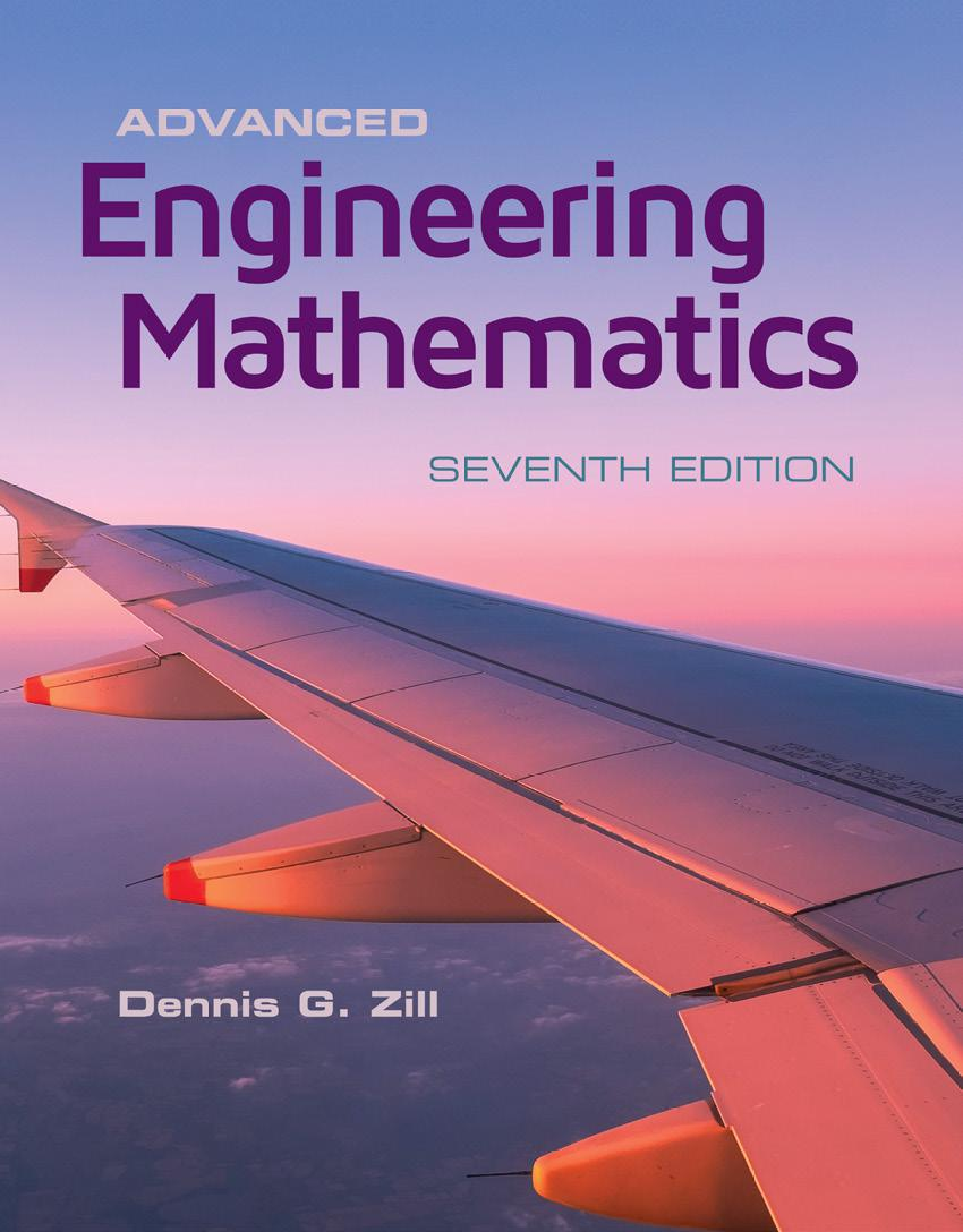
ADVANCED

Engineering Mathematics

SEVENTH EDITION

Dennis G. Zill



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SEVENTH EDITION

Dennis G. Zill

Loyola Marymount University

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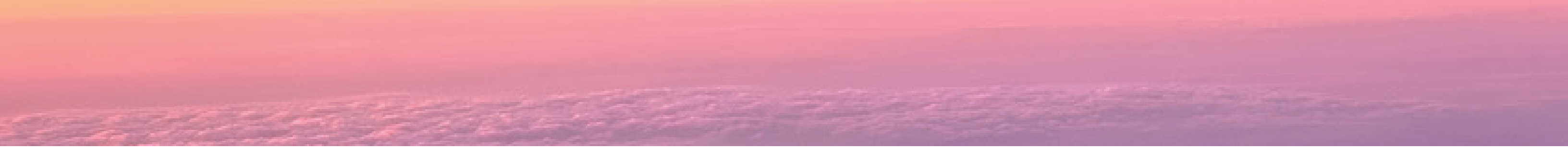

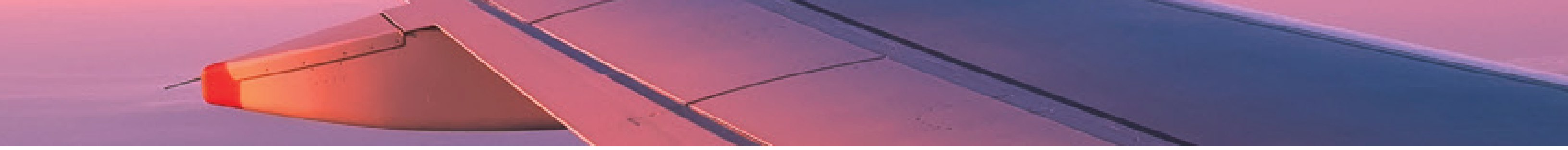

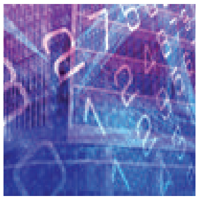

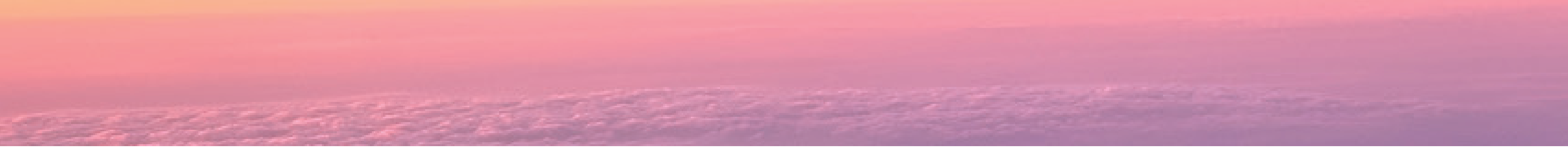

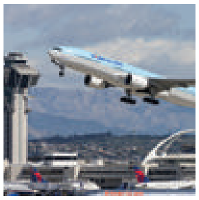

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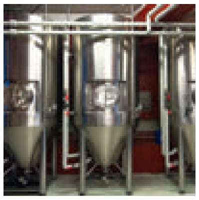

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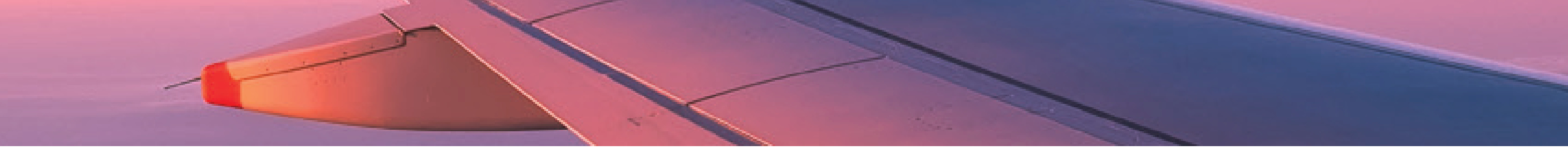

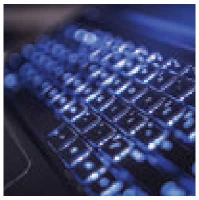

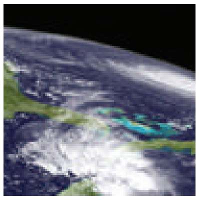

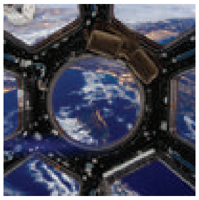

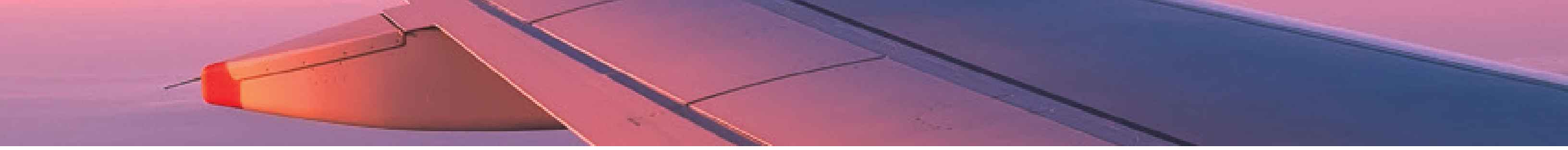

Preface

In courses such as *calculus* or *differential equations*, the content is fairly standardized, but the content of a course entitled *engineering mathematics* often varies considerably between two different academic institutions. Therefore a text entitled *Advanced Engineering Mathematics* is a compendium of many mathematical topics, all of which are loosely related by the expedient of either being needed or useful in courses in science and engineering or in subsequent careers in these areas. There is literally no upper bound to the number of topics that could be included in a text such as this. Consequently, this book represents the author's opinion of what constitutes *engineering mathematics*.

Content of the Text

For flexibility in topic selection this text is divided into five major parts. As can be seen from the titles of these various parts, it should be obvious that it is my belief that the backbone of science/engineering-related mathematics is the theory and applications of ordinary and partial differential equations.

PART 1: Ordinary Differential Equations (Chapters 1–6)

The six chapters in Part 1 constitute a complete short course in ordinary differential equations. These chapters, with some modifications, correspond to Chapters 1, 2, 3, 4, 5, 6, 7, and 9 in the text *A First Course in Differential Equations with Modeling Applications*, *Eleventh Edition*, by Dennis G. Zill (Cengage Learning). In Chapter 2 the focus is on methods for solving first-order differential equations and their applications. Chapter 3 deals mainly with linear second-order differential equations and their applications. Chapter 4 is devoted to the solution of differential equations and systems of differential equations by the important Laplace transform.

PART 2: Vectors, Matrices, and Vector Calculus (Chapters 7–9)

Chapter 7, *Vectors*, and Chapter 9, *Vector Calculus*, include the standard topics that are usually covered in the third semester of a calculus sequence: vectors in 2- and 3-space, vector functions, directional derivatives, line integrals, double and triple integrals, surface integrals, Green's theorem, Stokes' theorem, and the divergence theorem. In Section 7.6 the vector concept is generalized; by defining vectors analytically we lose their geometric interpretation but keep many of their properties in n-dimensional and infinite-dimensional vector spaces. Chapter 8, *Matrices*, is an introduction to systems of algebraic equations, determinants, and matrix algebra, with special emphasis on those types of matrices that are useful in solving systems of linear differential equations. Optional sections on cryptography, error correcting codes, the method of least squares, and discrete compartmental models are presented as applications of matrix algebra.

PART 3: Systems of Differential Equations (Chapters 10 and 11)

There are two chapters in Part 3. Chapter 10, *Systems of Linear Differential Equations*, and Chapter 11, *Systems of Nonlinear Differential Equations*, draw heavily on the matrix material presented in Chapter 8 of Part 2. In Chapter 10, systems of linear first-order equations are solved utilizing the concepts of eigenvalues and eigenvectors, diagonalization, and by means of a matrix exponential function. In Chapter 11, qualitative aspects of autonomous linear and nonlinear systems are considered in depth.

PART 4: Partial Differential Equations (Chapters 12–16)

The core material on Fourier series and boundary-value problems involving second-order partial differential equations was drawn from the text *Differential Equations with Boundary-Value Problems*, *Ninth Edition*, by Dennis G. Zill (Cengage Learning). In Chapter 12, *Fourier Series*, the fundamental topics of sets of orthogonal functions and expansions of functions in terms of an infinite series of orthogonal functions are presented. These topics are then utilized in Chapters 13 and 14 where boundary-value problems in rectangular, polar, cylindrical, and spherical coordinates are solved using the method of separation of variables. In Chapter 15, *Integral Transforms*, boundary-value problems are solved by means of the Laplace and Fourier integral transforms.

PART 5: Complex Analysis (Chapters 17–20)

The final four chapters of the hardbound text cover topics ranging from the basic complex number system through applications of conformal mappings in the solution of Dirichlet's problem. This material by itself could easily serve as a one quarter introductory course in complex variables. This material was taken from *Complex Analysis*: *A First Course with Applications*, *Third Edition*, by Dennis G. Zill and Patrick D. Shanahan (Jones & Bartlett Learning).

Design of the Text

For the benefit of those instructors and students who have not used the preceding edition, a word about the design of the text is in order. Each chapter opens with its own table of contents and a brief introduction to the material covered in that chapter. Because of the great number of figures, definitions, and theorems throughout this text, I use a double-decimal numeration system. For example, the interpretation of "Figure 1.2.3" is

Chapter Section of Chapter 1
↓ ↓
1.2.3 ← Third figure in Section 1.2

I think that this kind of numeration makes it easier to find, say, a theorem or figure when it is referred to in a later section or chapter. In addition, to better link a figure with the text, the *first* textual reference to each figure is done in the same font style and color as the figure number. For example, the first reference to the second figure in Section 5.7 is given as **FIGURE 5.7.2** and all subsequent references to that figure are written in the traditional style Figure 5.7.2.

Features of the Seventh Edition

- One of the goals of this revision was to emphasize applications throughout the text. So, the application problems contributed to previous editions

  Air Exchange, **Exercises 2.7**

  Potassium-40 Decay, **Exercises 2.9**

  Potassium–Argon Dating, **Exercises 2.9**

  Invasion of the Marine Toads, **Chapter 2 in Review**

  Temperature of a Fluid, **Exercises 3.6**

  Blowing in the Wind, **Exercises 3.9**

  The Paris Guns, **Chapter 3 in Review**

  have been retained.
- Section 15.5, *Finite Fourier Transforms*, is new to the text.
- New examples and many new problems (conceptual and applied) have been added throughout the text.
- New figures and photos have been added to highlight some of the older problems and discussions.
- New **REMARKS** have been added to some sections, and some of the older **REMARKS** have been expanded.
- A bit of history has been added following the proper name of a person associated with a certain differential equation or problem.
- Several application problems in chapter review exercises have been moved to the appropriate section exercises.
- Parts of several sections have been rewritten in an attempt to improve clarity.
- Appendix A, *Integral-Defined Functions,* is new to the text.
- The table of Laplace transforms in Appendix C has been expanded.

Supplements

For Instructors

- *Complete Solutions Manual* (*CSM*) by Warren S. Wright and Roberto Martinez
- Test Bank
- Slides in PowerPoint format
- Image Bank
- WebAssign: WebAssign is a flexible and fully customizable online instructional system that puts powerful tools in the hands of teachers, enabling them to deploy assignments, instantly assess individual student performance, and realize their teaching goals. Much more than just a homework grading system, WebAssign delivers secure online testing, customizable precoded questions directly from exercises in this textbook, and unparalleled customer service. Instructors who adopt this program for their classroom use will have access to a digital version of this textbook. Students who purchase an access code for WebAssign will also have access to the digital version of the printed text.

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For Students

- A WebAssign Student Access Code can be bundled with a copy of this text at a discount when requested by the adopting instructor. It may also be purchased separately online when WebAssign is required by the student's instructor or institution. The student access code provides the student with access to his or her specific classroom assignments in WebAssign and access to a digital version of this text.
- A *Student Solutions Manual* (*SSM*) prepared by Warren S. Wright and Roberto Martinez provides solutions to selected problems from the text.
- Access to the Student Companion Website and Projects Center, available at go.jblearning.com/ZillAEM7e, is included with each new copy of the text. This site includes the following resources to enhance student learning:
  - Chapter 21 Probability
  - Chapter 22 Statistics
  - Additional projects and essays that appeared in earlier editions of this text, including:

    Two Properties of the Sphere

    Vibration Control: Vibration Isolation

    Vibration Control: Vibration Absorbers

    Minimal Surfaces

    Road Mirages

    Two Ports in Electrical Circuits

    The Hydrogen Atom

    Instabilities of Numerical Methods

    A Matrix Model for Environmental Life Cycle Assessment

    Steady Transonic Flow Past Thin Airfoils

    Making Waves: Convection, Diffusion, and Traffic Flow

    When Differential Equations Invaded Geometry: Inverse Tangent Problem of the 17th Century

    Tricky Time: The Isochrones of Huygens and Leibniz

    The Uncertainty Inequality in Signal Processing

    Traffic Flow

    Temperature Dependence of Resistivity

    Fraunhofer Diffraction by a Circular Aperture

    The Collapse of the Tacoma Narrow Bridge: A Modern Viewpoint

    Atmospheric Drag and the Decay of Satellite Orbits

    Forebody Drag of Bluff Bodies

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The task of compiling a text this size is, to say the least, difficult and many people have put much time and energy into this revision. So I would like to take this opportunity to express my sincerest appreciation to everyone—most of them unknown to me—at Jones & Bartlett Learning who were involved in the publication of this edition. A special word of thanks goes to my editor Laura Pagluica and production editor Jennifer Risden for their guidance in putting all the pieces of a large puzzle together.

Over the years I have been very fortunate to receive valuable input, solicited and unsolicited, from students and my academic colleagues. An occasional word of support is always appreciated, but it is the criticisms and suggestions for improvement that have enhanced each edition. So it is fitting that I once again recognize and thank the following reviewers for sharing their expertise and insights:

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Although many eyes have scanned the thousands of symbols and hundreds of equations in the text, it is a surety that some errors persist. I apologize for this in advance and I would certainly appreciate hearing about any errors that you may find, either in the text proper or in the supplemental manuals. In order to expedite their correction, contact my editor at: **EHinman@jblearning.com**

Dennis G. Zill

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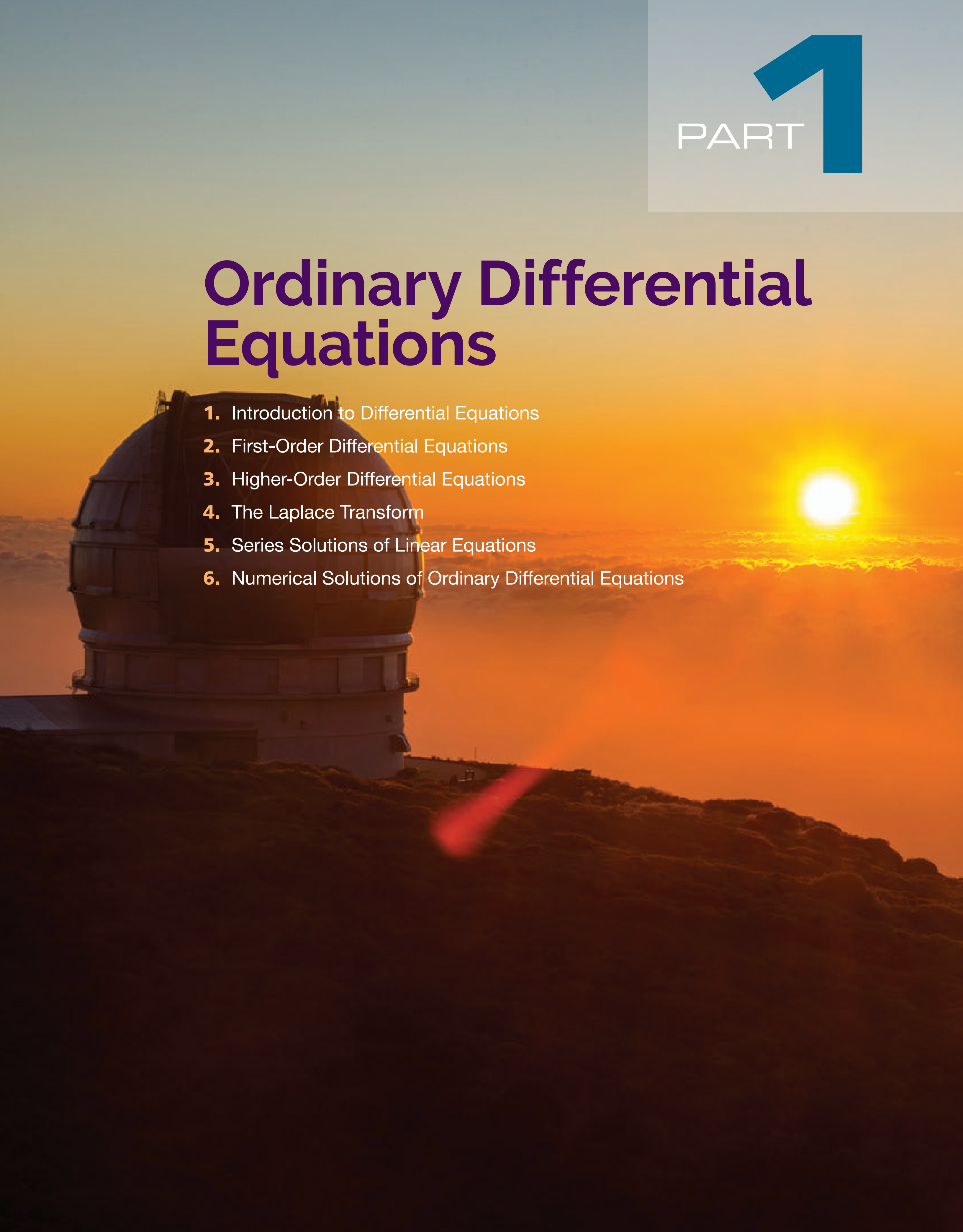
PART 1

Ordinary Differential Equations



Introduction to Differential Equations

The purpose of this short chapter is twofold: to introduce the basic terminology of **differential equations** and to briefly examine how differential equations arise in an attempt to describe or **model** physical phenomena in mathematical terms.

1.1 Definitions and Terminology

INTRODUCTION The words *differential* and *equation* certainly suggest solving some kind of equation that contains derivatives. But before you start solving anything, you must learn some of the basic definitions and terminology of the subject.

A Definition The derivative dy/dx of a function $y = \phi(x)$ is itself another function $\phi'(x)$ found by an appropriate rule. For example, the function $y = e^{0.1x^2}$ is differentiable on the interval $(-\infty, \infty)$, and its derivative is $dy/dx = 0.2xe^{0.1x^2}$. If we replace $e^{0.1x^2}$ in the last equation by the symbol y, we obtain

$$\frac{dy}{dx} = 0.2xy. \tag{1}$$

Now imagine that a friend of yours simply hands you the **differential equation** in (1), and that you have no idea how it was constructed. Your friend asks: "What is the function represented by the symbol y?" You are now face-to-face with *one* of the basic problems in a course in differential equations:

How do you solve such an equation for the unknown function $y = \phi(x)$?

The problem is loosely equivalent to the familiar reverse problem of differential calculus: Given a derivative, find an antiderivative.

Before proceeding any further, let us give a more precise definition of the concept of a differential equation.

DEFINITION 1.1.1 Differential Equation

An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation (DE)**.

In order to talk about them, we will classify a differential equation by **type**, **order**, and **linearity**.

Classification by Type If a differential equation contains only ordinary derivatives of one or more functions with respect to a *single* independent variable it is said to be an **ordinary differential equation (ODE)**. An equation involving only partial derivatives of one or more functions of two or more independent variables is called a **partial differential equation (PDE)**. Our first example illustrates several of each type of differential equation.

EXAMPLE 1 Types of Differential Equations

(a) The equations

an ODE can contain more than one dependent variable
↓ ↓

$$\frac{dy}{dx} + 6y = e^{-x}, \quad \frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = 0, \quad \text{and} \quad \frac{dx}{dt} + \frac{dy}{dt} = 3x + 2y \tag{2}$$

are examples of ordinary differential equations.

(b) The equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{3}$$

are examples of partial differential equations. Notice in the third equation that there are two dependent variables and two independent variables in the PDE. This indicates that u and v must be functions of *two or more* independent variables. ≡

Notation Throughout this text, ordinary derivatives will be written using either the **Leibniz notation** dy/dx, d^2y/dx^2, d^3y/dx^3, …, or the **prime notation** y', y'', y''', …. Using the latter notation, the first two differential equations in (2) can be written a little more compactly as $y' + 6y = e^{-x}$ and $y'' + y' - 12y = 0$, respectively. Actually, the prime notation is used to denote only the first three derivatives; the fourth derivative is written $y^{(4)}$ instead of y''''. In general, the nth derivative is d^ny/dx^n or $y^{(n)}$. Although less convenient to write and to typeset, the Leibniz notation has an advantage over the prime notation in that it clearly displays both the dependent and independent variables. For example, in the differential equation $d^2x/dt^2 + 16x = 0$, it is immediately seen that the symbol x now represents a dependent variable, whereas the independent variable is t. You should also be aware that in physical sciences and engineering, **Newton's dot notation** (derogatively referred to by some as the "flyspeck" notation) is sometimes used to denote derivatives with respect to time t. Thus the differential equation $d^2s/dt^2 = -32$ becomes $\ddot{s} = -32$. Partial derivatives are often denoted by a **subscript notation** indicating the independent variables. For example, the first and second equations in (3) can be written, in turn, as $u_{xx} + u_{yy} = 0$ and $u_{xx} = u_{tt} - u_t$.

Classification by Order The **order of a differential equation** (ODE or PDE) is the order of the highest derivative in the equation.

EXAMPLE 2 Order of a Differential Equation

The differential equations

$$\underset{\text{highest order}}{\frac{d^2y}{dx^2}} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x, \qquad 2\underset{\text{highest order}}{\frac{\partial^4 u}{\partial x^4}} + \frac{\partial^2 u}{\partial t^2} = 0$$

are examples of a second-order ordinary differential equation and a fourth-order partial differential equation, respectively. ≡

A first-order ordinary differential equation is sometimes written in the **differential form**

$$M(x, y)\,dx + N(x, y)\,dy = 0.$$

EXAMPLE 3 Differential Form of a First-Order ODE

If we assume that y is the dependent variable in a first-order ODE, then recall from calculus that the differential dy is defined to be $dy = y'dx$.

(a) By dividing by the differential dx an alternative form of the equation

$$(y - x)\,dx + 4x\,dy = 0$$

is given by

$$y - x + 4x\frac{dy}{dx} = 0 \quad \text{or equivalently} \quad 4x\frac{dy}{dx} + y = x.$$

(b) By multiplying the differential equation

$$6xy\frac{dy}{dx} + x^2 + y^2 = 0$$

by dx we see that the equation has the alternative differential form

$$(x^2 + y^2)\,dx + 6xy\,dy = 0.$$ ≡

In symbols, we can express an nth-order ordinary differential equation in one dependent variable by the **general form**

$$F(x, y, y', \dots, y^{(n)}) = 0, \tag{4}$$

where F is a real-valued function of $n + 2$ variables: $x, y, y', \ldots, y^{(n)}$. For both practical and theoretical reasons, we shall also make the assumption hereafter that it is possible to solve an ordinary differential equation in the form (4) uniquely for the highest derivative $y^{(n)}$ in terms of the remaining $n + 1$ variables. The differential equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}), \tag{5}$$

where f is a real-valued continuous function, is referred to as the **normal form** of (4). Thus, when it suits our purposes, we shall use the normal forms

$$\frac{dy}{dx} = f(x, y) \quad \text{and} \quad \frac{d^2y}{dx^2} = f(x, y, y')$$

to represent general first- and second-order ordinary differential equations.

EXAMPLE 4 Normal Form of an ODE

(a) By solving for the derivative dy/dx the normal form of the first-order differential equation

$$4x\frac{dy}{dx} + y = x \quad \text{is} \quad \frac{dy}{dx} = \frac{x - y}{4x}.$$

(b) By solving for the derivative y'' the normal form of the second-order differential equation

$$y'' - y' + 6y = 0 \quad \text{is} \quad y'' = y' - 6y.$$

≡

Classification by Linearity An nth-order ordinary differential equation (4) is said to be **linear** in the variable y if F is linear in $y, y', \ldots, y^{(n)}$. This means that an nth-order ODE is linear when (4) is $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y - g(x) = 0$ or

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x). \tag{6}$$

Two important special cases of (6) are **linear first-order** ($n = 1$) and **linear second-order** ($n = 2$) ODEs.

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \quad \text{and} \quad a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x). \tag{7}$$

In the additive combination on the left-hand side of (6) we see that the characteristic two properties of a linear ODE are

Remember these two characteristics of a linear ODE.

- The dependent variable y and all its derivatives $y', y'', \ldots, y^{(n)}$ are of the first degree; that is, the power of each term involving y is 1.
- The coefficients $a_0, a_1, \ldots, a_n$ of $y, y', \ldots, y^{(n)}$ depend at most on the independent variable x.

A **nonlinear** ordinary differential equation is simply one that is not linear. If the coefficients of $y, y', \ldots, y^{(n)}$ contain the dependent variable y or its derivatives or if powers of $y, y', \ldots, y^{(n)}$, such as $(y')^2$, appear in the equation, then the DE is nonlinear. Also, nonlinear functions of the dependent variable or its derivatives, such as $\sin y$ or $e^{y'}$, cannot appear in a linear equation.

EXAMPLE 5 Linear and Nonlinear Differential Equations

(a) The equations

$$(y - x)\,dx + 4x\,dy = 0, \quad y'' - 2y' + y = 0, \quad x^3\frac{d^3y}{dx^3} + 3x\frac{dy}{dx} - 5y = e^x$$

are, in turn, examples of *linear* first-, second-, and third-order ordinary differential equations. We have just demonstrated in part (a) of Example 3 that the first equation is linear in y by writing it in the alternative form $4xy' + y = x$.

(b) The equations

$$\underset{\substack{\text{nonlinear term:}\\ \text{coefficient depends on } y\\ \downarrow}}{(1-y)}y' + 2y = e^x, \qquad \frac{d^2y}{dx^2} + \underset{\substack{\text{nonlinear term:}\\ \text{nonlinear function of } y\\ \downarrow}}{\sin y} = 0, \qquad \frac{d^4y}{dx^4} + \underset{\substack{\text{nonlinear term:}\\ \text{power not 1}\\ \downarrow}}{y^2} = 0,$$

are examples of *nonlinear* first-, second-, and fourth-order ordinary differential equations, respectively.

(c) By using the quadratic formula the nonlinear first-order differential equation $(y')^2 + 2xy' - y = 0$ can be written as two nonlinear first-order equations in normal form

$$y' = -x + \sqrt{x^2 + y} \quad \text{and} \quad y' = -x - \sqrt{x^2 + y}.$$

≡

Solution

As stated before, one of our goals in this course is to solve—or find solutions of—differential equations. The concept of a solution of an ordinary differential equation is defined next.

DEFINITION 1.1.2 Solution of an ODE

Any function ϕ, defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an nth-order ordinary differential equation reduces the equation to an identity, is said to be a **solution** of the equation on the interval.

In other words, a solution of an nth-order ordinary differential equation (4) is a function ϕ that possesses at least n derivatives and

$$F(x, \phi(x), \phi'(x), \ldots, \phi^{(n)}(x)) = 0 \text{ for all } x \text{ in } I.$$

We say that ϕ *satisfies* the differential equation on I. For our purposes, we shall also assume that a solution ϕ is a real-valued function. In our initial discussion we have already seen that $y = e^{0.1x^2}$ is a solution of $dy/dx = 0.2xy$ on the interval $(-\infty, \infty)$.

Occasionally it will be convenient to denote a solution by the alternative symbol $y(x)$.

Interval of Definition

You can't think *solution* of an ordinary differential equation without simultaneously thinking *interval*. The interval I in Definition 1.1.2 is variously called the **interval of definition**, the **interval of validity**, or the **domain of the solution** and can be an open interval (a, b), a closed interval $[a, b]$, an infinite interval (a, ∞), and so on.

EXAMPLE 6 Verification of a Solution

Verify that the indicated function is a solution of the given differential equation on the interval $(-\infty, \infty)$.

(a) $\dfrac{dy}{dx} = xy^{1/2}$; $\quad y = \frac{1}{16}x^4$ **(b)** $y'' - 2y' + y = 0$; $\quad y = xe^x$

SOLUTION One way of verifying that the given function is a solution is to see, after substituting, whether each side of the equation is the same for every x in the interval $(-\infty, \infty)$.

(a) From *left-hand side*: $\dfrac{dy}{dx} = 4 \cdot \dfrac{x^3}{16} = \dfrac{x^3}{4}$

right-hand side: $xy^{1/2} = x \cdot \left(\dfrac{x^4}{16}\right)^{1/2} = x \cdot \dfrac{x^2}{4} = \dfrac{x^3}{4}$,

we see that each side of the equation is the same for every real number x. Note that $y^{1/2} = \frac{1}{4}x^2$ is, by definition, the nonnegative square root of $\frac{1}{16}x^4$.

(b) From the derivatives $y' = xe^x + e^x$ and $y'' = xe^x + 2e^x$ we have for every real number x,

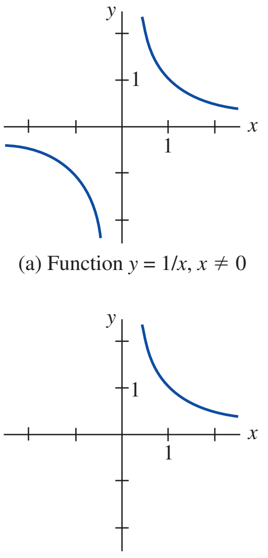
$$\textit{left-hand side:}\;\; y'' - 2y' + y = (xe^x + 2e^x) - 2(xe^x + e^x) + xe^x = 0$$

$$\textit{right-hand side:}\;\; 0. \qquad \equiv$$

Note, too, that in Example 6 each differential equation possesses the constant solution $y = 0$, defined on $(-\infty, \infty)$. A solution of a differential equation that is identically zero on an interval I is said to be a **trivial solution**.

Solution Curve

The graph of a solution ϕ of an ODE is called a **solution curve**. Since ϕ is a differentiable function, it is continuous on its interval I of definition. Thus there may be a difference between the graph of the *function* ϕ and the graph of the *solution* ϕ. Put another way, the domain of the function ϕ does not need to be the same as the interval I of definition (or domain) of the solution ϕ.



(a) Function $y = 1/x$, $x \neq 0$

(b) Solution $y = 1/x$, $(0, \infty)$

FIGURE 1.1.1 Example 7 illustrates the difference between the function $y = 1/x$ and the solution $y = 1/x$

EXAMPLE 7 Function vs. Solution

(a) Considered simply as a *function*, the domain of $y = 1/x$ is the set of all real numbers x except 0. When we graph $y = 1/x$, we plot points in the xy-plane corresponding to a judicious sampling of numbers taken from its domain. The rational function $y = 1/x$ is discontinuous at 0, and its graph, in a neighborhood of the origin, is given in **FIGURE 1.1.1(a)**. The function $y = 1/x$ is not differentiable at $x = 0$ since the y-axis (whose equation is $x = 0$) is a vertical asymptote of the graph.

(b) Now $y = 1/x$ is also a solution of the linear first-order differential equation $xy' + y = 0$ (verify). But when we say $y = 1/x$ is a *solution* of this DE we mean it is a function defined on an interval I on which it is differentiable and satisfies the equation. In other words, $y = 1/x$ is a solution of the DE on *any* interval not containing 0, such as $(-3, -1)$, $(\frac{1}{2}, 10)$, $(-\infty, 0)$, or $(0, \infty)$. Because the solution curves defined by $y = 1/x$ on the intervals $(-3, -1)$ and on $(\frac{1}{2}, 10)$ are simply segments or pieces of the solution curves defined by $y = 1/x$ on $(-\infty, 0)$ and $(0, \infty)$, respectively, it makes sense to take the interval I to be as large as possible. Thus we would take I to be either $(-\infty, 0)$ or $(0, \infty)$. The solution curve on the interval $(0, \infty)$ is shown in Figure 1.1.1(b). $\equiv$

Explicit and Implicit Solutions

You should be familiar with the terms *explicit* and *implicit functions* from your study of calculus. A solution in which the dependent variable is expressed solely in terms of the independent variable and constants is said to be an **explicit solution**. For our purposes, let us think of an explicit solution as an explicit formula $y = \phi(x)$ that we can manipulate, evaluate, and differentiate using the standard rules. We have just seen in the last two examples that $y = \frac{1}{16}x^4$, $y = xe^x$, and $y = 1/x$ are, in turn, explicit solutions of $dy/dx = xy^{1/2}$, $y'' - 2y' + y = 0$, and $xy' + y = 0$. Moreover, the trivial solution $y = 0$ is an explicit solution of all three equations. We shall see when we get down to the business of actually solving some ordinary differential equations that methods of solution do not always lead directly to an explicit solution $y = \phi(x)$. This is particularly true when attempting to solve nonlinear first-order differential equations. Often we have to be content with a relation or expression $G(x, y) = 0$ that defines a solution ϕ implicitly.

DEFINITION 1.1.3 Implicit Solution of an ODE

A relation $G(x, y) = 0$ is said to be an **implicit solution** of an ordinary differential equation (4) on an interval I provided there exists at least one function ϕ that satisfies the relation as well as the differential equation on I.

It is beyond the scope of this course to investigate the conditions under which a relation $G(x, y) = 0$ defines a differentiable function ϕ. So we shall assume that if the formal implementation of a method of solution leads to a relation $G(x, y) = 0$, then there exists at least one function ϕ that satisfies both the relation (that is, $G(x, \phi(x)) = 0$) and the differential equation on an interval I. If the implicit solution $G(x, y) = 0$ is fairly simple, we may be able to solve for y in terms of x and obtain one or more explicit solutions. See (*iv*) in the *Remarks*.

EXAMPLE 8 Verification of an Implicit Solution

The relation $x^2 + y^2 = 25$ is an implicit solution of the nonlinear differential equation

$$\frac{dy}{dx} = -\frac{x}{y} \tag{8}$$

on the interval defined by $-5 < x < 5$. By implicit differentiation we obtain

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}25 \qquad \text{or} \qquad 2x + 2y\frac{dy}{dx} = 0. \tag{9}$$

Solving the last equation in (9) for the symbol dy/dx gives (8). Moreover, solving $x^2 + y^2 = 25$ for y in terms of x yields $y = \pm\sqrt{25 - x^2}$. The two functions $y = \phi_1(x) = \sqrt{25 - x^2}$ and $y = \phi_2(x) = -\sqrt{25 - x^2}$ satisfy the relation (that is, $x^2 + \phi_1^2 = 25$ and $x^2 + \phi_2^2 = 25$) and are explicit solutions defined on the interval $(-5, 5)$. The solution curves given in **FIGURE 1.1.2(b)** and 1.1.2(c) are segments of the graph of the implicit solution in Figure 1.1.2(a).

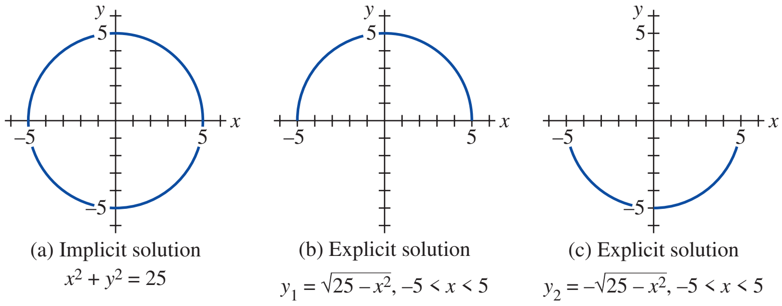


FIGURE 1.1.2 An implicit solution and two explicit solutions in Example 8

Any relation of the form $x^2 + y^2 - c = 0$ *formally* satisfies (8) for any constant c. However, it is understood that the relation should always make sense in the real number system; thus, for example, we cannot say that $x^2 + y^2 + 25 = 0$ is an implicit solution of the equation. Why not?

Because the distinction between an explicit solution and an implicit solution should be intuitively clear, we will not belabor the issue by always saying, "Here is an explicit (implicit) solution."

Families of Solutions

The study of differential equations is similar to that of integral calculus. When evaluating an antiderivative or indefinite integral in calculus, we use a single constant c of integration. Analogously, when solving a first-order differential equation $F(x, y, y') = 0$, we *usually* obtain a solution containing a single arbitrary constant or parameter c. A solution containing an arbitrary constant represents a set $G(x, y, c) = 0$ of solutions called a **one-parameter family of solutions**. When solving an nth-order differential equation $F(x, y, y', \ldots, y^{(n)}) = 0$, we seek an ***n*-parameter family of solutions** $G(x, y, c_1, c_2, \ldots, c_n) = 0$. This means that a single differential equation can possess an infinite number of solutions corresponding to the unlimited number of choices for the parameter(s). A solution of a differential equation that is free of arbitrary parameters is called a **particular solution**.

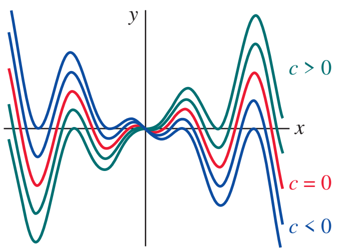


FIGURE 1.1.3 Some solutions of DE in part (a) of Example 9

EXAMPLE 9 Particular Solution

(a) For all values of c, the one-parameter family $y = cx - x\cos x$ is an explicit solution of the linear first-order differential equation

$$xy' - y = x^2 \sin x$$

on the interval $(-\infty, \infty)$. **FIGURE 1.1.3** shows the graphs of some particular solutions in this family for various choices of c. The solution $y = -x\cos x$, the red curve in the figure, is a particular solution corresponding to $c = 0$.

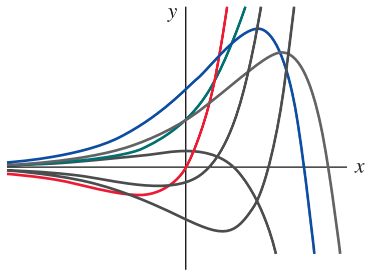


FIGURE 1.1.4 Some solutions of DE in part (b) of Example 9

(b) The two-parameter family $y = c_1e^x + c_2xe^x$ is an explicit solution of the linear second-order differential equation

$$y'' - 2y' + y = 0$$

in part (b) of Example 6. **FIGURE 1.1.4** shows seven of the "double infinity" of solutions in this family. The solution curves in red, green, and blue are the graphs of the particular solutions $y = 5xe^x$ $(c_1 = 0, c_2 = 5)$, $y = 3e^x$ $(c_1 = 3, c_2 = 0)$, and $y = 5e^x - 2xe^x$ $(c_1 = 5, c_2 = -2)$ respectively. ≡

In all the preceding examples, we have used x and y to denote the independent and dependent variables, respectively. But you should become accustomed to seeing and working with other symbols to denote these variables. For example, we could denote the independent variable by t and the dependent variable by x.

EXAMPLE 10 Using Different Symbols

The functions $x = c_1 \cos 4t$ and $x = c_2 \sin 4t$, where c_1 and c_2 are arbitrary constants or parameters, are both solutions of the linear differential equation

$$x'' + 16x = 0.$$

For $x = c_1 \cos 4t$, the first two derivatives with respect to t are $x' = -4c_1 \sin 4t$ and $x'' = -16c_1 \cos 4t$. Substituting x'' and x then gives

$$x'' + 16x = -16c_1 \cos 4t + 16(c_1 \cos 4t) = 0.$$

In like manner, for $x = c_2 \sin 4t$ we have $x'' = -16c_2 \sin 4t$, and so

$$x'' + 16x = -16c_2 \sin 4t + 16(c_2 \sin 4t) = 0.$$

Finally, it is straightforward to verify that the linear combination of solutions for the two-parameter family $x = c_1 \cos 4t + c_2 \sin 4t$ is also a solution of the differential equation. ≡

The next example shows that a solution of a differential equation can be a piecewise-defined function.

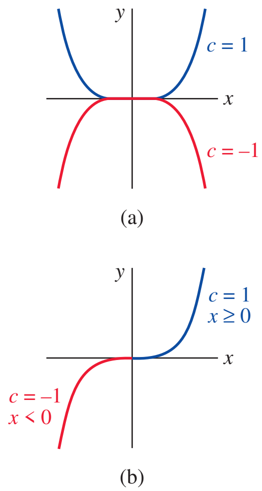


FIGURE 1.1.5 Some solutions of $xy' - 4y = 0$ in Example 11

EXAMPLE 11 A Piecewise-Defined Solution

You should verify that the one-parameter family $y = cx^4$ is a one-parameter family of solutions of the linear differential equation $xy' - 4y = 0$ on the interval $(-\infty, \infty)$. See **FIGURE 1.1.5(a)**. The piecewise-defined differentiable function

$$y = \begin{cases} -x^4, & x < 0 \\ x^4, & x \geq 0 \end{cases}$$

is a particular solution of the equation but cannot be obtained from the family $y = cx^4$ by a single choice of c; the solution is constructed from the family by choosing $c = -1$ for $x < 0$ and $c = 1$ for $x \geq 0$. See **FIGURE 1.1.5(b)**. ≡

Singular Solution Sometimes an nth-order differential equation possesses a solution that is not a member of an n-parameter family of solutions of the equation—that is, a solution that cannot be obtained by specializing *any* of the parameters in the family of solutions. Such a solution is called a **singular solution**.*

EXAMPLE 12 Singular Solution

We saw on pages 6 and 7 that the functions $y = \frac{1}{16}x^4$ and $y = 0$ are solutions of the differential equation $dy/dx = xy^{\frac{1}{2}}$ on $(-\infty, \infty)$. In Section 2.2 we shall demonstrate, by actually solving it, that the differential equation $dy/dx = xy^{\frac{1}{2}}$ possesses the one-parameter family of solutions

*There is a bit more to the definition of a singular solution, but it is beyond the intended level of this text.

$y = (\frac{1}{4}x^2 + c)^2$, $c \geq 0$. When $c = 0$ the resulting particular solution is $y = \frac{1}{16}x^4$. But the trivial solution $y = 0$ is a singular solution since it is not a member of the family $y = (\frac{1}{4}x^2 + c)^2$; there is no way of assigning a value to the constant c to obtain $y = 0$. ≡

Systems of Differential Equations

Up to this point we have been discussing single differential equations containing one unknown function. But often in theory, as well as in many applications, we must deal with systems of differential equations. A **system of ordinary differential equations** is two or more equations involving the derivatives of two or more unknown functions of a single independent variable. For example, if x and y denote dependent variables and t the independent variable, then a system of two first-order differential equations is given by

$$\frac{dx}{dt} = f(t, x, y)$$
$$\frac{dy}{dt} = g(t, x, y). \tag{10}$$

A **solution** of a system such as (10) is a pair of differentiable functions $x = \phi_1(t)$, $y = \phi_2(t)$ defined on a common interval I that satisfy each equation of the system on this interval. See Problems 49 and 50 in Exercises 1.1.

REMARKS

(*i*) It might not be apparent whether a first-order ODE written in differential form $M(x, y)\,dx + N(x, y)\,dy = 0$ is linear or nonlinear because there is nothing in this form that tells us which symbol denotes the dependent variable. See Problems 11 and 12 in Exercises 1.1.

(*ii*) We will see in the chapters that follow that a solution of a differential equation may involve an **integral-defined function**. One way of defining a function F of a single variable x by means of a definite integral is

$$F(x) = \int_a^x g(t)\,dt. \tag{11}$$

If the integrand g in (11) is continuous on an interval $[a, b]$ and $a \leq x \leq b$, then the derivative form of the Fundamental Theorem of Calculus states that F is differentiable on (a, b) and

$$F'(x) = \frac{d}{dx}\int_a^x g(t)\,dt = g(x). \tag{12}$$

The integral in (11) is often **nonelementary**, that is, an integral of a function g that does not have an elementary-function antiderivative. Elementary functions include the familiar functions studied in a typical precalculus course:

constant, polynomial, rational, exponential, logarithmic, trigonometric, and inverse trigonometric functions,

as well as rational powers of these functions, finite combinations of these functions using addition, subtraction, multiplication, division, and function compositions. For example, even though e^{-t^2}, $\sqrt{1 + t^3}$, and $\cos t^2$ are elementary functions, the integrals $\int e^{-t^2}\,dt$, $\int \sqrt{1 + t^3}\,dt$, and $\int \cos t^2\,dt$ are nonelementary. See Problems 27–30 in Exercises 1.1.

(*iii*) Although the concept of a solution of a differential equation has been emphasized in this section, you should be aware that a DE does not necessarily have to possess a solution. See Problem 51 in Exercises 1.1. The question of whether a solution exists will be touched on in the next section.

(*iv*) A few last words about implicit solutions of differential equations are in order. In Example 8 we were able to solve the relation $x^2 + y^2 = 25$ for y in terms of x to get two explicit solutions, $\phi_1(x) = \sqrt{25 - x^2}$ and $\phi_2(x) = -\sqrt{25 - x^2}$, of the differential equation (8). But don't read too much into this one example. Unless it is easy, obvious, or important, or you are instructed to, there is usually no need to try to solve an implicit solution $G(x, y) = 0$ for y explicitly in terms of x. Also do not misinterpret the second sentence following Definition 1.1.3. An implicit solution $G(x, y) = 0$ can define a perfectly good differentiable function ϕ that is a solution of a DE, but yet we may not be able to solve $G(x, y) = 0$ using analytical methods such as algebra. The solution curve of ϕ may be a segment or piece of the graph of $G(x, y) = 0$. See Problems 57 and 58 in Exercises 1.1.

(*v*) If *every* solution of an nth-order ODE $F(x, y, y', \ldots, y^{(n)}) = 0$ on an interval I can be obtained from an n-parameter family $G(x, y, c_1, c_2, \ldots, c_n) = 0$ by appropriate choices of the parameters c_i, $i = 1, 2, \ldots, n$, we then say that the family is the **general solution** of the DE. In solving linear ODEs, we shall impose relatively simple restrictions on the coefficients of the equation; with these restrictions one can be assured that not only does a solution exist on an interval but also that a family of solutions yields all possible solutions. Nonlinear equations, with the exception of some first-order DEs, are usually difficult or even impossible to solve in terms of familiar elementary functions. Furthermore, if we happen to obtain a family of solutions for a nonlinear equation, it is not evident whether this family contains all solutions. On a practical level, then, the designation "general solution" is applied only to linear DEs. Don't be concerned about this concept at this point but store the words *general solution* in the back of your mind—we will come back to this notion in Section 2.3 and again in Chapter 3.

1.1 Exercises

Answers to selected odd-numbered problems begin on page ANS-1.

In Problems 1–10, state the order of the given ordinary differential equation. Determine whether the equation is linear or nonlinear by matching it with (6).

1. $(1 - x)y'' - 4xy' + 5y = \cos x$

2. $x\dfrac{d^3y}{dx^3} - \left(\dfrac{dy}{dx}\right)^4 + y = 0$

3. $t^5y^{(4)} - t^3y'' + 6y = 0$

4. $\dfrac{d^2u}{dr^2} + \dfrac{du}{dr} + u = \cos(r + u)$

5. $\dfrac{d^2y}{dx^2} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$

6. $\dfrac{d^2R}{dt^2} = -\dfrac{k}{R^2}$

7. $(\sin\theta)y''' - (\cos\theta)y' = 2$

8. $\ddot{x} - (1 - \frac{1}{3}\dot{x}^2)\dot{x} + x = 0$

9. $\sin\left(\dfrac{dy}{dx}\right) = y + x$

10. $\dfrac{dx}{dy} + y^3x = \sin y$

In Problems 11 and 12, determine whether the given first-order differential equation is linear in the indicated dependent variable by matching it with the first differential equation given in (7).

11. $(y^2 - 1)\,dx + x\,dy = 0$; in y; in x

12. $u\,dv + (v + uv - ue^u)\,du = 0$; in v; in u

In Problems 13–16, verify that the indicated function is an explicit solution of the given differential equation. Assume an appropriate interval I of definition for each solution.

13. $2y' + y = 0$; $y = e^{-x/2}$

14. $\dfrac{dy}{dt} + 20y = 24$; $y = \frac{6}{5} - \frac{6}{5}e^{-20t}$

15. $y'' - 6y' + 13y = 0$; $y = e^{3x}\cos 2x$

16. $y'' + y = \tan x$; $y = -(\cos x)\ln(\sec x + \tan x)$

In Problems 17–20, verify that the indicated function $y = \phi(x)$ is an explicit solution of the given first-order differential equation. Proceed as in Example 7, by considering ϕ simply as a *function*, give its domain. Then by considering ϕ as a *solution* of the differential equation, give at least one interval I of definition.

17. $(y - x)y' = y - x + 8$; $y = x + 4\sqrt{x + 2}$

18. $y' = 25 + y^2$; $y = 5\tan 5x$

19. $y' = 2xy^2$; $y = 1/(4 - x^2)$

20. $2y' = y^3\cos x$; $y = (1 - \sin x)^{-1/2}$

In Problems 21 and 22, verify that the indicated expression is an implicit solution of the given first-order differential equation. Find at least one explicit solution $y = \phi(x)$ in each case. Use a graphing utility to obtain the graph of an explicit solution. Give an interval I of definition of each solution ϕ.

21. $\dfrac{dX}{dt} = (X - 1)(1 - 2X)$; $\ln\left(\dfrac{2X - 1}{X - 1}\right) = t$

22. $2xy\,dx + (x^2 - y)\,dy = 0$; $-2x^2y + y^2 = 1$

In Problems 23–26, verify that the indicated family of functions is a solution of the given differential equation. Assume an appropriate interval I of definition for each solution.

23. $\dfrac{dP}{dt} = P(1 - P); \quad P = \dfrac{c_1e^t}{1 + c_1e^t}$

24. $\dfrac{dy}{dx} + 4xy = 8x^3; \quad y = 2x^2 - 1 + c_1e^{-2x^2}$

25. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0; \quad y = c_1e^{2x} + c_2xe^{2x}$

26. $x^3\dfrac{d^3y}{dx^3} + 2x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 12x^2;$

$y = c_1x^{-1} + c_2x + c_3x\ln x + 4x^2$

In Problems 27–30, use (12) to verify that the indicated function is a solution of the given differential equation. Assume an appropriate interval I of definition of each solution.

27. $x\dfrac{dy}{dx} - 3xy = 1; \quad y = e^{3x}\displaystyle\int_1^x \frac{e^{-3t}}{t}\,dt$

28. $2x\dfrac{dy}{dx} - y = 2x\cos x; \quad y = \sqrt{x}\displaystyle\int_4^x \frac{\cos t}{\sqrt{t}}\,dt$

29. $x^2\dfrac{dy}{dx} + xy = 10\sin x; \quad y = \dfrac{5}{x} + \dfrac{10}{x}\displaystyle\int_1^x \frac{\sin t}{t}\,dt$

30. $\dfrac{dy}{dx} + 2xy = 1; \quad y = e^{-x^2} + e^{-x^2}\displaystyle\int_0^x e^{t^2}\,dt$

31. Verify that the piecewise-defined function

$$y = \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$$

is a solution of the differential equation $xy' - 2y = 0$ on the interval $(-\infty, \infty)$.

32. In Example 8 we saw that $y = \phi_1(x) = \sqrt{25 - x^2}$ and $y = \phi_2(x) = -\sqrt{25 - x^2}$ are solutions of $dy/dx = -x/y$ on the interval $(-5, 5)$. Explain why the piecewise-defined function

$$y = \begin{cases} \sqrt{25 - x^2}, & -5 < x < 0 \\ -\sqrt{25 - x^2}, & 0 \leq x < 5 \end{cases}$$

is not a solution of the differential equation on the interval $(-5, 5)$.

In Problems 33–36, find values of m so that the function $y = e^{mx}$ is a solution of the given differential equation.

33. $y' + 2y = 0$

34. $3y' = 4y$

35. $y'' - 5y' + 6y = 0$

36. $2y'' + 9y' - 5y = 0$

In Problems 37–40, find values of m so that the function $y = x^m$ is a solution of the given differential equation.

37. $xy'' + 2y' = 0$

38. $4x^2y'' + y = 0$

39. $x^2y'' - 7xy' + 15y = 0$

40. $x^2y''' - 3xy'' + 3y' = 0$

In Problems 41–44, use the concept that $y = c$, $-\infty < x < \infty$, is a constant function if and only if $y' = 0$ to determine whether the given differential equation possesses constant solutions.

41. $3xy' + 5y = 10$

42. $y' = y^2 + 2y - 3$

43. $(y - 1)y' = 1$

44. $y'' + 4y' + 6y = 10$

In Problems 45–48, verify that the one-parameter family is a solution of the given differential equation. Find at least one singular solution of the DE.

45. $y = (x + c_1)^2; \quad \left(\dfrac{dy}{dx}\right)^2 = 4y$

46. $y = 3\sin(x + c_1); \quad \left(\dfrac{dy}{dx}\right)^2 = 9 - y^2$

47. $x - \sqrt{16 - y^2} = c_1; \quad y\dfrac{dy}{dx} + \sqrt{16 - y^2} = 0$

48. $y = x - (x - c_1)^2; \quad \left(\dfrac{dy}{dx}\right)^2 - 2\dfrac{dy}{dx} + 4y = 4x - 1$

In Problems 49 and 50, verify that the indicated pair of functions is a solution of the given system of differential equations on the interval $(-\infty, \infty)$.

49. $\dfrac{dx}{dt} = x + 3y$

$\dfrac{dy}{dt} = 5x + 3y;$

$x = e^{-2t} + 3e^{6t},$

$y = -e^{-2t} + 5e^{6t}$

50. $\dfrac{d^2x}{dt^2} = 4y + e^t$

$\dfrac{d^2y}{dt^2} = 4x - e^t;$

$x = \cos 2t + \sin 2t + \frac{1}{5}e^t,$

$y = -\cos 2t - \sin 2t - \frac{1}{5}e^t$

Discussion Problems

51. Make up a differential equation that does not possess any real solutions.

52. Make up a differential equation that you feel confident possesses only the trivial solution $y = 0$. Explain your reasoning.

53. What function do you know from calculus is such that its first derivative is itself? Its first derivative is a constant multiple k of itself? Write each answer in the form of a first-order differential equation with a solution.

54. What function (or functions) do you know from calculus is such that its second derivative is itself? Its second derivative is the negative of itself? Write each answer in the form of a second-order differential equation with a solution.

55. Given that $y = \sin x$ is an explicit solution of the first-order differential equation $dy/dx = \sqrt{1 - y^2}$. Find an interval I of definition. [*Hint*: I is *not* the interval $(-\infty, \infty)$.]

56. Discuss why it makes intuitive sense to presume that the linear differential equation $y'' + 2y' + 4y = 5\sin t$ has a solution of the form $y = A\sin t + B\cos t$, where A and B are constants. Then find specific constants A and B so that $y = A\sin t + B\cos t$ is a particular solution of the DE.

In Problems 57 and 58, the given figure represents the graph of an implicit solution $G(x, y) = 0$ of a differential equation $dy/dx = f(x, y)$. In each case the relation $G(x, y) = 0$ implicitly defines several solutions of the DE. Carefully reproduce each figure on a piece of paper. Use different colored pencils to mark off segments, or pieces, on each graph that correspond to graphs of solutions. Keep in mind that a solution ϕ must be a function and differentiable. Use the solution curve to estimate the interval I of definition of each solution ϕ.

57.

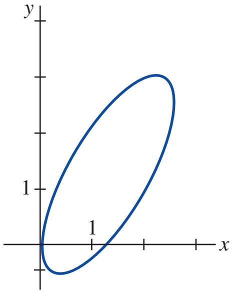


FIGURE 1.1.6 Graph for Problem 57

58.

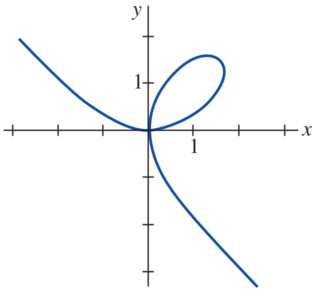


FIGURE 1.1.7 Graph for Problem 58

59. The graphs of the members of the one-parameter family $x^3 + y^3 = 3cxy$ are called **folia of Descartes** after the French mathematician and inventor of analytic geometry, **René Descartes** (1596–1650). Verify that this family is an implicit solution of the first-order differential equation.

$$\frac{dy}{dx} = \frac{y(y^3 - 2x^3)}{x(2y^3 - x^3)}.$$

60. The graph in **FIGURE 1.1.7** is the member of the family of folia in Problem 59 corresponding to $c = 1$. Discuss: How can the DE in Problem 59 help in finding points on the graph of $x^3 + y^3 = 3xy$ where the tangent line is vertical? How does knowing where a tangent line is vertical help in determining an interval I of definition of a solution ϕ of the DE? Carry out your ideas and compare with your estimates of the intervals in Problem 58.

61. In Example 8, the largest interval I over which the explicit solutions $y = \phi_1(x)$ and $y = \phi_2(x)$ are defined is the open interval $(-5, 5)$. Why can't the interval I of definition be the closed interval $[-5, 5]$?

62. In Problem 23, a one-parameter family of solutions of the DE $P' = P(1 - P)$ is given. Does any solution curve pass through the point $(0, 3)$? Through the point $(0, 1)$?

63. Discuss, and illustrate with examples, how to solve differential equations of the forms $dy/dx = f(x)$ and $d^2y/dx^2 = f(x)$.

64. The differential equation $x(y')^2 - 4y' - 12x^3 = 0$ has the form given in (4). Determine whether the equation can be put into the normal form $dy/dx = f(x, y)$.

65. The normal form (5) of an nth-order differential equation is equivalent to (4) whenever both forms have exactly the same solutions. Make up a first-order differential equation for which $F(x, y, y') = 0$ is not equivalent to the normal form $dy/dx = f(x, y)$.

66. Find a linear second-order differential equation $F(x, y, y', y'') = 0$ for which $y = c_1x + c_2x^2$ is a two-parameter family of solutions. Make sure that your equation is free of the arbitrary parameters c_1 and c_2.

Qualitative information about a solution $y = \phi(x)$ of a differential equation can often be obtained from the equation itself. Before working Problems 67–70, recall the geometric significance of the derivatives dy/dx and d^2y/dx^2.

67. Consider the differential equation $dy/dx = e^{-x^2}$.

(a) Explain why a solution of the DE must be an increasing function on any interval of the x-axis.

(b) What are $\lim_{x\to-\infty} dy/dx$ and $\lim_{x\to\infty} dy/dx$? What does this suggest about a solution curve as $x \to \pm\infty$?

(c) Determine an interval over which a solution curve is concave down and an interval over which the curve is concave up.

(d) Sketch the graph of a solution $y = \phi(x)$ of the differential equation whose shape is suggested by parts (a)–(c).

68. Consider the differential equation $dy/dx = 5 - y$.

(a) Either by inspection, or by the method suggested in Problems 41–44, find a constant solution of the DE.

(b) Using only the differential equation, find intervals on the y-axis on which a nonconstant solution $y = \phi(x)$ is increasing. Find intervals on the y-axis on which $y = \phi(x)$ is decreasing.

69. Consider the differential equation $dy/dx = y(a - by)$, where a and b are positive constants.

(a) Either by inspection, or by the method suggested in Problems 41–44, find two constant solutions of the DE.

(b) Using only the differential equation, find intervals on the y-axis on which a nonconstant solution $y = \phi(x)$ is increasing. On which $y = \phi(x)$ is decreasing.

(c) Using only the differential equation, explain why $y = a/2b$ is the y-coordinate of a point of inflection of the graph of a nonconstant solution $y = \phi(x)$.

(d) On the same coordinate axes, sketch the graphs of the two constant solutions found in part (a). These constant solutions partition the xy-plane into three regions. In each region, sketch the graph of a nonconstant solution $y = \phi(x)$ whose shape is suggested by the results in parts (b) and (c).

70. Consider the differential equation $y' = y^2 + 4$.

(a) Explain why there exist no constant solutions of the DE.

(b) Describe the graph of a solution $y = \phi(x)$. For example, can a solution curve have any relative extrema?

(c) Explain why $y = 0$ is the y-coordinate of a point of inflection of a solution curve.

(d) Sketch the graph of a solution $y = \phi(x)$ of the differential equation whose shape is suggested by parts (a)–(c).

Computer Lab Assignments

In Problems 71 and 72, use a CAS to compute all derivatives and to carry out the simplifications needed to verify that the indicated function is a particular solution of the given differential equation.

71. $y^{(4)} - 20y''' + 158y'' - 580y' + 841y = 0;$
$y = xe^{5x}\cos 2x$

72. $x^3y''' + 2x^2y'' + 20xy' - 78y = 0;$

$$y = 20\frac{\cos(5\ln x)}{x} - 3\frac{\sin(5\ln x)}{x}$$

1.2 Initial-Value Problems

INTRODUCTION We are often interested in problems in which we seek a solution $y(x)$ of a differential equation so that $y(x)$ satisfies prescribed side conditions—that is, conditions that are imposed on the unknown $y(x)$ or on its derivatives. In this section we examine one such problem called an *initial-value problem.*

Initial-Value Problem

On some interval I containing x_0, the problem

$$\textit{Solve:}\quad \frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}) \tag{1}$$

$$\textit{Subject to:}\quad y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots,\ y^{(n-1)}(x_0) = y_{n-1},$$

where $y_0, y_1, \ldots, y_{n-1}$ are arbitrarily specified real constants, is called an **initial-value problem (IVP)**. The values of $y(x)$ and its first $n-1$ derivatives at a single point x_0: $y(x_0) = y_0$, $y'(x_0) = y_1, \ldots,$ $y^{(n-1)}(x_0) = y_{n-1}$, are called **initial conditions (IC)**.

First- and Second-Order IVPs

The problem given in (1) is also called an ***n*th-order initial-value problem**. For example,

$$\textit{Solve:}\quad \frac{dy}{dx} = f(x, y) \qquad \textit{Subject to:}\quad y(x_0) = y_0 \tag{2}$$

and

$$\textit{Solve:}\quad \frac{d^2y}{dx^2} = f(x, y, y') \qquad \textit{Subject to:}\quad y(x_0) = y_0,\ y'(x_0) = y_1 \tag{3}$$

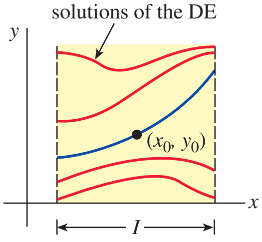


FIGURE 1.2.1 First-order IVP

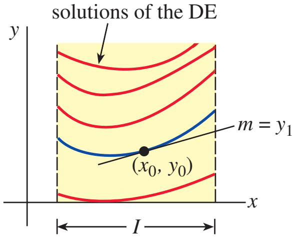


FIGURE 1.2.2 Second-order IVP

are **first-** and **second-order initial-value problems**, respectively. These two problems are easy to interpret in geometric terms. For (2) we are seeking a solution of the differential equation on an interval I containing x_0 so that a solution curve passes through the prescribed point (x_0, y_0). See **FIGURE 1.2.1**. For (3) we want to find a solution of the differential equation whose graph not only passes through (x_0, y_0) but passes through so that the slope of the curve at this point is y_1. See **FIGURE 1.2.2**. The term *initial condition* derives from physical systems where the independent variable is time t and where $y(t_0) = y_0$ and $y'(t_0) = y_1$ represent, respectively, the position and velocity of an object at some beginning, or initial, time t_0.

Solving an nth-order initial-value problem frequently entails using an n-parameter family of solutions of the given differential equation to find n specialized constants so that the resulting particular solution of the equation also "fits"—that is, satisfies—the n initial conditions.

EXAMPLE 1 First-Order IVPs

(a) It is readily verified that $y = ce^x$ is a one-parameter family of solutions of the simple first-order equation $y' = y$ on the interval $(-\infty, \infty)$. If we specify an initial condition, say, $y(0) = 3$, then substituting $x = 0$, $y = 3$ in the family determines the constant $3 = ce^0 = c$. Thus the function $y = 3e^x$ is a solution of the initial-value problem

$$y' = y, \quad y(0) = 3.$$

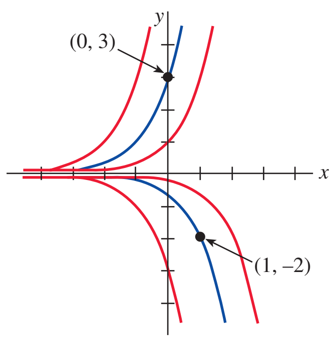


FIGURE 1.2.3 Solutions of IVPs in Example 1

(b) Now if we demand that a solution of the differential equation pass through the point $(1, -2)$ rather than $(0, 3)$, then $y(1) = -2$ will yield $-2 = ce$ or $c = -2e^{-1}$. The function $y = -2e^{x-1}$ is a solution of the initial-value problem

$$y' = y, \quad y(1) = -2.$$

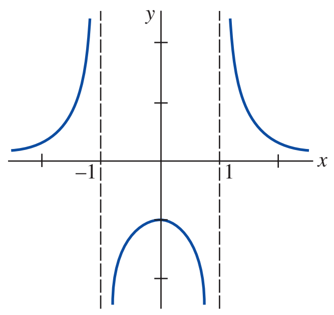
The graphs of these two solutions are shown in blue in **FIGURE 1.2.3**. ≡

The next example illustrates another first-order initial-value problem. In this example, notice how the interval I of definition of the solution $y(x)$ depends on the initial condition $y(x_0) = y_0$.

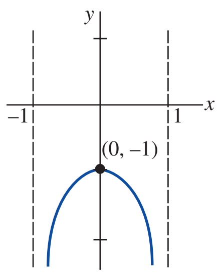
EXAMPLE 2 Interval *I* of Definition of a Solution

In Problem 6 of Exercises 2.2 you will be asked to show that a one-parameter family of solutions of the first-order differential equation $y' + 2xy^2 = 0$ is $y = 1/(x^2 + c)$. If we impose the initial condition $y(0) = -1$, then substituting $x = 0$ and $y = -1$ into the family of solutions gives $-1 = 1/c$ or $c = -1$. Thus, $y = 1/(x^2 - 1)$. We now emphasize the following three distinctions.

- Considered as a *function*, the domain of $y = 1/(x^2 - 1)$ is the set of real numbers x for which $y(x)$ is defined; this is the set of all real numbers except $x = -1$ and $x = 1$. See **FIGURE 1.2.4(a)**.
- Considered as a *solution of the differential equation* $y' + 2xy^2 = 0$, the interval I of definition of $y = 1/(x^2 - 1)$ could be taken to be any interval over which $y(x)$ is defined and differentiable. As can be seen in Figure 1.2.4(a), the largest intervals on which $y = 1/(x^2 - 1)$ is a solution are $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$.
- Considered as a *solution of the initial-value problem* $y' + 2xy^2 = 0$, $y(0) = -1$, the interval I of definition of $y = 1/(x^2 - 1)$ could be taken to be any interval over which $y(x)$ is defined, differentiable, and contains the initial point $x = 0$; the largest interval for which this is true is $(-1, 1)$. See Figure 1.2.4(b). ≡



(a) Function defined for all x except $x = \pm 1$



(b) Solution defined on interval containing $x = 0$

FIGURE 1.2.4 Graphs of function and solution of IVP in Example 2

See Problems 3–6 in Exercises 1.2 for a continuation of Example 2.

EXAMPLE 3 Second-Order IVP

In Example 10 of Section 1.1 we saw that $x = c_1 \cos 4t + c_2 \sin 4t$ is a two-parameter family of solutions of $x'' + 16x = 0$. Find a solution of the initial-value problem

$$x'' + 16x = 0, \quad x(\pi/2) = -2, \quad x'(\pi/2) = 1. \tag{4}$$

SOLUTION We first apply $x(\pi/2) = -2$ to the given family of solutions: $c_1 \cos 2\pi + c_2 \sin 2\pi = -2$. Since $\cos 2\pi = 1$ and $\sin 2\pi = 0$, we find that $c_1 = -2$. We next apply $x'(\pi/2) = 1$ to the one-parameter family $x(t) = -2 \cos 4t + c_2 \sin 4t$. Differentiating and then setting $t = \pi/2$ and $x' = 1$ gives $8 \sin 2\pi + 4c_2 \cos 2\pi = 1$, from which we see that $c_2 = \frac{1}{4}$. Hence $x = -2 \cos 4t + \frac{1}{4} \sin 4t$ is a solution of (4). ≡

Existence and Uniqueness

Two fundamental questions arise in considering an initial-value problem:

Does a solution of the problem exist? If a solution exists, is it unique?

For a first-order initial-value problem such as (2), we ask:

Existence { *Does the differential equation $dy/dx = f(x, y)$ possess solutions?*
Do any of the solution curves pass through the point (x_0, y_0)?

Uniqueness { *When can we be certain that there is precisely one solution curve passing through the point (x_0, y_0)?*

Note that in Examples 1 and 3, the phrase "*a* solution" is used rather than "*the* solution" of the problem. The indefinite article "a" is used deliberately to suggest the possibility that other solutions may exist. At this point it has not been demonstrated that there is a single solution of each problem. The next example illustrates an initial-value problem with two solutions.

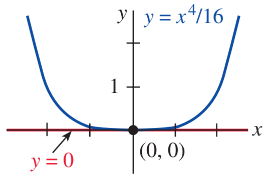


FIGURE 1.2.5 Two solutions of the same IVP in Example 4

EXAMPLE 4 An IVP Can Have Several Solutions

Each of the functions $y = 0$ and $y = \frac{1}{16}x^4$ satisfies the differential equation $dy/dx = xy^{1/2}$ and the initial condition $y(0) = 0$, and so the initial-value problem $dy/dx = xy^{1/2}$, $y(0) = 0$, has at least two solutions. As illustrated in **FIGURE 1.2.5**, the graphs of both functions pass through the same point (0, 0). ≡

Within the safe confines of a formal course in differential equations one can be fairly confident that *most* differential equations will have solutions and that solutions of initial-value problems will *probably* be unique. Real life, however, is not so idyllic. Thus it is desirable to know in advance of trying to solve an initial-value problem whether a solution exists and, when it does, whether it is the only solution of the problem. Since we are going to consider first-order differential equations in the next two chapters, we state here without proof a straightforward theorem that gives conditions that are sufficient to guarantee the existence and uniqueness of a solution of a first-order initial-value problem of the form given in (2). We shall wait until Chapter 3 to address the question of existence and uniqueness of a second-order initial-value problem.

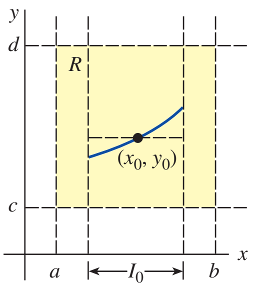


FIGURE 1.2.6 Rectangular region R

THEOREM 1.2.1 Existence of a Unique Solution

Let R be a rectangular region in the xy-plane defined by $a \le x \le b$, $c \le y \le d$, that contains the point (x_0, y_0) in its interior. If $f(x, y)$ and $\partial f/\partial y$ are continuous on R, then there exists some interval I_0: $(x_0 - h, x_0 + h)$, $h > 0$, contained in $[a, b]$, and a unique function $y(x)$ defined on I_0 that is a solution of the initial-value problem (2).

The foregoing result is one of the most popular existence and uniqueness theorems for first-order differential equations, because the criteria of continuity of $f(x, y)$ and $\partial f/\partial y$ are relatively easy to check. The geometry of Theorem 1.2.1 is illustrated in **FIGURE 1.2.6**.

EXAMPLE 5 Example 4 Revisited

We saw in Example 4 that the differential equation $dy/dx = xy^{1/2}$ possesses at least two solutions whose graphs pass through (0, 0). Inspection of the functions

$$f(x, y) = xy^{1/2} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{x}{2y^{1/2}}$$

shows that they are continuous in the upper half-plane defined by $y > 0$. Hence Theorem 1.2.1 enables us to conclude that through any point (x_0, y_0), $y_0 > 0$, in the upper half-plane there is some interval centered at x_0 on which the given differential equation has a unique solution. Thus, for example, even without solving it we know that there exists some interval centered at 2 on which the initial-value problem $dy/dx = xy^{1/2}$, $y(2) = 1$, has a unique solution. ≡

In Example 1, Theorem 1.2.1 guarantees that there are no other solutions of the initial-value problems $y' = y$, $y(0) = 3$, and $y' = y$, $y(1) = -2$, other than $y = 3e^x$ and $y = -2e^{x-1}$, respectively. This follows from the fact that $f(x, y) = y$ and $\partial f/\partial y = 1$ are continuous throughout the entire xy-plane. It can be further shown that the interval I on which each solution is defined is $(-\infty, \infty)$.

Interval of Existence/Uniqueness Suppose $y(x)$ represents a solution of the initial-value problem (2). The following three sets on the real x-axis may not be the same: the domain of the function $y(x)$, the interval I over which the solution $y(x)$ is defined or exists, and the interval I_0 of existence *and* uniqueness. In Example 7 of Section 1.1 we illustrated the difference between the domain of a function and the interval I of definition. Now suppose (x_0, y_0) is a point in the interior of the rectangular region R in Theorem 1.2.1. It turns out that the continuity of the function $f(x, y)$ on R by itself is sufficient to guarantee the existence of at least one solution of $dy/dx = f(x, y)$, $y(x_0) = y_0$, defined on some interval I. The interval I of definition for this initial-value problem is usually taken to be the largest interval containing x_0 over which the solution $y(x)$ is defined and differentiable. The interval I depends on both $f(x, y)$ and the initial condition $y(x_0) = y_0$. See Problems 31–34 in Exercises 1.2. The extra condition of continuity of the first partial derivative $\partial f/\partial y$ on R enables us to say that not only does a solution exist on some interval I_0 containing x_0, but it also is the *only* solution satisfying $y(x_0) = y_0$. However, Theorem 1.2.1 does not give any indication of the sizes of the intervals I and I_0; *the interval I of definition need not be as wide as the region R, and the interval I_0 of existence and uniqueness may not be as large as I.* The number $h > 0$ that defines the interval I_0: $(x_0 - h, x_0 + h)$, could be very small, and so it is best to think that the solution $y(x)$ is *unique in a local sense*, that is, a solution defined near the point (x_0, y_0). See Problem 50 in Exercises 1.2.

REMARKS

(*i*) The conditions in Theorem 1.2.1 are sufficient but not necessary. When $f(x, y)$ and $\partial f/\partial y$ are continuous on a rectangular region R, it must always follow that a solution of (2) exists and is unique whenever (x_0, y_0) is a point interior to R. However, if the conditions stated in the hypotheses of Theorem 1.2.1 do not hold, then anything could happen: Problem (2) *may* still have a solution and this solution *may* be unique, or (2) *may* have several solutions, or it may have no solution at all. A rereading of Example 4 reveals that the hypotheses of Theorem 1.2.1 do not hold on the line $y = 0$ for the differential equation $dy/dx = xy^{1/2}$, and so it is not surprising, as we saw in Example 4 of this section, that there are two solutions defined on a common interval $(-h, h)$ satisfying $y(0) = 0$. On the other hand, the hypotheses of Theorem 1.2.1 do not hold on the line $y = 1$ for the differential equation $dy/dx = |y - 1|$. Nevertheless, it can be proved that the solution of the initial-value problem $dy/dx = |y - 1|$, $y(0) = 1$, is unique. Can you guess this solution?

(*ii*) You are encouraged to read, think about, work, and then keep in mind Problem 49 in Exercises 1.2.

1.2 Exercises

Answers to selected odd-numbered problems begin on page ANS-1.

In Problems 1 and 2, $y = 1/(1 + c_1e^{-x})$ is a one-parameter family of solutions of the first-order DE $y' = y - y^2$. Find a solution of the first-order IVP consisting of this differential equation and the given initial condition.

1. $y(0) = -\frac{1}{3}$

2. $y(-1) = 2$

In Problems 3–6, $y = 1/(x^2 + c)$ is a one-parameter family of solutions of the first-order DE $y' + 2xy^2 = 0$. Find a solution of the first-order IVP consisting of this differential equation and the given initial condition. Give the largest interval I over which the solution is defined.

3. $y(2) = \frac{1}{3}$

4. $y(-2) = \frac{1}{2}$

5. $y(0) = 1$

6. $y\left(\frac{1}{2}\right) = -4$

In Problems 7–10, $x = c_1 \cos t + c_2 \sin t$ is a two-parameter family of solutions of the second-order DE $x'' + x = 0$. Find a solution of the second-order IVP consisting of this differential equation and the given initial conditions.

7. $x(0) = -1, \quad x'(0) = 8$

8. $x(\pi/2) = 0, \quad x'(\pi/2) = 1$

9. $x(\pi/6) = \frac{1}{2}, \quad x'(\pi/6) = 0$

10. $x(\pi/4) = \sqrt{2}, \quad x'(\pi/4) = 2\sqrt{2}$

In Problems 11–14, $y = c_1e^x + c_2e^{-x}$ is a two-parameter family of solutions of the second-order DE $y'' - y = 0$. Find a solution of the second-order IVP consisting of this differential equation and the given initial conditions.

11. $y(0) = 1, \quad y'(0) = 2$

12. $y(1) = 0, \quad y'(1) = e$

13. $y(-1) = 5, \quad y'(-1) = -5$

14. $y(0) = 0, \quad y'(0) = 0$

In Problems 15 and 16, determine by inspection at least two solutions of the given first-order IVP.

15. $y' = 3y^{2/3}, \quad y(0) = 0$

16. $xy' = 2y, \quad y(0) = 0$

In Problems 17–24, determine a region of the xy-plane for which the given differential equation would have a unique solution whose graph passes through a point (x_0, y_0) in the region.

17. $\dfrac{dy}{dx} = y^{2/3}$

18. $\dfrac{dy}{dx} = \sqrt{xy}$

19. $x\dfrac{dy}{dx} = y$

20. $\dfrac{dy}{dx} - y = x$

21. $(4 - y^2)y' = x^2$

22. $(1 + y^3)y' = x^2$

23. $(x^2 + y^2)y' = y^2$

24. $(y - x)y' = y + x$

In Problems 25–28, determine whether Theorem 1.2.1 guarantees that the differential equation $y' = \sqrt{y^2 - 9}$ possesses a unique solution through the given point.

25. $(1, 4)$

26. $(5, 3)$

27. $(2, -3)$

28. $(-1, 1)$

29. **(a)** By inspection, find a one-parameter family of solutions of the differential equation $xy' = y$. Verify that each member of the family is a solution of the initial-value problem $xy' = y$, $y(0) = 0$.
   (b) Explain part (a) by determining a region R in the xy-plane for which the differential equation $xy' = y$ would have a unique solution through a point (x_0, y_0) in R.
   (c) Verify that the piecewise-defined function

$$y = \begin{cases} 0, & x < 0 \\ x, & x \geq 0 \end{cases}$$

   satisfies the condition $y(0) = 0$. Determine whether this function is also a solution of the initial-value problem in part (a).

30. **(a)** Verify that $y = \tan(x + c)$ is a one-parameter family of solutions of the differential equation $y' = 1 + y^2$.
   (b) Since $f(x, y) = 1 + y^2$ and $\partial f/\partial y = 2y$ are continuous everywhere, the region R in Theorem 1.2.1 can be taken to be the entire xy-plane. Use the family of solutions in part (a) to find an explicit solution of the first-order initial-value problem $y' = 1 + y^2$, $y(0) = 0$. Even though $x_0 = 0$ is in the interval $(-2, 2)$, explain why the solution is not defined on this interval.
   (c) Determine the largest interval I of definition for the solution of the initial-value problem in part (b).

31. **(a)** Verify that $y = -1/(x + c)$ is a one-parameter family of solutions of the differential equation $y' = y^2$.
   (b) Since $f(x, y) = y^2$ and $\partial f/\partial y = 2y$ are continuous everywhere, the region R in Theorem 1.2.1 can be taken to be the entire xy-plane. Find a solution from the family in part (a) that satisfies $y(0) = 1$. Find a solution from the family in part (a) that satisfies $y(0) = -1$. Determine the largest interval I of definition for the solution of each initial-value problem.

32. **(a)** Find a solution from the family in part (a) of Problem 31 that satisfies $y' = y^2$, $y(0) = y_0$, where $y_0 \neq 0$. Explain why the largest interval I of definition for this solution is either $(-\infty, 1/y_0)$ or $(1/y_0, \infty)$.
   (b) Determine the largest interval I of definition for the solution of the first-order initial-value problem $y' = y^2$, $y(0) = 0$.

33. **(a)** Verify that $3x^2 - y^2 = c$ is a one-parameter family of solutions of the differential equation $y\,dy/dx = 3x$.
   (b) By hand, sketch the graph of the implicit solution $3x^2 - y^2 = 3$. Find all explicit solutions $y = \phi(x)$ of the DE in part (a) defined by this relation. Give the interval I of definition of each explicit solution.
   (c) The point $(-2, 3)$ is on the graph of $3x^2 - y^2 = 3$, but which of the explicit solutions in part (b) satisfies $y(-2) = 3$?

34. **(a)** Use the family of solutions in part (a) of Problem 33 to find an implicit solution of the initial-value problem $y\,dy/dx = 3x$, $y(2) = -4$. Then, by hand, sketch the graph of the explicit solution of this problem and give its interval I of definition.
   (b) Are there any explicit solutions of $y\,dy/dx = 3x$ that pass through the origin?

In Problems 35–38, the graph of a member of a family of solutions of a second-order differential equation $d^2y/dx^2 = f(x, y, y')$ is given. Match the solution curve with at least one pair of the following initial conditions.

(a) $y(1) = 1, y'(1) = -2$ **(b)** $y(-1) = 0, y'(-1) = -4$
(c) $y(1) = 1, y'(1) = 2$ **(d)** $y(0) = -1, y'(0) = 2$
(e) $y(0) = -1, y'(0) = 0$ **(f)** $y(0) = -4, y'(0) = -2$

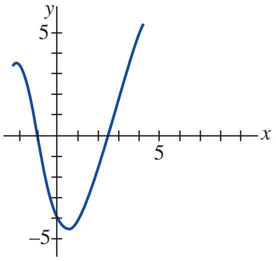
35.


FIGURE 1.2.7 Graph for Problem 35

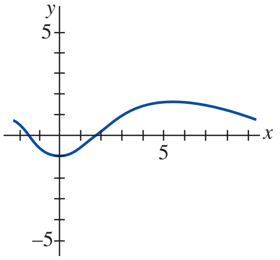
36.


FIGURE 1.2.8 Graph for Problem 36

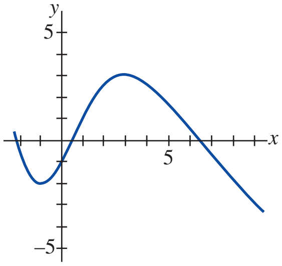
37.


FIGURE 1.2.9 Graph for Problem 37

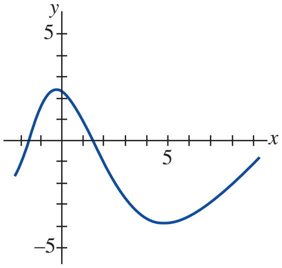
38.


FIGURE 1.2.10 Graph for Problem 38

In Problems 39–44, $y = c_1 \cos 3x + c_2 \sin 3x$ is a two-parameter family of solutions of the second-order DE $y'' + 9y = 0$. If possible, find a solution of the differential equation that satisfies the given side conditions. The conditions specified at two different points are called boundary conditions.

39. $y(0) = 0, y(\pi/6) = -1$
40. $y(0) = 0, y(\pi) = 0$
41. $y'(0) = 0, y'(\pi/4) = 0$
42. $y(0) = 1, y'(\pi) = 5$
43. $y(0) = 0, y(\pi) = 4$
44. $y'(\pi/3) = 1, y'(\pi) = 0$

Discussion Problems

In Problems 45 and 46, use Problem 63 in Exercises 1.1 and (2) and (3) of this section.

45. Find a function $y = f(x)$ whose graph at each point (x, y) has the slope given by $8e^{2x} + 6x$ and has the y-intercept $(0, 9)$.
46. Find a function $y = f(x)$ whose second derivative is $y'' = 12x - 2$ at each point (x, y) on its graph and $y = -x + 5$ is tangent to the graph at the point corresponding to $x = 1$.
47. Consider the initial-value problem $y' = x - 2y$, $y(0) = \frac{1}{2}$. Determine which of the two curves shown in **FIGURE 1.2.11** is the only plausible solution curve. Explain your reasoning.

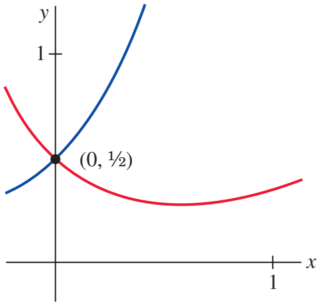


FIGURE 1.2.11 Graph for Problem 47

48. Without attempting to solve the initial-value problem $y' = x^2 + y^2$, $y(0) = 1$, find the values of $y'(0)$ and $y''(0)$.

49. Suppose that the first-order differential equation $dy/dx = f(x, y)$ possesses a one-parameter family of solutions and that $f(x, y)$ satisfies the hypotheses of Theorem 1.2.1 in some rectangular region R of the xy-plane. Explain why two different solution curves cannot intersect or be tangent to each other at a point (x_0, y_0) in R.

50. The functions

$$y(x) = \tfrac{1}{16}x^4, \quad -\infty < x < \infty$$

and

$$y(x) = \begin{cases} 0, & x < 0 \\ \tfrac{1}{16}x^4, & x \ge 0 \end{cases}$$

have the same domain but are clearly different. See **FIGURES 1.2.12(a)** and 1.2.12(b), respectively. Show that both functions are solutions of the initial-value problem $dy/dx = xy^{1/2}$, $y(2) = 1$ on the interval $(-\infty, \infty)$. Resolve the apparent contradiction between this fact and the last sentence in Example 5.

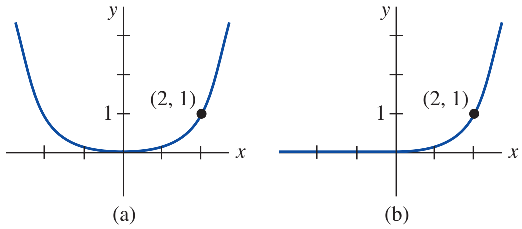


FIGURE 1.2.12 Two solutions of the IVP in Problem 50

51. Show that

$$x = \int_0^y \frac{1}{\sqrt{t^3 + 1}}\, dt$$

is an implicit solution of the initial-value problem

$$2\frac{d^2y}{dx^2} - 3y^2 = 0, \quad y(0) = 0,\ y'(0) = 1.$$

Assume that $y \ge 0$. [*Hint*: The integral is nonelementary. See (*ii*) in the *Remarks* at the end of Section 1.1.]

1.3 Differential Equations as Mathematical Models

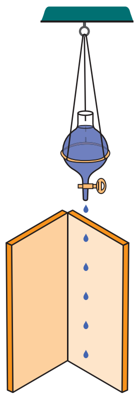


FIGURE 1.3.1 Da Vinci's apparatus for determining the speed of falling body

INTRODUCTION In this section we introduce the notion of a *mathematical model.* Roughly speaking, a mathematical model is a mathematical description of something. This description could be as simple as a function. For example, **Leonardo da Vinci** (1452–1519) was able to deduce the speed v of a falling body by examining a sequence. Leonardo allowed water drops to fall, at equally spaced intervals of time, between two boards covered with blotting paper. When a spring mechanism was disengaged, the boards were clapped together. See **FIGURE 1.3.1**. By carefully examining the sequence of water blots, Leonardo discovered that the distances between consecutive drops increased in "a continuous arithmetic proportion." In this manner he discovered the formula $v = gt$.

Although there are many kinds of mathematical models, in this section we focus only on differential equations and discuss some specific differential-equation models in biology, physics, and chemistry. Once we have studied some methods for solving DEs, in Chapters 2 and 3 we return to, and solve, some of these models.

Mathematical Models

It is often desirable to describe the behavior of some real-life system or phenomenon, whether physical, sociological, or even economic, in mathematical terms. The mathematical description of a system or a phenomenon is called a **mathematical model** and is constructed with certain goals in mind. For example, we may wish to understand the mechanisms of a certain ecosystem by studying the growth of animal populations in that system, or we may wish to date fossils by means of analyzing the decay of a radioactive substance either in the fossil or in the stratum in which it was discovered.

Construction of a mathematical model of a system starts with *identification of the variables* that are responsible for changing the system. We may choose not to incorporate all these variables into the model at first. In this first step we are specifying the **level of resolution** of the model. Next, *we make a set of reasonable assumptions* or hypotheses about the system we are trying to describe. These assumptions will also include any empirical laws that may be applicable to the system.

For some purposes it may be perfectly within reason to be content with low-resolution models. For example, you may already be aware that in modeling the motion of a body falling near the surface of the Earth, the retarding force of air friction is sometimes ignored in beginning physics courses; but if you are a scientist whose job it is to accurately predict the flight path of a long-range projectile, air resistance and other factors such as the curvature of the Earth have to be taken into account.

Since the assumptions made about a system frequently involve *a rate of change* of one or more of the variables, the mathematical depiction of all these assumptions may be one or more equations involving *derivatives*. In other words, the mathematical model may be a differential equation or a system of differential equations.

Once we have formulated a mathematical model that is either a differential equation or a system of differential equations, we are faced with the not insignificant problem of trying to solve it. *If* we can solve it, then we deem the model to be reasonable if its solution is consistent with either experimental data or known facts about the behavior of the system. But if the predictions produced by the solution are poor, we can either increase the level of resolution of the model or make alternative assumptions about the mechanisms for change in the system. The steps of the modeling process are then repeated as shown in **FIGURE 1.3.2**.

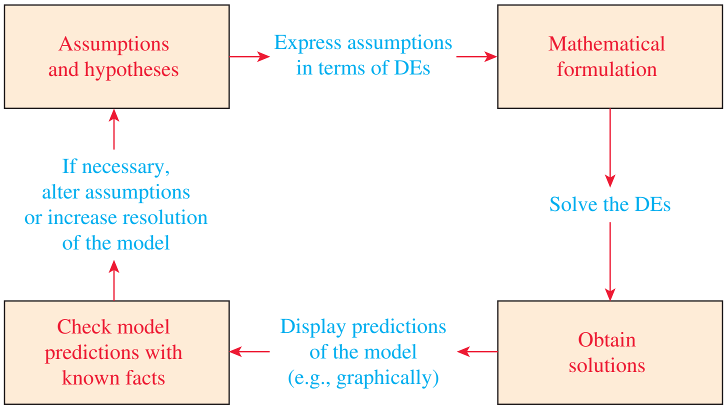


FIGURE 1.3.2 Steps in the modeling process

Of course, by increasing the resolution we add to the complexity of the mathematical model and increase the likelihood that we cannot obtain an explicit solution.

A mathematical model of a physical system will often involve the variable time t. A solution of the model then gives the **state of the system**; in other words, for appropriate values of t, the values of the dependent variable (or variables) describe the system in the past, present, and future.

Population Dynamics

One of the earliest attempts to model human **population growth** by means of mathematics was by the English economist **Thomas Robert Malthus** (1776–1834) in 1798. Basically, the idea of the Malthusian model is the assumption that the rate at which a population of a country grows at a certain time is proportional* to the total population of the country at that time. In other words, the more people there are at time t, the more there are going to be in the future. In mathematical terms, if $P(t)$ denotes the total population at time t, then this assumption can be expressed as

$$\frac{dP}{dt} \propto P \quad \text{or} \quad \frac{dP}{dt} = kP, \tag{1}$$

where k is a constant of proportionality. This simple model, which fails to take into account many factors (immigration and emigration, for example) that can influence human populations to either grow or decline, nevertheless turned out to be fairly accurate in predicting the population of the United States during the years 1790–1860. Populations that grow at a rate described by (1) are rare; nevertheless, (1) is still used to model *growth of small populations over short intervals of time*, for example, bacteria growing in a petri dish.

*If two quantities u and v are proportional, we write $u \propto v$. This means one quantity is a constant multiple of the other: $u = kv$.

Radioactive Decay The nucleus of an atom consists of combinations of protons and neutrons. Many of these combinations of protons and neutrons are unstable; that is, the atoms decay or transmute into the atoms of another substance. Such nuclei are said to be radioactive. For example, over time, the highly radioactive radium, Ra-226, transmutes into the radioactive gas radon, Rn-222. In modeling the phenomenon of **radioactive decay**, it is assumed that the rate dA/dt at which the nuclei of a substance decay is proportional to the amount (more precisely, the number of nuclei) $A(t)$ of the substance remaining at time t:

$$\frac{dA}{dt} \propto A \quad \text{or} \quad \frac{dA}{dt} = kA. \tag{2}$$

Of course equations (1) and (2) are exactly the same; the difference is only in the interpretation of the symbols and the constants of proportionality. For growth, as we expect in (1), $k > 0$, and in the case of (2) and decay, $k < 0$.

The model (1) for growth can be seen as the equation $dS/dt = rS$, which describes the growth of capital S when an annual rate of interest r is compounded continuously. The model (2) for decay also occurs in a biological setting, such as determining the half-life of a drug—the time that it takes for 50% of a drug to be eliminated from a body by excretion or metabolism. In chemistry, the decay model (2) appears as the mathematical description of a first-order chemical reaction. The point is this:

A single differential equation can serve as a mathematical model for many different phenomena.

Mathematical models are often accompanied by certain side conditions. For example, in (1) and (2) we would expect to know, in turn, an initial population P_0 and an initial amount of radioactive substance A_0 that is on hand. If this initial point in time is taken to be $t = 0$, then we know that $P(0) = P_0$ and $A(0) = A_0$. In other words, a mathematical model can consist of either an initial-value problem or, as we shall see later in Section 3.9, a boundary-value problem.

Newton's Law of Cooling/Warming According to **Newton's empirical law of cooling**—or warming—the rate at which the temperature of a body changes is proportional to the difference between the temperature of the body and the temperature of the surrounding medium, the so-called ambient temperature. If $T(t)$ represents the temperature of a body at time t, T_m the temperature of the surrounding medium, and dT/dt the rate at which the temperature of the body changes, then Newton's law of cooling/warming translates into the mathematical statement

$$\frac{dT}{dt} \propto T - T_m \quad \text{or} \quad \frac{dT}{dt} = k(T - T_m), \tag{3}$$

where k is a constant of proportionality. In either case, cooling or warming, if T_m is a constant, it stands to reason that $k < 0$.

Spread of a Disease A contagious disease—for example, a flu virus—is spread throughout a community by people coming into contact with other people. Let $x(t)$ denote the number of people who have contracted the disease and $y(t)$ the number of people who have not yet been exposed. It seems reasonable to assume that the rate dx/dt at which the disease spreads is proportional to the number of encounters or *interactions* between these two groups of people. If we assume that the number of interactions is jointly proportional to $x(t)$ and $y(t)$, that is, proportional to the product xy, then

$$\frac{dx}{dt} = kxy, \tag{4}$$

where k is the usual constant of proportionality. Suppose a small community has a fixed population of n people. If one infected person is introduced into this community, then it could be argued that $x(t)$ and $y(t)$ are related by $x + y = n + 1$. Using this last equation to eliminate y in (4) gives us the model

$$\frac{dx}{dt} = kx(n + 1 - x). \tag{5}$$

An obvious initial condition accompanying equation (5) is $x(0) = 1$.

Chemical Reactions The disintegration of a radioactive substance, governed by the differential equation (2), is said to be a **first-order reaction**. In chemistry, a few reactions follow this same empirical law: If the molecules of substance A decompose into smaller molecules, it is a natural assumption that the rate at which this decomposition takes place is proportional to the amount of the first substance that has not undergone conversion; that is, if $X(t)$ is the amount of substance A remaining at any time, then $dX/dt = kX$, where k is a negative constant since X is decreasing. An example of a first-order chemical reaction is the conversion of t-butyl chloride into t-butyl alcohol:

$$(CH_3)_3CCl + NaOH \rightarrow (CH_3)_3COH + NaCl.$$

Only the concentration of the t-butyl chloride controls the rate of reaction. But in the reaction

$$CH_3Cl + NaOH \rightarrow CH_3OH + NaCl,$$

for every molecule of methyl chloride, one molecule of sodium hydroxide is consumed, thus forming one molecule of methyl alcohol and one molecule of sodium chloride. In this case the rate at which the reaction proceeds is proportional to the product of the remaining concentrations of CH_3Cl and of NaOH. If X denotes the amount of CH_3OH formed and α and β are the given amounts of the first two chemicals A and B, then the instantaneous amounts not converted to chemical C are $\alpha - X$ and $\beta - X$, respectively. Hence the rate of formation of C is given by

$$\frac{dX}{dt} = k(\alpha - X)(\beta - X), \tag{6}$$

where k is a constant of proportionality. A reaction whose model is equation (6) is said to be **second order**.

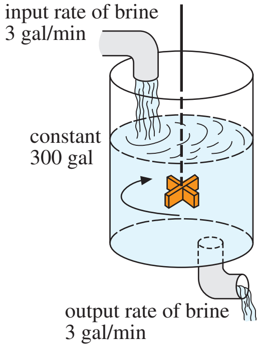


FIGURE 1.3.3 Mixing tank

Mixtures The mixing of two salt solutions of differing concentrations gives rise to a first-order differential equation for the amount of salt contained in the mixture. Let us suppose that a large mixing tank initially holds 300 gallons of brine (that is, water in which a certain number of pounds of salt has been dissolved). Another brine solution is pumped into the large tank at a rate of 3 gallons per minute; the concentration of the salt in this inflow is 2 pounds of salt per gallon. When the solution in the tank is well stirred, it is pumped out at the same rate as the entering solution. See **FIGURE 1.3.3**. If $A(t)$ denotes the amount of salt (measured in pounds) in the tank at time t, then the rate at which $A(t)$ changes is a net rate:

$$\frac{dA}{dt} = \begin{pmatrix}\text{input rate}\\ \text{of salt}\end{pmatrix} - \begin{pmatrix}\text{output rate}\\ \text{of salt}\end{pmatrix} = R_{in} - R_{out}. \tag{7}$$

The input rate R_{in} at which the salt enters the tank is the product of the inflow concentration of salt and the inflow rate of fluid. Note that R_{in} is measured in pounds per minute:

$$R_{in} = \overset{\substack{\text{concentration}\\ \text{of salt}\\ \text{in inflow}\\ \downarrow}}{(2\text{ lb/gal})} \cdot \overset{\substack{\text{input rate}\\ \text{of brine}\\ \downarrow}}{(3\text{ gal/min})} = \overset{\substack{\text{input rate}\\ \text{of salt}\\ \downarrow}}{(6\text{ lb/min})}.$$

Now, since the solution is being pumped out of the tank at the same rate that it is pumped in, the number of gallons of brine in the tank at time t is a constant 300 gallons. Hence the concentration of the salt in the tank, as well as in the outflow, is $c(t) = A(t)/300$ lb/gal, and so the output rate R_{out} of salt is

$$R_{out} = \overset{\substack{\text{concentration}\\ \text{of salt}\\ \text{in outflow}\\ \downarrow}}{\left(\frac{A(t)}{300}\text{ lb/gal}\right)} \cdot \overset{\substack{\text{output rate}\\ \text{of brine}\\ \downarrow}}{(3\text{ gal/min})} = \overset{\substack{\text{output rate}\\ \text{of salt}\\ \downarrow}}{\frac{A(t)}{100}\text{ lb/min}}.$$

The net rate (7) then becomes

$$\frac{dA}{dt} = 6 - \frac{A}{100} \quad \text{or} \quad \frac{dA}{dt} + \frac{1}{100}A = 6. \tag{8}$$

If r_{in} and r_{out} denote general input and output rates of the brine solutions,* respectively, then there are three possibilities: $r_{in} = r_{out}$, $r_{in} > r_{out}$, and $r_{in} < r_{out}$. In the analysis leading to (8) we have assumed that $r_{in} = r_{out}$. In the latter two cases, the number of gallons of brine in the tank is

*Don't confuse these symbols with R_{in} and R_{out}, which are input and output rates of *salt*.

either increasing ($r_{in} > r_{out}$) or decreasing ($r_{in} < r_{out}$) at the net rate $r_{in} - r_{out}$. See Problems 10–12 in Exercises 1.3.

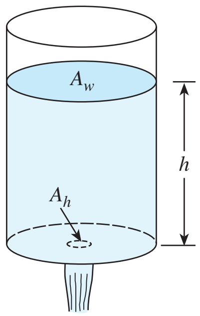


FIGURE 1.3.4 Water draining from a tank

Draining a Tank

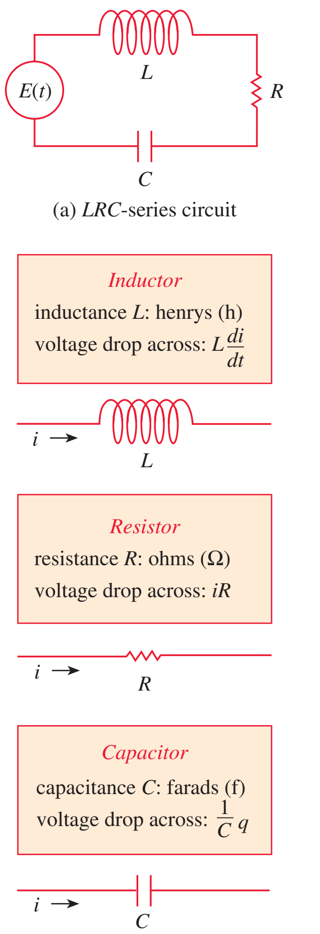
Evangelista Torricelli (1608–1647) was an Italian physicist who invented the barometer and was a student of Galileo Galilei. In hydrodynamics, **Torricelli's law** states that the speed v of efflux of water through a sharp-edged hole at the bottom of a tank filled to a depth h is the same as the speed that a body (in this case a drop of water) would acquire in falling freely from a height h; that is, $v = \sqrt{2gh}$, where g is the acceleration due to gravity. This last expression comes from equating the kinetic energy $\frac{1}{2}mv^2$ with the potential energy mgh and solving for v. Suppose a tank filled with water is allowed to drain through a hole under the influence of gravity. We would like to find the depth h of water remaining in the tank at time t. Consider the tank shown in **FIGURE 1.3.4**. If the area of the hole is A_h (in ft^2) and the speed of the water leaving the tank is $v = \sqrt{2gh}$ (in ft/s), then the volume of water leaving the tank per second is $A_h\sqrt{2gh}$ (in ft^3/s). Thus if $V(t)$ denotes the volume of water in the tank at time t,

$$\frac{dV}{dt} = -A_h\sqrt{2gh}, \tag{9}$$

where the minus sign indicates that V is decreasing. Note here that we are ignoring the possibility of friction at the hole that might cause a reduction of the rate of flow there. Now if the tank is such that the volume of water in it at time t can be written $V(t) = A_w h$, where A_w (in ft^2) is the *constant* area of the upper surface of the water (see Figure 1.3.4), then $dV/dt = A_w dh/dt$. Substituting this last expression into (9) gives us the desired differential equation for the height of the water at time t:

$$\frac{dh}{dt} = -\frac{A_h}{A_w}\sqrt{2gh}. \tag{10}$$

It is interesting to note that (10) remains valid even when A_w is not constant. In this case we must express the upper surface area of the water as a function of h; that is, $A_w = A(h)$. See Problem 14 in Exercises 1.3.



(b) Symbols and voltage drops

FIGURE 1.3.5 Current $i(t)$ and charge $q(t)$ are measured in amperes (A) and coulombs (C), respectively

Series Circuits

The mathematical analysis of electrical circuits and networks is relatively straightforward, using two laws formulated by the German physicist **Gustav Robert Kirchhoff** (1824–1887) in 1845 while he was still a student. Consider the single-loop LRC-series circuit containing an inductor, resistor, and capacitor shown in **FIGURE 1.3.5(a)**. The current in a circuit after a switch is closed is denoted by $i(t)$; the charge on a capacitor at time t is denoted by $q(t)$. The letters L, R, and C are known as inductance, resistance, and capacitance, respectively, and are generally constants. Now according to **Kirchhoff's second law**, the impressed voltage $E(t)$ on a closed loop must equal the sum of the voltage drops in the loop. Figure 1.3.5(b) also shows the symbols and the formulas for the respective voltage drops across an inductor, a resistor, and a capacitor. Since current $i(t)$ is related to charge $q(t)$ on the capacitor by $i = dq/dt$, by adding the three voltage drops

Inductor	Resistor	Capacitor
$L\frac{di}{dt} = L\frac{d^2q}{dt^2}$,	$iR = R\frac{dq}{dt}$,	$\frac{1}{C}q$

and equating the sum to the impressed voltage, we obtain a second-order differential equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t). \tag{11}$$

We will examine a differential equation analogous to (11) in great detail in Section 3.8.

Falling Bodies

In constructing a mathematical model of the motion of a body moving in a force field, one often starts with Newton's second law of motion. Recall from elementary physics that **Newton's first law of motion** states that a body will either remain at rest or will continue to move with a constant velocity unless acted upon by an external force. In each case this is equivalent to saying that when the sum of the forces $F = \Sigma F_k$—that is, the *net* or resultant force—acting on the body is zero, then the acceleration a of the body is zero. **Newton's second law of motion** indicates that when the net force acting on a body is not zero, then the net force is proportional to its acceleration a, or more precisely, $F = ma$, where m is the mass of the body.

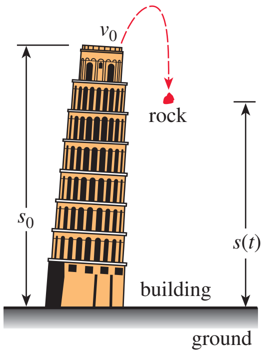


FIGURE 1.3.6 Position of rock measured from ground level

Now suppose a rock is tossed upward from a roof of a building as illustrated in **FIGURE 1.3.6**. What is the position $s(t)$ of the rock relative to the ground at time t? The acceleration of the rock is the second derivative d^2s/dt^2. If we assume that the upward direction is positive and that no force acts on the rock other than the force of gravity, then Newton's second law gives

$$m\frac{d^2s}{dt^2} = -mg \qquad \text{or} \qquad \frac{d^2s}{dt^2} = -g. \tag{12}$$

In other words, the net force is simply the weight $F = F_1 = -W$ of the rock near the surface of the Earth. Recall that the magnitude of the weight is $W = mg$, where m is the mass of the body and g is the acceleration due to gravity. The minus sign in (12) is used because the weight of the rock is a force directed downward, which is opposite to the positive direction. If the height of the building is s_0 and the initial velocity of the rock is v_0, then s is determined from the second-order initial-value problem

$$\frac{d^2s}{dt^2} = -g, \quad s(0) = s_0, \quad s'(0) = v_0. \tag{13}$$

Although we have not stressed solutions of the equations we have constructed, we note that (13) can be solved by integrating the constant $-g$ twice with respect to t. The initial conditions determine the two constants of integration. You might recognize the solution of (13) from elementary physics as the formula $s(t) = -\frac{1}{2}gt^2 + v_0t + s_0$.

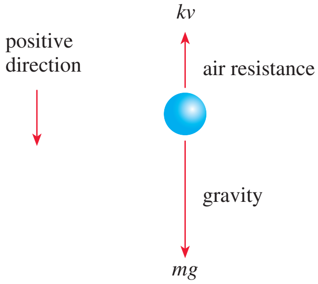


FIGURE 1.3.7 Falling body of mass m

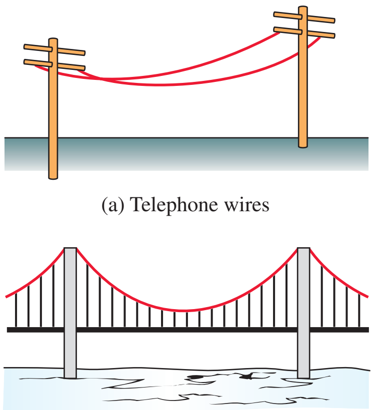
Falling Bodies and Air Resistance

Prior to the famous experiment by Italian mathematician and physicist **Galileo Galilei** (1564–1642) from the Leaning Tower of Pisa, it was generally believed that heavier objects in free fall, such as a cannonball, fell with a greater acceleration than lighter objects, such as a feather. Obviously a cannonball and a feather, when dropped simultaneously from the same height, *do* fall at different rates, but it is not because a cannonball is heavier. The difference in rates is due to air resistance. The resistive force of air was ignored in the model given in (13). Under some circumstances a falling body of mass m—such as a feather with low density and irregular shape—encounters air resistance proportional to its instantaneous velocity v. If we take, in this circumstance, the positive direction to be oriented downward, then the net force acting on the mass is given by $F = F_1 + F_2 = mg - kv$, where the weight $F_1 = mg$ of the body is a force acting in the positive direction and air resistance $F_2 = -kv$ is a force, called **viscous damping**, or **drag**, acting in the opposite or upward direction. See **FIGURE 1.3.7**. Now since v is related to acceleration a by $a = dv/dt$, Newton's second law becomes $F = ma = m\,dv/dt$. By equating the net force to this form of Newton's second law, we obtain a first-order differential equation for the velocity $v(t)$ of the body at time t,

$$m\frac{dv}{dt} = mg - kv. \tag{14}$$

Here k is a positive constant of proportionality called the **drag coefficient**. If $s(t)$ is the distance the body falls in time t from its initial point of release, then $v = ds/dt$ and $a = dv/dt = d^2s/dt^2$. In terms of s, (14) is a second-order differential equation

$$m\frac{d^2s}{dt^2} = mg - k\frac{ds}{dt} \qquad \text{or} \qquad m\frac{d^2s}{dt^2} + k\frac{ds}{dt} = mg. \tag{15}$$



(a) Telephone wires

(b) Suspension bridge

FIGURE 1.3.8 Cables suspended between vertical supports

Suspended Cables

Suppose a flexible cable, wire, or heavy rope is suspended between two vertical supports. Physical examples of this could be a long telephone wire strung between two posts as shown in red in **FIGURE 1.3.8(a)**, or one of the two cables supporting the roadbed of a suspension bridge shown in red in Figure 1.3.8(b). Our goal is to construct a mathematical model that describes the shape that such a cable assumes.

To begin, let's agree to examine only a portion or element of the cable between its lowest point P_1 and any arbitrary point P_2. As drawn in blue in **FIGURE 1.3.9**, this element of the cable is the curve in a rectangular coordinate system with the y-axis chosen to pass through the lowest point P_1 on the curve and the x-axis chosen a units below P_1. Three forces are acting on the cable: the tensions $\mathbf{T}_1$ and $\mathbf{T}_2$ in the cable that are tangent to the cable at P_1 and P_2, respectively, and the portion $\mathbf{W}$ of the total vertical load between the points P_1 and P_2. Let $T_1 = |\mathbf{T}_1|$, $T_2 = |\mathbf{T}_2|$, and $W = |\mathbf{W}|$ denote the magnitudes of these vectors. Now the tension $\mathbf{T}_2$ resolves

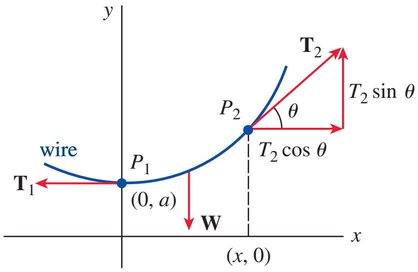


FIGURE 1.3.9 Element of cable

into horizontal and vertical components $T_2 \cos\theta$ and $T_2 \sin\theta$. Because of static equilibrium, we can write

$$T_1 = T_2\cos\theta \quad \text{and} \quad W = T_2\sin\theta.$$

By dividing the last equation by the first, we eliminate T_2 and get $\tan\theta = W/T_1$. But since $dy/dx = \tan\theta$, we arrive at

$$\frac{dy}{dx} = \frac{W}{T_1}. \tag{16}$$

This simple first-order differential equation serves as a model for both the shape of a flexible wire, such as a telephone wire hanging under its own weight, as well as the shape of the cables that support the roadbed. We will come back to equation (16) in Exercises 2.2 and in Section 3.11.

REMARKS

Except for equation (16), the differential equations derived in this section have described a *dynamical system*—a system that changes or evolves over time. Since the study of dynamical systems is a branch of mathematics currently in vogue, we shall occasionally relate the terminology of that field to the discussion at hand.

In more precise terms, a **dynamical system** consists of a set of time-dependent variables, called **state variables**, together with a rule that enables us to determine (without ambiguity) the state of the system (this may be past, present, or future states) in terms of a state prescribed at some time t_0. Dynamical systems are classified as either discrete-time systems or continuous-time systems. In this course we shall be concerned only with continuous-time dynamical systems—systems in which *all* variables are defined over a continuous range of time. The rule or the mathematical model in a continuous-time dynamical system is a differential equation or a system of differential equations. The **state of the system** at a time t is the value of the state variables at that time; the specified state of the system at a time t_0 is simply the initial conditions that accompany the mathematical model. The solution of the initial-value problem is referred to as the **response of the system**. For example, in the preceding case of radioactive decay, the rule is $dA/dt = kA$. Now if the quantity of a radioactive substance at some time t_0 is known, say $A(t_0) = A_0$, then by solving the rule, the response of the system for $t \geq t_0$ is found to be $A(t) = A_0 e^{t-t_0}$ (see Section 2.7). The response $A(t)$ is the single-state variable for this system. In the case of the rock tossed from the roof of the building, the response of the system, the solution of the differential equation $d^2s/dt^2 = -g$ subject to the initial state $s(0) = s_0$, $s'(0) = v_0$, is the function $s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0$, $0 \leq t \leq T$, where the symbol T represents the time when the rock hits the ground. The state variables are $s(t)$ and $s'(t)$, which are, respectively, the vertical position of the rock above ground and its velocity at time t. The acceleration $s''(t)$ is *not* a state variable since we only have to know any initial position and initial velocity at a time t_0 to uniquely determine the rock's position $s(t)$ and velocity $s'(t) = v(t)$ for any time in the interval $[t_0, T]$. The acceleration $s''(t) = a(t)$ is, of course, given by the differential equation $s''(t) = -g$, $0 < t < T$.

One last point: Not every system studied in this text is a dynamical system. We shall also examine some static systems in which the model is a differential equation.

1.3 Exercises

Answers to selected odd-numbered problems begin on page ANS-1.

Population Dynamics

1. Under the same assumptions underlying the model in (1), determine a differential equation governing the growing population $P(t)$ of a country when individuals are allowed to immigrate into the country at a constant rate $r > 0$. What is the differential equation for the population $P(t)$ of the country when individuals are allowed to emigrate at a constant rate $r > 0$?

2. The population model given in (1) fails to take death into consideration; the growth rate equals the birth rate. In another model of a changing population of a community, it is assumed that the rate at which the population changes is a *net* rate—that is, the difference between the rate of births and the rate of deaths in the community. Determine a model for the population $P(t)$ if both the birth rate and the death rate are proportional to the population present at time t.

3. Using the concept of a net rate introduced in Problem 2, determine a differential equation governing a population $P(t)$ if the birth rate is proportional to the population present at time t but the death rate is proportional to the square of the population present at time t.

4. Modify the model in Problem 3 for the net rate at which the population $P(t)$ of a certain kind of fish changes by also assuming that the fish are harvested at a constant rate $h > 0$.

Newton's Law of Cooling/Warming

5. A cup of coffee cools according to Newton's law of cooling (3). Use data from the graph of the temperature $T(t)$ in **FIGURE 1.3.10** to estimate the constants T_m, T_0, and k in a model of the form of the first-order initial-value problem

$$\frac{dT}{dt} = k(T - T_m), \quad T(0) = T_0.$$

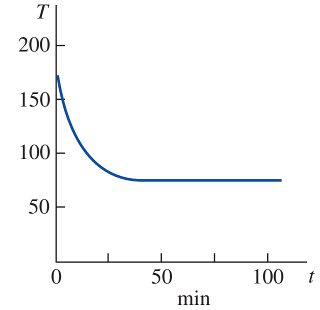


FIGURE 1.3.10 Cooling curve in Problem 5

6. The ambient temperature T_m in (3) could be a function of time t. Suppose that in an artificially controlled environment, $T_m(t)$ is periodic with a 24-hour period, as illustrated in **FIGURE 1.3.11**. Devise a mathematical model for the temperature $T(t)$ of a body within this environment.

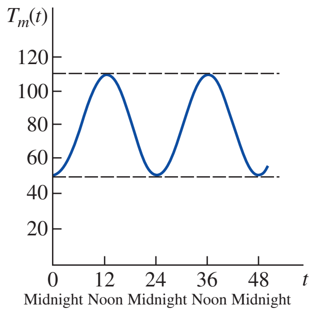


FIGURE 1.3.11 Ambient temperature in Problem 6

Spread of a Disease/Technology

7. Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. Determine a differential equation governing the number of students $x(t)$ who have contracted the flu if the rate at which the disease spreads is proportional to the number of interactions between the number of students with the flu and the number of students who have not yet been exposed to it.

8. At a time $t = 0$, a technological innovation is introduced into a community with a fixed population of n people. Determine a differential equation governing the number of people $x(t)$ who have adopted the innovation at time t if it is assumed that the rate at which the innovation spreads through the community is jointly proportional to the number of people who have adopted it and the number of people who have not adopted it.

Mixtures

9. Suppose that a large mixing tank initially holds 300 gallons of water in which 50 pounds of salt has been dissolved. Pure water is pumped into the tank at a rate of 3 gal/min, and when the solution is well stirred, it is pumped out at the same rate. Determine a differential equation for the amount $A(t)$ of salt in the tank at time t. What is $A(0)$?

10. Suppose that a large mixing tank initially holds 300 gallons of water in which 50 pounds of salt has been dissolved. Another brine solution is pumped into the tank at a rate of 3 gal/min, and when the solution is well stirred, it is pumped out at a *slower* rate of 2 gal/min. If the concentration of the solution entering is 2 lb/gal, determine a differential equation for the amount $A(t)$ of salt in the tank at time t.

11. What is the differential equation in Problem 10 if the well-stirred solution is pumped out at a *faster* rate of 3.5 gal/min?

12. Generalize the model given in (8) of this section by assuming that the large tank initially contains N_0 number of gallons of brine, r_{in} and r_{out} are the input and output rates of the brine, respectively (measured in gallons per minute), c_{in} is the concentration of the salt in the inflow, $c(t)$ is the concentration of the salt in the tank as well as in the outflow at time t (measured in pounds of salt per gallon), and $A(t)$ is the amount of salt in the tank at time t.

Draining a Tank

13. Suppose water is leaking from a tank through a circular hole of area A_h at its bottom. When water leaks through a hole, friction and contraction of the stream near the hole reduce the volume of the water leaving the tank per second to $cA_h\sqrt{2gh}$ where $c\,(0 < c < 1)$ is an empirical constant. Determine a differential equation for the height h of water at time t for the cubical tank in **FIGURE 1.3.12**. The radius of the hole is 2 in. and $g = 32$ ft/s^2.

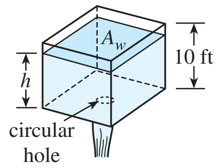


FIGURE 1.3.12 Cubical tank in Problem 13

14. The right-circular conical tank shown in **FIGURE 1.3.13** loses water out of a circular hole at its bottom. Determine a differential equation for the height of the water h at time t. The radius of the hole is 2 in., $g = 32$ ft/s^2, and the friction/contraction factor introduced in Problem 13 is $c = 0.6$.

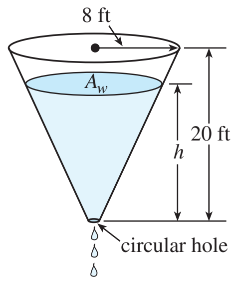


FIGURE 1.3.13 Conical tank in Problem 14

Series Circuits

15. A series circuit contains a resistor and an inductor as shown in **FIGURE 1.3.14**. Determine a differential equation for the current $i(t)$ if the resistance is R, the inductance is L, and the impressed voltage is $E(t)$.

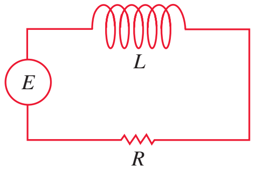


FIGURE 1.3.14 *LR*-series circuit in Problem 15

16. A series circuit contains a resistor and a capacitor as shown in **FIGURE 1.3.15**. Determine a differential equation for the charge $q(t)$ on the capacitor if the resistance is R, the capacitance is C, and the impressed voltage is $E(t)$.

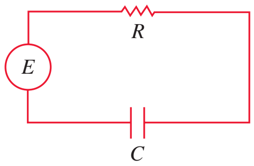


FIGURE 1.3.15 *RC*-series circuit in Problem 16

Falling Bodies and Air Resistance

17. For high-speed motion through the air—such as the skydiver shown in **FIGURE 1.3.16** falling before the parachute is opened—air resistance is closer to a power of the instantaneous velocity $v(t)$. Determine a differential equation for the velocity $v(t)$ of a falling body of mass m if air resistance is proportional to the square of the instantaneous velocity.

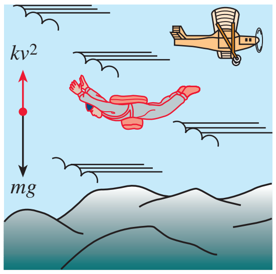


FIGURE 1.3.16 Air resistance proportional to square of velocity in Problem 17

Newton's Second Law and Archimedes' Principle

18. A cylindrical barrel s ft in diameter of weight w lb is floating in water as shown in **FIGURE 1.3.17(a)**. After an initial depression, the barrel exhibits an up-and-down bobbing motion along a vertical line. Using Figure 1.3.17(b), determine a differential equation for the vertical displacement $y(t)$ if the origin is taken to be on the vertical axis at the surface of the water when the barrel is at rest. Assume the downward direction is positive, that the weight density of the water is 62.4 lb/ft^3, and that there is no resistance between the barrel and the water. Use **Archimedes' principle**: Buoyancy, or the upward force of the water on the barrel, is equal to the weight of the water displaced. **Archimedes of Syracuse** (287 BCE–212 BCE) was arguably one of the greatest scientists/mathematicians of antiquity. Using his approximation of the number π, he found the area of a circle as well as the surface area and volume of a sphere.

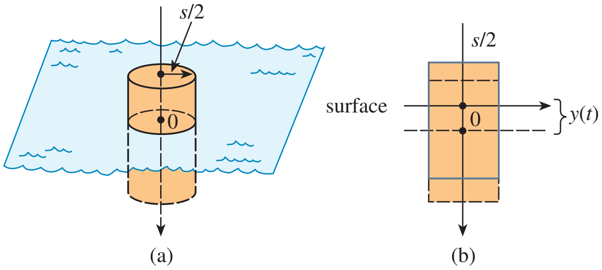


FIGURE 1.3.17 Bobbing motion of floating barrel in Problem 18

Newton's Second Law and Hooke's Law

19. After a mass m is attached to a spring, it stretches s units and then hangs at rest in the equilibrium position as shown in **FIGURE 1.3.18(b)**. After the spring/mass system has been set in motion, let $x(t)$ denote the directed distance of the mass beyond the equilibrium position. As indicated in Figure 1.3.18(c), assume that the downward direction is positive, that the motion takes place in a vertical straight line through the center of gravity of the mass, and that the only forces acting on the system are the weight of the mass and the restoring force of the stretched spring. Use **Hooke's law**: The restoring force of a spring is proportional to its total elongation. Determine a differential equation for the displacement $x(t)$ at time t.

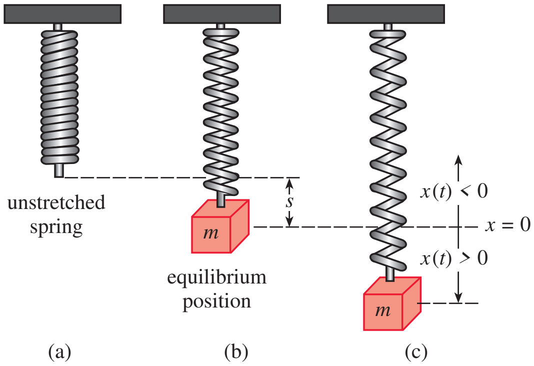


FIGURE 1.3.18 Spring/mass system in Problem 19

20. In Problem 19, what is a differential equation for the displacement $x(t)$ if the motion takes place in a medium that imparts a damping force on the spring/mass system that is proportional to the instantaneous velocity of the mass and acts in a direction opposite to that of motion?

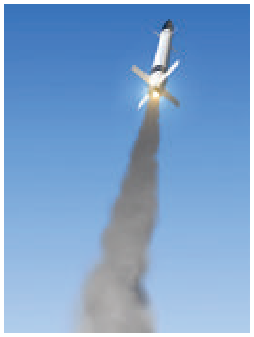
Newton's Second Law and Variable Mass

When the mass m of a body moving through a force field is variable, **Newton's second law of motion** takes on the following form: If the net force acting on a body is not zero, then the net force F is equal to the time rate of change of momentum of the body. That is,

$$F = \frac{d}{dt}(mv)^*, \tag{17}$$

where mv is momentum. Use this formulation of Newton's second law in Problems 21 and 22.

21. Consider a single-stage rocket that is launched vertically upward as shown in the accompanying photo. Let $m(t)$ denote the total mass of the rocket at time t (which is the sum of three masses: the constant mass of the payload, the constant mass of the vehicle, and the variable amount of fuel). Assume that the positive direction is upward, air resistance is proportional to the instantaneous velocity v of the rocket, and R is the upward thrust or force generated by the propulsion system. Use (17) to find a mathematical model for the velocity $v(t)$ of the rocket.



Rocket in Problem 21

22. In Problem 21, suppose $m(t) = m_p + m_v + m_f(t)$ where m_p is constant mass of the payload, m_v is the constant mass of the vehicle, and $m_f(t)$ is the variable amount of fuel.

(a) Show that the rate at which the total mass of the rocket changes is the same as the rate at which the mass of the fuel changes.

(b) If the rocket consumes its fuel at a constant rate λ, find $m(t)$. Then rewrite the differential equation in Problem 21 in terms of λ and the initial total mass $m(0) = m_0$.

(c) Under the assumption in part (b), show that the burnout time $t_b > 0$ of the rocket, or the time at which all the fuel is consumed, is $t_b = m_f(0)/\lambda$, where $m_f(0)$ is the initial mass of the fuel.

Newton's Second Law and the Law of Universal Gravitation

23. By **Newton's law of universal gravitation**, the free-fall acceleration a of a body, such as the satellite shown in **FIGURE 1.3.19**, falling a great distance to the surface is *not* the constant g. Rather, the acceleration a is inversely proportional to the square of the distance from the center of the Earth, $a = k/r^2$, where k is the constant of proportionality. Use the fact that at the surface of the Earth $r = R$ and $a = g$ to determine k. If the positive direction is upward, use Newton's second law and his universal law of gravitation to find a differential equation for the distance r.

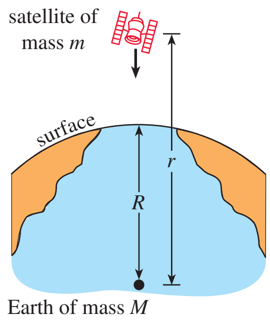


FIGURE 1.3.19 Satellite in Problem 23

24. Suppose a hole is drilled through the center of the Earth and a bowling ball of mass m is dropped into the hole, as shown in **FIGURE 1.3.20**. Construct a mathematical model that describes the motion of the ball. At time t let r denote the distance from the center of the Earth to the mass m, M denote the mass of the Earth, M_r denote the mass of that portion of the Earth within a sphere of radius r, and δ denote the constant density of the Earth.

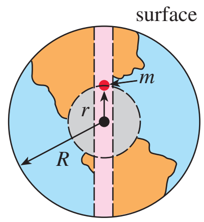


FIGURE 1.3.20 Hole through Earth in Problem 24

Additional Mathematical Models

25. Learning Theory In the theory of learning, the rate at which a subject is memorized is assumed to be proportional to the amount that is left to be memorized. Suppose M denotes the total amount of a subject to be memorized and $A(t)$ is the amount memorized in time t. Determine a differential equation for the amount $A(t)$.

26. Forgetfulness In Problem 25, assume that the rate at which material is *forgotten* is proportional to the amount memorized in time t. Determine a differential equation for $A(t)$ when forgetfulness is taken into account.

27. Infusion of a Drug A drug is infused into a patient's bloodstream at a constant rate of r grams per second. Simultaneously, the drug is removed at a rate proportional to the amount $x(t)$ of the drug present at time t. Determine a differential equation governing the amount $x(t)$.

28. Tractrix A motorboat starts at the origin and moves in the direction of the positive x-axis, pulling a waterskier along a curve C called a **tractrix**. See **FIGURE 1.3.21**. The waterskier, initially located on the y-axis at the point $(0, s)$, is pulled by keeping a rope of constant length s, which is kept taut throughout the motion. At time $t > 0$ the waterskier is at the point $P(x, y)$. Find the differential equation of the path of motion C.

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*Note that when m is constant, this is the same as $F = ma$.

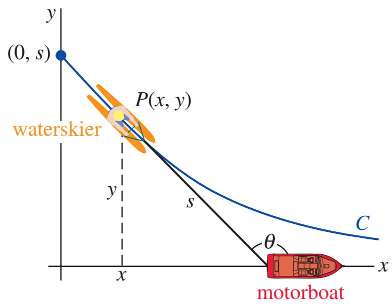


FIGURE 1.3.21 Tractrix curve in Problem 28

29. Reflecting Surface Assume that when the plane curve C shown in **FIGURE 1.3.22** is revolved about the x-axis it generates a surface of revolution with the property that all light rays L parallel to the x-axis striking the surface are reflected to a single point O (the origin). Use the fact that the angle of incidence is equal to the angle of reflection to determine a differential equation that describes the shape of the curve C. Such a curve C is important in applications ranging from construction of telescopes to satellite antennas, automobile headlights, and solar collectors. [*Hint*: Inspection of the figure shows that we can write $\phi = 2\theta$. Why? Now use an appropriate trigonometric identity.]

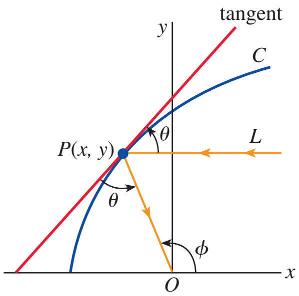


FIGURE 1.3.22 Reflecting surface in Problem 29

Satellite dish antenna

Discussion Problems

30. Reread Problem 53 in Exercises 1.1 and then give an explicit solution $P(t)$ for equation (1). Find a one-parameter family of solutions of (1).

31. Reread the sentence following equation (3) and assume that T_m is a positive constant. Discuss why we would expect $k < 0$ in (3) in both cases of cooling and warming. You might start by interpreting, say, $T(t) > T_m$ in a graphical manner.

32. Reread the discussion leading up to equation (8). If we assume that initially the tank holds, say, 50 lb of salt, it stands to reason that since salt is being added to the tank continuously for $t > 0$, that $A(t)$ should be an increasing function. Discuss how you might determine from the DE, without actually solving it, the number of pounds of salt in the tank after a long period of time.

33. Population Model The differential equation $dP/dt = (k \cos t)P$, where k is a positive constant, is a model of human population $P(t)$ of a certain community. Discuss an interpretation for the solution of this equation; in other words, what kind of population do you think the differential equation describes?

34. Rotating Fluid As shown in **FIGURE 1.3.23(a)**, a right-circular cylinder partially filled with fluid is rotated with a constant angular velocity ω about a vertical y-axis through its center. The rotating fluid is a surface of revolution S. To identify S, we first establish a coordinate system consisting of a vertical plane determined by the y-axis and an x-axis drawn perpendicular to the y-axis such that the point of intersection of the axes (the origin) is located at the lowest point on the surface S. We then seek a function $y = f(x)$, which represents the curve C of intersection of the surface S and the vertical coordinate plane. Let the point $P(x, y)$ denote the position of a particle of the rotating fluid of mass m in the coordinate plane. See Figure 1.3.23(b).

(a) At P, there is a reaction force of magnitude F due to the other particles of the fluid, which is normal to the surface S. By Newton's second law the magnitude of the net force acting on the particle is $m\omega^2 x$. What is this force? Use Figure 1.3.23(b) to discuss the nature and origin of the equations

$$F \cos \theta = mg, \qquad F \sin \theta = m\omega^2 x.$$

(b) Use part (a) to find a first-order differential equation that defines the function $y = f(x)$.

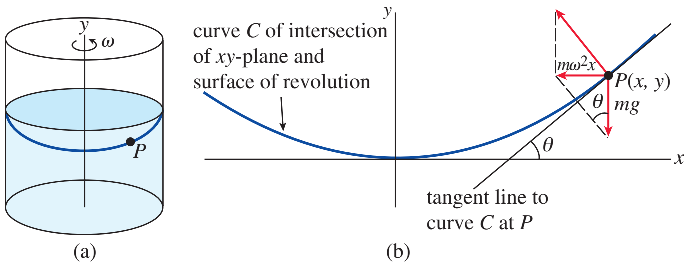


FIGURE 1.3.23 Rotating fluid in Problem 34

35. Falling Body In Problem 23, suppose $r = R + s$, where s is the distance from the surface of the Earth to the falling body. What does the differential equation obtained in Problem 23 become when s is very small compared to R?

36. Raindrops Keep Falling In meteorology, the term *virga* refers to falling raindrops or ice particles that evaporate before they reach the ground. Assume that a typical raindrop is spherical in shape. Starting at some time, which we can designate as $t = 0$, the raindrop of radius r_0 falls from rest from a cloud and begins to evaporate.

(a) If it is assumed that a raindrop evaporates in such a manner that its shape remains spherical, then it also makes sense to assume that the rate at which the raindrop evaporates—that is, the rate at which it loses mass—is proportional to its surface area. Show that this latter assumption implies that the rate at which the radius r of the raindrop decreases is a constant. Find $r(t)$. [*Hint*: See Problem 63 in Exercises 1.1.]

(b) If the positive direction is downward, construct a mathematical model for the velocity v of the falling raindrop at time t. Ignore air resistance. [*Hint*: Use the form of Newton's second law as given in (17).]

37. Let It Snow The "snowplow problem" is a classic and appears in many differential equations texts but was probably made famous by Ralph Palmer Agnew:

One day it started snowing at a heavy and steady rate. A snowplow started out at noon, going 2 *miles the first hour and* 1 *mile the second hour. What time did it start snowing?*

If possible, find the text *Differential Equations*, Ralph Palmer Agnew, McGraw-Hill, and then discuss the construction and solution of the mathematical model.

Snowplow in Problem 37

38. Reread this section and classify each mathematical model as linear or nonlinear.

39. Population Dynamics Suppose that $P'(t) = 0.15\,P(t)$ represents a mathematical model for the growth of a certain cell culture, where $P(t)$ is the size of the culture (measured in millions of cells) at time t (measured in hours). How fast is the culture growing at the time t when the size of the culture reaches 2 million cells?

40. Radioactive Decay Suppose that

$$A'(t) = -0.0004332\,A(t)$$

represents a mathematical model for the decay of radium-226, where $A(t)$ is the amount of radium (measured in grams) remaining at time t (measured in years). How much of the radium sample remains at time t when the sample is decaying at a rate of 0.002 grams per year?

1 Chapter in Review

Answers to selected odd-numbered problems begin on page ANS-1.

In Problems 1 and 2, fill in the blank and then write this result as a linear first-order differential equation that is free of the symbol c_1 and has the form $dy/dx = f(x, y)$. The symbols c_1 and k represent constants.

1. $\dfrac{d}{dx}\,c_1e^{kx} =$ ________________

2. $\dfrac{d}{dx}(5 + c_1e^{-2x}) =$ ________________

In Problems 3 and 4, fill in the blank and then write this result as a linear second-order differential equation that is free of the symbols c_1 and c_2 and has the form $F(y, y'') = 0$. The symbols c_1, c_2, and k represent constants.

3. $\dfrac{d^2}{dx^2}(c_1\cos kx + c_2\sin kx) =$ ________________

4. $\dfrac{d^2}{dx^2}(c_1\cosh kx + c_2\sinh kx) =$ ________________

In Problems 5 and 6, compute y' and y'' and then combine these derivatives with y as a linear second-order differential equation that is free of the symbols c_1 and c_2 and has the form $F(y, y', y'') = 0$. The symbols c_1 and c_2 represent constants.

5. $y = c_1e^x + c_2xe^x$

6. $y = c_1e^x\cos x + c_2e^x\sin x$

In Problems 7–12, match each of the given differential equations with one or more of these solutions:
(a) $y = 0$, **(b)** $y = 2$, **(c)** $y = 2x$, **(d)** $y = 2x^2$.

7. $xy' = 2y$

8. $y' = 2$

9. $y' = 2y - 4$

10. $xy' = y$

11. $y'' + 9y = 18$

12. $xy'' - y' = 0$

In Problems 13 and 14, determine by inspection at least one solution of the given differential equation.

13. $y'' = y'$

14. $y' = y(y - 3)$

In Problems 15 and 16, interpret each statement as a differential equation.

15. On the graph of $y = \phi(x)$, the slope of the tangent line at a point $P(x, y)$ is the square of the distance from $P(x, y)$ to the origin.

16. On the graph of $y = \phi(x)$, the rate at which the slope changes with respect to x at a point $P(x, y)$ is the negative of the slope of the tangent line at $P(x, y)$.

17. **(a)** Give the domain of the function $y = x^{2/3}$.
(b) Give the largest interval I of definition over which $y = x^{2/3}$ is a solution of the differential equation $3xy' - 2y = 0$.

18. **(a)** Verify that the one-parameter family $y^2 - 2y = x^2 - x + c$ is an implicit solution of the differential equation $(2y - 2)y' = 2x - 1$.
(b) Find a member of the one-parameter family in part (a) that satisfies the initial condition $y(0) = 1$.
(c) Use your result in part (b) to find an explicit *function* $y = \phi(x)$ that satisfies $y(0) = 1$. Give the domain of ϕ. Is $y = \phi(x)$ a *solution* of the initial-value problem? If so, give its interval I of definition; if not, explain.

19. Given that $y = -\dfrac{2}{x} + x$ is a solution of the DE $xy' + y = 2x$. Find x_0 and the largest interval I for which $y(x)$ is a solution of the IVP

$$xy' + y = 2x, \qquad y(x_0) = 1.$$

20. Suppose that $y(x)$ denotes a solution of the initial-value problem $y' = x^2 + y^2$, $y(1) = -1$ and that $y(x)$ possesses at least a second derivative at $x = 1$. In some neighborhood of $x = 1$, use the DE to determine whether $y(x)$ is increasing or decreasing, and whether the graph $y(x)$ is concave up or concave down.

21. A differential equation may possess more than one family of solutions.
(a) Plot different members of the families $y = \phi_1(x) = x^2 + c_1$ and $y = \phi_2(x) = -x^2 + c_2$.
(b) Verify that $y = \phi_1(x)$ and $y = \phi_2(x)$ are two solutions of the nonlinear first-order differential equation $(y')^2 = 4x^2$.
(c) Construct a piecewise-defined function that is a solution of the nonlinear DE in part (b) but is not a member of either family of solutions in part (a).

22. What is the slope of the tangent line to the graph of the solution of $y' = 6\sqrt{y} + 5x^3$ that passes through $(-1, 4)$?

In Problems 23–26, verify that the indicated function is an explicit solution of the given differential equation. Give an interval of definition I for each solution.

23. $y'' + y = 2\cos x - 2\sin x; \quad y = x\sin x + x\cos x$

24. $y'' + y = \sec x; \quad y = x\sin x + (\cos x)\ln(\cos x)$

25. $x^2y'' + xy' + y = 0; \quad y = \sin(\ln x)$

26. $x^2y'' + xy' + y = \sec(\ln x);$
$y = \cos(\ln x)\ln(\cos(\ln x)) + (\ln x)\sin(\ln x)$

In Problems 27–30, use (12) of Section 1.1 to verify that the indicated function is a solution of the given differential equation. Assume an appropriate interval I of definition of each solution.

27. $\dfrac{dy}{dx} + (\sin x)y = x; \quad y = e^{\cos x}\displaystyle\int_0^x te^{-\cos t}\,dt$

28. $\dfrac{dy}{dx} - 2xy = e^x; \quad y = e^{x^2}\displaystyle\int_0^x e^{t-t^2}\,dt$

29. $x^2y'' + (x^2 - x)y' + (1 - x)y = 0; \quad y = x\displaystyle\int_1^x \frac{e^{-t}}{t}\,dt$

30. $y'' + y = e^{x^2}; \quad y = \sin x\displaystyle\int_0^x e^{t^2}\cos t\,dt - \cos x\int_0^x e^{t^2}\sin t\,dt$

In Problems 31–34, verify that the indicated expression is an implicit solution of the given differential equation.

31. $x\dfrac{dy}{dx} + y = \dfrac{1}{y^2}; \quad x^3y^3 = x^3 + 5$

32. $\left(\dfrac{dy}{dx}\right)^2 + 1 = \dfrac{1}{y^2}; \quad (x - 7)^2 + y^2 = 1$

33. $y'' = 2y(y')^3; \quad y^3 + 3y = 2 - 3x$

34. $(1 + xy)y' + y^2 = 0; \quad y = e^{-xy}$

35. Find a constant c_1 such that $y = c_1 + \cos 3x$ is a solution of the differential equation $y'' + 9y = 5$.

36. Find constants c_1 and c_2 such that $y = c_1 + c_2x$ is a solution of the differential equation $y' + 2y = 3x$.

37. If c is an arbitrary constant, find a first-order differential equation for which $y = ce^{-x} + 4x - 6$ is a solution. [*Hint*: Differentiate and eliminate c between the two equations.]

38. Find a function $y = f(x)$ whose graph passes through $(0, 0)$ and whose slope at any point (x, y) in the xy-plane is $6 - 2x$.

In Problems 39–42, $y = c_1e^{-3x} + c_2e^x + 4x$ is a two-parameter family of the second-order differential equation $y'' + 2y' - 3y = -12x + 8$. Find a solution of the second-order initial-value problem consisting of this differential equation and the given initial conditions.

39. $y(0) = 0, \ y'(0) = 0$ **40.** $y(0) = 5, \ y'(0) = -11$

41. $y(1) = -2, \ y'(1) = 4$ **42.** $y(-1) = 1, \ y'(-1) = 1$

In Problems 43 and 44, verify that the function defined by the definite integral is a particular solution of the given differential equation. In both problems, use **Leibniz's rule** for the derivative of an integral:

$$\frac{d}{dx}\int_{u(x)}^{v(x)} F(x, t)\,dt = F(x, v(x))\frac{dv}{dx} - F(x, u(x))\frac{du}{dx} + \int_{u(x)}^{v(x)} \frac{\partial}{\partial x}F(x, t)\,dt.$$

43. $y'' + 9y = f(x); \quad y(x) = \dfrac{1}{3}\displaystyle\int_0^x f(t)\sin 3(x - t)\,dt$

44. $xy'' + y' - xy = 0; \quad y = \int_0^{\pi} e^{x\cos t}\,dt$ [*Hint*: After computing y' use integration by parts with respect to t.]

45. The graph of a solution of a second-order initial-value problem $d^2y/dx^2 = f(x, y, y')$, $y(2) = y_0$, $y'(2) = y_1$, is given in **FIGURE 1.R.1**. Use the graph to estimate the values of y_0 and y_1.

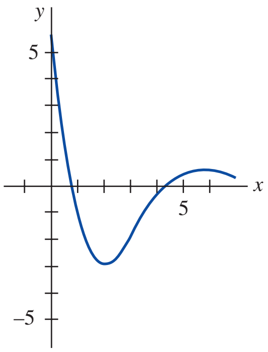


FIGURE 1.R.1 Graph for Problem 45

46. A tank in the form of a right-circular cylinder of radius 2 ft and height 10 ft is standing on end. If the tank is initially full of water, and water leaks from a circular hole of radius $\frac{1}{2}$ in. at its bottom, determine a differential equation for the height h of the water at time t. Ignore friction and contraction of water at the hole.

47. A uniform 10-foot-long heavy rope is coiled loosely on the ground. As shown in **FIGURE 1.R.2** one end of the rope is pulled vertically upward by means of a constant force of 5 lb. The rope weighs 1 lb/ft. Use Newton's second law in the form given in (17) in Exercises 1.3 to determine a differential equation for the height $x(t)$ of the end above ground level at time t. Assume that the positive direction is upward.

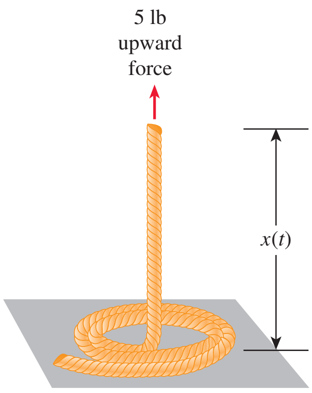


FIGURE 1.R.2 Rope pulled upward in Problem 47

CHAPTER 2

First-Order Differential Equations

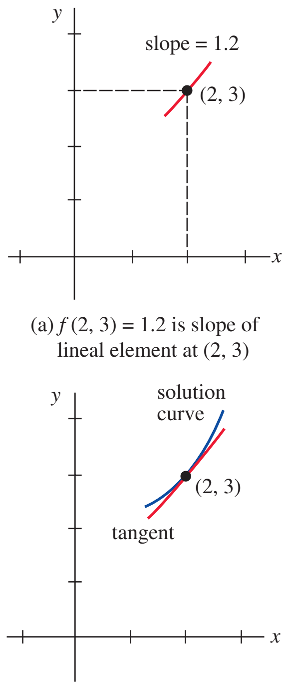
We begin our study of differential equations (DEs) with first-order equations. In this chapter we illustrate the three different ways DEs can be studied: qualitatively (Section 2.1), analytically (Sections 2.2–2.5), and numerically (Section 2.6). The chapter ends with an introduction to mathematical modeling with DEs (Sections 2.7–2.9).

2.1 Solution Curves Without a Solution

INTRODUCTION Some differential equations do not possess any solutions. For example, there is no real function that satisfies $(y')^2 + 1 = 0$. Some differential equations possess solutions that can be found **analytically**, that is, solutions in explicit or implicit form found by implementing an equation-specific method of solution. These solution methods may involve certain manipulations, such as a substitution, and procedures, such as integration. Some differential equations possess solutions, but the differential equation cannot be solved analytically. In other words, when we say that a solution of a DE exists, we do not mean that there also exists a method of solution that will produce explicit or implicit solutions. Over a time span of centuries, mathematicians have devised ingenious procedures for solving some very specialized equations, so there are, not surprisingly, a large number of differential equations that can be solved analytically. Although we shall study some of these methods of solution for first-order equations in the subsequent sections of this chapter, let us imagine for the moment that we have in front of us a first-order differential equation in normal form $dy/dx = f(x, y)$, and let us further imagine that we can neither find nor invent a method for solving it analytically. This is not as bad a predicament as one might think, since the differential equation itself can sometimes "tell" us specifics about how its solutions "behave." We have seen in Section 1.2 that whenever $f(x, y)$ and $\partial f/\partial y$ satisfy certain continuity conditions, **qualitative** questions about existence and uniqueness of solutions can be answered. In this section we shall see that other qualitative questions about properties of solutions—such as, How does a solution behave near a certain point? or How does a solution behave as $x \to \infty$?—can often be answered when the function f depends solely on the variable y.

We begin our study of first-order differential equations with two ways of analyzing a DE qualitatively. Both these ways enable us to determine, in an approximate sense, what a solution curve must look like without actually solving the equation.

2.1.1 Direction Fields



(a) $f(2, 3) = 1.2$ is slope of lineal element at (2, 3)

(b) A solution curve passing through (2, 3)

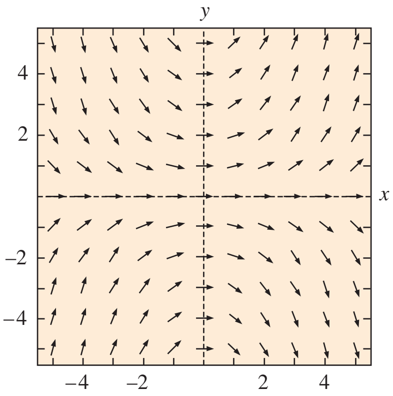
FIGURE 2.1.1 Solution curve is tangent to lineal element at (2, 3)

Slope

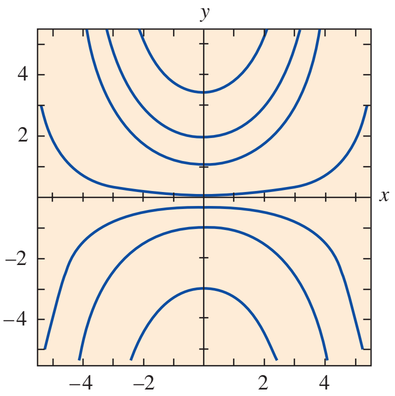
We begin with a simple concept from calculus: A derivative dy/dx of a differentiable function $y = y(x)$ gives slopes of tangent lines at points on its graph. Because a solution $y = y(x)$ of a first-order differential equation $dy/dx = f(x, y)$ is necessarily a differentiable function on its interval I of definition, it must also be continuous on I. Thus the corresponding solution curve on I must have no breaks and must possess a tangent line at each point $(x, y(x))$. The slope of the tangent line at $(x, y(x))$ on a solution curve is the value of the first derivative dy/dx at this point, and this we know from the differential equation $f(x, y(x))$. Now suppose that (x, y) represents any point in a region of the xy-plane over which the function f is defined. The value $f(x, y)$ that the function f assigns to the point represents the slope of a line, or as we shall envision it, a line segment called a **lineal element**. For example, consider the equation $dy/dx = 0.2xy$, where $f(x, y) = 0.2xy$. At, say, the point (2, 3), the slope of a lineal element is $f(2, 3) = 0.2(2)(3) = 1.2$. **FIGURE 2.1.1(a)** shows a line segment with slope 1.2 passing through (2, 3). As shown in Figure 2.1.1(b), *if* a solution curve also passes through the point (2, 3), it does so tangent to this line segment; in other words, the lineal element is a miniature tangent line at that point.

Direction Field

If we systematically evaluate f over a rectangular grid of points in the xy-plane and draw a lineal element at each point (x, y) of the grid with slope $f(x, y)$, then the collection of all these lineal elements is called a **direction field** or a **slope field** of the differential equation $dy/dx = f(x, y)$. Visually, the direction field suggests the appearance or shape of a family of solution curves of the differential equation, and consequently it may be possible to see at a glance certain qualitative aspects of the solutions—regions in the plane, for example, in which a solution exhibits an unusual behavior. A single solution curve that passes through a direction field must follow the flow pattern of the field; it is tangent to a lineal element when it intersects a point in the grid.



(a) Direction field for $dy/dx = 0.2xy$



(b) Some solution curves in the family $y = ce^{0.1x^2}$

FIGURE 2.1.2 Direction field and solution curves in Example 1

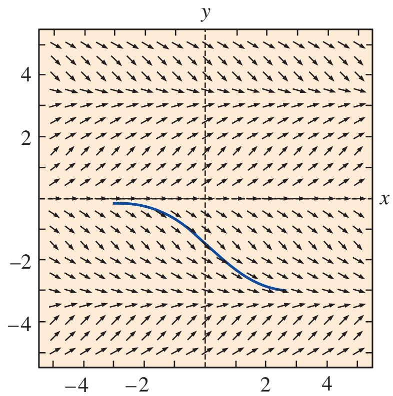


FIGURE 2.1.3 Direction field for $dy/dx = \sin y$ in Example 2

EXAMPLE 1 Direction Field

The direction field for the differential equation $dy/dx = 0.2xy$ shown in **FIGURE 2.1.2(a)** was obtained using computer software in which a 5×5 grid of points (mh, nh), m and n integers, was defined by letting $-5 \le m \le 5$, $-5 \le n \le 5$ and $h = 1$. Notice in Figure 2.1.2(a) that at any point along the x-axis ($y = 0$) and the y-axis ($x = 0$) the slopes are $f(x, 0) = 0$ and $f(0, y) = 0$, respectively, so the lineal elements are horizontal. Moreover, observe in the first quadrant that for a fixed value of x, the values of $f(x, y) = 0.2xy$ increase as y increases; similarly, for a fixed y, the values of $f(x, y) = 0.2xy$ increase as x increases. This means that as both x and y increase, the lineal elements become almost vertical and have positive slope ($f(x, y) = 0.2xy > 0$ for $x > 0$, $y > 0$). In the second quadrant, $|f(x, y)|$ increases as $|x|$ and y increase, and so the lineal elements again become almost vertical but this time have negative slope ($f(x, y) = 0.2xy < 0$ for $x < 0$, $y > 0$). Reading left to right, imagine a solution curve starts at a point in the second quadrant, moves steeply downward, becomes flat as it passes through the y-axis, and then as it enters the first quadrant moves steeply upward—in other words, its shape would be concave upward and similar to a horseshoe. From this it could be surmised that $y \to \infty$ as $x \to \pm\infty$. Now in the third and fourth quadrants, since $f(x, y) = 0.2xy > 0$ and $f(x, y) = 0.2xy < 0$, respectively, the situation is reversed; a solution curve increases and then decreases as we move from left to right.

We saw in (1) of Section 1.1 that $y = e^{0.1x^2}$ is an explicit solution of the differential equation $dy/dx = 0.2xy$; you should verify that a one-parameter family of solutions of the same equation is given by $y = ce^{0.1x^2}$. For purposes of comparison with Figure 2.1.2(a) some representative graphs of members of this family are shown in Figure 2.1.2(b). ≡

EXAMPLE 2 Direction Field

Use a direction field to sketch an approximate solution curve for the initial-value problem $dy/dx = \sin y$, $y(0) = -\frac{3}{2}$.

SOLUTION Before proceeding, recall that from the continuity of $f(x, y) = \sin y$ and $\partial f/\partial y = \cos y$, Theorem 1.2.1 guarantees the existence of a unique solution curve passing through any specified point (x_0, y_0) in the plane. Now we set our computer software again for a 5×5 rectangular region and specify (because of the initial condition) points in that region with vertical and horizontal separation of $\frac{1}{2}$ unit—that is, at points (mh, nh), $h = \frac{1}{2}$, m and n integers such that $-10 \le m \le 10$, $-10 \le n \le 10$. The result is shown in **FIGURE 2.1.3**. Since the right-hand side of $dy/dx = \sin y$ is 0 at $y = 0$ and at $y = -\pi$, the lineal elements are horizontal at all points whose second coordinates are $y = 0$ or $y = -\pi$. It makes sense then that a solution curve passing through the initial point $(0, -\frac{3}{2})$ has the shape shown in color in the figure. ≡

Increasing/Decreasing Interpretation of the derivative dy/dx as a function that gives slope plays the key role in the construction of a direction field. Another telling property of the first derivative will be used next, namely, if $dy/dx > 0$ (or $dy/dx < 0$) for all x in an interval I, then a differentiable function $y = y(x)$ is increasing (or decreasing) on I.

REMARKS

Sketching a direction field by hand is straightforward but time consuming; it is probably one of those tasks about which an argument can be made for doing it once or twice in a lifetime, but it is overall most efficiently carried out by means of computer software. Prior to calculators, PCs, and software, the **method of isoclines** was used to facilitate sketching a direction field by hand. For the DE $dy/dx = f(x, y)$, any member of the family of curves $f(x, y) = c$, c a constant, is called an **isocline**. Lineal elements drawn through points on a specific isocline, say, $f(x, y) = c_1$, all have the same slope c_1. In Problem 15 in Exercises 2.1, you have your two opportunities to sketch a direction field by hand.

2.1.2 Autonomous First-Order DEs

DEs Free of the Independent Variable In Section 1.1 we divided the class of ordinary differential equations into two types: linear and nonlinear. We now consider briefly another kind of classification of ordinary differential equations, a classification that is of particular importance in the qualitative investigation of differential equations. An ordinary differential equation in which the independent variable does not appear explicitly is said to be **autonomous**. If the symbol x denotes the independent variable, then an autonomous first-order differential equation can be written in general form as $F(y, y') = 0$ or in normal form as

$$\frac{dy}{dx} = f(y). \tag{1}$$

We shall assume throughout the discussion that follows that f in (1) and its derivative f' are continuous functions of y on some interval I. The first-order equations

$$\frac{dy}{dx} = \overset{\overset{f(y)}{\downarrow}}{1 + y^2} \quad \text{and} \quad \frac{dy}{dx} = \overset{\overset{f(x,\,y)}{\downarrow}}{0.2xy}$$

are autonomous and nonautonomous, respectively.

Many differential equations encountered in applications, or equations that are models of physical laws that do not change over time, are autonomous. As we have already seen in Section 1.3, in an applied context, symbols other than y and x are routinely used to represent the dependent and independent variables. For example, if t represents time, then inspection of

$$\frac{dA}{dt} = kA, \quad \frac{dx}{dt} = kx(n + 1 - x), \quad \frac{dT}{dt} = k(T - T_m), \quad \frac{dA}{dt} = 6 - \frac{1}{100}A,$$

where k, n, and T_m are constants, shows that each equation is time-independent. Indeed, *all* of the first-order differential equations introduced in Section 1.3 are time-independent and so are autonomous.

Critical Points The zeros of the function f in (1) are of special importance. We say that a real number c is a **critical point** of the autonomous differential equation (1) if it is a zero of f, that is, $f(c) = 0$. A critical point is also called an **equilibrium point** or **stationary point**. Now observe that if we substitute the constant function $y(x) = c$ into (1), then both sides of the equation equal zero. This means

If c is a critical point of (1), then $y(x) = c$ is a constant solution of the autonomous differential equation.

A constant solution $y(x) = c$ of (1) is called an **equilibrium solution**; equilibria are the *only* constant solutions of (1).

As already mentioned, we can tell when a nonconstant solution $y = y(x)$ of (1) is increasing or decreasing by determining the algebraic sign of the derivative dy/dx; in the case of (1) we do this by identifying the intervals on the y-axis over which the function $f(y)$ is positive or negative.

EXAMPLE 3 An Autonomous DE

The differential equation

$$\frac{dP}{dt} = P(a - bP),$$

where a and b are positive constants, has the normal form $dP/dt = f(P)$, which is (1) with t and P playing the parts of x and y, respectively, and hence is autonomous. From $f(P) = P(a - bP) = 0$, we see that 0 and a/b are critical points of the equation and so the equilibrium solutions are $P(t) = 0$ and $P(t) = a/b$. By putting the critical points on a vertical line, we divide the line into three intervals defined by $-\infty < P < 0$, $0 < P < a/b$, $a/b < P < \infty$. The arrows on the line shown in **FIGURE 2.1.4** indicate the algebraic sign of $f(P) = P(a - bP)$ on these intervals and whether a nonconstant solution $P(t)$ is increasing or decreasing on an interval. The following table explains the figure.

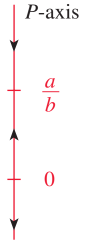


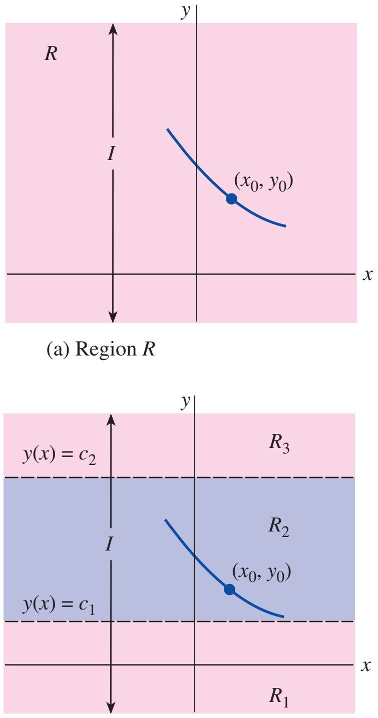
FIGURE 2.1.4 Phase portrait for Example 3

Interval	Sign of $f(P)$	$P(t)$	Arrow
$(-\infty, 0)$	minus	decreasing	points down
$(0, a/b)$	plus	increasing	points up
$(a/b, \infty)$	minus	decreasing	points down

Figure 2.1.4 is called a **one-dimensional phase portrait**, or simply **phase portrait**, of the differential equation $dP/dt = P(a - bP)$. The vertical line is called a **phase line**. ≡

Solution Curves

Without solving an autonomous differential equation, we can usually say a great deal about its solution curves. Since the function f in (1) is independent of the variable x, we can consider f defined for $-\infty < x < \infty$ or for $0 \le x < \infty$. Also, since f and its derivative f' are continuous functions of y on some interval I of the y-axis, the fundamental results of Theorem 1.2.1 hold in some horizontal strip or region R in the xy-plane corresponding to I, and so through any point (x_0, y_0) in R there passes only one solution curve of (1). See **FIGURE 2.1.5(a)**. For the sake of discussion, let us suppose that (1) possesses exactly two critical points, c_1 and c_2, and that $c_1 < c_2$. The graphs of the equilibrium solutions $y(x) = c_1$ and $y(x) = c_2$ are horizontal lines, and these lines partition the region R into three subregions R_1, R_2, and R_3 as illustrated in Figure 2.1.5(b). Without proof, here are some conclusions that we can draw about a nonconstant solution $y(x)$ of (1):



(a) Region R

(b) Subregions R_1, R_2, and R_3

FIGURE 2.1.5 Lines $y(x) = c_1$ and $y(x) = c_2$ partition R into three horizontal subregions

- If (x_0, y_0) is in a subregion R_i, $i = 1, 2, 3$, and $y(x)$ is a solution whose graph passes through this point, then $y(x)$ remains in the subregion R_i for all x. As illustrated in Figure 2.1.5(b), the solution $y(x)$ in R_2 is bounded below by c_1 and above by c_2; that is, $c_1 < y(x) < c_2$ for all x. The solution curve stays within R_2 for all x because the graph of a nonconstant solution of (1) cannot cross the graph of either equilibrium solution $y(x) = c_1$ or $y(x) = c_2$. See Problem 33 in Exercises 2.1.
- By continuity of f we must then have either $f(y) > 0$ or $f(y) < 0$ for all x in a subregion R_i, $i = 1, 2, 3$. In other words, $f(y)$ cannot change signs in a subregion. See Problem 33 in Exercises 2.1.
- Since $dy/dx = f(y(x))$ is either positive or negative in a subregion R_i, $i = 1, 2, 3$, a solution $y(x)$ is strictly monotonic—that is, $y(x)$ is either increasing or decreasing in a subregion R_i. Therefore $y(x)$ cannot be oscillatory, nor can it have a relative extremum (maximum or minimum). See Problem 33 in Exercises 2.1.
- If $y(x)$ is *bounded above* by a critical point c_1 (as in subregion R_1 where $y(x) < c_1$ for all x), then the graph of $y(x)$ must approach the graph of the equilibrium solution $y(x) = c_1$ either as $x \to \infty$ or as $x \to -\infty$. If $y(x)$ is *bounded*, that is, bounded above and below by two consecutive critical points (as in subregion R_2 where $c_1 < y(x) < c_2$ for all x), then the graph of $y(x)$ must approach the graphs of the equilibrium solutions $y(x) = c_1$ and $y(x) = c_2$, one as $x \to \infty$ and the other as $x \to -\infty$. If $y(x)$ is *bounded below* by a critical point (as in subregion R_3 where $c_2 < y(x)$ for all x), then the graph of $y(x)$ must approach the graph of the equilibrium solution $y(x) = c_2$ either as $x \to \infty$ or as $x \to -\infty$. See Problem 34 in Exercises 2.1.

With the foregoing facts in mind, let us reexamine the differential equation in Example 3.

EXAMPLE 4 Example 3 Revisited

The three intervals determined on the P-axis or phase line by the critical points $P = 0$ and $P = a/b$ now correspond in the tP-plane to three subregions:

$$R_1: -\infty < P < 0, \qquad R_2: 0 < P < a/b, \qquad R_3: a/b < P < \infty,$$

where $-\infty < t < \infty$. The phase portrait in Figure 2.1.4 tells us that $P(t)$ is decreasing in R_1, increasing in R_2, and decreasing in R_3. If $P(0) = P_0$ is an initial value, then in R_1, R_2, and R_3, we have, respectively, the following:

(*i*) For $P_0 < 0$, $P(t)$ is bounded above. Since $P(t)$ is decreasing, $P(t)$ decreases without bound for increasing t and so $P(t) \to 0$ as $t \to -\infty$. This means the negative t-axis, the graph of the equilibrium solution $P(t) = 0$, is a horizontal asymptote for a solution curve.

(*ii*) For $0 < P_0 < a/b$, $P(t)$ is bounded. Since $P(t)$ is increasing, $P(t) \to a/b$ as $t \to \infty$ and $P(t) \to 0$ as $t \to -\infty$. The graphs of the two equilibrium solutions, $P(t) = 0$ and $P(t) = a/b$, are horizontal lines that are horizontal asymptotes for any solution curve starting in this subregion.

(*iii*) For $P_0 > a/b$, $P(t)$ is bounded below. Since $P(t)$ is decreasing, $P(t) \to a/b$ as $t \to \infty$. The graph of the equilibrium solution $P(t) = a/b$ is a horizontal asymptote for a solution curve.

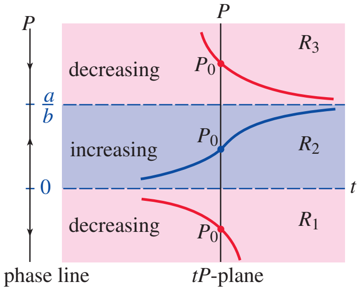


FIGURE 2.1.6 Phase portrait and solution curves in each of the three subregions in Example 4

In **FIGURE 2.1.6**, the phase line is the P-axis in the tP-plane. For clarity, the original phase line from Figure 2.1.4 is reproduced to the left of the plane in which the subregions R_1, R_2, and R_3 are shaded. The graphs of the equilibrium solutions $P(t) = a/b$ and $P(t) = 0$ (the t-axis) are shown in the figure as blue dashed lines; the solid graphs represent typical graphs of $P(t)$ illustrating the three cases just discussed. ≡

In a subregion such as R_1 in Example 4, where $P(t)$ is decreasing and unbounded below, we must necessarily have $P(t) \to -\infty$. Do *not* interpret this last statement to mean $P(t) \to -\infty$ as $t \to \infty$; we could have $P(t) \to -\infty$ as $t \to T$, where $T > 0$ is a finite number that depends on the initial condition $P(t_0) = P_0$. Thinking in dynamic terms, $P(t)$ could "blow up" in finite time; thinking graphically, $P(t)$ could have a vertical asymptote at $t = T > 0$. A similar remark holds for the subregion R_3.

The differential equation $dy/dx = \sin y$ in Example 2 is autonomous and has an infinite number of critical points since $\sin y = 0$ at $y = n\pi$, n an integer. Moreover, we now know that because the solution $y(x)$ that passes through $(0, -\frac{3}{2})$ is bounded above and below by two consecutive critical points $(-\pi < y(x) < 0)$ and is decreasing ($\sin y < 0$ for $-\pi < y < 0$), the graph of $y(x)$ must approach the graphs of the equilibrium solutions as horizontal asymptotes: $y(x) \to -\pi$ as $x \to \infty$ and $y(x) \to 0$ as $x \to -\infty$.

EXAMPLE 5 Solution Curves of an Autonomous DE

The autonomous equation $dy/dx = (y - 1)^2$ possesses the single critical point 1. From the phase portrait in **FIGURE 2.1.7(a)**, we conclude that a solution $y(x)$ is an increasing function in the subregions defined by $-\infty < y < 1$ and $1 < y < \infty$, where $-\infty < x < \infty$. For an initial condition $y(0) = y_0 < 1$, a solution $y(x)$ is increasing and bounded above by 1, and so $y(x) \to 1$ as $x \to \infty$; for $y(0) = y_0 > 1$, a solution $y(x)$ is increasing and unbounded.

Now $y(x) = 1 - 1/(x + c)$ is a one-parameter family of solutions of the differential equation. (See Problem 4 in Exercises 2.2.) A given initial condition determines a value for c. For the initial conditions, say, $y(0) = -1 < 1$ and $y(0) = 2 > 1$, we find, in turn, that $y(x) = 1 - 1/(x + \frac{1}{2})$ and so $y(x) = 1 - 1/(x - 1)$. As shown in Figure 2.1.7(b) and 2.1.7(c), the graph of each of these rational functions possesses a vertical asymptote. But bear in mind that the solutions of the IVPs

$$\frac{dy}{dx} = (y - 1)^2, \quad y(0) = -1 \quad \text{and} \quad \frac{dy}{dx} = (y - 1)^2, \quad y(0) = 2$$

are defined on special intervals. The two solutions are, respectively,

$$y(x) = 1 - \frac{1}{x + \frac{1}{2}}, \quad -\frac{1}{2} < x < \infty \quad \text{and} \quad y(x) = 1 - \frac{1}{x - 1}, \quad -\infty < x < 1.$$

The solution curves are the portions of the graphs in Figures 2.1.7(b) and 2.1.7(c) shown in blue. As predicted by the phase portrait, for the solution curve in Figure 2.1.7(b), $y(x) \to 1$ as $x \to \infty$; for the solution curve in Figure 2.1.7(c), $y(x) \to \infty$ as $x \to 1$ from the left.

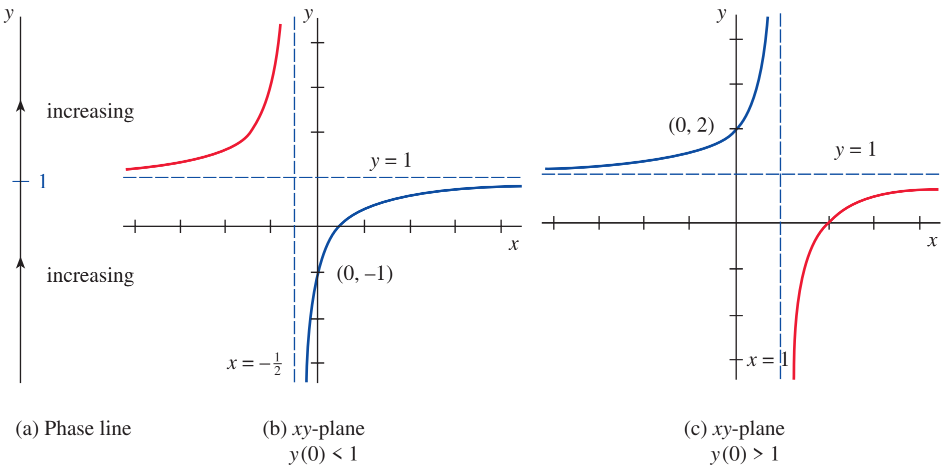


FIGURE 2.1.7 Behavior of solutions near $y = 1$ in Example 5 ≡

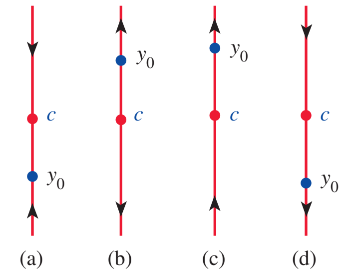


FIGURE 2.1.8 Critical point c is an attractor in (a), a repeller in (b), and semi-stable in (c) and (d)

Attractors and Repellers

Suppose $y(x)$ is a nonconstant solution of the autonomous differential equation given in (1) and that c is a critical point of the DE. There are basically three types of behavior $y(x)$ can exhibit near c. In **FIGURE 2.1.8** we have placed c on four vertical phase lines. When both arrowheads on either side of the dot labeled c point *toward* c, as in Figure 2.1.8(a), all solutions $y(x)$ of (1) that start from an initial point (x_0, y_0) sufficiently near c exhibit the asymptotic behavior $\lim_{x\to\infty} y(x) = c$. For this reason the critical point c is said to be **asymptotically stable**. Using a physical analogy, a solution that starts near c is like a charged particle that, over time, is drawn to a particle of opposite charge, and so c is also referred to as an **attractor**. When both arrowheads on either side of the dot labeled c point *away* from c, as in Figure 2.1.8(b), all solutions $y(x)$ of (1) that start from an initial point (x_0, y_0) move away from c as x increases. In this case the critical point c is said to be **unstable**. An unstable critical point is also called a **repeller**, for obvious reasons. The critical point c illustrated in Figures 2.1.8(c) and 2.1.8(d) is neither an attractor nor a repeller. But since c exhibits characteristics of both an attractor and a repeller—that is, a solution starting from an initial point (x_0, y_0) sufficiently near c is attracted to c from one side and repelled from the other side—we say that the critical point c is **semi-stable**. In Example 3, the critical point a/b is asymptotically stable (an attractor) and the critical point 0 is unstable (a repeller). The critical point 1 in Example 5 is semi-stable.

EXAMPLE 6 Classifying Critical Points

Locate and classify all critical points of $\dfrac{dy}{dx} = 4y - y^3$.

SOLUTION Rewriting the differential equation as

$$\frac{dy}{dx} = y(4 - y^2) = y(2 - y)(2 + y)$$

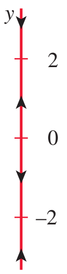


FIGURE 2.1.9 Phase portrait of DE in Example 6

we see from $y(2 - y)(2 + y) = 0$ that $y = 0$, $y = 2$, and $y = -2$ are critical points of the DE. Now by examining, in turn, the algebraic signs of dy/dx on intervals of the y-axis determined by the critical points, we see from the phase portrait in **FIGURE 2.1.9** that:

$$\overbrace{(-\infty, -2)}^{dy/dx>0},\ \overbrace{(-2, 0)}^{dy/dx<0}, \text{ implies } y = -2 \text{ is asymptotically stable (attractor)},$$

$$\overbrace{(-2, 0)}^{dy/dx<0},\ \overbrace{(0, 2)}^{dy/dx>0}, \text{ implies } y = 0 \text{ is unstable (repeller)},$$

$$\overbrace{(0, 2)}^{dy/dx>0},\ \overbrace{(2, \infty)}^{dy/dx<0}, \text{ implies } y = 2 \text{ is asymptotically stable (attractor)}.$$

See Problems 21–28 in Exercises 2.1. ≡