcomes in these subjects.

What Then Are the Major Differences Between These Books and Other Engineering Mechanics Texts?

A Consistent and Systematic Approach to Problem Solving One of the main objectives of these texts is to foster the habit of solving problems using a systematic approach. Therefore, the example problems in these texts follow a structured four-step problem-solving methodology that will help you develop your problem-solving skills not only in statics and dynamics, but also in all other mechanics subjects that follow. This structured problem-solving approach consists of the following steps: Road Map & Modeling, Governing Equations, Computation, and Discussion & Verification. The Road Map provides some of the general objectives of the problem and develops a strategy for how the solution will be developed. Modeling is next, where a real-life problem is idealized by a model. This step results in the creation of a free body diagram and the selection of the balance laws needed to solve the problem. The Governing Equations step is devoted to writing all the equations needed to solve the problem. These equations typically include the equilibrium equations, and, depending upon the particular problem, force laws (e.g., spring laws or frictional laws) and kinematic equations. In the Computation step, the governing equations are solved. In the final step, Discussion & Verification, the solution is interrogated to ensure that it is meaningful and accurate. This four-step problem-solving methodology is followed for all examples that involve a balance principle such as Newton's second law or the work-energy principle. Some problems (e.g., kinematics problems) do not involve balance principles, and for these the Modeling step is not needed.

Contemporary Examples, Problems, and Applications The examples, homework problems, and design problems were carefully constructed to help show you how the various topics of statics and dynamics are used in engineering practice. Statics and dynamics are immensely important subjects in modern engineering and science, and one of our goals is to excite you about these subjects and the career that lies ahead of you.

A Focus on Design A major difference between these texts and other books is the systematic incorporation of design and modeling of real-life problems throughout. In statics, topics include important discussions on design, ethics, and professional responsibility. In dynamics, the emphasis is on parametric analysis and motion over ranges of time and space. These books show you that meaningful engineering design is possible using the concepts of statics and dynamics. Not only is the ability to develop a design very satisfying, but it also helps you develop a greater understanding of basic concepts and helps sharpen your ability to apply these concepts. Because the main focus of statics and dynamics textbooks should be the establishment of a firm understanding of basic concepts and correct problem-solving techniques, design topics do not have an overbearing presence in the books. Rather, design topics are included where they are most appropriate. While some of the discussions on design could be described as "common sense," such a characterization trivializes the importance and necessity for discussing pertinent issues such as safety, uncertainty in determining loads, the designer's responsibility to anticipate uses, even unintended uses, communications, ethics, and uncertainty in workmanship. Perhaps the most important feature of our inclusion of design and modeling topics is that you get a glimpse of what engineering is about and where your career in engineering is headed. The book is structured so that design topics and design problems are offered in a variety of places, and it is possible to pick when and where the coverage of design is most effective.

Computational Tools Some examples and problems are appropriate for solution using computer software. The use of computers extends the types of problems that can be solved while alleviating the burden of solving equations. Such examples and problems give you insight into the power of computer tools and further insight into how statics and dynamics are used in engineering practice.

Modern Pedagogy Numerous modern pedagogical elements have been included. These elements are designed to reinforce concepts, and they provide additional information to help you make meaningful connections with real-world applications. Marginal notes (i.e., Helpful Information, Common Pitfalls, Interesting Facts, and Concept Alerts) help you place topics, ideas, and examples in a larger context. These notes will help you study (e.g., Helpful Information and Common Pitfalls), will provide real-world examples of how different aspects of statics and dynamics are used (e.g., Interesting Facts), and will drive home important concepts or help dispel misconceptions (e.g., Concepts Alerts and Common Pitfalls). Mini-Examples are used throughout the text to immediately and quickly illustrate a point or concept without making readers wait for the worked-out examples at the end of the section.

Answers to Problems The answers to most even-numbered problems have been included in the back matter for ease of use as Appendix B. Providing answers in this manner allows for the inclusion of more complex information than would otherwise be possible. In addition to final numerical and/or symbolic answers, plots for Computer Problems are included.

Changes to the Third Edition

The third editions of *Engineering Mechanics: Statics* and *Engineering Mechanics: Dynamics* retain all of the major pedagogical features of the previous editions, including a structured problem-solving methodology for all example problems, contemporary engineering applications in the example problems and homework exercises, the inclusion of engineering design and its implications for problem solving and applications, and use of computational tools where applicable. In addition, as a result of the author-based typesetting process, the outstanding accuracy of the earlier editions has been preserved, leading to books whose accuracy is unrivaled among textbooks.

The third editions contain revised and enhanced textual discussions and example problems, additional figures where effective, and new homework exercises. In Connect, the online homework system, there are significant updates, including an auto-graded FBD tool and interactive learning tools. These interactive assignments help reinforce what is being covered in the text and show students how to tie the material to real-world situations. These tools complement the hundreds of auto-graded, algorithmic-exercises that are included in Connect from the text.

The following individuals have been instrumental in ensuring the highest standard of content and accuracy. We are deeply indebted to them for their tireless efforts.

Third Edition

Reviewers

David Ancalle-Reyes
Kennesaw State University
Hossein Ataei
University of Illinois at Chicago
Sarah Baxter
University of St. Thomas
John Burkhardt
U.S. Naval Academy
Amiya Chatterjee
University of California at Los Angeles
Mona Eskandari
University of California at Riverside
Serhan Guner
University of Toledo
Anant Honkan
Perimeter College, Georgia State University
Yufeng Hu
Western Michigan University
Hesam Moghaddam
Northern Arizona University
Karim Nohra
University of South Florida
Anurag Purwar
Stony Brook University
Adrian Rodriguez
University of Texas at Austin
Scott D. Schif
Kansas State University
Frank Sup
University of Massachusetts at Amherst
Jekan Thangavelautham
University of Arizona
Vimal Viswanathan
San Jose State University

Second Edition

Reviewers

George G. Adams
Northeastern University
Stephen Bechtel
The Ohio State University
J. A. M. Boulet
University of Tennessee–Knoxville
Janet Brelin-Fornari
Kettering University
Suren Chen
Colorado State University
Nicola Ferrier
University of Wisconsin–Madison
Michael W. Keller
The University of Tulsa
Yohannes Ketema
University of Minnesota
Dragomir C. Marinkovich
Milwaukee School of Engineering
Tom Mase
California Polytechnic State University–San Luis Obispo
Richard McNitt
Penn State University
William R. Murray
California Polytechnic State University
Chris Passerello
Michigan Technological University
Gordon R. Pennock
Purdue University
Vincent C. Prantil
Milwaukee School of Engineering
Bidhan C. Roy
University of Wisconsin–Platteville
David A. Rubenstein
Oklahoma State University
John Schmitt
Oregon State University
Larry M. Silverberg
North Carolina State University
Richard E. Stanley
Kettering University
T. W. Wu
University of Kentucky
Jack Xin
Kansas State University

Focus Group Participants

Brock E. Barry
United States Military Academy
Daniel Dickrell, III
University of Florida
Ali Gordon
University of Central Florida–Orlando
Stephanie Magleby
Brigham Young University
Tom Mase
California Polytechnic State University–San Luis Obispo
Gregory Miller
University of Washington
Carisa H. Ramming
Oklahoma State University

First Edition

Board of Advisors

Janet Brelin-Fornari
Kettering University
Manoj Chopra
University of Central Florida
Pasquale Cinnella
Mississippi State University
Ralph E. Flori
Missouri University of Science and Technology
Christine B. Masters
Penn State University
Mark Nagurka
Marquette University
David W. Parish
North Carolina State University
Gordon R. Pennock
Purdue University
Michael T. Shelton
California State Polytechnic University–Pomona
Joseph C. Slater
Wright State University
Arun R. Srinivasa
Texas A&M University
Carl R. Vilmann
Michigan Technological University
Ronald W. Welch
The University of Texas at Tyler

Reviewers

Makola M. Abdullah
Florida Agricultural and Mechanical University
Murad Abu-Farsakh
Louisiana State University
George G. Adams
Northeastern University
Farid Amirouche
University of Illinois at Chicago
Stephen Bechtel
Ohio State University
Kenneth Belanus
Oklahoma State University
Glenn Beltz
University of California–Santa Barbara
Haym Benaroya
Rutgers University
Sherrill B. Biggers
Clemson University
James Blanchard
University of Wisconsin–Madison
Janet Brelin-Fornari
Kettering University
Pasquale Cinnella
Mississippi State University
Ted A. Conway
University of Central Florida
Joseph Cusumano
Penn State University
Bogdan I. Epureanu
University of Michigan
Ralph E. Flori
Missouri University of Science and Technology
Barry Goodno
Georgia Institute of Technology
Kurt Gramoll
University of Oklahoma
Hartley T. Grandin, Jr.
Professor Emeritus, Worcester Polytechnic Institute
Roy J. Hartfield, Jr.
Auburn University
Paul R. Heyliger
Colorado State University
James D. Jones
Purdue University
Yohannes Ketema
University of Minnesota
Carl R. Knospe
University of Virginia
Sang-Joon John Lee
San Jose State University
Jia Lu
The University of Iowa
Ron McClendon
University of Georgia
Paul Mitiguy
Consulting Professor, Stanford University
William R. Murray
California Polytechnic State University–San Luis Obispo
Mark Nagurka
Marquette University
Robert G. Oakberg
Montana State University
James J. Olsen
Wright State University
Chris Passerello
Michigan Technological University
Gary A. Pertmer
University of Maryland
David Richardson
University of Cincinnati
William C. Schneider
Texas A&M University
Sorin Siegler
Drexel University
Joseph C. Slater
Wright State University
Ahmad Sleiti
University of Central Florida
Arun R. Srinivasa
Texas A&M University
Josef S. Torok
Rochester Institute of Technology
John J. Uicker
Professor Emeritus, University of Wisconsin–Madison
David G. Ullman
Professor Emeritus, Oregon State University
Carl R. Vilmann
Michigan Technological University
Claudia M. D. Wilson
Florida State University
C. Ray Wimberly
University of Texas at Arlington
Robert J. Witt
University of Wisconsin–Madison

T. W. Wu
University of Kentucky
X. J. Xin
Kansas State University

Joseph R. Zaworski
Oregon State University

M. A. Zikry
North Carolina State University

Symposium Attendees

Farid Amirouche
University of Illinois at Chicago
Subhash C. Anand
Clemson University
Manohar L. Arora
Colorado School of Mines
Stephen Bechtel
Ohio State University
Sherrill B. Biggers
Clemson University
J. A. M. Boulet
University of Tennessee
Janet Brelin-Fornari
Kettering University
Louis M. Brock
University of Kentucky
Amir Chaghajerdi
Colorado School of Mines
Manoj Chopra
University of Central Florida
Pasquale Cinnella
Mississippi State University
Adel ElSafty
University of North Florida
Ralph E. Flori
Missouri University of Science and Technology
Walter Haisler
Texas A&M University
Kimberly Hill
University of Minnesota
James D. Jones
Purdue University

Yohannes Ketema
University of Minnesota
Charles Krousgrill
Purdue University
Jia Lu
The University of Iowa
Mohammad Mahinfalah
Milwaukee School of Engineering
Tom Mase
California Polytechnic State University–San Luis Obispo
Christine B. Masters
Penn State University
Daniel A. Mendelsohn
The Ohio State University
Faissal A. Moslehy
University of Central Florida
LTC Mark Orwat
United States Military Academy at West Point
David W. Parish
North Carolina State University
Arthur E. Peterson
Professor Emeritus, University of Alberta
W. Tad Pfeffer
University of Colorado at Boulder
David G. Pollock
Washington State University
Robert L. Rankin
Professor Emeritus, Arizona State University

Mario Rivera-Borrero
University of Puerto Rico at Mayaguez
Hani Salim
University of Missouri
Michael T. Shelton
California State Polytechnic University–Pomona
Lorenz Sigurdson
University of Alberta
Larry Silverberg
North Carolina State University
Joseph C. Slater
Wright State University
Arun R. Srinivasa
Texas A&M University
David G. Ullman
Professor Emeritus, Oregon State University
Carl R. Vilmann
Michigan Technological University
Anthony J. Vizzini
Mississippi State University
Andrew J. Walters
Mississippi State University
Ronald W. Welch
The University of Texas at Tyler
T. W. Wu
University of Kentucky
Musharraf Zaman
University of Oklahoma–Norman
Joseph R. Zaworski
Oregon State University

Focus Group Attendees

Janet Brelin-Fornari
Kettering University
Yohannes Ketema
University of Minnesota
Mark Nagurka
Marquette University
C. Ray Wimberly
University of Texas at Arlington
M. A. Zikry
North Carolina State University

Accuracy Checkers

Walter Haisler
Texas A&M University
Richard McNitt
Penn State University
Mark Nagurka
Marquette University

ACKNOWLEDGMENTS

The authors are grateful to the many instructors who have provided valuable feedback on the previous editions of this book. We are especially thankful to Professor Christine B. Masters of Penn State University, Professor Carl R. Vilmann of Michigan Technological University, Professor Mark L. Nagurka of Marquette University, and Dr. Antonio Hernandez, University of Wisconsin–Madison for the guidance they have provided over the years these books have been in development and use. The authors are also grateful for helpful feedback from Professor Rani El-Hajjar of the University of Wisconsin–Milwaukee, Professor Richard Keles of the Center for Technology, Innovation and Community Engagement, New York, Professor Charles E. Bakis of Penn State University, and Dr. Suzannah Sandrik of the University of Wisconsin–Madison.

Special thanks go to Andrew Miller for infrastructure he created to keep the authors, manuscript, and solutions manual in sync. His knowledge of programming, scripting, subversion, and many other computer technologies made a gargantuan task feel just a little more manageable.

Mini-Examples

Mini-examples are used throughout the text to immediately and quickly illustrate a point or concept without having to wait for the worked-out examples at the end of the section.

force of attraction between two bodies. The gravitational force on a mass m_1 due to a mass m_2 a distance r away from m_1 is

$$\vec{F}_{12} = \frac{Gm_1 m_2}{r^2}\,\hat{u}, \tag{11.5}$$

where $\hat{u}$ is a unit vector pointing from m_1 to m_2 and G is the *universal gravitational constant** (sometimes called the *constant of gravitation* or *constant of universal gravitation*). The following example demonstrates the application of this law.

Mini-Example

Using the planets Jupiter and Neptune as an example, the force on Jupiter due to the gravitational attraction of Neptune, $\vec{F}_{JN}$, is given by (see Fig. 11.2)

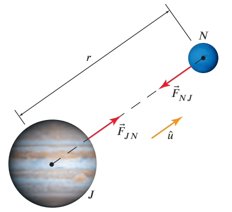
$$\vec{F}_{JN} = \frac{Gm_J m_N}{r^2}\,\hat{u}, \tag{11.6}$$

where r is the distance between the two bodies, m_J is the mass of Jupiter, m_N is the mass of Neptune, and $\hat{u}$ is a unit vector pointing from the center of Jupiter to the center of Neptune. The mass of Jupiter is 1.9×10^{27} kg, and that of Neptune is 1.02×10^{26} kg. Since the mean radius of Jupiter's orbit is 778,300,000 km and that of Neptune is 4,505,000,000 km, we assume that their closest approach to one another is approximately 3,727,000,000 km. Thus, at their closest approach, the magnitude of the force between these two planets is

$$|\vec{F}_{JN}| = \left(6.674\times10^{-11}\,\frac{\text{m}^3}{\text{kg}\cdot\text{s}^2}\right)\frac{(1.9\times10^{27}\,\text{kg})(1.02\times10^{26}\,\text{kg})}{(3.727\times10^{12}\,\text{m})^2} \tag{11.7}$$
$$= 9.312\times10^{17}\,\text{N}.$$

We can compare this force with the force of gravitation between Jupiter and the Sun. The Sun's mass is 1.989×10^{30} kg, and we have already stated that the mean radius of Jupiter's orbit is 778,300,000 km. Applying Eq. (11.5) between Jupiter and the Sun gives 4.164×10^{23} N, which is almost 450,000 times larger.

Acceleration due to gravity. Equation (11.5) allows us to determine the force of Earth's gravity on an object of mass m on the surface of the Earth. This is done by noting that the radius of the Earth is 6371.0 km (see the marginal note) and the mass of the Earth is 5.9736×10^{24} kg and then applying Eq. (11.5):



(left) JPL/University of Arizona/NASA; (right) NASA/JPL

Figure 11.2
The gravitational force between the planets Jupiter J and Neptune N. The relative sizes of the planets are accurate, but their separation distance is not.

Interesting Fact

The radius of the Earth. The Earth is not a perfect sphere. Therefore, there are different notions of "radius of the Earth." The

EXAMPLE 13.6 *Tension in a Wrecking Ball Cable*

The wrecking ball A shown in Fig. 1 is released from rest when $\theta = \theta_0 = 30°$, and it swings freely about the fixed point at O. Assuming that the weight of the ball is $W = 2500$ lb and $L = 30$ ft, determine the tension in the cable to which the ball is attached when the ball reaches $\theta = 0°$.

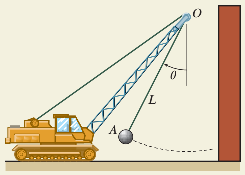


Figure 1

SOLUTION

Road Map & Modeling Modeling the wrecking ball as a particle and neglecting all forces except the weight force W and the cable tension T, the FBD is as shown in Fig. 2. Applying Newton's second law in the polar component system shown should allow us to find the tension in the cable as a function of its swing angle and thus, find its tension when $\theta = 0°$.

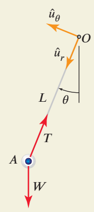


Figure 2
FBD of the wrecking ball as it swings downward.

Governing Equations

Balance Principles Referring to the FBD in Fig. 2 and applying Newton's second law, we obtain

$$\sum F_\theta: \quad -W\sin\theta = ma_\theta, \tag{1}$$
$$\sum F_r: \quad W\cos\theta - T = ma_r, \tag{2}$$

where $m = W/g$.

Force Laws All forces are accounted for on the FBD.

Kinematic Equations Writing a_θ and a_r in polar components gives

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = L\ddot{\theta} \quad \text{and} \quad a_r = \ddot{r} - r\dot{\theta}^2 = -L\dot{\theta}^2, \tag{3}$$

where we have replaced r with the constant length L.

Computation Substituting Eqs. (3) into Eqs. (1) and (2), we obtain

$$-W\sin\theta = mL\ddot{\theta} \quad \text{and} \quad W\cos\theta - T = -mL\dot{\theta}^2$$
$$\Rightarrow \quad \ddot{\theta} = -\frac{g}{L}\sin\theta \quad \text{and} \quad T = W\cos\theta + mL\dot{\theta}^2. \tag{4}$$

Notice that the tension is a function of $\dot{\theta}$, so we need to integrate $\ddot{\theta}(\theta)$ to find $\dot{\theta}(\theta)$ using the chain rule, that is,

$$\ddot{\theta} = \dot{\theta}\frac{d\dot{\theta}}{d\theta} = -\frac{g}{L}\sin\theta \quad \Rightarrow \quad \int_0^{\dot{\theta}} \dot{\theta}\,d\dot{\theta} = -\frac{g}{L}\int_{\theta_0}^{\theta}\sin\theta\,d\theta$$
$$\Rightarrow \quad \dot{\theta}^2 = 2\frac{g}{L}(\cos\theta - \cos\theta_0). \tag{5}$$

Substituting Eq. (5) into the expression for T in Eq. (4) gives $T(\theta)$ as

$$T = W(3\cos\theta - 2\cos\theta_0) \quad \Rightarrow \quad \boxed{T(\theta = 0) = W(3 - 2\cos\theta_0) = 3170\,\text{lb},} \tag{6}$$

where we have used $W = 2500$ lb and $\theta_0 = 30°$ to obtain the final numerical result.

Discussion & Verification The final result in Eq. (6) is dimensionally correct, and the magnitude of the tension seems reasonable. Interestingly, the tension does not depend on the length of the supporting cable. That is, if the initial angle is 30° and the wrecking ball is released from rest, the tension in the cable will always be 3170 lb, regardless of the length of the suspending cable.

Examples

Consistent Problem-Solving Methodology

Every problem in the text employs a carefully defined problem-solving methodology to encourage systematic problem formulation, while reinforcing the steps needed to arrive at correct and realistic solutions.

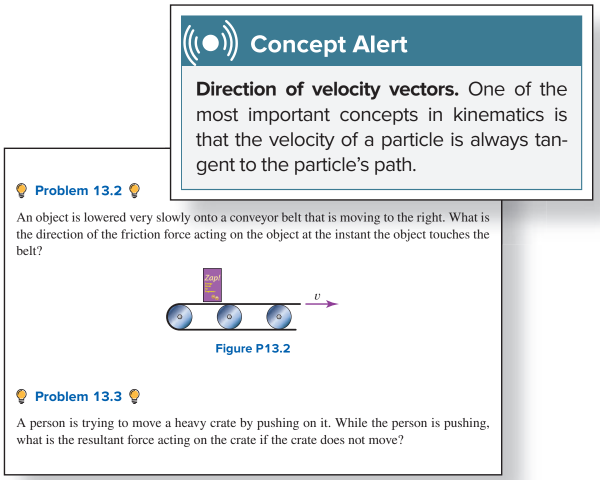
Each example problem contains these four steps:

- **Road Map & Modeling**
- **Governing Equations**
- **Computation**
- **Discussion & Verification**

Some examples include a Closer Look (noted with a magnifying glass icon) that offers additional insight into the example.

Concept Alerts and Concept Problems

Two additional features are the Concept Alerts and the Concept Problems. These have been included because research has shown (and it has been our experience) that even though you may do quite well in a science or engineering course, your conceptual understanding may be lacking. **Concept Alerts** are marginal notes and are used to drive home important concepts (or help dispel misconceptions) that are related to the material being developed at that point in the text. **Concept Problems** are mixed in with the problems that appear at the end of each section. These are questions designed to get you thinking about the application of a concept or idea presented within that section. They should never require calculation and should require answers of no more than a few sentences.



Concept Alert

Direction of velocity vectors. One of the most important concepts in kinematics is that the velocity of a particle is always tangent to the particle's path.

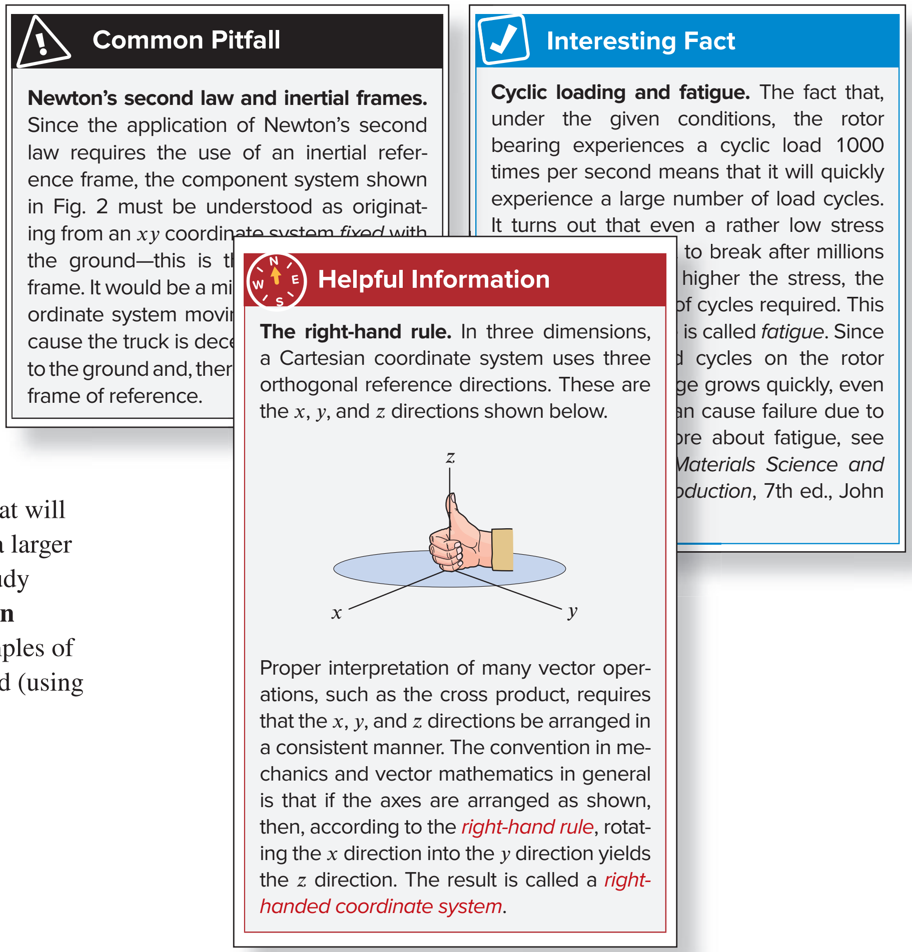
Problem 13.2

An object is lowered very slowly onto a conveyor belt that is moving to the right. What is the direction of the friction force acting on the object at the instant the object touches the belt?

Figure P13.2

Problem 13.3

A person is trying to move a heavy crate by pushing on it. While the person is pushing, what is the resultant force acting on the crate if the crate does not move?



Common Pitfall

Newton's second law and inertial frames. Since the application of Newton's second law requires the use of an inertial reference frame, the component system shown in Fig. 2 must be understood as originating from an xy coordinate system *fixed* with the ground—this is t frame. It would be a mi ordinate system movi cause the truck is dece to the ground and, ther frame of reference.

Interesting Fact

Cyclic loading and fatigue. The fact that, under the given conditions, the rotor bearing experiences a cyclic load 1000 times per second means that it will quickly experience a large number of load cycles. It turns out that even a rather low stress to break after millions higher the stress, the of cycles required. This is called *fatigue*. Since d cycles on the rotor ge grows quickly, even an cause failure due to ore about fatigue, see *Materials Science and oduction*, 7th ed., John

Helpful Information

The right-hand rule. In three dimensions, a Cartesian coordinate system uses three orthogonal reference directions. These are the x, y, and z directions shown below.

Proper interpretation of many vector operations, such as the cross product, requires that the x, y, and z directions be arranged in a consistent manner. The convention in mechanics and vector mathematics in general is that if the axes are arranged as shown, then, according to the *right-hand rule*, rotating the x direction into the y direction yields the z direction. The result is called a *right-handed coordinate system*.

Marginal Notes

Marginal notes have been implemented that will help place topics, ideas, and examples in a larger context. This feature will help students study (using **Helpful Information** and **Common Pitfalls**) and will provide real-world examples of how different aspects of dynamics are used (using **Interesting Facts**).

Sections and End of Section Summary

Each chapter is organized into several sections. There is a wealth of information and features within each section, including examples, problems, marginal notes, and other pedagogical aids. Each section concludes with an end of section summary that succinctly summarizes that section. In many cases, cross-referenced important equations are presented again for review and reinforcement before the student proceeds to the examples and homework problems.

19.2 Undamped Forced Vibration

Many systems are *forced* to vibrate by an external excitation. This section is devoted to the forced vibration of mechanical systems.

Standard form of the forced harmonic oscillator

A standard forced harmonic oscillator is shown in Fig. 19.11, in which the block of mass m is attached to a fixed support by a linear spring of constant k and is also being driven by the time-dependent force $P(t) = F_0 \sin\omega_0 t$. Modeling the block as a particle, its FBD is as shown in Fig. 19.12, where F_s is the spring force acting on the block. Summing forces in the x direction, we obtain

$$\sum F_x: \quad P(t) - F_s = ma_x, \tag{19.29}$$

where the force law is given by $F_s = kx$ and the kinematic equation is $a_x = \ddot{x}$. Substituting these relations, as well as $P(t)$ into Eq. (19.29), we obtain

$$F_0 \sin\omega_0 t - kx = m\ddot{x} \quad \Rightarrow \quad \ddot{x} + \frac{k}{m}x = \frac{F_0}{m}\sin\omega_0 t. \tag{19.30}$$

Noting that $\omega_n^2 = k/m$, this last equation becomes

$$\ddot{x} + \omega_n^2 x = \frac{F_0}{m}\sin\omega_0 t, \tag{19.31}$$

which is the *standard form of the forced harmonic oscillator equation*. It is a *nonhomogeneous* version of Eq. (19.12) on p. 1291 as a result of the term $(F_0/m)\sin\omega_0 t$. The term on the right-hand side of Eq. (19.31) is a function of *only* the independent variable t. It is often called a *forcing function* because it forces the system to vibrate. This particular type of forcing is harmonic because it is a harmonic function of time.

The theory of differential equations tells us that the *general solution* of Eq. (19.31) is the sum of the *complementary solution* $x_c(t)$ and a *particular solution* $x_p(t)$. The *complementary solution** is the solution of the associated homogeneous equation

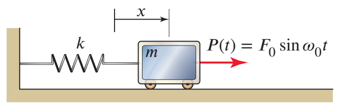


Figure 19.11
A forced harmonic oscillator whose equation of motion is given by Eq. (19.31) with $\omega_n = \sqrt{k/m}$. The position x is measured from the equilibrium position of the system when $F_0 = 0$.

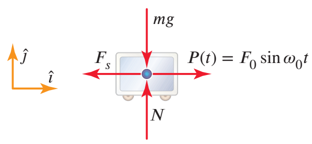


Figure 19.12
FBD of the forced harmonic oscillator in Fig. 19.11.

Interesting Fact

How practical is harmonic forcing? The answer to this question lies in an amazing result due to Jean Baptiste Joseph Fourier ... contributors. It ...iecewise smooth ...ted by an infinite ...es (called *Fourier* ...ier). *This means* ...*can be regarded* ...*functions!* In ad... ...f the left side of ...), it turns out that ...tion for a sum of ...simply the sum of ...r each individual ...ese results taken ... to easily obtain ...with *any* periodic ... forced harmonic ...se periodic forc... ...neering systems, ...portant results in

End of Section Summary

When a harmonic oscillator is subject to harmonic forcing, the standard form of the equation of motion is

Eq. (19.31), p. 1307

$$\ddot{x} + \omega_n^2 x = \frac{F_0}{m}\sin\omega_0 t,$$

where F_0 is the amplitude of the forcing and ω_0 is its frequency (see Fig. 19.16). The general solution to this equation consists of the sum of the complementary solution and a particular solution. The *complementary solution* x_c is the solution of the associated homogeneous equation, which is given by, for example, Eq. (19.13). For $\omega_0 \neq \omega_n$, a particular solution was found to be

Eq. (19.35), p. 1308

$$x_p = \frac{F_0/k}{1 - (\omega_0/\omega_n)^2}\sin\omega_0 t,$$

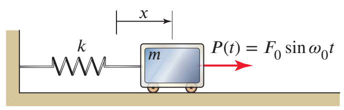


Figure 19.16
A forced harmonic oscillator whose equation of motion is given by Eq. (19.31) with $\omega_n = \sqrt{k/m}$. The position x is measured from the equilibrium position of the block.

Problems

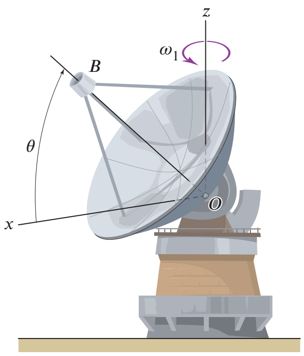


Figure P20.1 and P20.2

Problems 20.1 and 20.2

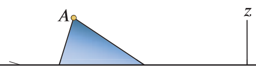
The radar dish can rotate about the vertical z axis at rate ω_1 and about the horizontal y axis (not shown in the figure) at rate $\dot{\theta}$. The distance between the center of rotation at O and the subreflector at B is ℓ.

Problem 20.1 If ω_1 and $\dot{\theta}$ are both constant, determine the velocity and acceleration of the subreflector B in terms of the elevation angle θ.

Problem 20.2 If $\omega_1(t)$ and $\dot{\theta}(t)$ are known functions of time, determine the velocity and acceleration of the subreflector B in terms of the elevation angle θ.

Problems 20.3 and 20.4

The truncated cone rolls without slipping on the xy plane. At the instant shown, the angular speed about the z axis is ω_1, and it is changing at $\dot{\omega}_1$.



Problems 12.178 and 12.179

A micro spiral pump* consists of a spiral channel attached to a stationary plate. This plate has two ports, one for fluid inlet and another for outlet, the outlet being farther from the center of the plate than the inlet. The system is capped by a rotating disk. The fluid trapped between the rotating disk and the stationary plate is put in motion by the rotation of the top disk, which pulls the fluid through the spiral channel.

Problem 12.178 Consider a spiral channel with the geometry given by the equation $r = \eta\theta + r_0$, where $r_0 = 146\,\mu$m is the starting radius, r is the distance from the spin axis, and θ, measured in radians, is the angular position of a point in the spiral channel. Assume that the radius at the outlet is $r_{out} = 190\,\mu$m, that the top disk rotates with a constant angular speed ω, and that the fluid particles in contact with the rotating disk are essentially stuck to it. Determine the constant η and the value of ω (in rpm) such that after 1.25 rev of the top disk, the speed of the particles in contact with this disk is $v = 0.5$ m/s at the outlet.

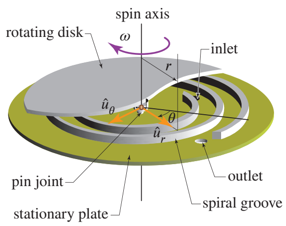


Figure P12.178 and P12.179

Modern Problems

Problems of varying difficulty follow each section. These problems allow students to develop their ability to apply concepts of dynamics on their own. The most common question asked by students is "How do I set this problem up?" What is really meant by this question is "How do I develop a good mathematical model for this problem?" The only way to develop this ability is by practicing numerous problems.

Answers to most even-numbered problems appear in Appendix B. Providing answers in this manner allows for more complex information than would otherwise be possible. Each problem in the book is accompanied by a thermometer icon that indicates the approximate level of difficulty. Those considered to be "introductory" are indicated with the symbol . Problems considered to be "representative" are indicated with the symbol , and problems that are considered to be "challenging" are indicated with the symbol .

Engineering Design and Design Problems

Several design problems are presented where appropriate throughout the book. These problems can be tackled with the knowledge and skill set that are typical of introductory-level courses, although the use of mathematical software is strongly recommended. These problems are open ended, and their solution requires the definition of a parameter space in which the dynamics of the system must be analyzed. In dynamics we have chosen to emphasize the role played by *parametric analyses* in the overall design process, as opposed to cost-benefit analyses or the choice of specific materials and/or components.

Design Problems

Design Problem 17.1

Revisit the calculations done at the beginning of the chapter concerning the determination of the maximum acceleration that can be achieved by a motorcycle without causing the front wheel to lift off the ground. Specifically, construct a new model of the motorcycle by selecting a real-life motorcycle and researching its geometry and inertia properties, including the inertia properties of the wheels. Then analyze your model to determine how the maximum acceleration in question depends on the horizontal and vertical positions of the center of mass with respect to the points of contact between the ground and the wheels. Include in your analysis a comparison of results that account for the inertia of the front wheel with results that neglect the inertia of the front wheel.

Guitar Studio/Shutterstock

Figure DP17.1

What Resources Support This Textbook?

Proctorio

Remote Proctoring and Browser-Locking Capabilities

Remote proctoring and browser-locking capabilities, hosted by Proctorio within Connect, provide control of the assessment environment by enabling security options and verifying the identity of the student.

Seamlessly integrated within Connect, these services allow instructors to control students' assessment experience by restricting browser activity, recording students' activity, and verifying that students are doing their own work.

Instant and detailed reporting gives instructors an at-a-glance view of potential academic integrity concerns, thereby avoiding personal bias and supporting evidence-based claims.

ReadAnywhere. Read or study when it's convenient for you with McGraw Hill's free ReadAnywhere app. Available for iOS or Android smartphones or tablets, ReadAnywhere gives users access to McGraw Hill tools, including the eBook and SmartBook 2.0 or Adaptive Learning Assignments in Connect. Take notes, highlight, and complete assignments offline—all of your work will sync when you open the app with WiFi access. Log in with your McGraw Hill Connect username and password to start learning—anytime, anywhere!

OLC-Aligned Courses

Implementing High-Quality Online Instruction and Assessment through Preconfigured Courseware. In consultation with the Online Learning Consortium (OLC) and our certified Faculty Consultants, McGraw Hill has created preconfigured courseware using OLC's quality scorecard to align with best practices in online course delivery. This turnkey courseware contains a combination of formative assessments, summative assessments, homework, and application activities, and can easily be customized to meet an individual's needs and course outcomes. For more information, visit https://www.mheducation.com/highered/olc.

Tegrity: Lectures 24/7 Tegrity in Connect is a tool that makes class time available 24/7 by automatically capturing every lecture. With a simple one-click start-and-stop process, you capture all computer screens and corresponding audio in a format that is easy to search, frame by frame. Students can replay any part of any class with easy-to-use, browser-based viewing on a PC, Mac, or any mobile device.

Educators know that the more students can see, hear, and experience class resources, the better they learn. In fact, studies prove it. Tegrity's unique search feature helps students efficiently find what they need, when they need it, across an entire semester of class recordings. Help turn your students' study time into learning moments immediately supported by your lecture. With Tegrity, you also increase intent listening and class participation by easing students' concerns about note-taking. Using Tegrity in Connect will make it more likely you will see students' faces, not the tops of their heads.

Test Builder in Connect. Available within Connect, Test Builder is a cloud-based tool that enables instructors to format tests that can be printed, administered within a Learning Management System, or exported as a Word document of the test bank. Test Builder offers a modern, streamlined interface for easy content configuration that matches course needs, without requiring a download.

Test Builder allows you to:

- Access all test bank content from a particular title.
- Easily pinpoint the most relevant content through robust filtering options.
- Manipulate the order of questions or scramble questions and/or answers.
- Pin questions to a specific location within a test.
- Determine your preferred treatment of algorithmic questions.

- Choose the layout and spacing.
- Add instructions and configure default settings.

Test Builder provides a secure interface for better protection of content and allows for just-in-time updates to flow directly into assessments.

Writing Assignment. Available within Connect and Connect Master, the Writing Assignment tool delivers a learning experience to help students improve their written communication skills and conceptual understanding. As an instructor you can assign, monitor, grade, and provide feedback on writing more efficiently and effectively.

Application-Based Activities in Connect. Application-Based Activities in Connect are highly interactive, assignable exercises that provide students with a safe space to apply the concepts they have learned to real-world, course-specific problems. Each Application-Based Activity involves the application of multiple concepts, allowing students to synthesize information and use critical thinking skills to solve realistic scenarios.

Free-Body Diagram Tool. The Free-Body Diagram Tool allows students to draw free-body diagrams that are auto-graded by the system. Students receive immediate feedback on their diagrams to help students solidify their understanding of the physical situation presented in the problem.

Create

Your Book, Your Way. McGraw Hill's Content Collections Powered by Create® is a self-service website that enables instructors to create custom course materials—print and eBooks—by drawing upon McGraw Hill's comprehensive, cross-disciplinary content. Choose what you want from our high-quality textbooks, articles, and cases. Combine it with your own content quickly and easily, and tap into other rights-secured, third-party content such as readings, cases, and articles. Content can be arranged in a way that makes the most sense for your course, and you can include the course name and information as well. Choose the best format for your course: color print, black-and-white print, or eBook. The eBook can be included in your Connect course and is available on the free ReadAnywhere app for smartphone or tablet access as well. When you are finished customizing, you will receive a free digital copy to review in just minutes! Visit McGraw Hill Create®—www.mcgrawhillcreate.com—today and begin building!

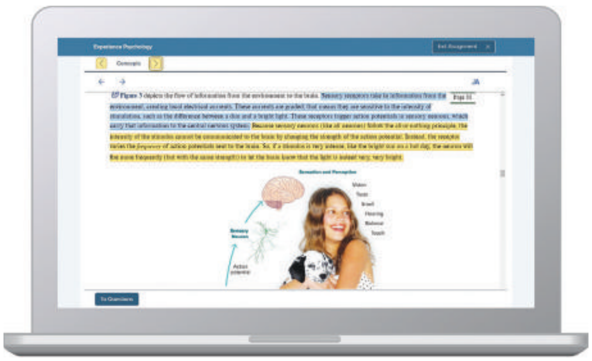
McGraw Hill connect®

Instructors: Student Success Starts with You

Tools to enhance your unique voice

Want to build your own course? No problem. Prefer to use an OLC-aligned, prebuilt course? Easy. Want to make changes throughout the semester? Sure. And you'll save time with Connect's auto-grading too.

65%
Less Time Grading



Laptop: McGraw Hill; Woman/dog: George Doyle/Getty Images

Study made personal

Incorporate adaptive study resources like SmartBook® 2.0 into your course and help your students be better prepared in less time. Learn more about the powerful personalized learning experience available in SmartBook 2.0 at **www.mheducation.com/highered/connect/smartbook**

Affordable solutions, added value

Make technology work for you with LMS integration for single sign-on access, mobile access to the digital textbook, and reports to quickly show you how each of your students is doing. And with our Inclusive Access program you can provide all these tools at a discount to your students. Ask your McGraw Hill representative for more information.

Padlock: Jobalou/Getty Images

Solutions for your challenges

A product isn't a solution. Real solutions are affordable, reliable, and come with training and ongoing support when you need it and how you want it. Visit **www.supportateverystep.com** for videos and resources both you and your students can use throughout the semester.

Checkmark: Jobalou/Getty Images

SUPPORT AT every step

Students: Get Learning that Fits You

Effective tools for efficient studying

Connect is designed to help you be more productive with simple, flexible, intuitive tools that maximize your study time and meet your individual learning needs. Get learning that works for you with Connect.

Study anytime, anywhere

Download the free ReadAnywhere app and access your online eBook, SmartBook 2.0, or Adaptive Learning Assignments when it's convenient, even if you're offline. And since the app automatically syncs with your Connect account, all of your work is available every time you open it. Find out more at **www.mheducation.com/readanywhere**

> ***"I really liked this app—it made it easy to study when you don't have your text-book in front of you."***
>
> - Jordan Cunningham, Eastern Washington University

Calendar: owattaphotos/Getty Images

Everything you need in one place

Your Connect course has everything you need—whether reading on your digital eBook or completing assignments for class, Connect makes it easy to get your work done.

Learning for everyone

McGraw Hill works directly with Accessibility Services Departments and faculty to meet the learning needs of all students. Please contact your Accessibility Services Office and ask them to email accessibility@mheducation.com, or visit **www.mheducation.com/about/accessibility** for more information.

Top: Jenner Images/Getty Images, Left: Hero Images/Getty Images, Right: Hero Images/Getty Images

Introduction to Dynamics

11

In Section 11.1, we introduce Isaac Newton's (1643–1727) laws of motion and his universal law of gravitation. In Section 11.2, we review those elements of physics and vector algebra needed to develop the material in the remainder of the book. In Section 11.3, we touch upon the role of dynamics in engineering design.

MShieldsPhotos/Alamy Stock Photo

Raphael's *School of Athens* depicts ancient Greek philosophers, such as Aristotle, Plato, Euclid, and Pythagoras. This fresco celebrates the kinship that the renaissance humanists felt with the great minds from antiquity as they explored new ways of thinking about the arts, sciences, and engineering.

11.1 The Newtonian Equations

The *dynamics* we study in this book is the part of mechanics concerned with the motion of bodies, the forces causing their motion, and/or the forces caused by their motion. Dynamics builds upon statics in that the ability to draw free body diagrams and to write the corresponding balance equations for particles and rigid bodies are fundamental to dynamics. Dynamics also complements mechanics of materials in that it develops your ability to find forces due to the acceleration of objects, forces that can be used to find stresses using mechanics of materials.

Since the middle of the 20th century, dynamics has also included the study and analysis of any time-varying process, be it mechanical, electrical, chemical, or biological. While we focus on mechanical processes, much of what we study is also applicable to other time-varying phenomena. Our goal is to provide an introduction to the science, skill, and art involved in modeling mechanical systems to predict their motion.

Newton's laws of motion

Newton's three laws of motion are

First Law. A particle remains at rest, or moves in a straight line with a constant speed, as long as the total force acting on the particle is zero.

Second Law. The time rate of change of momentum of a particle is equal to the resultant force acting on that particle.

Third Law. The forces of action and reaction between interacting particles are equal in magnitude, opposite in direction, and collinear.

The second law, stated mathematically, is

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt}, \tag{11.1}$$

where $\vec{F}$ is the net force acting on the particle, $\vec{p}$ is the momentum of the particle, m is the mass of the particle, and $\vec{v}$ is the velocity of the particle. We have used the definition of momentum, which is $\vec{p} = m\vec{v}$. Throughout this book, we will denote vectors by using either a superposed arrow ($\vec{}$) or a superposed caret or hat ($\hat{}$) if the vector is a *unit vector*. Newton's second law is often written as

$$\vec{F} = m\vec{a}, \tag{11.2}$$

which explicitly accounts for the fact that a particle is generally understood to have constant mass.* We will learn in Chapter 13 that the first law is simply a special case of the second. The second and third laws, along with the ideas developed by Leonhard Euler (1707–1783) for rigid body dynamics, are all that is needed to solve a broad spectrum of problems involving particles and rigid bodies.

The third law, stated mathematically, is

$$\vec{F}_{ij} = -\vec{F}_{ji}, \tag{11.3}$$

$$\vec{F}_{ij} \times (\vec{r}_i - \vec{r}_j) = \vec{0}, \tag{11.4}$$

where, for any interacting particles i and j, $\vec{F}_{ij}$ is the force on particle i due to particle j and $\vec{r}_i$ is the position of the ith particle (Fig. 11.1). Some people refer to Newton's third law as just Eq. (11.3), while others require both Eqs. (11.3) and (11.4). Requiring both equations is sometimes referred to as the *strong form of Newton's third law*.

Interesting Fact

Newton's third law in modern mechanics. Modern mechanics generally discards Newton's third law and replaces it with a much more general result based on the concept of *angular momentum*. Since the 1950s, it has been proposed that an even more general notion called *the principle of material frame indifference* could be used to replace Newton's third law. This latter principle states that the properties of materials and the actions of bodies on one another are the same for all observers.

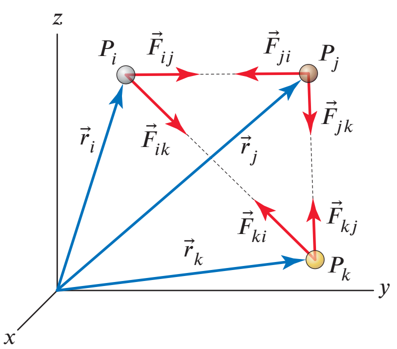


Figure 11.1
A system of particles interacting with one another.

Newton's universal law of gravitation

Newton used his laws of dynamics along with the laws postulated by Johannes Kepler (1571–1630) to deduce the *universal law of gravitation*, which describes the

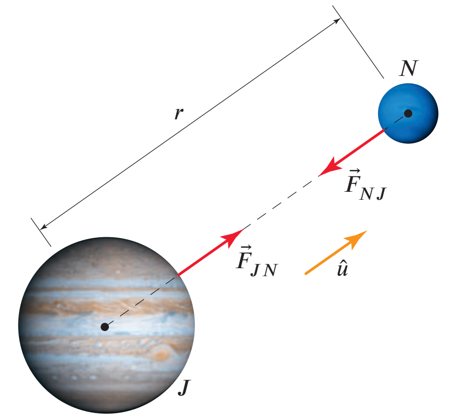
---

* The application of Eq. (11.1) to variable mass systems will be considered in Section 15.5.

force of attraction between two bodies. The gravitational force on a mass m_1 due to a mass m_2 a distance r away from m_1 is

$$\vec{F}_{12} = \frac{Gm_1 m_2}{r^2}\,\hat{u}, \tag{11.5}$$

where $\hat{u}$ is a unit vector pointing from m_1 to m_2 and G is the *universal gravitational constant** (sometimes called the *constant of gravitation* or *constant of universal gravitation*). The following example demonstrates the application of this law.



(left) JPL/University of Arizona/NASA; (right) NASA/JPL

Figure 11.2
The gravitational force between the planets Jupiter J and Neptune N. The relative sizes of the planets are accurate, but their separation distance is not.

Mini-Example

Using the planets Jupiter and Neptune as an example, the force on Jupiter due to the gravitational attraction of Neptune, $\vec{F}_{JN}$, is given by (see Fig. 11.2)

$$\vec{F}_{JN} = \frac{Gm_J m_N}{r^2}\,\hat{u}, \tag{11.6}$$

where r is the distance between the two bodies, m_J is the mass of Jupiter, m_N is the mass of Neptune, and $\hat{u}$ is a unit vector pointing from the center of Jupiter to the center of Neptune. The mass of Jupiter is 1.9×10^{27} kg, and that of Neptune is 1.02×10^{26} kg. Since the mean radius of Jupiter's orbit is 778,300,000 km and that of Neptune is 4,505,000,000 km, we assume that their closest approach to one another is approximately 3,727,000,000 km. Thus, at their closest approach, the magnitude of the force between these two planets is

$$|\vec{F}_{JN}| = \left(6.674\times10^{-11}\,\frac{\text{m}^3}{\text{kg}\cdot\text{s}^2}\right)\frac{(1.9\times10^{27}\,\text{kg})(1.02\times10^{26}\,\text{kg})}{(3.727\times10^{12}\,\text{m})^2} \tag{11.7}$$
$$= 9.312\times10^{17}\,\text{N}.$$

We can compare this force with the force of gravitation between Jupiter and the Sun. The Sun's mass is 1.989×10^{30} kg, and we have already stated that the mean radius of Jupiter's orbit is 778,300,000 km. Applying Eq. (11.5) between Jupiter and the Sun gives 4.164×10^{23} N, which is almost 450,000 times larger.

Acceleration due to gravity. Equation (11.5) allows us to determine the force of Earth's gravity on an object of mass m on the surface of the Earth. This is done by noting that the radius of the Earth is 6371.0 km (see the marginal note) and the mass of the Earth is 5.9736×10^{24} kg and then applying Eq. (11.5):

$$F_s = \left(6.674\times10^{-11}\,\frac{\text{m}^3}{\text{kg}\cdot\text{s}^2}\right)\frac{(5.9736\times10^{24}\,\text{kg})\,m}{(6371.0\times10^3\,\text{m})^2} \tag{11.8}$$
$$= (9.8222\,\text{m/s}^2)\,m.$$

This result† tells us that the force of gravity (in N) on an object on the Earth's surface is about 9.8 times the object's mass (in kg). This factor of 9.8 is so prevalent in engineering that it is given the label g, and it is called the *acceleration due to*

Interesting Fact

The radius of the Earth. The Earth is not a perfect sphere. Therefore, there are different notions of "radius of the Earth." The given value of 6371.0 km is the *volumetric radius* when rounded to 5 significant digits. The Earth's volumetric radius is the radius of a perfect sphere with volume equal to that of the Earth. Other measures of the Earth's radius, rounded to 5 significant digits, are the *quadratic mean radius*, the *authalic mean radius*, and the *meridional Earth radius*, which are equal to 6372.8, 6371.0, and 6367.4 km, respectively.

* Henry Cavendish (1731–1810) was the first to measure G and did so in 1798. The generally accepted value is $G = 6.674\times10^{-11}\,\text{m}^3/(\text{kg}\cdot\text{s}^2) = 3.439\times10^{-8}\,\text{ft}^3/(\text{slug}\cdot\text{s}^2)$.

† We will normally round the result of all calculations to 4 significant digits. Here we are using 5 significant figures because the data used in this particular calculation is known to that degree of accuracy.

gravity because it has units of acceleration and its value is the acceleration of objects in free fall near the surface of the Earth. We will take the value of g to be 9.81 m/s^2 in SI units and 32.2 ft/s^2 in U.S. Customary units. Notice that the value of g obtained in Eq. (11.8) is slightly greater than the 9.81 m/s^2 that we will use in this book. The difference between these values has several causes, including that the Earth is not perfectly spherical, does not have uniform mass distribution, and is rotating. Because of these factors, the actual acceleration due to gravity is about 0.27% lower at the equator, and 0.26% higher at the poles, relative to the standard value of $g = 9.81\,\text{m/s}^2$, which is for a north or south latitude of 45° at sea level. There may also be small local variations in gravity due to geological formations. Nonetheless, throughout this book we will use the standard value of g stated above.

Change in acceleration due to altitude. There is a formula that allows us to find how the acceleration due to gravity changes with altitude. To find it, we begin by equating Eqs. (11.2) and (11.5) to determine the acceleration a at a height h above the surface of the Earth

$$a = \frac{Gm_e}{(r_e + h)^2}, \tag{11.9}$$

where r_e is the radius of the Earth, m_e is the mass of the Earth, and we have canceled the mass of the object on both sides of the equation. Now, at the surface of the Earth, we know that $a = g$ and $h = 0$, so Eq. (11.9) becomes

$$g = Gm_e/r_e^2 \quad \Rightarrow \quad Gm_e = gr_e^2. \tag{11.10}$$

Substituting Eq. (11.10) into Eq. (11.9), we see that a is given by

$$\boxed{a = g\frac{r_e^2}{(r_e + h)^2},} \tag{11.11}$$

where g is the acceleration due to gravity at the surface of the Earth. Equation (11.11) is very handy because it requires knowledge of only the radius of the Earth to get the acceleration due to gravity rather than having to know both the radius of the Earth *and* the universal gravitational constant G.

Building on statics to develop mastery in dynamics

Students undertaking dynamics have often just finished a prior course in statics. It is useful to point out the similarities and differences between the two subjects before outlining the content in dynamics. The approach to solving a large subset of statics problems can be summarized as follows:

1. Draw one or more Free Body Diagrams (FBDs) representing the loads and reactions acting on rigid bodies.
2. Write the equations of static equilibrium informed by those FBDs.
3. Solve the resulting equations for reactions or internal loads.

Unless we examine problems involving distributed loads, mathematical proficiency required for such problems is restricted to geometry, trigonometry, and algebra.

FBDs are as important in dynamics as they are in statics, but we will use more than Cartesian components to solve problems. Our governing equation, as embodied in Eq. (11.1), now contains the time rate of change of the linear momentum, so force imbalance implies motion.

Because velocity is time rate of change of position, and acceleration is time rate of change in velocity, *we will find ourselves in need of calculus from the outset of dynamics*. An additional complication arises from our differentiating a vector with respect to time. Because the velocity vector has a direction as well as a magnitude, $\vec{v} = v\,\hat{u}_v$, where $v = |\vec{v}|$ and $\hat{u}_v$ is a unit vector in the direction of $\vec{v}$,* the acceleration in Eq. (11.2) generally requires the product rule from calculus:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv}{dt}\,\hat{u}_v + v\frac{d\hat{u}_v}{dt}. \tag{11.12}$$

This tells us that acceleration can occur not just from the familiar change in speed, as embodied in the first term, but from change in direction, as embodied in $d\hat{u}_v/dt$. As an example, a satellite in a circular orbit about the Earth has a constant speed but is always accelerating because the direction of the velocity unit vector is constantly changing. There is no contradiction between a satellite at altitude h above the surface of the Earth having the acceleration as described by Eq. (11.11) and stating that the satellite moves at constant speed. The ability to account for changing directions of unit vectors in different component systems is of the utmost importance in describing the motion of particles.

In statics, we generally begin from simpler 2D and 3D particle equilibrium problems before moving on to 2D and 3D equilibrium of individual rigid bodies and then, finally, 2D and 3D equilibrium of assemblies of rigid bodies in the form of trusses, frames, and machines. Once we transition from particles to rigid bodies, we need to impose rotational as well as translational equilibrium, and the former requires a rotational constraint in the form that the sum of moments about a point, on or off the rigid body, must equal zero.

In dynamics, we likewise begin with an examination of particle behavior. Because particle motion is an entirely new subject for us, we devote an entire chapter (Chapter 12) to descriptions of motion in different component systems. One of the challenges is selecting the appropriate component system for each problem. In some problems, the forces are naturally described in one component system, while the motion is most appropriately described in a different component system. This requires us to map either the forces or the motion into the other component system before writing our equations of motion.

Once we understand how to represent particle motion in different component systems, we turn to the examination of force imbalance as embodied in Eq. (11.2). In Chapters 13, 14, and 15 we provide different methods for examining force imbalance on particles. In statics, we found many ways to draw FBDs and solve for internal loads and reactions, but some processes were more efficient than others. In dynamics, we might begin with Eq. (11.2) and follow the evolution of a particle's motion from some initial state to a final state. This is an approach we use in Chapter 13, and this can be characterized as a *path-dependent* approach to particle dynamics. In contrast, we will see that some problems are amenable to a *path-independent* approach, where the final state can be inferred from information about the initial state, and without having to follow all the details from initial to final state. This is the subject of Chapters 14 and 15. In Chapter 14, we examine transitions over distance, while in Chapter 15, we examine transitions through time. All three chapters begin from the

* We will cover this in detail in Section 12.4.

same fundamental equation, Eq. (11.2), and the choice of problem-solving method is often telegraphed by the form of the problem statement.

Chapters 16, 17, and 18 follow a similar development for the dynamics of rigid bodies. Chapter 16 is devoted to a description of rigid body motion, and is to rigid bodies what Chapter 12 is to particles. To keep problems relatively simple, we restrict our attention in these chapters to planar motion, where all rotational motion is in the out-of-plane direction. For rigid bodies in general plane motion, we will see that every material point in the body has, in general, a different velocity and acceleration at any one instant in time. We will show that the treatment of a rigid body as an assembly of particles leads to a translational dynamics equation similar to Eq. (11.2), but with the acceleration on the right-hand side being equal to that of the body's center of mass. We will also be able to formulate an equation describing the rotational dynamics of the body. Unlike statics, the rotational dynamics equation is not equally simple at all points. Rotational dynamics is most simply expressed when taking moments about the center of mass. Chapter 17 is to rigid bodies as Chapter 13 is to particles. Chapter 18 articulates path-independent processes for planar rigid body dynamics, and is to rigid bodies as Chapters 14 and 15 are to particles.

Chapter 19 is devoted to an introduction to mechanical vibrations, where we examine the interplay of inertia and compliance. Vibrations is an important subject all its own, providing a foundation for structural dynamics. The subject matter examined in this chapter is confined to single degree-of-freedom problems, where the vibration of a single translational or rotational degree-of-freedom is of interest.

Finally, Chapter 20 relaxes the constraint on planar motion and presents 3D dynamics of rigid bodies. Rotational inertia must now be treated as a full tensor, as rotation about any axis is possible. To the uninitiated, it may seem as though the transition from 2D to 3D rigid body motion is relatively straightforward. But there is an order of magnitude increase in complexity associated with this transition, and a quick skim of the equations in Chapter 20 should disabuse the reader of the notion that this is a simple transition.

For now, we devote a bit of space to discussing some fundamental concepts in dynamics as well as reviewing important vector mechanics operations. The ability to resolve a vector into orthogonal components in any coordinate system is especially useful when we go back and forth between different coordinate systems in Chapter 12.

11.2 Fundamental Concepts in Dynamics

Space and time

Space

Space is the environment in which objects move, and we consider it to be a collection of locations or *points*. The position of a point is indicated by the point's coordinates in a chosen coordinate system. Figure 11.3 shows a three-dimensional *Cartesian coordinate system* with *origin* at O and mutually orthogonal axes x, y, and z. The *Cartesian coordinates* of the point P are x_P, y_P, and z_P, which are scalars obtained by measuring the distance between O and the perpendicular projections of the point P onto the axes x, y, and z, respectively. Note that x_P, y_P, and z_P have a positive or negative sign depending on whether, in going from O to the projections of P along each axis, one moves in the positive or negative direction of these axes. In Chapter 12, we will introduce additional coordinate systems.

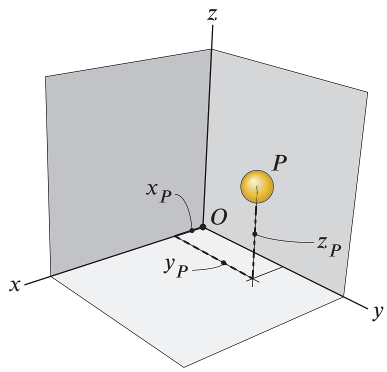


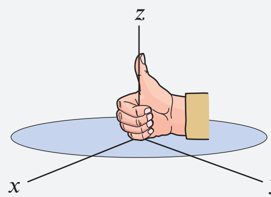
Figure 11.3
A point in a three-dimensional space.

Time

Time is a scalar variable that allows us to specify the order of a sequence of events. In classical mechanics and in this book, the most important assumption about time is that it is *absolute*. We assume that the duration of an event is independent of the motion of the observer making time measurements. Einstein's theory of relativity rejects this assumption.

Helpful Information

The right-hand rule. In three dimensions, a Cartesian coordinate system uses three orthogonal reference directions. These are the x, y, and z directions shown below.



Proper interpretation of many vector operations, such as the cross product, requires that the x, y, and z directions be arranged in a consistent manner. The convention in mechanics and vector mathematics in general is that if the axes are arranged as shown, then, according to the *right-hand rule*, rotating the x direction into the y direction yields the z direction. The result is called a *right-handed coordinate system*.

Force, mass, and inertia

Force

The *force* acting on an object is the interaction between that object and its environment. A more precise description of this interaction requires that we know something about the interaction in question. For example, if two objects collide or slide against one another, we say that they interact via *contact* forces. Regardless of the type, the characteristics of a force are its magnitude, its line of action, and its orientation or direction. This is why we use *vectors* to represent forces.

Mass

The *mass* of an object is a measure of the amount of matter in the object. Along with the concept of force, the concept of mass is considered a *primitive concept*—that is, not explainable via more elementary ideas. Newton's second law postulates that the force acting on a body is *proportional* to the body's acceleration—the constant of proportionality is the *mass* of the body.

Inertia

Inertia is commonly understood as a body's resistance to changing its state of motion in response to the application of a force system. In this book, we use *inertia* as an umbrella term encompassing both the idea of mass and that of mass distribution over a region of space. The *inertia properties* of an object are its mass and a quantitative description of the mass distribution.

Particle and rigid body

Particle

A *particle* is an object whose mass is concentrated at a point; therefore, it is also called a point mass. The inertia properties of a particle consist only of its mass. A particle is generally understood to have zero volume. It is meaningless to talk about the rotation of a particle whose position is held fixed, although we do say that a particle can "rotate about a point," meaning that a particle can move along a path around a point. Regardless of its volume, when we choose to model an object as a particle, we neglect the possibility that the object might rotate in the sense of "change its orientation" relative to some chosen reference.

Rigid body

A *rigid body* is an object whose mass is (1) distributed over a region of space and (2) such that the distance between any two points on it never changes. Since its mass is not concentrated at a point, the *rigid body* is the simplest model for the study of motions that include the possibility of rotation—that is, a change of orientation relative to a chosen reference. We model objects as rigid bodies when we want to account for the possibility of rotation while neglecting the effects of deformation. Finally, the mass distribution of a rigid body does not change relative to an observer moving with the body. This fact makes it possible to describe the inertia properties of a three-dimensional rigid body with seven pieces of information consisting of the body's mass and six mass moments of inertia.*

Vectors and their Cartesian representation

Notation

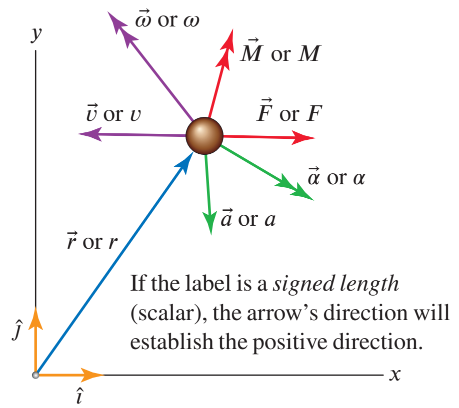


Figure 11.4
Notation and colors for commonly used vectors. Position vectors will always be blue ■ ($\vec{r}$), velocity (linear and angular) vectors purple ■ ($\vec{v}$ and $\vec{\omega}$), and acceleration (linear and angular) vectors green ■ ($\vec{a}$ and $\vec{\alpha}$). Forces and moments will always be red ■ ($\vec{F}$ and $\vec{M}$) and unit vectors orange ■ ($\hat{\imath}$ and $\hat{\jmath}$). Vectors with no particular physical significance will be black ■, magenta ■, or gray ■.

Scalars. By *scalar* we mean a *real number*. Scalars will be denoted by italic roman characters (e.g., a, h, or W) or by Greek letters (e.g., α, ω, or δ).

Vectors. We will *always* denote vectors by placing arrows or carets (in the case of unit vectors) over letters, such as $\vec{F}$ or $\hat{\imath}$. The conventions we use to depict vectors in figures are shown in Fig. 11.4. The color scheme used in the figure is defined in the caption. Depending on what we want or need to emphasize in a figure, a vector will be labeled with a letter that has an arrow placed above it (e.g., $\vec{a}$ or $\vec{\omega}$) or with just a letter (e.g., a or ω) according to the following conventions:

- In figures, a vector will be labeled with arrows over letters when it is important to emphasize the arbitrary directional nature of the vector (e.g., a velocity) or the vectorial nature of the quantity (e.g., a unit vector).
- Base vectors in Cartesian components will be designated using the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$. A *unit vector* is a vector with magnitude equal to 1. In any other context, e.g., in other component systems, unit vectors will be designated using a caret over the letter u, that is, $\hat{u}$, often accompanied by a subscript indicating the direction of the vector, such as $\hat{u}_r$.
- In figures, the label of a vector with known direction will generally not be a letter with an arrow over it. A vector with known direction will usually be

* For a definition of the mass moments of inertia of a rigid body, see Section 10.3 on p. 593 of M. E. Plesha, G. L. Gray, R. J. Witt, and F. Costanzo, *Engineering Mechanics: Statics*, McGraw-Hill, Dubuque, IA, 2023.

labeled as a *signed length* (i.e., a scalar component) whose positive direction is that of the arrow in the figure.

- Double-headed arrows will designate vectors associated with "rotational" quantities—that is, moments, angular velocities, and angular accelerations (angular velocities and accelerations will be discussed in Chapter 12).

Cartesian vector representation

We now review those aspects of vectors in two dimensions that are most important for our applications. This presentation is easily extended to three dimensions.

Figure 11.5 shows the position of a point P with respect to the origin O of a

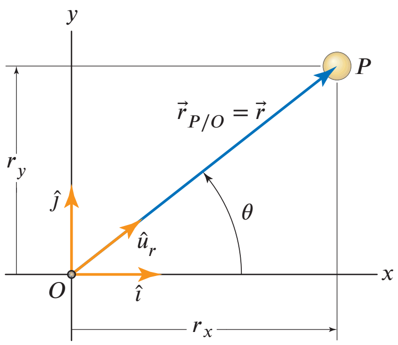


Figure 11.5. Description of the position of a point P. The curved arrows indicating an angle with a single arrowhead designate an angle's positive direction.

rectangular coordinate system. The position of P is represented by the arrow that starts at O and ends at P, which we call the vector $\vec{r}_{P/O}$. The subscript "P/O" is read "P relative to O," or "P as seen by an observer at O," or "P with respect to O." When only one point is being discussed, we typically drop the "P/O" part of the notation and simply indicate position as $\vec{r}$.

The *Cartesian representation* of $\vec{r}$ is

$$\vec{r} = r_x\,\hat{\imath} + r_y\,\hat{\jmath}, \tag{11.13}$$

where $\hat{\imath}$ and $\hat{\jmath}$ are unit vectors in the x and y directions, respectively. The quantities r_x and r_y are the *(scalar) Cartesian components* of $\vec{r}$. Using trigonometry, we have

$$r_x = |\vec{r}|\cos\theta \quad \text{and} \quad r_y = |\vec{r}|\sin\theta, \tag{11.14}$$

where θ is the orientation of the segment $\overline{OP}$ (the bar over the letters O and P designates the line segment connecting the points O and P) relative to the x axis and $|\vec{r}|$, called the *magnitude of $\vec{r}$* or *length of $\vec{r}$*, is the length of $\overline{OP}$. Equation (11.13) could be written as $\vec{r} = \vec{r}_x + \vec{r}_y$, where the vectors $\vec{r}_x = r_x\,\hat{\imath}$ and $\vec{r}_y = r_y\,\hat{\jmath}$ are called the x and y *vector* components of $\vec{r}$, respectively. In this book, *component* will always mean *scalar component*. When talking about *vector components*, we will explicitly say *vector components*.

Generalizing what we said about $\vec{r}_{P/O}$, given points A and B with coordinates (x_A, y_A) and (x_B, y_B), respectively, the vector

$$\vec{r}_{A/B} = (x_A - x_B)\,\hat{\imath} + (y_A - y_B)\,\hat{\jmath} \tag{11.15}$$

will be called *the position of A with respect to B*, or position of A relative to B (Fig. 11.6).

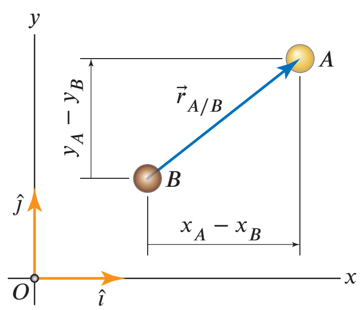


Figure 11.6
Vector representation of the position of A relative to B.

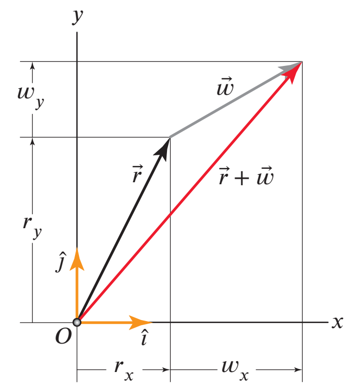


Figure 11.7
Graphical representation of the vector addition of $\vec{r}$ and $\vec{w}$ showing the "triangle law."

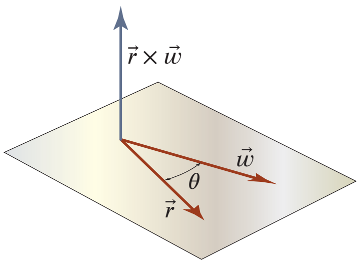


Figure 11.8
Graphical representation of the vector cross product of $\vec{r}$ and $\vec{w}$.

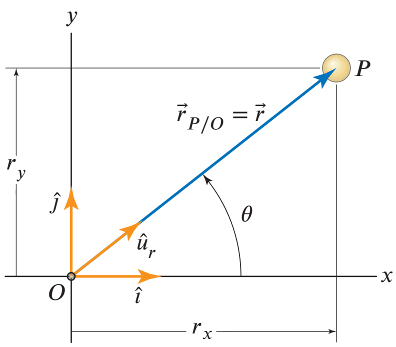


Figure 11.9
Description of the position of a particle P.

Vector operations

Here are the vector operations we will use:

1. A vector $\vec{r}$ can be multiplied by a scalar a in the following way:

$$a\vec{r} = ar_x\,\hat{\imath} + ar_y\,\hat{\jmath}. \tag{11.16}$$

   This *scales* the magnitude of $\vec{r}$ by the factor $|a|$. The object $a\vec{r}$ is a *vector* (not a scalar) with the same line of action as $\vec{r}$; the direction of $a\vec{r}$ is the same as that of $\vec{r}$ if $a > 0$, whereas it is opposite to $\vec{r}$ if $a < 0$.

2. Two vectors can be summed to obtain another vector as follows:

$$\vec{r}+\vec{w} = \left(r_x + w_x\right)\hat{\imath} + \left(r_y + w_y\right)\hat{\jmath}, \tag{11.17}$$

   which conforms to the triangle law of *vector addition* (see Fig. 11.7).

3. The operation of summing a scalar with a vector is not defined.

4. Referring to Fig. 11.8, the *dot* or *scalar product* of two vectors $\vec{r}$ and $\vec{w}$ is denoted by $\vec{r}\cdot\vec{w}$ and yields the following *scalar* quantity:

$$\vec{r}\cdot\vec{w} = |\vec{r}||\vec{w}|\cos\theta. \tag{11.18}$$

5. Referring to Fig. 11.8, the *cross product* of two vectors $\vec{r}$ and $\vec{w}$ is the *vector* denoted by $\vec{r}\times\vec{w}$ with

   (a) magnitude

$$|\vec{r}\times\vec{w}| = |\vec{r}||\vec{w}|\sin\theta, \tag{11.19}$$

   (b) line of action perpendicular to the plane containing $\vec{r}$ and $\vec{w}$, and

   (c) direction determined by the right-hand rule.

In Eqs. (11.18) and (11.19), θ is the smallest angle that will rotate one of the vectors into the other. For the cross product, this choice of θ ensures that Eq. (11.19) always yields a nonnegative value. For the dot product, since $\cos\theta = \cos(2\pi - \theta)$, θ can be replaced by $2\pi - \theta$. Finally, the definition of cross product implies that the cross product is *anticommutative*, that is,

$$\vec{r}\times\vec{w} = -\vec{w}\times\vec{r}. \tag{11.20}$$

Referring to Fig. 11.9, we recall that $\vec{r}$ represents the length and orientation of the segment $\overline{OP}$. Using the Pythagorean theorem and trigonometry, and expressing angles in radians, we have

$$\text{length of } \vec{r} = |\vec{r}| = \sqrt{r_x^2 + r_y^2}, \tag{11.21}$$

and

$$\text{direction of } \vec{r} = \theta = \tan^{-1}\left(\frac{r_y}{r_x}\right) \pm n\pi, \quad n = 0, 1, 2, \ldots \tag{11.22}$$

where n is found by identifying the quadrant containing P (for the case in Fig. 11.9, $n = 0$). Finally, Eqs. (11.13) and (11.14) allow us to rewrite $\vec{r}$ as

$$\vec{r} = |\vec{r}|\,\hat{u}_r, \quad \text{where} \quad \hat{u}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}. \tag{11.23}$$

Since $\hat{u}_r$ is a unit vector in the direction of $\vec{r}$, Eq. (11.23) implies that

> *The information carried by any vector can be written as the product of its magnitude and a unit vector pointing in the direction of that vector.*

Useful vector "tips and tricks"

Components of a vector

We now review how to find the components of a vector, since this operation occurs often in dynamics.

Figure 11.10 shows two perpendicular and oriented lines ℓ_1 and ℓ_2, where by *oriented* we mean that they have a positive and a negative direction. The lines are oriented using the unit vector $\hat{u}_1$ for ℓ_1 and $\hat{u}_2$ for ℓ_2. We also have a vector $\vec{q}$ oriented arbitrarily relative to ℓ_1 and ℓ_2. Our goal is to find the scalar components of $\vec{q}$ along ℓ_1 and ℓ_2.

If we apply Eq. (11.18) to Fig. 11.10 and let $\vec{r}$ be $\vec{q}$, $\vec{w}$ be $\hat{u}_1$, and θ be θ_1, we see that the dot product gives us q_1 directly, that is,

$$q_1 = \vec{q} \cdot \hat{u}_1 = |\vec{q}||\hat{u}_1| \cos\theta_1 = |\vec{q}| \cos\theta_1. \tag{11.24}$$

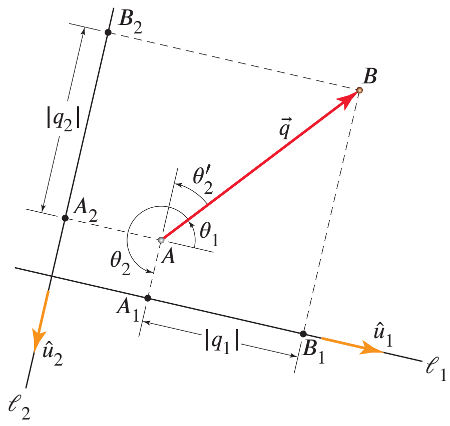


Figure 11.10
Diagram showing the components of $\vec{q}$ in the directions of $\hat{u}_1$ and $\hat{u}_2$.

The quantity q_1 in Eq. (11.24) is what we were looking for because, according to the definition of scalar component of a vector and Fig. 11.10,

1. $|q_1|$ is the distance between A_1 and B_1.
2. The sign of $\vec{q} \cdot \hat{u}_1$ is determined by the sign of $\cos\theta_1$, which is positive if $0° \le \theta_1 < 90°$ and negative if $90° < \theta_1 \le 180°$ (if $\theta_1 = 90°$, A_1 and B_1 coincide so that $q_1 = 0$).

In summary,

$$\boxed{\text{component of } \vec{q} \text{ along } \ell_1 = q_1 = \vec{q} \cdot \hat{u}_1.} \tag{11.25}$$

Repeating the foregoing discussion in the case of q_2 we have that the

$$\text{component of } \vec{q} \text{ along } \ell_2 = q_2 = \vec{q} \cdot \hat{u}_2 = |\vec{q}| \cos\theta_2 = -|\vec{q}| \cos\theta_2', \tag{11.26}$$

since $\theta_2 = \theta_2' + \pi$ and $\cos(\theta_2' + \pi) = -\cos\theta_2'$.

In dynamics we often face the situation depicted in Fig. 11.11, in which we need to calculate the Cartesian components of two mutually orthogonal vectors $\vec{q}$ and $\vec{r}$. If the angle θ defining the orientation of $\vec{q}$ relative to the y axis is given, to express $\vec{q}$ and $\vec{r}$ in components, we can write $\vec{q}$ as

$$\begin{aligned}\vec{q} = (\vec{q} \cdot \hat{\imath})\,\hat{\imath} + (\vec{q} \cdot \hat{\jmath})\,\hat{\jmath} &= |\vec{q}| \cos\left(\theta + \tfrac{\pi}{2}\right)\hat{\imath} + |\vec{q}| \cos\theta\,\hat{\jmath} \\ &= -|\vec{q}| \sin\theta\,\hat{\imath} + |\vec{q}| \cos\theta\,\hat{\jmath} \\ &= |\vec{q}|\underbrace{(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath})}_{\text{unit vector}},\end{aligned} \tag{11.27}$$

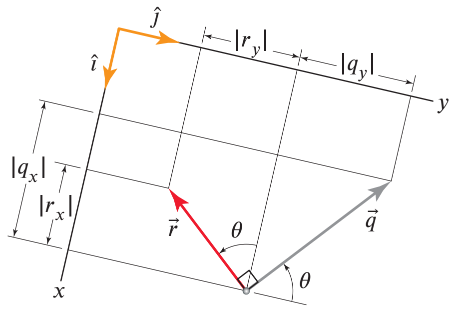


Figure 11.11
Diagram showing the Cartesian components of the vector $\vec{q}$ as well as the vector $\vec{r}$ that is orthogonal to $\vec{q}$.

and then we can write $\vec{r}$ as

$$\begin{aligned}\vec{r} = (\vec{r} \cdot \hat{\imath})\,\hat{\imath} + (\vec{r} \cdot \hat{\jmath})\,\hat{\jmath} &= |\vec{r}| \cos(\pi - \theta)\,\hat{\imath} + |\vec{r}| \cos\left(\theta + \tfrac{\pi}{2}\right)\hat{\jmath} \\ &= -|\vec{r}| \cos\theta\,\hat{\imath} - |\vec{r}| \sin\theta\,\hat{\jmath} \\ &= |\vec{r}|\underbrace{(-\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath})}_{\text{unit vector}}.\end{aligned} \tag{11.28}$$

Equations (11.27) and (11.28) demonstrate that the two equations have the following structure:

- Each vector is equal to its magnitude times a unit vector with one sine and one cosine term.

- The argument of the sine and cosine terms is the angle orienting one of the vectors with respect to one of the component directions.

- In the two equations there will *always* be three positive terms and one negative term or three negative terms and one positive term. As a final check that the decomposition has been performed correctly, note that since $\vec{q}$ and $\vec{r}$ are orthogonal, their dot product should yield zero. The dot product of the unit vectors in (11.27) and (11.28) give us $\sin\theta\cos\theta - \sin\theta\cos\theta = 0$.

Cross products

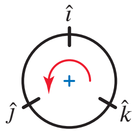


Figure 11.12
A little "trick" to help remember the cross products between Cartesian unit vectors.

Since we will often encounter cross products in dynamics, here is a useful procedure to help us evaluate them. Consider three unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ such that, according to the right-hand rule, $\hat{\imath}\times\hat{\jmath} = \hat{k}$. Arrange the three vectors as shown in Fig. 11.12. To calculate the product of, say, $\hat{\jmath}\times\hat{k}$, just move around the circle, starting from $\hat{\jmath}$ and going toward $\hat{k}$. Now notice that (1) the next vector on the circle is $\hat{\imath}$ and (2) in going from $\hat{\jmath}$ to $\hat{k}$ we move with the arrow (counterclockwise). Hence, $\hat{\jmath}\times\hat{k} = +\hat{\imath}$. Now consider $\hat{k}\times\hat{\jmath}$ and notice that in going from $\hat{k}$ toward $\hat{\jmath}$, the next vector along the circle is $\hat{\imath}$, and we move opposite to the arrow. Therefore, the result is negative, and we have $\hat{k}\times\hat{\jmath} = -\hat{\imath}$.

Helpful Information

Cross products using determinants. You may be familiar with the following *determinant method* of evaluating the cross product of two vectors:

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}.$$

For vectors in 3D, this method provides a very efficient evaluation. As an alternative, the cross product may be evaluated on a term-by-term basis by expanding the following product:

$$\vec{a}\times\vec{b} = \left(a_x\,\hat{\imath} + a_y\,\hat{\jmath} + a_z\,\hat{k}\right) \times \left(b_x\,\hat{\imath} + b_y\,\hat{\jmath} + b_z\,\hat{k}\right).$$

When expanded, nine terms such as $a_x\,\hat{\imath}\times b_x\,\hat{\imath}$ and $a_x\,\hat{\imath}\times b_y\,\hat{\jmath}$ must be evaluated. This is accomplished quickly using Fig. 11.12. In this book, we primarily do cross products of vectors in 2D, and we will likely find the term-by-term evaluation to be quicker.

Units

Units are essential to any quantifiable measure. Newton's second law in scalar form, $F = ma$, provides for the formulation of a consistent and unambiguous system of units. We will use both U.S. Customary units and SI units (International System*) as shown in Table 11.1. Each system has three *base dimensions* and a fourth *derived dimension*. In the U.S. Customary system, the base dimensions are force, length, and time, whose corresponding *base units* are lb (pounds), ft (feet), and s (seconds), respectively. The corresponding derived dimension is mass, which is obtained from the equation $m = F/a$. This gives the mass unit as $\text{lb}\cdot\text{s}^2/\text{ft}$. This unit of mass is often called the *slug*.

Table 11.1. U.S. Customary and SI unit systems.

	System of units	
Base dimension	*U.S. Customary*	*SI*
force	pound (lb)	newton[a] (N) $\equiv \text{kg}\cdot\text{m/s}^2$
mass	slug[a] $\equiv \text{lb}\cdot\text{s}^2/\text{ft}$	kilogram (kg)
length	foot (ft)	meter (m)
time	second (s)	second (s)

[a] derived unit

In the SI system, the base dimensions are mass, length, and time, whose corresponding base units are kg (kilogram), m (meter), and s (second), respectively. The corresponding derived dimension is force, the unit of which is obtained from the

* SI has been adopted as the abbreviation for the French *Le Système International d'Unités*.

equation $F = ma$, which gives the force unit as $\text{kg}\cdot\text{m/s}^2$. This unit of force is referred to as a *newton*, and its abbreviation is N.

Because of the difference in base dimensions between the U.S. Customary system and the SI system, when using the U.S. Customary system, we normally specify the weight of an object (typically in lb) instead of its mass; and, conversely, when using the SI system, we normally specify the mass of an object (typically in kg) instead of its weight.

For both systems, we may occasionally use different, but consistent, units for some dimensions. For example, we may use minutes rather than seconds, inches instead of feet, grams instead of kilograms.

Plane angles are dimensionless quantities (they are defined as the ratio of two lengths). In both the U.S. Customary and SI systems, angles are expressed in radians, abbreviated rad. Another commonly used unit to express angle measurements is the degree, indicated by the symbol °. Angle measurements in degrees and in radians are related as follows:

$$180^\circ = \pi \text{ rad.} \tag{11.29}$$

Common Pitfall

Weight and mass. Unfortunately, it is common to refer to weight using mass units. For example, the person who says, "I weigh 70 kg" really means "My mass is 70 kg." In science and engineering it is essential that accurate nomenclature be used. Weights and forces must be reported using appropriate force units, and masses must be reported using appropriate mass units.

Dimensional homogeneity and unit conversions

Equations must be dimensionally homogeneous. This means that the quantities on the two sides of the equal sign must have the same dimensions. Our strong recommendation is that appropriate units always be used in all equations during a calculation to make sure that the results are dimensionally correct. Such practice helps avoid catastrophic blunders and provides a useful check on a solution, for if an equation is found to be dimensionally inconsistent, then an error has certainly been made. In September 1999, NASA (National Aeronautics and Space Administration) lost a $125 million Mars orbiter because the climate orbiter spacecraft team at the contractor who built the spacecraft used U.S. Customary units when computing rocket thrust, while the mission navigation team at NASA used metric units for this key spacecraft operation. This units error, which came into play when the spacecraft was to be inserted into orbit around Mars, caused the spacecraft to approach Mars at too low an altitude, thus causing it to burn up in Mars' atmosphere.

Unit conversions are often needed and are easily done using conversion factors, such as those shown in Table 11.2, and rules of algebra. The basic idea is to multiply either or both sides of an equation by dimensionless factors of unity, where each factor of unity embodies an appropriate unit conversion. This procedure is illustrated in the examples at the end of this section.

Table 11.2
Conversion factors between U.S. Customary and SI unit systems.

	U.S. Customary	SI
length	1 in.	0.0254 m (25.4 mm)
	1 ft (12 in.)	0.3048 m
	1 mi (5280 ft)	1.609 km
force	1 lb	4.448 N
	1 kip (1000 lb)	4.448 kN
mass	1 slug ($1\ \text{lb}\cdot\text{s}^2/\text{ft}$)	14.59 kg

Prefixes

Prefixes are a useful alternative to scientific notation for representing numbers that are very large or very small. Common prefixes and a summary of rules for their use are given in Table 11.3.

Here is a list of common rules for correct prefix use:

1. With few exceptions, prefixes should be used only in the numerator of unit combinations; for example, use the unit km/s (kilometer per second) and avoid the unit m/ms (meter per millisecond). One common exception to this rule is kg, which may appear in numerator or denominator; for example, use the unit kW/kg (kilowatt per kilogram) and avoid the unit W/g (watt per gram).

Table 11.3. Common prefixes used in the SI unit systems.

Multiplication factor		Prefix	Symbol
1 000 000 000 000 000 000 000 000	10^{24}	yotta	Y
1 000 000 000 000 000 000 000	10^{21}	zetta	Z
1 000 000 000 000 000 000	10^{18}	exa	E
1 000 000 000 000 000	10^{15}	peta	P
1 000 000 000 000	10^{12}	tera	T
1 000 000 000	10^{9}	giga	G
1 000 000	10^{6}	mega	M
1 000	10^{3}	kilo	k
100	10^{2}	hecto	h
10	10^{1}	deka	da
0.1	10^{-1}	deci	d
0.01	10^{-2}	centi	c
0.001	10^{-3}	milli	m
0.000 001	10^{-6}	micro	μ
0.000 000 001	10^{-9}	nano	n
0.000 000 000 001	10^{-12}	pico	p
0.000 000 000 000 001	10^{-15}	femto	f
0.000 000 000 000 000 001	10^{-18}	atto	a
0.000 000 000 000 000 000 001	10^{-21}	zepto	z
0.000 000 000 000 000 000 000 001	10^{-24}	yocto	y

2. Double prefixes must be avoided; for example, use the unit GHz (gigahertz) and avoid the unit kMHz (kilo-megahertz).

3. Use a center dot or dash to denote multiplication of units, for example, N·m or N-m. In this book, we denote multiplication of units by a dot, for example, N·m.

4. Exponentiation applies to both the unit and prefix, for example, $\text{mm}^2 = (\text{mm})^2$.

5. If the number of digits on either side of a decimal point exceeds 4, it is common to group the digits into groups of 3, with the groups separated by commas or thin spaces. Since many countries use a comma to represent a decimal point, the thin space is sometimes preferable; for example, 1234.0 could be written as is, but by contrast, 12345.0 should be written as 12,345.0 or as 12 345.0.

While prefixes can often be incorporated into an expression by inspection, the rules for doing this are identical to those for performing unit transformations.

Accuracy of numbers in calculations

Throughout this book, we will generally assume that the data given for problems is accurate to three significant digits. When calculations are performed, such as in example problems, all intermediate results are stored in the memory of a calculator or computer using the full precision these machines offer. However, when these intermediate results are reported in this book, they are rounded to four significant digits. Final answers are usually reported with three significant digits. If you verify the calculations described in this book using the rounded numbers that are reported, you may occasionally calculate results that are slightly different from those shown.

End of Section Summary

Review of vector operations. Referring to Fig. 11.13, the Cartesian representation of a two-dimensional vector $\vec{r}$ takes the form

Eq. (11.13), p. 627

$$\vec{r} = r_x\,\hat{\imath} + r_y\,\hat{\jmath},$$

where $\hat{\imath}$ and $\hat{\jmath}$ are unit vectors in the positive x and y directions, respectively, and where r_x and r_y are the x and y (*scalar*) *components* of $\vec{r}$, respectively. Using trigonometry, r_x and r_y are given by

Eqs. (11.14), p. 627

$$r_x = |\vec{r}|\cos\theta \quad \text{and} \quad r_y = |\vec{r}|\sin\theta,$$

where θ is the orientation of the segment $\overline{OP}$ relative to the x axis and $|\vec{r}|$, called the *magnitude of* $\vec{r}$ or *length of* $\vec{r}$, is the length of $\overline{OP}$.

The *dot* or *scalar product* of the vectors $\vec{r}$ and $\vec{w}$ gives the *scalar*

Eq. (11.18), p. 628

$$\vec{r}\cdot\vec{w} = |\vec{r}||\vec{w}|\cos\theta,$$

with θ being the smallest angle that will rotate one of the vectors into the other.

Referring to Fig. 11.14, the *cross product* of two vectors $\vec{r}$ and $\vec{w}$ is denoted by $\vec{r}\times\vec{w}$ and gives a *vector*

1. With magnitude equal to

   Eq. (11.19), p. 628

   $$|\vec{r}\times\vec{w}| = |\vec{r}||\vec{w}|\sin\theta,$$

   where θ is the smallest angle that will rotate one of the vectors into the other (to ensure that $|\vec{r}||\vec{w}|\sin\theta \ge 0$).

2. Whose line of action is perpendicular to the plane containing $\vec{r}$ and $\vec{w}$ and whose direction is determined by the right-hand rule.

3. For which the cross product is anticommutative; i.e., $\vec{r}\times\vec{w} = -\vec{w}\times\vec{r}$.

Referring to Fig 11.13, given r_x and r_y, we compute $|\vec{r}|$ and θ as follows:

Eqs. (11.21) and (11.22), p. 628

$$\text{length of } \vec{r} = |\vec{r}| = \sqrt{r_x^2 + r_y^2},$$

$$\text{direction of } \vec{r} = \theta = \tan^{-1}\left(\frac{r_y}{r_x}\right).$$

Another useful representation of a vector $\vec{r}$ is as follows:

Eq. (11.23), p. 628

$$\vec{r} = |\vec{r}|\,\hat{u}_r, \quad \text{where} \quad \hat{u}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath},$$

where $\hat{u}_r$ is the unit vector in the direction of $\vec{r}$.

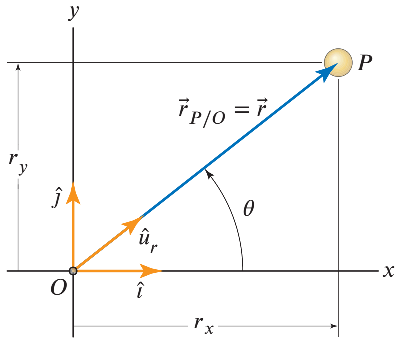


Figure 11.13
Description of the position of a particle P.

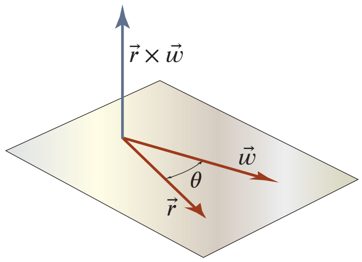


Figure 11.14
Graphical representation of the vector cross product of $\vec{r}$ and $\vec{w}$.

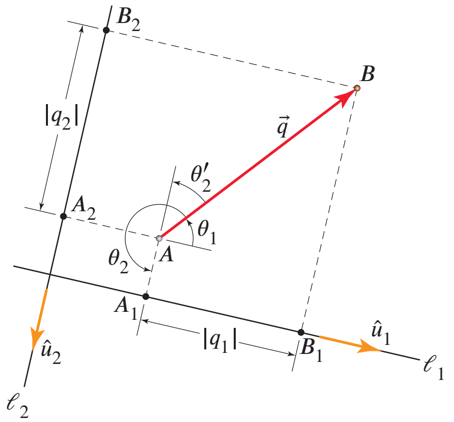


Figure 11.15
Diagram showing the components of $\vec{q}$ in the directions of $\hat{u}_1$ and $\hat{u}_2$.

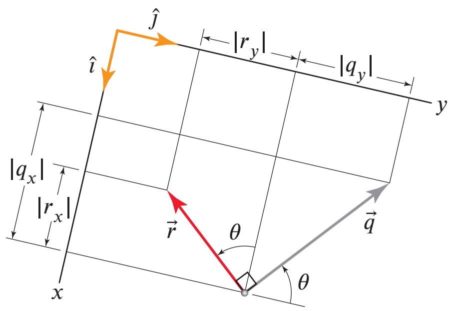


Figure 11.16
Diagram showing the Cartesian components of the vector $\vec{q}$ as well as the vector $\vec{r}$ that is orthogonal to $\vec{q}$.

Useful vector "tips and tricks." Referring to Fig. 11.15, the component of a vector in a given direction can be computed using the dot product. For example,

Eq. (11.24), p. 629

$$q_1 = \vec{q} \cdot \hat{u}_1 = |\vec{q}||\hat{u}_1| \cos\theta_1 = |\vec{q}| \cos\theta_1,$$

that is,

Eq. (11.25), p. 629

$$\text{component of } \vec{q} \text{ along } \ell_1 = q_1 = \vec{q} \cdot \hat{u}_1.$$

Referring to Fig. 11.16, for two mutually orthogonal vectors $\vec{r}$ and $\vec{q}$ the following relations hold:

Eqs. (11.27) and (11.28), p. 629

$$\begin{aligned}
\vec{q} = (\vec{q} \cdot \hat{\imath})\,\hat{\imath} + (\vec{q} \cdot \hat{\jmath})\,\hat{\jmath} &= |\vec{q}| \cos\left(\theta + \tfrac{\pi}{2}\right)\hat{\imath} + |\vec{q}| \cos\theta\,\hat{\jmath} \\
&= -|\vec{q}| \sin\theta\,\hat{\imath} + |\vec{q}| \cos\theta\,\hat{\jmath} \\
&= |\vec{q}|(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}), \\
\vec{r} = (\vec{r} \cdot \hat{\imath})\,\hat{\imath} + (\vec{r} \cdot \hat{\jmath})\,\hat{\jmath} &= |\vec{r}| \cos(\pi - \theta)\,\hat{\imath} + |\vec{r}| \cos\left(\theta + \tfrac{\pi}{2}\right)\hat{\jmath} \\
&= -|\vec{r}| \cos\theta\,\hat{\imath} - |\vec{r}| \sin\theta\,\hat{\jmath} \\
&= |\vec{r}|(-\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath}).
\end{aligned}$$

EXAMPLE 11.1 *Components of Vectors*

At the instant shown, the acceleration of the airplane in Fig. 1 is the vector

$$\vec{a} = (5.63\,\hat{u}_t + 37.2\,\hat{u}_n)\,\text{m/s}^2, \tag{1}$$

where the mutually perpendicular unit vectors $\hat{u}_t$ and $\hat{u}_n$ are tangent and perpendicular to the airplane's path, respectively. The angle between $\hat{u}_t$ and the horizontal direction is $\theta = 26°$. Determine ϕ, the angle between $\vec{a}$ and $\hat{u}_t$, and the expression of $\vec{a}$ relative to the unit vectors $\hat{\imath}$ and $\hat{\jmath}$, which are horizontal and vertical, respectively.

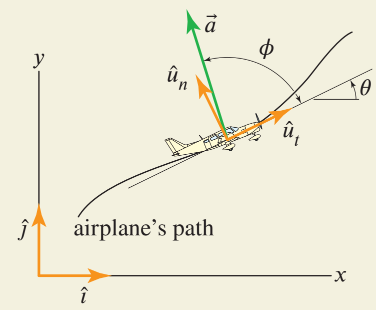


Figure 1
An airplane performing a maneuver.

SOLUTION

Road Map Letting a_t and a_n be the components of $\vec{a}$ in the $(\hat{u}_t, \hat{u}_n)$ component system, Eq. (1) implies that $a_t = 5.63\,\text{m/s}^2$ and $a_n = 37.2\,\text{m/s}^2$. Since a_t and a_n are known, ϕ can be found via Eq. (11.22) on p. 628 after replacing the quantities r_x and r_y in that equation with a_t and a_n, respectively. To express $\vec{a}$ using $\hat{\imath}$ and $\hat{\jmath}$, we need to determine the components of $\vec{a}$ in the $\hat{\imath}$ and $\hat{\jmath}$ directions. This can be done via Eq. (11.27) on p. 629.

Determination of ϕ

Computation Replacing r_x and r_y in Eq. (11.22) on p. 628 with a_t and a_n, respectively, we have

$$\phi = \tan^{-1}\left(\frac{a_n}{a_t}\right) \pm n\pi, \quad n = 0, 1, 2, \ldots. \tag{2}$$

Since the components a_n and a_t are both positive, the vector $\vec{a}$ lies in the first quadrant of the $(\hat{u}_t, \hat{u}_n)$ system. Therefore, we can choose $n = 0$ in Eq. (2). Recalling that $a_t = 5.63\,\text{m/s}^2$ and $a_n = 37.2\,\text{m/s}^2$, Eq. (2) can be evaluated to obtain

$$\boxed{\phi = 81.39°.} \tag{3}$$

Expression of $\vec{a}$ via $\hat{\imath}$ and $\hat{\jmath}$

Computation Replacing the vector $\vec{q}$ with $\vec{a} = a_t\,\hat{u}_t + a_n\,\hat{u}_n$ in the first equality in Eq. (11.27) on p. 629, we have

$$\begin{aligned}\vec{a} &= \left[(a_t\,\hat{u}_t + a_n\,\hat{u}_n)\cdot\hat{\imath}\right]\hat{\imath} + \left[(a_t\,\hat{u}_t + a_n\,\hat{u}_n)\cdot\hat{\jmath}\right]\hat{\jmath}\\ &= (a_t\hat{u}_t\cdot\hat{\imath} + a_n\hat{u}_n\cdot\hat{\imath})\,\hat{\imath} + (a_t\hat{u}_t\cdot\hat{\jmath} + a_n\hat{u}_n\cdot\hat{\jmath})\,\hat{\jmath}.\end{aligned} \tag{4}$$

Referring to Fig. 2, the angles between $\hat{u}_t$ and the unit vectors $\hat{\imath}$ and $\hat{\jmath}$ are θ and $90° - \theta$, respectively. Similarly, the angles between $\hat{u}_n$ and the unit vectors $\hat{\imath}$ and $\hat{\jmath}$ are $\theta + 90°$ and θ, respectively. Therefore, we have

$$\hat{u}_t\cdot\hat{\imath} = \cos\theta, \qquad \hat{u}_t\cdot\hat{\jmath} = \cos(90° - \theta) = \sin\theta, \tag{5}$$

$$\hat{u}_n\cdot\hat{\imath} = \cos(\theta + 90°) = -\sin\theta, \quad \hat{u}_n\cdot\hat{\jmath} = \cos\theta. \tag{6}$$

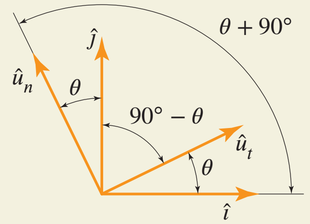


Figure 2
Orientation of the unit vectors $\hat{u}_t$ and $\hat{u}_n$ relative to the unit vectors $\hat{\imath}$ and $\hat{\jmath}$.

Using Eqs. (5) and (6), Eq. (4) can be simplified to

$$\vec{a} = (a_t\cos\theta - a_n\sin\theta)\,\hat{\imath} + (a_t\sin\theta + a_n\cos\theta)\,\hat{\jmath}. \tag{7}$$

Recalling that $a_t = 5.63\,\text{m/s}^2$, $a_n = 37.2\,\text{m/s}^2$, and $\theta = 26°$, Eq. (7) gives

$$\boxed{\vec{a} = (-11.25\,\hat{\imath} + 35.90\,\hat{\jmath})\,\text{m/s}^2.} \tag{8}$$

Discussion & Verification The value of ϕ seems reasonable given how much larger a_n is relative to a_t. To verify the result in Eq. (8), we can calculate the magnitude of $\vec{a}$ using its components relative to both the $(\hat{\imath}, \hat{\jmath})$ and the $(\hat{u}_t, \hat{u}_n)$ systems and check that we obtain the same value. In the $(\hat{\imath}, \hat{\jmath})$ system, we have $|\vec{a}| = \left(-11.25^2 + 35.90^2\right)^{1/2}\,\text{m/s}^2 = 37.62\,\text{m/s}^2$. In the $(\hat{u}_t, \hat{u}_n)$ system we have $|\vec{a}| = \left(5.63^2 + 37.2^2\right)^{1/2}\,\text{m/s}^2 = 37.62\,\text{m/s}^2$. Since the result is as expected, we can say that the result in Eq. (8) appears to be correct.

EXAMPLE 11.2 *Position Vectors, Relative Position Vectors, and Components*

Figure 1
A map of the state of Minnesota with the locations of four of its cities. An east-north or xy coordinate frame with origin at O is also defined.

The map of the state of Minnesota in Fig. 1 shows four of its cities and defines a coordinate system whose origin is at O. The coordinates of the four cities relative to O are given in Table 1. Assuming the Earth is flat and ignoring errors due to the map projection used, the x and y directions can be considered to be east and north, respectively. Using the information in Table 1, determine

(a) The position of Duluth (D) relative to Minneapolis/St. Paul (M), $\vec{r}_{D/M}$.

(b) The orientation, relative to north, of the position of International Falls (I) relative to Fargo/Moorhead (F).

(c) The east and north (scalar) components of the position of Fargo/Moorhead relative to Minneapolis/St. Paul.

(d) The position of the point H halfway between Minneapolis/St. Paul and International Falls.

Table 1. Coordinates of the four cities in the state of Minnesota shown in Fig. 1. All coordinates are relative to the origin O.

City	x, east (mi)	y, north (mi)
Minneapolis/St. Paul (M)	216	130
Duluth (D)	259	267
Fargo/Moorhead (F)	12	278
International Falls (I)	195	413

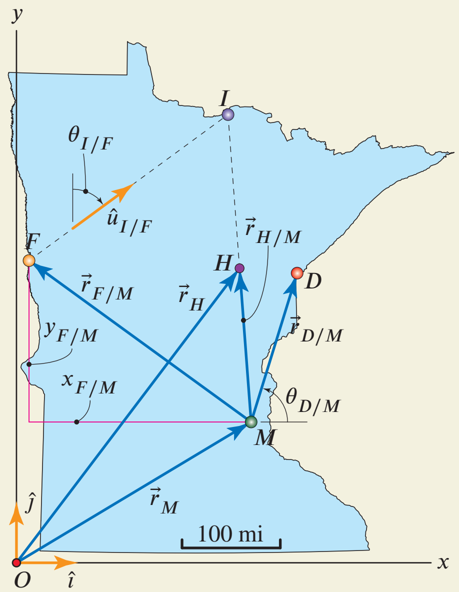


Figure 2
Vectors, projections, and angles needed to compute the quantities of interest.

SOLUTION

Part (a)

Road Map Since the coordinates of points D and M are available, the components of $\vec{r}_{D/M}$ can be found by using Eq. (11.15) on p. 627 to compute the difference between the coordinates of D and M.

Computation Referring to Fig. 2 and Table 1 and then taking the difference between $\vec{r}_D$ and $\vec{r}_M$, we obtain

$$\begin{aligned}\vec{r}_{D/M} &= \vec{r}_D - \vec{r}_M = (x_D - x_M)\,\hat{\imath} + (y_D - y_M)\,\hat{\jmath}\\ &= \big[(259 - 216)\,\hat{\imath} + (267 - 130)\,\hat{\jmath}\big]\text{ mi}\\ &= (43.00\,\hat{\imath} + 137.0\,\hat{\jmath})\text{ mi} = 143.6\text{ mi @ }72.57^\circ\measuredangle,\end{aligned} \tag{1}$$

where $\theta_{D/M} = 72.57°$.

Discussion & Verification Referring to Fig. 2 and taking advantage of the scale indicated on the map, we can graphically verify that our answers are correct. We note that in this problem, the answer given as a length (143.6 mi) and a direction ($\theta_{D/M} = 72.57°$) is probably more straightforward to verify than the answer given in terms of Cartesian components, since we could directly measure the distance from Minneapolis/St. Paul to Duluth on the map itself.

Part (b)

Road Map To determine the orientation of I relative to F, finding either the unit vector $\hat{u}_{I/F}$ or the angle $\theta_{I/F}$ will suffice. To find $\hat{u}_{I/F}$, we can find $\vec{r}_{I/F}$ and then divide by its magnitude [Eq. (11.21) on page 628].

Computation Again referring to Fig. 2 and Table 1 and using Eq. (11.15), the vector $\hat{u}_{I/F}$ is given by

$$\hat{u}_{I/F} = \frac{\vec{r}_{I/F}}{|\vec{r}_{I/F}|} = \frac{(195-12)\,\hat{\imath} + (413-278)\,\hat{\jmath}}{\sqrt{(195-12)^2 + (413-278)^2}} = 0.8047\,\hat{\imath} + 0.5936\,\hat{\jmath}. \tag{2}$$

Now, we can find the angle $\theta_{I/F}$ by applying Eq. (11.22), which gives

$$\boxed{\theta_{I/F} = \tan^{-1}\left(\frac{x_{I/F}}{y_{I/F}}\right) = \tan^{-1}\left(\frac{183}{135}\right) = 53.58°,} \tag{3}$$

which would be approximately northeast.

Discussion & Verification As with Part (a), we have expressed the answer in two different ways, and the version given in terms of an angle [i.e., Eq. (3)] probably allows for an easier verification that the solution is reasonable.

Part (c)

Road Map To find the east (x) and north (y) components of $\vec{r}_{F/M}$, we can use Eq. (11.25) on p. 629.

Computation Referring to Fig. 2 and using Eq. (11.25), we find that

$$\boxed{x_{F/M} = \vec{r}_{F/M} \cdot \hat{\imath} = \big[(12-216)\,\hat{\imath} + (278-130)\,\hat{\jmath}\big] \cdot \hat{\imath}\,\text{mi} = -204.0\,\text{mi},} \tag{4}$$

$$\boxed{y_{F/M} = \vec{r}_{F/M} \cdot \hat{\jmath} = \big[(12-216)\,\hat{\imath} + (278-130)\,\hat{\jmath}\big] \cdot \hat{\jmath}\,\text{mi} = 148.0\,\text{mi},} \tag{5}$$

where we have used the fact that $\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = 1$ and $\hat{\imath} \cdot \hat{\jmath} = \hat{\jmath} \cdot \hat{\imath} = 0$.

Discussion & Verification We calculated that Fargo/Moorhead is 148.0 mi north of Minneapolis/St. Paul, and it is 204.0 mi *west* (i.e., −204 mi east), which certainly seem reasonable given the figure.

Part (d)

Road Map As we can see from Fig. 2, the position of the point H, which is halfway between Minneapolis/St. Paul and International Falls, is given by the vector $\vec{r}_H$. The key to the solution is then seeing that we can write this position as $\vec{r}_H = \vec{r}_M + \vec{r}_{H/M}$.

Computation We begin with the decomposition of $\vec{r}_H$, which is given by

$$\vec{r}_H = \vec{r}_M + \vec{r}_{H/M}. \tag{6}$$

We can write $\vec{r}_{H/M}$ as

$$\vec{r}_{H/M} = \tfrac{1}{2}\vec{r}_{I/M} = \tfrac{1}{2}\big[(195-216)\,\hat{\imath} + (413-130)\,\hat{\jmath}\big]\,\text{mi} = (-10.50\,\hat{\imath} + 141.5\,\hat{\jmath})\,\text{mi}, \tag{7}$$

which, when substituted into Eq. (6), gives

$$\boxed{\vec{r}_H = \big[(216\,\hat{\imath} + 130\,\hat{\jmath}) + (-10.50\,\hat{\imath} + 141.5\,\hat{\jmath})\big]\,\text{mi} = (205.5\,\hat{\imath} + 271.5\,\hat{\jmath})\,\text{mi},} \tag{8}$$

where we have used $\vec{r}_M = (216\,\hat{\imath} + 130\,\hat{\jmath})$ mi from Table 1.

Discussion & Verification Again referring to Fig. 2, the result in Eq. (8) looks reasonable. In addition, the power of vectors starts to come into focus in this part of the example. That is, once we knew the position of International Falls relative to Minneapolis/St. Paul, computing the position at any fraction of the distance in between was trivial. Once that was computed, vector addition allowed us to easily find the position of the halfway point relative to the origin at O.

EXAMPLE 11.3 Dimensional Analysis and Unit Usage

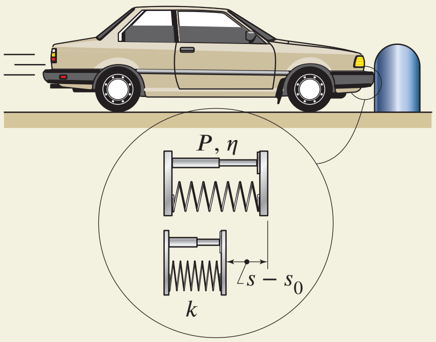


Figure 1
A car about to collide with a concrete block.

In a collision, the bumper of a car can undergo both elastic (i.e., reversible) and permanent deformation. A model for the force F transmitted by the bumper to the car is given by $F = P + k(s - s_0) + \eta\, ds/dt$, where $s - s_0$ is the compression experienced by the bumper and has the dimension of length, s_0 is a constant with dimension of length, and ds/dt is the time rate of change of s, which has the dimension of length over time. The quantity P is the force needed to permanently deform the bumper, k is the stiffness of the system, and η is a constant that relates the overall force to the speed of deformation. Determine

(a) The dimensions of P, k, and η, and

(b) The units that these quantities would have in the SI and the U.S. Customary systems.

SOLUTION

Part (a)

Road Map The first step in dimensional analysis is the identification of a basic relation, such as a law of nature, containing the quantities to analyze and for which the dimensions are known. Since we are dealing with the expression of a force, we can use Eq. (11.2) on p. 620 as the basic relation. Notice that the given force law consists of the sum of three terms. For this sum to be meaningful, each of the terms in question must have dimensions of force. We call this property *dimensional homogeneity*, and it is the key to solving the given problem.

Computation Let L, M, and T denote length, mass, and time, respectively. Writing "[*something*]" to mean the "dimensions of *something*," for Eq. (11.2), we have

$$[F] = [ma] = [m]\,[a] = M\frac{L}{T^2}. \tag{1}$$

Considering the given force law, we have

$$[F] = \left[P + k(s - s_0) + \eta\frac{ds}{dt}\right] = [P] + [k(s - s_0)] + \left[\eta\frac{ds}{dt}\right]. \tag{2}$$

Comparing Eq. (1) with Eq. (2) and enforcing dimensional homogeneity between them, we see that $[P]$, $[k(s - s_0)]$, and $[\eta\, ds/dt]$ must each be ML/T^2. Therefore, the quantity P must have the dimensions of force:

$$\boxed{[P] = M\frac{L}{T^2}.} \tag{3}$$

For the term k, we have

$$[k(s - s_0)] = [k]\,[s - s_0] = [k]L = M\frac{L}{T^2}, \tag{4}$$

where we have used the fact that the dimension of s and s_0 is L. Simplifying the last equality in Eq. (4), we have

$$\boxed{[k] = \frac{M}{T^2}.} \tag{5}$$

Next, considering the term with η in Eq. (2), we have

$$\left[\eta\frac{ds}{dt}\right] = [\eta]\left[\frac{ds}{dt}\right] = [\eta]\frac{L}{T} = M\frac{L}{T^2}, \tag{6}$$

where we have used the fact that the dimensions of ds/dt are L/T. Simplifying the last equality in Eq. (6), we have

$$\boxed{[\eta] = \frac{M}{T}.} \tag{7}$$

> **Concept Alert**
>
> **Dimensional homogeneity.** In writing Eq. (2), we have used a property stating that "[something + something else] = [something] + [something else]." This property expresses the requirement that the sum of two physical quantities makes sense only when these quantities have the same dimensions. Using a more formal language, we say that the quantities in question must satisfy *dimensional homogeneity* or, equivalently, must be *dimensionally homogeneous*.

Discussion & Verification The correctness of our dimensional analysis in the case of P is apparent, since P must have the dimensions of a force. In the case of k and η, we can verify the correctness of our solution by substituting these quantities back into the expression for F. Doing so confirms that the dimensions of k and η are correct.

Part (b)

Road Map To solve Part (b) of the problem, we need to match the dimensions obtained in Part (a) with their corresponding units according to the conventions established by the SI and U.S. Customary systems. To do this, we need only look at Table 11.1 on p. 630.

Computation Since P has the same dimensions as a force, its SI units can be simply taken to be N (newtons) or, using the SI system base units, $\text{kg}\cdot\text{m/s}^2$. In the U.S. Customary system, P is measured in lb (pounds).

Since the dimensions of k are mass over time squared, the corresponding units are kg/s^2 in the SI system and lb/ft in the U.S. Customary system. Notice that in the SI system, the units of k can also be expressed as N/m.

Recalling that η has dimensions of mass over time, the units of η are kg/s in the SI system and slug/s in the U.S. Customary system. Using the base units of the U.S. Customary system, the units of η are $\text{lb}\cdot\text{s/ft}$.

All of these results are summarized in Table 1.

Table 1. Summary of the solution to the second part of the problem.

Quantity	SI units	U.S. Customary units
P	N or $\text{kg}\cdot\text{m/s}^2$	lb
k	kg/s^2 or N/m	lb/ft
η	kg/s	$\text{lb}\cdot\text{s/ft}$

Discussion & Verification The correctness of our results can be verified by replacing the units with the dimensions they correspond to. For example, for P we have that the dimensions corresponding to the unit of pound are those of a force, i.e., ML/T^2, which are the dimensions of P obtained in Part (a) of the problem solution. Repeating this process for the other quantities and for both the SI system and the U.S. Customary system, we see that our results are correct.

EXAMPLE 11.4 *Dimensional Analysis and Unit Conversion*

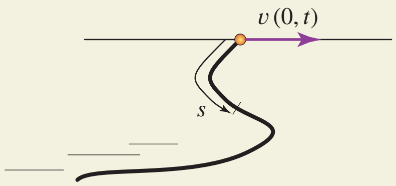


Figure 1
A moving string with one end being dragged.

In studying the motion of a string, it is determined that the speed at various points along the string is given by the function

$$v(s,t) = \alpha + \beta t^2 - \gamma s + \delta \frac{s}{t}, \tag{1}$$

where s is the coordinate of points along the string, t is time, and α, β, γ, and δ are constants.

(a) What are the dimensions of α, β, γ, and δ?

(b) If α, β, γ, and δ are all equal to 1 in SI units, what are they in U.S. Customary units?

> **Interesting Fact**
>
> **Are engineers really interested in the motion of strings?** The answer is "Actually, yes!" Highly detailed models of real physical objects tend to be quite complicated, difficult to tackle from a numerical standpoint, and often difficult to interpret. Therefore, before delving into the complexity of highly detailed models, both physicists and engineers tend to "simplify things" and model real systems as simple objects, such as strings (or beams). Simple models can be very effective in capturing the essential physical behavior of real physical objects and, in that way, give us useful insight into the physical world.

SOLUTION

Part (a)

Road Map Since the dimensions of speed are L/T (with corresponding units of m/s in SI and ft/s in U.S. Customary), the dimensions of every term on the right-hand side of Eq. (1) must also be L/T.

Computation Begin with $[\alpha]$, which must have the same dimensions as those of the speed v, so they are simply L/T. As for $[\beta]$, we know that

$$\left[\beta t^2\right] = [\beta]T^2 = L/T \quad \Rightarrow \quad \boxed{[\beta] = L/T^3.} \tag{2}$$

To get $[\gamma]$, we proceed as in Eq. (2) to obtain

$$[\gamma s] = [\gamma]L = L/T \quad \Rightarrow \quad \boxed{[\gamma] = 1/T.} \tag{3}$$

Finally, $[\delta]$ is obtained similarly as

$$\left[\delta \frac{s}{t}\right] = [\delta]L/T = L/T \quad \Rightarrow \quad \boxed{[\delta] \text{ is dimensionless.}} \tag{4}$$

Discussion & Verification The verification of the results for α is immediate since no calculations were performed to obtain them. In the case of β, γ, and δ, substituting the results in Eqs. (2)–(4), we see that our results are correct.

Part (b)

Road Map After expressing α, β, and γ in SI units, we need to convert them into U.S. Customary units. Note that no conversion is needed for δ since it is dimensionless.

Computation Based on Part (a), the SI units of α, β, and γ are m/s, m/s^3, and s^{-1}, respectively. Converting unit values for α, β, and γ to U.S. Customary units gives

$$\alpha = 1\,\text{m/s} = 1\frac{\not{\text{m}}}{\text{s}}\left(\frac{\text{ft}}{0.3048\,\not{\text{m}}}\right) = 3.281\,\text{ft/s}, \tag{5}$$

$$\beta = 1\,\text{m/s}^3 = 1\frac{\not{\text{m}}}{\text{s}^3}\left(\frac{\text{ft}}{0.3048\,\not{\text{m}}}\right) = 3.281\,\text{ft/s}^3, \tag{6}$$

$$\gamma = 1\,\text{s}^{-1}, \tag{7}$$

where no conversion is needed for γ since $1\,\text{s}^{-1}$ is the same in either SI or U.S. Customary units.

Discussion & Verification The only results to verify are those for α and β, which have dimensions of L/T and L/T^3, respectively. Therefore, since the base unit for time is the same in both the SI and U.S. Customary systems, the unit conversion was expected to affect only the L dimension and yield the same value (namely, 3.281) for both α and β. Finally, the value in question was expected to be close to 3 since a meter is a little over 3 ft. Thus, our results appear to be correct.