The *Nature* of Mathematics

13th Edition

KARL J. SMITH

Getting Help with the material in this text.

- Important terms are in **boldface** and are listed at the end of each chapter and in the **glossary.**
- Important ideas are reviewed at the end of each chapter.
- Types of problems are listed at the end of each chapter.
- The *Student's Survival Manual* lists the new terms for each chapter and enumerates the types of problems in each chapter.
- I also use this special font to speak to you directly out of the context of the regular textual material. I call these author's notes, and they are comments that I might say to you if we were chatting in my office about the content in this text.
- Road signs are used to help you with your journey through the text:

This stop sign means that you should stop and pay attention to this idea, since it will be used as you travel through the rest of the text.

Caution means that you will need to proceed more slowly to understand this material.

A bump symbol means to watch out, because you are coming to some difficult material.

This logo means you should check this out on the Web. The website for this text is **www.mathnature.com.** You will find homework hints, essential ideas, search engines, projects, and links to related topics.

Free mathematics tutorials, problems, and worksheets (with applets). This is an invaluable link:

http://www.analyzemath.com/

Webmath: Instant solution to your math problems:

http://www.webmath.com/

Ask Dr. Math is a question and answer service for K-12 math students and their teachers. A searchable archive is available by level and topic, together with frequently asked questions (FAQ).

http://mathforum.org/dr.math/

Online Math Calculators and Solvers. Easy-to-use online math calculators and solvers are available for various mathematics topics. These may be used to check homework solutions, as well as to practice on your own.

http://www.analyzemath.com/Calculators.html

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http://archives.math.utk.edu/

The Math Forum@Drexel is a great source of mathematical information that offers Math Talk, Math Problems and Puzzles, Math Help, and other Math Resources and Tools:

http://mathforum.org/

You can contact the author at: smithkjs@mathnature.com.

Thirteenth Edition

THE
Nature of Mathematics

KARL J. SMITH

Santa Rosa Junior College

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I dedicate this book, with love,
to my wife, Linda.

Preface

I frequently encounter people who tell me about their unpleasant experiences with mathematics. I have a true sympathy for those people, and I recall one of my elementary school teachers who assigned additional arithmetic problems as punishment. This can only create negative attitudes toward mathematics, which is indeed unfortunate. If elementary school teachers and parents have positive attitudes toward mathematics, their children cannot help but see some of the beauty of the subject. I want students to come away from this course with the feeling that mathematics can be pleasant, useful, and practical—and enjoyed for its own sake.

Since the first edition, my goal has been, and continues to be, to create a positive attitude toward mathematics. But the world, the students, and the professors are very different today than they were when I began writing this text. This is a very different text from its first printing, and this edition is very different from the previous edition. The world of knowledge is more accessible today (via the Internet) than at any time in history. Supplementary help is available on the Internet, and can be accessed at the following Web address: **www.mathnature.com**

All of the Web addresses mentioned in this text are linked to the above Web address. If you have access to a computer and the Internet, check out this Web address. You will find links to several search engines, history, and reference topics. You will find, for each section, homework hints, and a listing of essential ideas, projects, and links to related information on the Web.

This text was written for students who need a mathematics course to satisfy the general university competency requirement in mathematics. Because of the university requirement, many students enrolling in a course that uses my text have postponed taking this course as long as possible. They dread the experience, and come to class with a great deal of anxiety. Rather than simply presenting the technical details needed to proceed to the next course, I have attempted to give insight into what mathematics is, what it accomplishes, and how it is pursued as a human enterprise. However, at the same time, in this thirteenth edition I have included a great deal of material to help students estimate, calculate, and solve problems *outside* the classroom or textbook setting.

This text was written to meet the needs of all students and schools. How did I accomplish that goal? First, the chapters are almost independent of one another, and can be covered in any order appropriate to a particular audience. For example, in this edition, you will see the order of the chapters on measurement and on networks has been reversed from the previous edition. This shift was made in response to many of you who had been teaching these chapters for years.

Second, the problems are designed as the core of the course. There are problems that every student will find easy and this will provide the opportunity for success; there are also problems that are very challenging. Much interesting material appears in the problems, and students should get into the habit of reading (not necessarily working) all the problems whether or not they are assigned.

Level 1: Mechanical or drill problems

Level 2: Problems that require understanding of the concepts

Level 3: Problems that require problem-solving skills or original thinking

Like almost every subject of human interest, mathematics is as easy or as difficult as we choose to make it. At the beginning of Chapter 1, I have included a Fable, and have addressed it directly to you, the student. I hope you will take the time to read it, and then ponder why I call it a fable.

You will notice street sign symbols used throughout this text. I use this stop sign to mean that you should stop and pay attention to this idea, since it will be used as you travel through the rest of the text.

Caution means that you will need to proceed more slowly to understand this material.

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I also use this special font to speak to you directly out of the context of the regular textual material. I call these **author's notes;** they are comments that I might say to you if we were chatting in my office about the content in this text.

What Are the Major Themes of This Text?

The major themes of this text are *problem solving* and estimation in the context of presenting the great ideas in the history of mathematics.

I believe that *learning to solve problems is the principal reason for studying mathematics.* Problem solving is the process of applying previously acquired knowledge to new and unfamiliar situations. Solving word problems in most textbooks is one form of problem solving, but students also should be faced with non-text-type problems. In the first section of this edition, I introduce students to Pólya's problem-solving techniques, and these techniques are used throughout the text to solve non-text-type problems. Problem-solving examples are found throughout the text (marked as **PÓLYA'S METHOD** examples).

YinYang/Getty Images

Example 2 Retirement calculation

Pólya's Method

Suppose you are 21 years old and will make monthly deposits to a bank account paying 4% annual interest compounded monthly.

***Option I*:** Pay yourself \$200 per month for 5 years and then leave the balance in the bank until age 65. (Total amount of deposits is \$200 × 5 × 12 = \$12,000.)

***Option II*:** Wait until you are 50 years old (the age most of us start thinking seriously about retirement) and then deposit \$200 per month until age 65. (Total amount of deposits is \$200 × 15 × 12 = \$36,000.)

Compare the amounts you would have from each of these options.

Solution We use Pólya's problem-solving guidelines for this example.

Understand the Problem. When most of us are 21 years old, we do not think about retirement. However, if we do, the results can be dramatic. With this example, we investigate the differences if we save early (for 5 years), or later (for 15 years).

Devise a Plan. We calculate the value of the annuity for the 5 years of the first option, and then calculate the effect of leaving the value at the end of 5 years (the annuity) in a savings account until retirement. This part of the problem is a future value problem because it becomes a lump-sum problem when deposits are no longer made (after 5 years). For the second option, we calculate the value of the annuity for 15 years.

Carry Out the Plan.

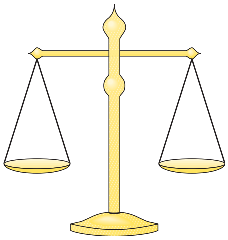
***Option I*:** \$200 per month for 5 years at 4% annual interest is an annuity with $m = 200$, $r = 0.04$, $t = 5$, and $n = 12$; this is an *annuity*.

$$A = 200\left[\frac{\left(1 + \frac{0.04}{12}\right)^{12(5)} - 1}{\frac{0.04}{12}}\right] \approx 13,259.80$$

You will find problems in *each* section that require Pólya's method for problem solving, and then you can practice your problem-solving skills with problems that are marked **Level 3, Problem Solving.**

Problem Solving Level 3

57. You have 9 coins, but you are told that one of the coins is counterfeit and weighs just a little more than an authentic coin. How can you determine the counterfeit with 2 weighings on a two-pan balance scale? (This problem is discussed in Chapter 3.)



Students should learn the language and notation of mathematics. Most students who have trouble with mathematics do not realize that mathematics *does require hard work.* The usual pattern for most mathematics students is to open the text to the assigned page of problems, and begin working. Only after getting "stuck" is an attempt made to "find it in the text." The final resort is reading the text. In this text, students are asked not only to "do math problems," but also to "experience mathematics." This means it is necessary to become involved with the **concepts** being presented, not "just get answers." In fact, the advertising slogan "Math Is Not a Spectator Sport" is an invitation that suggests that the only way to succeed in mathematics is to become involved with it.

PACIFIC GROVE, CALIFORNIA
★ MATH ★
is not a
spectator
sport
BROOKS/COLE PUBLISHING COMPANY

Students will learn to receive mathematical ideas through listening, reading, and visualizing. They are expected to present mathematical ideas by speaking, writing, drawing pictures and graphs, and demonstrating with concrete models. There is a category of problems in each section that is designated **IN YOUR OWN WORDS,** and that provides practice in communication skills.

1. **IN YOUR OWN WORDS** Discuss the difference between solving word problems in textbooks and problem solving outside the classroom.
2. **IN YOUR OWN WORDS** In Problem Set 1.1, we asked you to discuss Pólya's problem-solving model. Now that you have spent some time reading this text, we ask the same question again to see whether your perspective has changed at all. Discuss Pólya's problem-solving model.

Students should view mathematics in historical perspective. There is no argument that mathematics has been a driving force in the history of civilization. In order to bring students closer to this history, I've included not only Historical Notes, but also a category of problems called **HISTORICAL QUEST**.

Historical Note

Karl Smith Library

Karl Gauss
(1777–1855)

Along with Archimedes and Isaac Newton, Gauss is considered one of the three greatest mathematicians of all time. When he was 3 years old, he corrected an error in his father's payroll calculations. By the time he was 21, he had contributed more to mathematics than most do in a lifetime.

56. **HISTORICAL QUEST** The Historical Note on page 218 introduces the great mathematician Karl Gauss. Gauss kept a scientific diary containing 146 entries, some of which were independently discovered and published by others. On July 10, 1796, he wrote

EUREKA!
NUM = △ + △ + △

What do you think this meant? Illustrate with some numerical examples.

Students should learn to think critically. Many colleges have a broad educational goal of increasing critical thinking skills. Wikipedia defines **critical thinking** as "purposeful and reflective judgment about what to believe or do in response to observations, experience, verbal or written expressions or arguments." Critical thinking might involve determining the meaning and significance of what is observed or expressed, or, concerning a given inference or argument, determining whether there is adequate justification to accept the conclusion as true. Critical thinking begins in earnest in Section 1.1 when we introduce Pólya's problem-solving method. These Pólya examples found throughout the text are not the usual "follow-the-leader"-type problems, but attempt, slowly, but surely, to teach critical thinking. The Problem Solving problems in almost every section continue this theme. The following sections are especially appropriate to teaching critical thinking skills: Problem Solving (1.1), Problem Solving Using Logic (3.5), Cryptography (5.8), Modeling Uncategorized Problems (6.9), Summary of Financial Formulas (11.7), Probability Models (13.3), Voting Dilemmas (17.2), Apportionment Paradoxes (17.4), and What Is Calculus? (18.1).

A Note for Instructors

The prerequisites for this course vary considerably, as do the backgrounds of students. Some schools have no prerequisites, while other schools have an intermediate algebra prerequisite. The students, as well, have heterogeneous backgrounds. Some have little or no mathematics skills; others have had a great deal of mathematics. Even though the usual prerequisite for using this text is intermediate algebra, a careful selection of topics and chapters would allow a class with a beginning algebra prerequisite to study the material effectively.

Feel free to arrange the material in a different order from that presented in the text. I have written the chapters to be as independent of one another as possible. There is much more material than could be covered in a single course. This text can be used in classes designed for liberal arts, teacher training, finite mathematics, college algebra, or a combination of these.

Over the years, many instructors from all over the country have told me that they love the material, love to teach from this text, but complain that there is just too much material in this text to cover in one, or even two, semesters. In response to these requests, I have divided some of the material into two separate volumes:

The Nature of Problem Solving in Geometry and Probability
The Nature of Problem Solving in Algebra

The first volume, *The Nature of Geometry and Probability,* includes Chapters 1, 2, 3, 7, 8, 9, 11, 12, and 13 from this text.

The second volume, *The Nature of Algebra,* includes Chapters 1, 4, 5, 6, 8, 10, 14, 15, 16, and 17 from this text.

Since the first edition of this text, I have attempted to make the chapters as independent as possible to allow instructors to "pick and choose" the chapters to custom design the course. It is possible to customize your own text for class. Details are available from your sales representative.

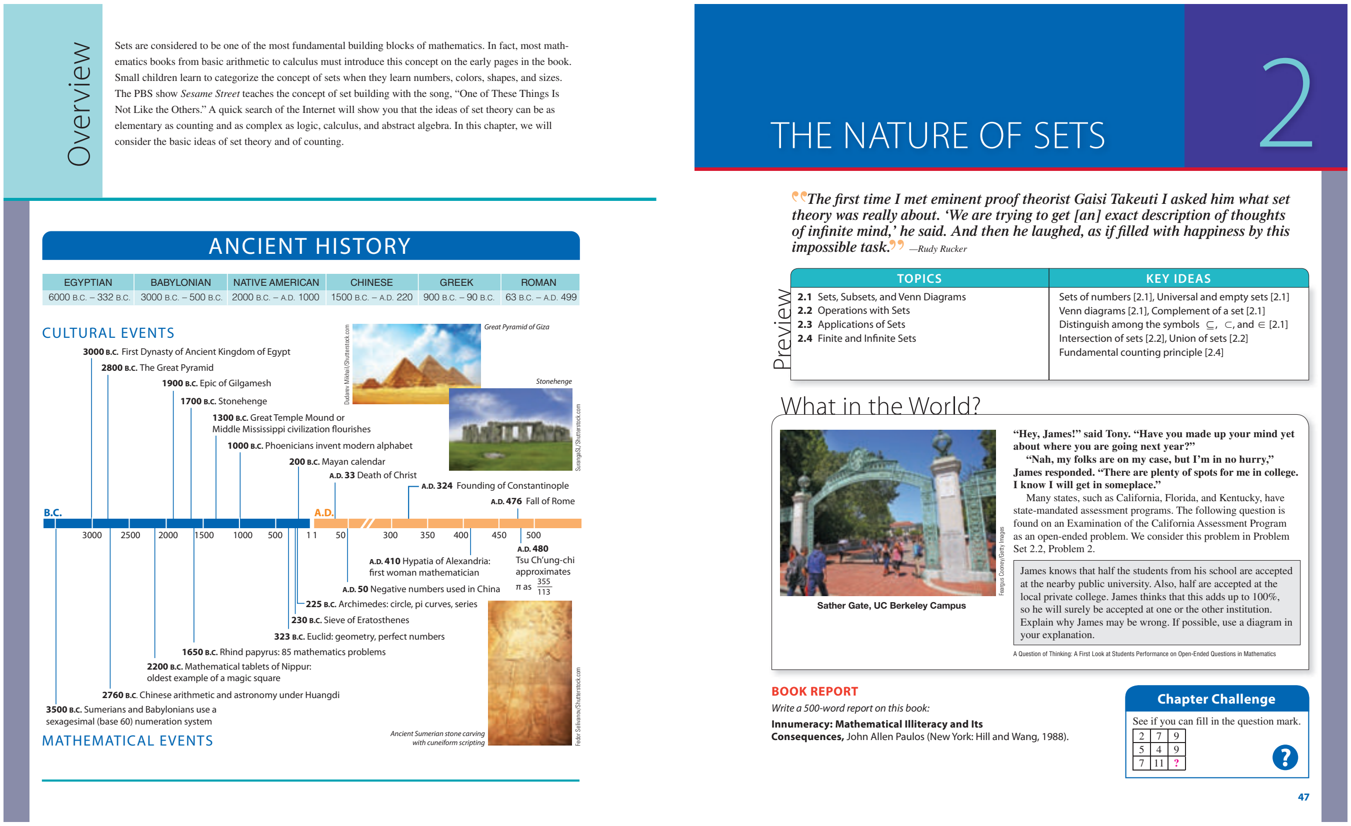
One of the advantages of using a textbook that has traveled through many editions is that it is well seasoned. Errors are minimal, pedagogy is excellent, and it is easy to use; in other words, it works. For example, you will find that the sections and chapters are about the right length . . . each section will take about one classroom day. The problem sets are graded so that you can teach the course at different levels of difficulty, depending on the assigned problems. The problem sets are uniform in length (60 problems each), which facilitates the assigning of problems from day to day. The chapter reviews are complete and lead students to the type of review they will need to prepare for an examination.

Changes from the Previous Edition

As a result of extensive reviewer feedback, there are many new ideas and changes in this edition.

- Mathematical history has been an integral feature of this text since its inception, and I have long used Historical Notes to bring the human story into our venture through this text. In the last edition, I experimented with a new type of problem called an **HISTORICAL QUEST**, and it has proved to be an overwhelming success, so I have greatly expanded its use in this edition. These problems are designed to *involve* the student in the historical development of the great ideas in mathematical history.

- Added a new feature called **FAST FACTS** to provide some essential information at exactly the right time and place.
- The chapter openings have been completely redesigned into two-page openers.



Overview

Sets are considered to be one of the most fundamental building blocks of mathematics. In fact, most mathematics books from basic arithmetic to calculus must introduce this concept on the early pages in the book. Small children learn to categorize the concept of sets when they learn numbers, colors, shapes, and sizes. The PBS show *Sesame Street* teaches the concept of set building with the song, "One of These Things Is Not Like the Others." A quick search of the Internet will show you that the ideas of set theory can be as elementary as counting and as complex as logic, calculus, and abstract algebra. In this chapter, we will consider the basic ideas of set theory and of counting.

ANCIENT HISTORY

EGYPTIAN	BABYLONIAN	NATIVE AMERICAN	CHINESE	GREEK	ROMAN
6000 B.C. – 332 B.C.	3000 B.C. – 500 B.C.	2000 B.C. – A.D. 1000	1500 B.C. – A.D. 220	900 B.C. – 90 B.C.	63 B.C. – A.D. 499

CULTURAL EVENTS

3000 B.C. First Dynasty of Ancient Kingdom of Egypt
2800 B.C. The Great Pyramid
1900 B.C. Epic of Gilgamesh
1700 B.C. Stonehenge
1300 B.C. Great Temple Mound or Middle Mississippi civilization flourishes
1000 B.C. Phoenicians invent modern alphabet
200 B.C. Mayan calendar
A.D. 33 Death of Christ
A.D. 324 Founding of Constantinople
A.D. 476 Fall of Rome

A.D. 480 Tsu Ch'ung-chi approximates π as $\frac{355}{113}$
A.D. 410 Hypatia of Alexandria: first woman mathematician
A.D. 50 Negative numbers used in China
225 B.C. Archimedes: circle, pi curves, series
230 B.C. Sieve of Eratosthenes
323 B.C. Euclid: geometry, perfect numbers
1650 B.C. Rhind papyrus: 85 mathematics problems
2200 B.C. Mathematical tablets of Nippur: oldest example of a magic square
2760 B.C. Chinese arithmetic and astronomy under Huangdi
3500 B.C. Sumerians and Babylonians use a sexagesimal (base 60) numeration system

MATHEMATICAL EVENTS

THE NATURE OF SETS 2

"The first time I met eminent proof theorist Gaisi Takeuti I asked him what set theory was really about. 'We are trying to get [an] exact description of thoughts of infinite mind,' he said. And then he laughed, as if filled with happiness by this impossible task." —*Rudy Rucker*

Preview

TOPICS	KEY IDEAS
2.1 Sets, Subsets, and Venn Diagrams **2.2** Operations with Sets **2.3** Applications of Sets **2.4** Finite and Infinite Sets	Sets of numbers [2.1], Universal and empty sets [2.1] Venn diagrams [2.1], Complement of a set [2.1] Distinguish among the symbols ⊆, ⊂, and ∈ [2.1] Intersection of sets [2.2], Union of sets [2.2] Fundamental counting principle [2.4]

What in the World?

Sather Gate, UC Berkeley Campus

"Hey, James!" said Tony. "Have you made up your mind yet about where you are going next year?"

"Nah, my folks are on my case, but I'm in no hurry," James responded. "There are plenty of spots for me in college. I know I will get in someplace."

Many states, such as California, Florida, and Kentucky, have state-mandated assessment programs. The following question is found on an Examination of the California Assessment Program as an open-ended problem. We consider this problem in Problem Set 2.2, Problem 2.

James knows that half the students from his school are accepted at the nearby public university. Also, half are accepted at the local private college. James thinks that this adds up to 100%, so he will surely be accepted at one or the other institution. Explain why James may be wrong. If possible, use a diagram in your explanation.

A Question of Thinking: A First Look at Students Performance on Open-Ended Questions in Mathematics

BOOK REPORT

Write a 500-word report on this book:

Innumeracy: Mathematical Illiteracy and Its Consequences, John Allen Paulos (New York: Hill and Wang, 1988).

Chapter Challenge

See if you can fill in the question mark.

2	7	9
5	4	9
7	11	?

47

- Reversed the order of Chapters 8 and 9, so that The Nature of Measurement now follows directly behind The Nature of Geometry, since those two chapters are closely linked.
- Added a section on stochastic processes and tree diagrams.
- Added a new section on Bayes' theorem.
- Added Laffer curves to Section 15.4.
- Reworked the group research projects; added 5 new substantial group projects.
- Reworked the individual research projects; added 40 new projects.
- Reworked and updated the problem sets; all problem sets now have 60 problems. The problem sets are uniform in length (60 problems each), which facilitates the assigning of problems from day to day. (For example, 3-60, multiples of 3 work well with the paired nature of the problems.)
- Over 700 problems have been added or revised.
- The prologue and epilogue have been redesigned and rewritten to offer unique "bookends" to the material in the text. The prologue asks the question, "**Why Math?**". This prologue not only puts mathematics into a historical perspective, but also is designed to help students to begin thinking about problem solving. The problems accompanying this prologue could serve as a pre-test or diagnostic test, but I use these prologue problems to let the students know that this text will not be like other math books they may have used in the past. The epilogue, "**Why Not Math?**", is designed to tie together many parts of the text (which may or may not have been "covered" in the class) to show that there are many rooms in the mansion known as mathematics. The problems accompanying this epilogue could serve as a review to show that it would be difficult to choose a course of study in college without somehow being touched by mathematics. When have you seen a mathematics textbook that asks the question "Why study mathematics?" and then actually produces an example to show it?*

---

*See Example 2, Section 11.5.

Help for the Instructor

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Instructor's Manual (978-1-305-65777-9)

Karl Smith has written an extensive *Instructor's Manual* (over 500 pages) to accompany this text. It includes the complete solutions to all the problems (including the "Problem Solving" problems), as well as teaching suggestions and transparency masters. These transparencies are included (in color) as a PDF file at **www.mathnature.com** so that you can download them for use in your classroom.

Cognero TestBank for *Nature of Mathematics,* 13e

Cognero® is a flexible, online system that allows you to author, edit, and manage test bank content; create multiple test versions in an instant; and deliver tests from your LMS, your classroom, or wherever you want. This is available online via **www.cengage.com/login**.

Help for the Student

MindTap for Mathematics

MindTap for Mathematics is a digital-learning solution that places learning at the center of the experience. In addition to algorithmically generated problems, immediate feedback, and a powerful answer evaluation and grading system, MindTap for Mathematics gives you a personalized path of dynamic assignments, a focused improvement plan, and just-in-time, integrated remediation that turns cookie cutter into cutting edge, apathy into engagement, and memorizers into higher-level thinkers.

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Student Survival and Solutions Manual (978-1-305-65776-2)

Authored by Karl Smith, this manual is intended to help students with their success. It contains the complete solution of approximately one-half of the problems in the text. Also available are sample tests, not only in hard copy form, but also in electronic form for both Windows and Macintosh formats.

Text Companion Website at www.mathnature.com

Created and updated by Karl Smith, the website offers supplementary help and practice for students. All of the Web addresses mentioned in the text are linked to the above Web address. You will find links to several search engines, history, and reference topics. You will find, for each section, homework hints, and a listing of essential ideas, projects, and links to related information on the Web.

Acknowledgments

I'd especially like to thank Arian Hojat for his input about the great pioneers of Persian/Iranian heritage and their contribution to human civilization.

Many people have contributed to the evolution of this text over the years. I very much appreciate the suggestions and contributions of the following people who contributed to this and previous editions:

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Prologue: Why Math?

A HISTORICAL OVERVIEW

> *"If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is."* —*John van Neumann*

Whether you love or loathe mathematics, it is hard to deny its importance in the development of the main ideas of this world! Read the BON VOYAGE invitation in the Overview to Chapter 1. The goal of this text is to help you to discover an answer to the question, "Why study math?"

The study of mathematics can be organized as a history or story of the development of mathematical ideas, or it can be organized by topic. The intended audience of this text dictates that the development should be by topic, but mathematics involves real people with real stories, so you will find this text to be very historical in its presentation. This overview rearranges the material you will encounter in the text into a historical timeline. It is not intended to be read as a history of mathematics, but rather as an overview to make you want to do further investigation. Sit back, relax, and use this overview as a *starting place* to expand your knowledge about the beginnings of some of the greatest ideas in the history of the world!

We have divided this history of mathematics into five chronological periods:

Ancient History	Chapter 2 overview	6000 B.C. to A.D. 499
Hindu and Persian Period	Chapter 6 overview	500 to 1199
Transition Period	Chapter 10 overview	1200 to 1599
Age of Reason	Chapter 14 overview	1600 to 1799
Modern Period	Chapter 18 overview	1800 to present

Ancient History: 6000 B.C. to A.D. 499

The ancient period includes the Egyptian (6000 B.C.–332 B.C.), Babylonian (3000 B.C.–500 B.C.), Native American (2000 B.C.–A.D. 1000), Chinese (1500 B.C.–A.D. 220), Greek (900 B.C.–90 B.C.), and Roman (63 B.C.–A.D. 499) civilizations.


Gianni Dagli Orti/Fine Art/Corbis

Sumerian clay tablet

Mesopotamia is an ancient region located in southwest Asia between the lower Tigris and Euphrates rivers and is historically known as the birthplace of civilization. It is part of modern Iraq. Mesopotamian mathematics refers to the mathematics of the ancient Babylonians, and this mathematics is sometimes referred to as Sumerian mathematics. Over 50,000 tablets from Mesopotamia have been found and are exhibited at major museums around the world.

Interesting readings about Babylon can be found in a book on the history of mathematics, such as *An Introduction to the History of Mathematics,* 6th edition, by Howard Eves (New York: Saunders, 1990), or by looking at the many sources on the World Wide Web. You can find links to such websites, as well as all websites in this text, by looking at the Web page for this text:

www.mathnature.com

This Web page allows you to access a world of information by using the links provided.

The mathematics of this period was very practical, and it was used in construction, surveying, record keeping, and the creation of calendars. The culture of the Babylonians reached its height about 2500 B.C., and about 1700 B.C. King Hammurabi formulated a famous code of law. In 330 B.C., Alexander the Great conquered Asia Minor, ending the great Persian (Achaemenid) Empire. Even though there was a great deal of political and social upheaval during this period, there was a continuity in the development of mathematics from ancient times to the time of Alexander.

The main information we have about the civilization and mathematics of the Babylonians is their numeration system, which we introduce in Section 4.1 of this text. The Babylonian numeration system was positional with base 60. It did not have a 0 symbol, but it did represent fractions, squares, square roots, cubes, and cube roots. We have evidence that the Babylonians knew the quadratic formula and that they had stated algebraic problems verbally. The base 60 system of the Babylonians led to the division of a circle into 360 equal parts that today we call degrees, and each degree was in turn divided into 360 parts that today we call seconds. The Greek astronomer Ptolemy (A.D. 85–165) used this Babylonian system, which no doubt is why we have minutes, seconds, and degree measurement today.

The Egyptian civilization existed from about 4000 B.C., and was less influenced by foreign powers than was the Babylonian civilization. Egypt was divided into two kingdoms


Bettmann/CORBIS

Egyptian hieroglyphics: Inscription and relief from the grave of Prince Rahdep (ca. 2800 B.C.)

until about 3000 B.C., when the ruler Menes unified Egypt and consequently became known as the founder of the first dynasty in 2500 B.C. This was the Egyptians' pyramid-building period, and the Great Pyramid of Cheops was built around 2600 B.C. (Chapter 7, page 362; see The Riddle of the Pyramids in the Historical Note in Problem Set 7.5).

The Egyptians developed their own pictorial way of writing, called *hieroglyphics,* and their numeration system was consequently very pictorial (Chapter 4).

The Egyptian numeration system is an example of a simple grouping system. Although the Egyptians were able to write fractions, they used only unit fractions. Like the Babylonians, they had not developed a symbol for zero. Since the writing of the Egyptians was on papyrus, and not on tablets as with the Babylonians, most of the written history has been lost. Our information comes from the Rhind papyrus, discovered in 1858 and dated to about 1700 B.C., and the Moscow papyrus, which has been dated to about the same time period.

The mathematics of the Egyptians remained remarkably unchanged from the time of the first dynasty to the time of Alexander the Great, who conquered Egypt in 332 B.C. The Egyptians did surveying using a unique method of stretching rope, so they referred to their surveyors as "rope stretchers." The basic unit used by the Egyptians for measuring length was the *cubit*, which was the distance from a person's elbow to the end of the middle finger. A *khet* was defined to equal 100 cubits; khets were used by the Egyptians when land was surveyed. The Egyptians did not have the concept of a variable, and all of their problems were verbal or arithmetic. Even though they solved many equations, they used the word *AHA* or *heap* in place of the variable. For an example of an Egyptian problem, see Ahmes' dilemma in Chapter 1 and the statement of the problem in terms of Thoth, an ancient Egyptian god of wisdom and learning.

The Egyptians had formulas for the area of a circle and the volume of a cube, box, cylinder, and other figures. Particularly remarkable is their formula for the volume of a truncated pyramid of a square base, which in modern notation is

$$V = \frac{h}{3}(a^2 + ab + b^2)$$

where h is the height and a and b are the sides of the top and bottom. Even though we are not certain the Egyptians knew of the Pythagorean theorem, we believe they did because the rope stretchers had knots on their ropes that would form right triangles. They had a very good reckoning of the calendar and knew that a solar year was approximately $365\frac{1}{4}$ days long. They chose as the first day of their year the day on which the Nile would flood.

Contemporaneous with the great civilizations in Mesopotamia was the great Mayan civilization in what is now Mexico. Just as with the Mesopotamian civilizations, the Olmeca and Mayan civilizations lay between two great rivers, in this case the Grijalva and Papaloapan rivers. Sometimes the Olmecas are referred to as the Tenocelome. The Olmeca culture is considered the mother culture of the Americas. What we know about the Olmecas centers around their art. We do know they were a farming community. The Mayan civilization began around 2600 B.C. and gave rise to the Olmecas. The Olmecas had developed a written hieroglyphic language by 700 B.C., and they had a very accurate solar calendar. The Mayan culture had developed a positional numeration system. You will find the influences from this period discussed throughout this text.

Greek mathematics began in 585 B.C. when Thales, the first of the Seven Sages of Greece (625–547 B.C.), traveled to Egypt.*

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*The Seven Sages in Greek history refer to Thales of Miletus, Bias of Priene, Chilo of Sparta, Cleobulus of Rhodes, Periander of Corinth, Pittacus of Mitylene, and Solon of Athens; they were famous because of their practical knowledge about the world and how things work.

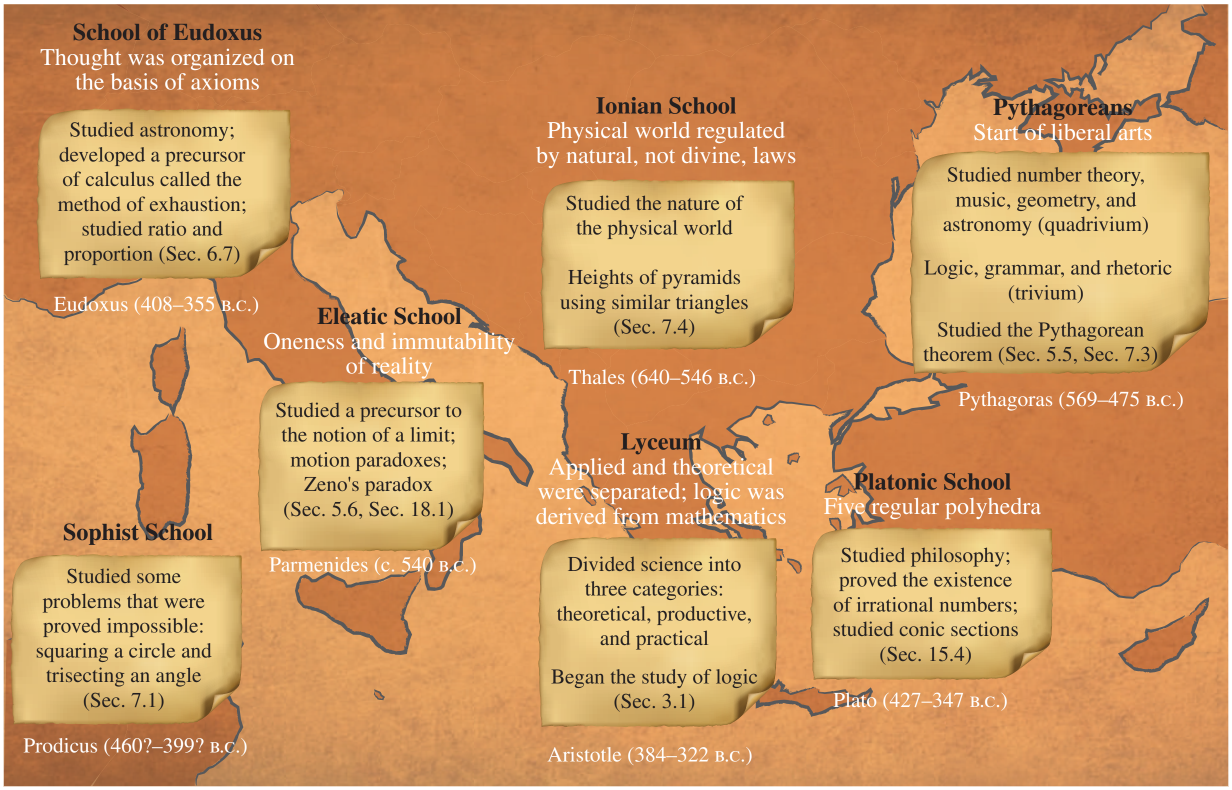

FIGURE 1 **Greek schools from 585 B.C. to 352 B.C.**

The Greek civilization was most influential in our history of mathematics. So striking was its influence that the historian Morris Kline declares, "One of the great problems of the history of civilization is how to account for the brilliance and creativity of the ancient Greeks."* The Greeks settled in Asia Minor, modern Greece, southern Italy, Sicily, Crete, and North Africa. They replaced the various hieroglyphic systems with the Phoenician alphabet, and with that they were able to become more literate and more capable of recording history and ideas. The Greeks had their own numeration system. They had fractions and some irrational numbers, including π.

The great mathematical contributions of the Greeks are Euclid's *Elements* and Apollonius' *Conic Sections* (page 735, Figure 15.28). Greek knowledge developed in several centers or schools. (See Figure 2 on page P4 for a depiction of one of these centers of learning.) The first was founded by Thales (ca. 640–546 B.C.) and known as the Ionian in Miletus. It is reported that while he was traveling and studying in Egypt, Thales calculated the heights of the pyramids by using similar triangles (see Section 7.4). You can read about these great Greek mathematicians in *Mathematical Thought from Ancient to Modern Times,* by Morris Kline.† You can also refer to the World Wide Web at **www.mathnature.com.**

Between 585 B.C. and 352 B.C., schools flourished and established the foundations for the way knowledge is organized today. Figure 1 shows each of the seven major schools, along with each school's most notable contribution. References to textual discussion are shown within each school of thought, along with the principal person for each of these schools. Books have been written about the importance of each of these Greek schools, and several links can be found at **www.mathnature.com.**

One of the three greatest mathematicians in the entire history of mathematics was Archimedes (287–212 B.C.). His accomplishments are truly remarkable, and you should seek out other sources about the magnitude of his accomplishments. He invented a pump (the Archimedean screw), military engines and weapons, and catapults; in addition, he used a parabolic mirror as a weapon by concentrating the sun's rays on the invading Roman ships. "The most famous of the stories about Archimedes is his discovery of the method of testing the debasement of a crown of gold. The king of Syracuse had ordered the crown. When it was delivered, he suspected that it was filled with baser metals and sent it to Archimedes to devise some method of testing the contents without, of course, destroying the workmanship. Archimedes pondered the problem; one day while bathing he observed that his body was partly buoyed up by the water and suddenly grasped the principle that enabled him to solve the problem. He was so excited by this discovery that he ran out into the street naked shouting, 'Eureka!' ('I have found it!').

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*p. 24, *Mathematical Thought from Ancient to Modern Times* by Morris Kline (New York: Oxford University Press, 1972).

†New York: Oxford University Press, 1972.

Erich Lessing/Art Resource, NY

FIGURE 2 ***The School at Athens*** **by Raphael, 1509. This fresco includes portraits of Raphael's contemporaries and demonstrates the use of perspective. Note the figures in the lower right, who are, no doubt, discussing mathematics.**

He had discovered that a body immersed in water is buoyed up by a force equal to the weight of the water displaced, and by means of this principle was able to determine the contents of the crown."*

The Romans conquered the world, but their mathematical contributions were minor. We introduce the Roman numerals in Section 4.1; Romans' fractions were based on a duodecimal (base 12) system and are still used today in certain circumstances. The unit of weight was the *as* and one-twelfth of this was the *uncia,* from which we get our measurements of *ounce* and *inch*, respectively.

The Romans improved on our calendar and set up the notion of leap year every four years. The Julian calendar was adopted in 45 B.C. The Romans conquered Greece and Mesopotamia, and in 47 B.C., they set fire to the Egyptian fleet in the harbor of Alexandria. The fire spread to the city and burned the library, destroying two and a half centuries of book-collecting, including all the important knowledge of the time.

Another great world civilization existed in China and also developed a decimal numeration system and used a decimal system with symbols 1, 2, 3,· · ·, 9, 10, 100, 1000, and 10,000.

Calculations were performed using small bamboo counting rods, which eventually evolved into the abacus. Our first historical reference to the Chinese culture is the yin-yang

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FIGURE 3 **Tiantan Temple, Beijing, China, constructed from A.D. 1406 to 1420**

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*pp. 105–106, *Mathematical Thought from Ancient to Modern Times* by Morris Kline (New York: Oxford University Press, 1972).

symbol, which has its roots in ancient cosmology. The original meaning is representative of the mountains, both the bright side and the dark side. The "yin" represents the female, or shaded, aspect; the earth, the darkness, the moon, and passivity. The "yang" represents the male, light, sun, heaven, and the active principle in nature. These words can be traced back to the Shang and Chou dynasties (1550–1050 B.C.), but most scholars credit them to the Han Dynasty (220–206 B.C.). One of the first examples of a magic square comes from the Lo River around 200 B.C., where legend tells us that the emperor Yu of the Shang Dynasty received a magic square on the back of a tortoise's shell.

From 100 B.C. to A.D. 100, the Chinese described the motion of the planets, as well as what is the earliest known proof of the Pythagorean theorem. The longest surviving and most influential Chinese math book is dated from the beginning of the Han Dynasty around A.D. 50. It includes measurement and area problems, proportions, volumes, and some approximations for π. Sun Zi (ca. A.D. 250) wrote his mathematical manual, which included the "Chinese remainder problem": Find n so that upon division by 3 you obtain a remainder of 2; upon division by 5 a remainder of 3; and upon division by 7 you get a remainder of 2. His solution: Add 140, 63, 30 to obtain 233, and subtract 210 to obtain 23. Zhang Qiujian (ca. A.D. 450) wrote a mathematics manual that included a formula for summing the terms of an arithmetic sequence, along with the solution to a system of two linear equations in three unknowns. The problem is the "One Hundred Fowl Problem," and is included in Problem Set 5.7 (page 235). At the end of this historic period, the mathematician and astronomer Wang Xiaotong (ca. A.D. 626) solved cubic equations by generalization of an algorithm for finding the cube root.

Check **www.mathnature.com** for links to many excellent sites on Greek mathematics.

Hindu and Persian Period: 500 to 1199

Much of the mathematics that we read in contemporary mathematics textbooks ignores the rich history of this period. Included on the World Wide Web are some very good sources for this period. Check our website **www.mathnature.com** for some links. The Hindu civilization dates back to 2000 B.C., but the first recorded mathematics was during the Śulvasūtra period from 800 B.C. to 200 A.D. In the third century, Brahmi symbols were used for 1, 2, 3, $\cdots$, 9 and are significant because there was a single symbol for each number. There was no zero or positional notation at this time, but by A.D. 600 the Hindus used the Brahmi symbols with positional notation. In Chapter 4, we will discuss a numeration system that eventually evolved from these Brahmi symbols. For fractions, the Hindus used sexagesimal positional notation in astronomy, but in other applications they used a ratio of integers and wrote $\begin{smallmatrix}3\\4\end{smallmatrix}$ (without the fractional bar we use today). The first mathematically important period was the second period, A.D. 200–1200. The important mathematicians of this period are Āryabhata (A.D. 476–550), Brahmagupta (A.D. 598–670), Mahāvīra (9th century), and Bhāskara (1114–1185). In Chapter 6, we include some historical questions from Bhāskara and Brahmagupta.

The Hindus developed arithmetic independently of geometry and had a fairly good knowledge of rudimentary algebra. They knew that quadratic equations had two solutions, and they had a fairly good approximation for π. Astronomy motivated their study of trigonometry. Around 1200, scientific activity in India declined, and mathematical progress ceased and did not revive until the British conquered India in the 18th century.

The Persians invited Hindu scientists to settle in Baghdad, and when Plato's Academy closed in A.D. 529, many scholars traveled to Persia and became part of the Iranian tradition of science and mathematics. Omar Khayyám (1048–1122) and Nasîr-Eddin (1201–1274), both renowned Persian scholars, worked freely with irrationals, which contrasts with the Greek idea of number. What we call Pascal's triangle dates back to this period.

The word *algebra* comes from the Persians in a book by the Persian astronomer Mohammed ibn Musa al-Khwârizmî (780–850) entitled *Hisâb al-jabr w'al muqâbala.* Due to the Arab conquest of Persia, Persian scholars (notably Nasir-Eddin and al-Khwârizmî) were obliged to publish their works in the Arabic language and not Persian, causing many historians to falsely label the texts as products of Arab scholars. Al-Khwârizmî solved quadratic equations and knew there are two roots, and even though the Persians gave algebraic solutions of quadratic equations, they explained their work geometrically. They solved some cubics, but could solve only simple trigonometric problems.

Check **www.mathnature.com** for links to many excellent sites on Hindu and Arabian mathematics.

Lordprice Collection / Alamy

FIGURE 4 Omar Khayyám was a Persian mathematician, astronomer, philosopher, and poet. He is the symbol of pure love in the history of oriental wisdom.

Transition Period: 1200 to 1599

Mathematics during the Middle Ages was transitional between the great early civilizations and the Renaissance.

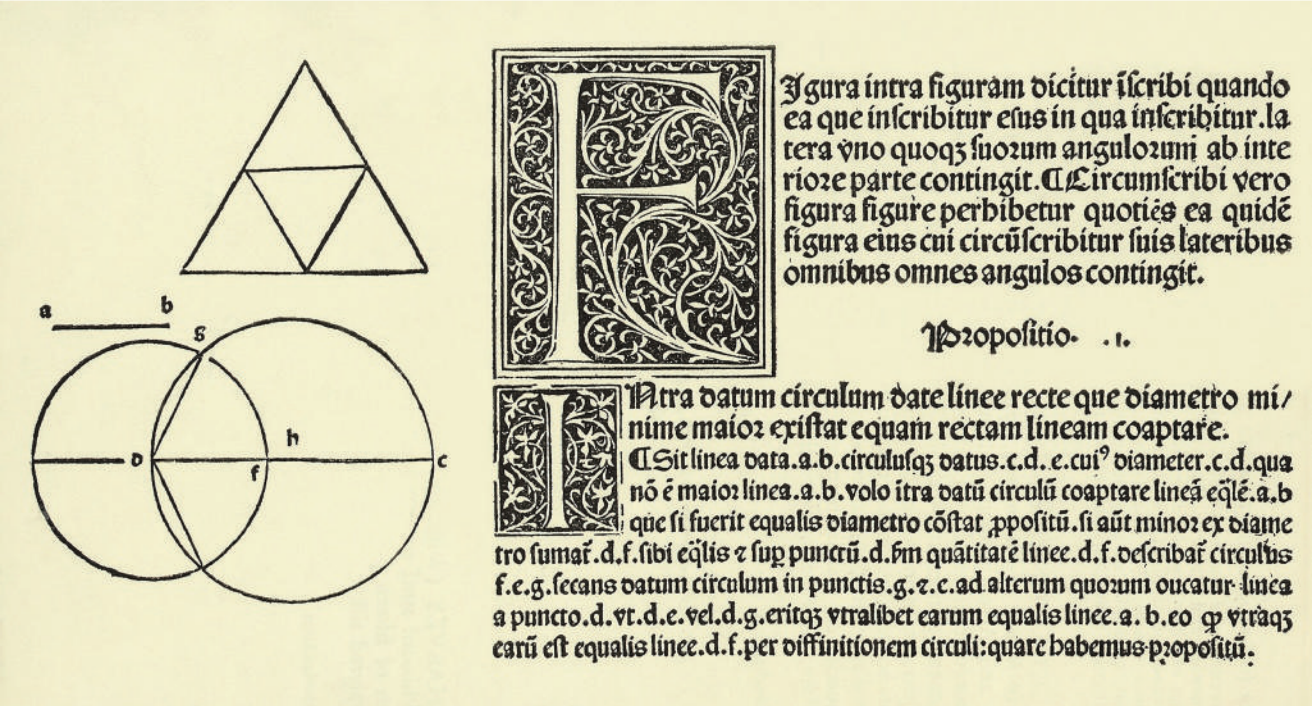
In the 1400s the Black Death killed over 70% of the European population. The Turks conquered Constantinople, and many Eastern scholars traveled to Europe, spreading Greek knowledge as they traveled. The period from 1400 to 1600, known as the Renaissance, forever changed the intellectual outlook in Europe and raised up mathematical thinking to new levels. Johann Gutenberg's invention of printing with movable type in 1450 changed the complexion of the world. Linen and cotton paper, which the Europeans learned about from the Chinese through the Arabians, came at precisely the right historical moment. The first printed edition of Euclid's *Elements* in a Latin translation appeared in 1482. Other early printed books were Apollonius' *Conic Sections*, Pappus' works, and Diophantus' *Arithmetica.*

The first breakthrough in mathematics was by artists who discovered mathematical perspective. The theoretical genius in mathematical perspective was Leone Alberti (1404–1472). He was a secretary in the Papal Chancery writing biographies of the saints, but his work *Della Pictura* on the laws of perspective (1435) was a masterpiece. He said, "Nothing pleases me so much as mathematical investigations and demonstrations, especially when I can turn them into some useful practice drawing from mathematics and the principles of painting perspective and some amazing propositions on the moving of weights." He collaborated with Toscanelli, who supplied Columbus with maps for his first voyage. The most famous mathematician among the Renaissance artists was Albrecht Dürer (1471–1528). The most significant development of the Renaissance was the breakthrough in astronomical theory by Nicolaus Copernicus (1473–1543) and Johannes Kepler (1571–1630). There were no really significant new results in mathematics during this period of history.

It is interesting to tie together some of the previous timelines to trace the history of algebra. It began around 2000 B.C. in Egypt and Babylon. This knowledge was incorporated into the mathematics of Greece between 500 B.C. and A.D. 320, as well as into the Persian civilization and Indian mathematics around A.D. 1000. By the Transition Period, the great ideas of algebra had made their way to Europe, as shown in Figure 6. Additional information can be found on the World Wide Web; check our Web page at **www.mathnature.com.**

Age of Reason: 1600 to 1799

From Shakespeare and Galileo to Peter the Great and the great Bernoulli family, this period, also called the Age of Genius, marks the growth of intellectual endeavors; both technology and knowledge grew as never seen before in history. A great deal of the content of this book focuses on discoveries from this period of time, so instead of providing a commentary in this overview, we will simply list the references to this period in world history. Other sources and links are found on our Web page at **www.mathnature.com.**



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Bettmann/Corbis

FIGURE 5 **Euclid's *Elements* contained the cumulative knowledge of the ancient Greek mathematicians.**

Modern Period: 1800 to Present

This period marks the dawn of modern mathematics, and it includes all of the discoveries of the last two centuries. The Early Modern Period was characterized by experimentation and formalization of the ideas germinated in the previous century. There is so much that we could say about the period from 1700 to 1799. The mathematics that you studied in high school represents, for the most part, the ideas formulated during this period. Take a look at the mathematical events in the following timeline, and you will see an abundance of discoveries, often embodied in the contents of entire books. One of the best sources of information about this period is found at these websites:

www.mathnature.com and **http://www-history.mcs.st-and.ac.uk/~history/Indexes/Full_Chron.html**

Students often think that all the important mathematics has been done, and there is nothing new to be discovered, but this is not true. Mathematics is alive and constantly changing. This textbook was first published in 1973, and as this 13th edition is prepared in 2015, we are struck with all the significant mathematics that has been discovered during the life of this text:

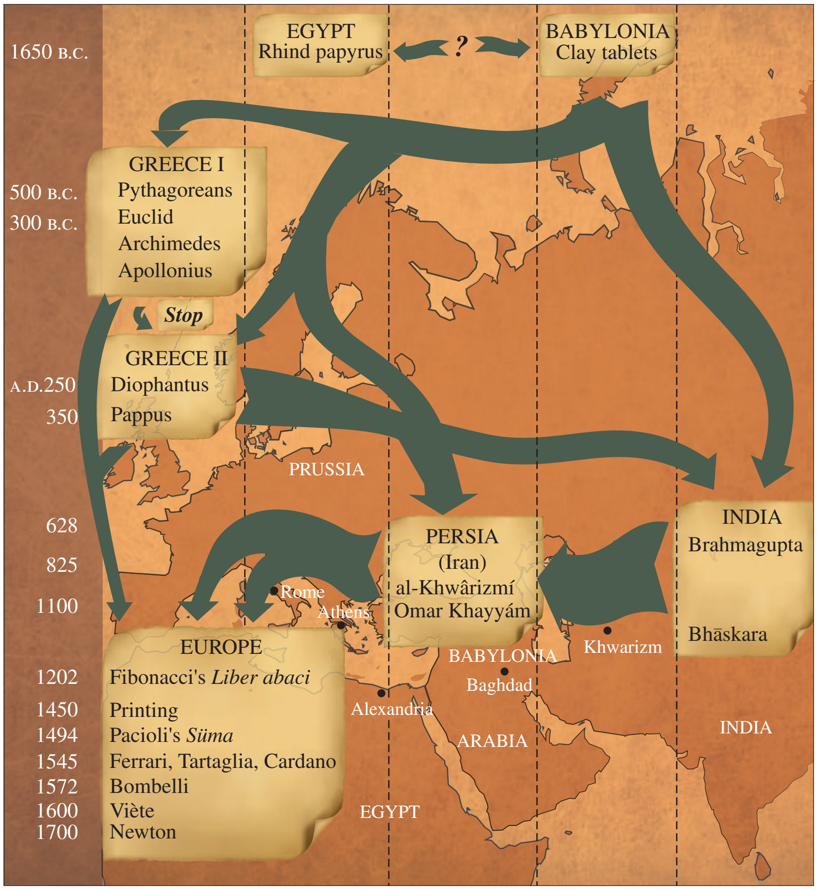


FIGURE 6 **Mainstreams in the flow of algebra**

Date	Topic	Where to Look in This Edition
1976	Appel and Haken solve the four-color problem.	Project 1.6 and Section 9.3
1977	Apple II personal computer introduced.	Section 4.5
1992	World Wide Web released by CREN.	Figure 4.13
1993	Andrew Wiles proves Fermat's last theorem.	Project 1.6 and Project G24
1994	RSA unbreakable code encryption invented.	Section 5.8
1998	Kepler's sphere-packing problem solved.	Project 1.6
2003	Poincaré conjecture proved by Grigori Perelman.	Section 9.3
2004	Mark Zuckerberg writes Facebook.	Section 4.5
2008	Edison Smith, George Woltman, et al. find largest known prime.	Table 5.3
2010	Grigori Perelman rejects the $1,000,000 Millennium Prize.	Section 9.3
2012	Shouryya Ray calculates the path of a projectile under gravity and subject to air resistance.	Epilogue

A quick search of the current mathematical journals shows a plethora of new topics and discoveries: algebraic geometry, category theory, combinatorics (Chapter 12), dynamical systems, homology, information theory, *k*-theory, logic, number theory, representation theory, symbolic geometry, and topology (Chapter 9). Some of these topics are included (as shown) in this edition, but the others are beyond the scope of this text.

There is no way a short commentary or overview can convey the richness or implications of the mathematical discoveries of this period. As we enter the new millennium, we can only imagine and dream about what is to come!

One of the major themes of this text is problem solving. The following problem set is a potpourri of problems that should give you a foretaste of the variety of ideas and concepts that we will consider in this text. Although none of these problems is to be considered routine, you might wish to attempt to work some of them before you begin, and then return to these problems at the end of your study in this text.

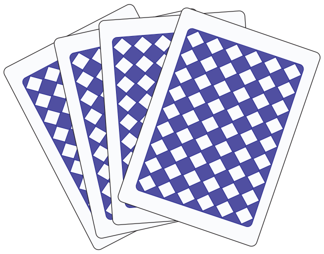
Prologue Problem Set

1. **IN YOUR OWN WORDS** The prologue provides a historical overview and asks the question, “Why Math?” This question seems appropriate at the beginning of a college mathematics course. Why do you think it is important to study mathematics?
2. **IN YOUR OWN WORDS** The epilogue for this text provides an essay of how the material of this text relates specifically to the natural sciences, social sciences, and humanities. In a sense, this is a recap of the question in Problem 1. In the prologue you are looking forward to the text and in the epilogue you are looking backward at what you did in the text. In the epilogue, we ask the question, “Why Not Math?” To answer this question, write a few paragraphs about why someone would not study mathematics.
3. **HISTORICAL QUEST** What are the five chronological historical periods into which the prologue is divided? Which period seems the most interesting to you, and why?
4. **HISTORICAL QUEST** Select what you believe to be the most interesting cultural event and the most interesting mathematical event of the Ancient Period.
5. **HISTORICAL QUEST** Select what you believe to be the most interesting cultural event and the most interesting mathematical event of the Hindu and Persian Period.
6. **HISTORICAL QUEST** Select what you believe to be the most interesting cultural event and the most interesting mathematical event of the Transition Period.
7. **HISTORICAL QUEST** Select what you believe to be the most interesting cultural event and the most interesting mathematical event of the Age of Reason.
8. **HISTORICAL QUEST** Select what you believe to be the most interesting cultural event and the most interesting mathematical event of the Modern Period.
9. **IN YOUR OWN WORDS** This prologue has 60 problems (as does every problem set in this text), but at this point you have no basis for working the problems in this set. Read, but DO NOT WORK, Problems 11–60.
    - **a.** From this set of problems, name 10 problems that you think you might be able to answer correctly.
    - **b.** From this set of problems, name 10 problems that you know you would not be able to answer correctly.
10. **IN YOUR OWN WORDS** Problems 11–60 are included to give you a quick overview of what this text is about. Select five problems you would like to learn how to solve.

In mathematics, it is important to read the directions before attempting to work any problems. You are NOT expected to be able to work Problems 11–60 at this time. You ARE expected to be able to work those problems AFTER reading this text. So for now, just place each assigned question into one of the following categories:

Yes. I know how to start this problem.
No. I don’t think I know how to start this problem.
Hopeful. If someone shows me how to proceed, I think I could answer this question.
Hopeless. I don’t think I’ll ever be able to answer this question.

11. A long, straight fence having a pole every 8 ft is 1,440 ft long. How many fence poles are needed for the fence? 181
12. How many cards must you draw from a deck of 52 playing cards to be sure that at least two are from the same suit? 5 cards

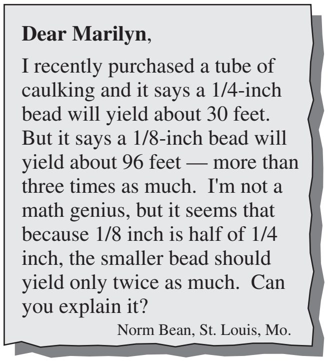


13. How many people must be in a room to be sure that at least four of them have the same birthday (not necessarily the same year)? 1,099
14. Find the units digit of $3^{2015} - 2^{2015}$. 9
15. If a year had two consecutive months with a Friday the thirteenth, which months would they have to be? Feb and Mar
16. On Saturday evenings, a favorite pastime of the high school students in Santa Rosa, California, is to cruise certain streets. The selected routes are shown in the following illustration. Is it possible to choose a route so that all of the permitted streets are traveled exactly once? impossible to trace out this circuit

Santa Rosa street problem

17. What is the largest number that is a divisor of both 210 and 330? 30

18. The news clip shows a letter printed in the "Ask Marilyn" column of *Parade Magazine* (Sept. 27, 1992). How would you answer it?



Dear Marilyn,

I recently purchased a tube of caulking and it says a 1/4-inch bead will yield about 30 feet. But it says a 1/8-inch bead will yield about 96 feet — more than three times as much. I'm not a math genius, but it seems that because 1/8 inch is half of 1/4 inch, the smaller bead should yield only twice as much. Can you explain it?

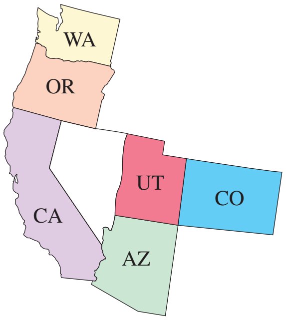
Norm Bean, St. Louis, Mo.

Hint: We won't give you the answer, but we will quote one line from Marilyn's answer: "So the question should be not why the smaller one yields that much, but why it yields that little." Marilyn is correct.

19. If the population of the world on October 12, 2002, was 6.248 billion, when do you think the world population should have reached 7 billion? Calculate the date (to the nearest month) using the information that the world population reached March 2011 6 billion on October 12, 1999. The world population actually reached 7 billion on October 31, 2011. What does this say about your prediction? Actual growth rate is slower.

20. The Pacific 12 football conference consists of the following schools:

Arizona
Arizona State
Cal Berkeley
Colorado
Oregon
Oregon State
Stanford (CA)
UCLA
USC
Utah
Washington
Washington State



a. Is it possible to visit each of these schools by crossing each common state border exactly once? If so, show the path. yes

b. Is it possible to start the trip in any given state, cross each common state border exactly once, and end the trip in the state in which you started? no

21. If $(a, b) = a \times b + a + b$, what is the value of $((1, 2), (3, 4))$? 119

22. If it is known that all Angelenos are Venusians and all Venusians are Los Angeles residents, then what must necessarily be the conclusion? All Angelenos are Los Angeles residents.

23. If 1 is the first odd number, what is the 473rd odd number? 945

24. If $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$, what is the sum of the first 100,000 counting numbers beginning with 1? 5,000,050,000

25. A four-inch cube is painted red on all sides. It is then cut into one-inch cubes. What fraction of all the one-inch cubes are painted on one side only? 3/8

26. If slot machines had two arms and people had one arm, then it is probable that our number system would be based on the digits 0, 1, 2, 3, and 4 only. How would the number we know as 18 be written in such a number system? 33

27. If $M(a, b)$ stands for the larger number in the parentheses, and $m(a, b)$ stands for the lesser number in the parentheses, what is the value of $M(m(1, 2), m(2, 3))$? 2

28. If a group of 50 persons consists of 20 males, 12 children, and 25 women, how many men are in the group? 13

29. There are only five regular polyhedra, and Figure 7 shows the patterns that give those polyhedra. Name the polyhedron obtained from each of the patterns shown.

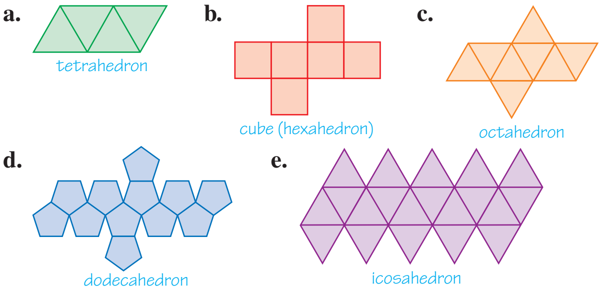


FIGURE 7 Five regular polyhedra patterns

30. Jack and Jill decide to exercise together. Jack walks around their favorite lake in 16 minutes and Jill jogs around the lake in 10 minutes. If Jack and Jill start at the same time and at the same place and continue to exercise around the lake until they return to the starting point at the same time, how long will they be exercising? 80 min

31. What is the 1,000th positive integer that is not divisible by 3? 1,499

32. A frugal man allows himself a glass of wine before dinner on every third day, an after-dinner chocolate every fifth day, and a steak dinner once a week. If it happens that he enjoys all three luxuries on March 31, what will be the date of the next steak dinner that is preceded by wine and followed by an after-dinner chocolate? July 14

33. How many trees must be cut to make a trillion one-dollar bills? To answer this question, you need to make some assumptions. Assume that a pound of paper is equal to a pound of wood, and also assume that a dollar bill weighs about one gram. This implies that a pound of wood yields about 450 dollar bills. Furthermore, estimate that an average tree has a height of 50 ft and a diameter of 12 inches. Finally, assume that wood yields about 50 lb/ft^3. more than a million trees

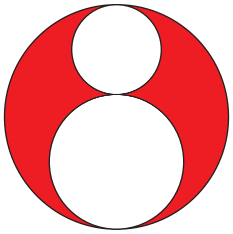
34. Estimate the volume of beer in the six-pack shown in the photograph. about 750,000 gal or almost 8 million standard beer cans

Courtesy Pabst Brewing Company

35. Critique the statement given in the news clip. Sample was not randomly selected.

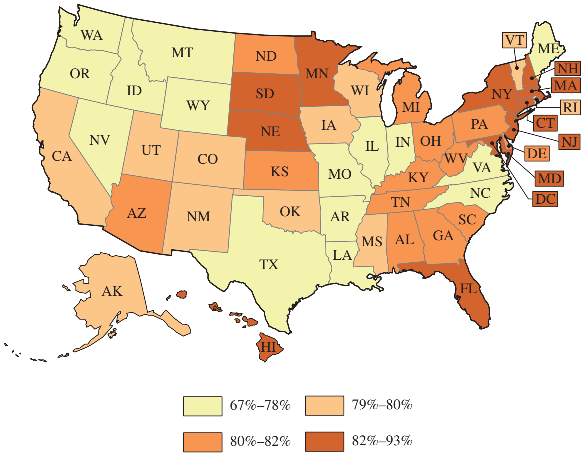
> **Smoking ban**
> Judy Green, owner of the White Restaurant and an adamant opponent of a smoking ban, went so far as to survey numerous restaurants. She cited one restaurant that suffered a 75% decline in business after the smoking ban was activated.

36. The two small circles have radii of 2 and 3. Find the ratio of the area of the smallest circle to the area of the shaded region. 1 to 3



37. A gambler went to the horse races two days in a row. On the first day, she doubled her money and spent $30. On the second day, she tripled her money and spent $20, after which she had what she started with the first day. How much did she start with? $22

38. The map shows the percent of children ages 19–35 months who are immunized by the state. What conclusions can you draw from this map? The percents range from a low of 67% to a high of 93%.



Source: www.cdc.gov/Vaccines/states-surv/his/tables/07/tab03_atigen_state.xls

39. A charter flight has signed up 100 travelers. The travelers are told that if they can sign up an additional 25 persons, they can save $78 each. What is the cost per person if 100 persons make the trip? $390

40. Find $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. e

41. Suppose that it costs $450 to enroll your child in a 10-week summer recreational program. If this cost is prorated (that is, reduced linearly over the 10-week period), express the cost as a function of the number of weeks that have elapsed since the start of the 10-week session. Draw a graph to show the cost at any time for the duration of the session. See IM.

42. Candidates Ramirez (R), Smith (S), and Tillem (T) are running for office. According to public opinion polls, the preferences are (percentages rounded to the nearest percent):

Ranking	38%	29%	24%	10%
1st choice	R	S	T	R
2nd choice	S	R	S	T
3rd choice	T	T	R	S

a. Who will win the plurality vote? Ramirez (R)
b. Who will win the Borda count? Ramirez (R)
c. Does a strategy exist that the voters in the 24% column could use to vote insincerely to keep Ramirez from winning? If the people in the 24% column change to (S, T, R), then Smith will win.

43. Suppose the percentage of alcohol in the blood t hours after consumption is given by

$$C(t) = 0.3e^{-t/2}$$

What is the rate at which the percentage of alcohol is changing with respect to time? $-0.15e^{-t/2}$

44. If a megamile is one million miles and a kilomile is one thousand miles, how many kilomiles are there in 2.376 megamiles? 2,376 kilomiles

45. A map of a small village is shown in Figure 8. To walk from *A* to *B*, Sarah obviously must walk at least 7 blocks (all the blocks are the same length). What is the number of shortest paths from *A* to *B*? 33 paths

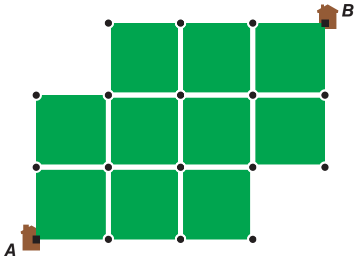


FIGURE 8 A village map

46. A hospital wishes to provide for its patients a diet that has a minimum of 100 g of carbohydrates, 60 g of protein, and 40 g of fats per day. These requirements can be met with two foods:

Food	Carbohydrates	Protein	Fats
A	6 g/oz	3 g/oz	1 g/oz
B	2 g/oz	2 g/oz	2 g/oz

It is also important to minimize costs; food A costs $0.14 per ounce and food B costs $0.06 per ounce. How many ounces of each food should be bought for each patient per day to meet the minimum daily requirements at the lowest cost? The minimum cost is $2.52 with 12 g of food A and 14 g of food B.

47. On December 8, 2015, the U.S. national debt hit $18 trillion and on that date there were 319.4 million people living in the United States. How long would it take to pay off this debt if *every* person paid $1 per day? About 154 years

48. Find the smallest number of operations needed to build up to the number 100 if you start at 0 and use only two operations: doubling or increasing by 1. *Challenge*: Answer the same question if you want to build up any positive integer n. 9 steps

49. If $\log_2 x + \log_4 x = \log_b x$, what is b? $2^{2/3}$

50. Supply the missing number in the following sequence: 10, 11, 12, 13, 14, 15, 16, 17, 20, 22, 24, __, 100, 121, 10,000. 31

51. How many different configurations can you see in Figure 9? at least 3

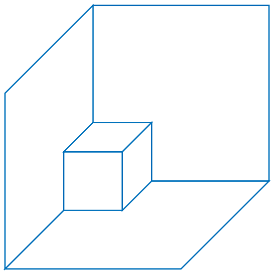
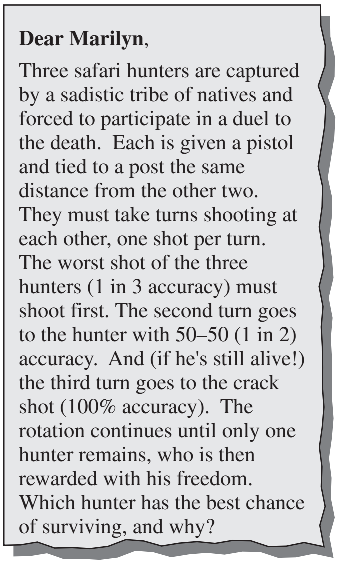


FIGURE 9 **Count the cubes**

52. Answer the question asked in the news clip from the "Ask Marilyn" column of *Parade Magazine* (July 16, 1995). The first hunter has the best chance.



Dear Marilyn,

Three safari hunters are captured by a sadistic tribe of natives and forced to participate in a duel to the death. Each is given a pistol and tied to a post the same distance from the other two. They must take turns shooting at each other, one shot per turn. The worst shot of the three hunters (1 in 3 accuracy) must shoot first. The second turn goes to the hunter with 50–50 (1 in 2) accuracy. And (if he's still alive!) the third turn goes to the crack shot (100% accuracy). The rotation continues until only one hunter remains, who is then rewarded with his freedom. Which hunter has the best chance of surviving, and why?

From "Ask Marilyn," by Marilyn vos Savant, *Parade Magazine*, July 16, 1995.

53. Five cards are drawn at random from a pack of cards that have been numbered consecutively from 1 to 104 and have been thoroughly shuffled. What is the probability that the numbers on the cards as they are drawn are in increasing order of magnitude? 1/120

54. What is the sum of the counting numbers from 1 to 104? 5,460

55. The Kabbalah is a body of mystical teachings from the Torah. One medieval inscription is shown on the left:

ד	ט	ב
ג	ה	ז
ח	א	ו

4	9	2
3	5	7
8	1	6

The inscription on the left shows Hebrew characters that can be translated into numbers, as shown at the right. What can you say about this pattern of numbers? The sum of the numbers in every row, column, and diagonal is 15.

56. What is the maximum number of points of intersection of n distinct lines? If s_n is the number of intersection points for n lines, then $s_n = s_{n-1} + (n - 1)$.

57. The equation $P = 153{,}000e^{0.05t}$ represents the population of a city t years after 2000. What is the population of the city in the year 2000? Show a graph of the city's population for the next 20 years. See IM.

58. The Egyptians had an interesting, pictorial numeration system. Here is how you would count using Egyptian numerals:

|, ||, |||, ||||, |||||, ||||||, |||||||, ||||||||, |||||||||, ∩, ∩|, ∩||, ∩|||, . . .

Write down your age using Egyptian numerals. The symbol "|" is called a stroke, and "∩" is called a heel bone. The Egyptians used a scroll for 100, a lotus flower for 1,000, a pointing finger for 10,000, a polliwog for 100,000, and an astonished man for the number 1,000,000. *Without* doing any research, write what you think today's date would look like using Egyptian numerals. Answers vary.

59. If you start with \$1 and double the amount received on the previous day, how much money will you have in 30 days? $2^{30} - 1$ dollars or more than a billion dollars

60. Consider two experiments and events defined as follows:

Experiment A: Roll one die 4 times and keep a record of how many times you obtain at least one 6. Event $E = \{$obtain at least one 6 in 4 rolls of a single die$\}$

Experiment B: Roll a pair of dice 24 times and keep a record of how many times you obtain at least one 12. Event $F = \{$obtain at least one 12 in 24 rolls of a pair of dice$\}$

Do you think event E or event F is more likely? You might wish to experiment by rolling dice. E is more likely.

THE NATURE OF PROBLEM SOLVING

1

> *"The idea that aptitude for mathematics is rarer than aptitude for other subjects is merely an illusion which is caused by belated or neglected beginners."* —*J. F. Herbart*

Preview

TOPICS	KEY IDEAS
1.1 Problem Solving	Guidelines for problem solving [1.1]
1.2 Inductive and Deductive Reasoning	Order of operations [1.2]
1.3 Scientific Notation and Estimation	Euler circles [1.2]
	Extended order of operations [1.3]
	Laws of exponents [1.3]
	Inductive vs. deductive reasoning [1.3]

What in the World?

Doug Menuez/Photodisc/Getty Images

"Hey, Tom, what are you taking this semester?" asked Susan. "I'm taking English, history, and math. I can't believe my math teacher," responded Tom. "The first day we were there, she walked in, wrote her name on the board, and then she asked, 'How much space would you have if you, along with everyone else in the world, moved to California?' What a stupid question . . . I would not have enough room to turn around!"

"Oh, I had that math class last semester," said Susan. "It isn't so bad. The whole idea is to give you the ability to solve problems *outside* the class. I want to get a good job when I graduate, and I've read that because of the economy, employers are looking for people with problem-solving skills. I hear that working smarter is more important than working harder."

BOOK REPORTS

Write a 500-word report on one of these books:

Mathematical Magic Show, Martin Gardner (New York: Alfred A. Knopf, 1977).

How to Solve It: A New Aspect of Mathematical Method, George Pólya (New Jersey: Princeton University Press, 1945, 1973).

Chapter Challenge*

$A + B = C$

$A + C = D$

$B + C = E$

$F + H = N$

$G + J = ?$

---

*At the beginning of each chapter, we present a puzzle which represents some pattern. In each case, see if you can fill in the question mark.

Overview

There are many reasons for reading a book, but the best reason is because you want to read it. Although you are probably reading this first page because you were required to do so by your instructor, it is my hope that in a short while you will be reading this text because you *want* to read it. It was written for people who think they don't like mathematics, or people who think they can't work math problems, or people who think they are never going to use math. The common thread in this text is *problem solving*—that is, strengthening your ability to solve problems—not in the classroom, but outside the classroom. This first chapter is designed to introduce you to the nature of problem solving. Notice the first thing you see on this page is the question, "What in the World?" Each chapter begins with such a real-world question that appears later in the chapter. This first one is considered in Problem 59, page 39.

As you begin your trip through this text, I wish you a BON VOYAGE!

K J Smith

A FABLE

Wallnarez/Shutterstock.com

Two young ladies, Shelley and Cindy, came to a town called Mathematics. People had warned them that this was a particularly confusing town. Many people who arrived in Mathematics were very enthusiastic, but could not find their way around, became frustrated, gave up, and left town.

Shelley was strongly determined to succeed. She was going to learn her way through the town. For example, in order to learn how to go from her dorm to class, she concentrated on memorizing this clearly essential information: she had to walk 325 steps south, then 253 steps west, then 129 steps in a diagonal (south-west), and finally 86 steps north. It was not easy to remember all of that, but fortunately she had a very good instructor who helped her to walk this same path 50 times. In order to stick to the strictly necessary information, she ignored much of the beauty along the route, such as the color of the adjacent buildings or the existence of trees, bushes, and nearby flowers. She always walked blindfolded. After repeated exercising, she succeeded in learning her way to class and also to the cafeteria. But she could not learn the way to the grocery store, the bus station, or a nice restaurant; there were just too many routes to memorize. It was so overwhelming! Finally, she gave up and left town; Mathematics was too complicated for her.

Cindy, on the other hand, was of a much less serious nature. To the dismay of her instructor, she did not even intend to memorize the number of steps of her walks. Neither did she use the standard blindfold which students need for learning. She was always curious, looking at the different buildings, trees, bushes, and nearby flowers or anything else not necessarily related to her walk. Sometimes she walked down dead-end alleys in order to find out where they were leading, even if this was obviously superfluous. Curiously, Cindy succeeded in learning how to walk from one place to another. She even found it easy and enjoyed the scenery. She eventually built a building on a vacant lot in the city of Mathematics.*

---

*My thanks to Emilio Roxin of the University of Rhode Island for the idea for this fable.

1.1 Problem Solving

A Word of Encouragement

Do you think of mathematics as a difficult, foreboding subject that was invented hundreds of years ago? Do you think that you will never be able (or even want) to use mathematics? If you answered “yes” to either of these questions, then I want you to know that I have written this text for you. I have tried to give you some insight into how mathematics is developed and to introduce you to some of the people behind the mathematics. In this text, I will present some of the great ideas of mathematics, and then we will look at how these ideas can be used in an everyday setting to build your problem-solving abilities. *The most important prerequisite for this course is an openness to try out new ideas—a willingness to experience the suggested activities rather than to sit on the sideline as a spectator.* I have attempted to make this material interesting by putting it together differently from the way you might have had mathematics presented in the past. You will find this text difficult if you wait for the text or the teacher to give you answers—instead, *be willing to guess, experiment, estimate, and manipulate*, and try out problems *without fear of being wrong!*

PACIFIC GROVE, CALIFORNIA
★ MATH ★
is not a
spectator
sport
BROOKS/COLE PUBLISHING COMPANY

There is a common belief that mathematics is to be pursued only in a clear-cut logical fashion. This belief is perpetuated by the way mathematics is presented in most textbooks. Often it is reduced to a series of definitions, methods to solve various types of problems, and theorems. These theorems are justified by means of proofs and deductive reasoning. I do not mean to minimize the importance of proof in mathematics, for it is the very thing that gives mathematics its strength. But the power of the imagination is every bit as important as the power of deductive reasoning. As the mathematician Augustus De Morgan once said, “The power of mathematical invention is not reasoning but imagination.”

Let’s begin with a short (nonmathematical) quiz. Write down each answer before you look at the answers in the footnote.

Silly Challenge

1. How long was the Hundred Years’ War?

2. Which country makes Panama hats?

3. In which month do the Russians celebrate the October Revolution?

OK, now you are starting to suspect trick questions . . . let’s continue.

4. From which animal(s) do we get catgut?

5. The Canary Islands in the Pacific are named after what animal?

6. What is a camel’s hair brush made of?

Now, change your goal . . . can you research these questions to obtain one correct answer?

7. What color is a purple finch?

8. Where are Chinese gooseberries from?

9. What was King George VI’s first name?

And now, for the bonus question:

10. How long was the Thirty Years’ War?*

---

*Answers to the quiz: **1.** 116 years (1337 to 1453) **2.** Ecuador **3.** November (The Russian calendar at the time was 13 days behind ours.) **4.** From sheep and horses **5.** Dogs (the Latin name is *Insularia Canaria*—Island of the Dogs) **6.** Squirrel fur **7.** Crimson **8.** New Zealand **9.** Albert (Queen Victoria said that no king should ever be called Albert, so he took the name George when he ascended to the throne.) **10.** Thirty years, of course (1618 to 1648)—we included this so that you would get at least one correct answer!

Success in this course is not dependent on what you know or do not know . . . it is based on your willingness to try, to guess, to experiment, to estimate, and to manipulate (we just said that in the opening paragraph). This "Silly Challenge" should be viewed as a quick self-test to see if you are ready to follow this simple rule for success. We continue with other hints for success.

> Mathematics is one component of any plan for liberal education. Mother of all the sciences, it is a builder of the imagination, a weaver of patterns of sheer thought, an intuitive dreamer, a poet. The study of mathematics cannot be replaced by any other activity...

American Mathematical Monthly, Volume 56, 1949, p. 19.

Hints for Success

Mathematics is different from other subjects. One topic builds upon another, and you need to make sure that you understand *each* topic before progressing to the next one.

You must make a commitment to attend each class. Obviously, unforeseen circumstances can come up, but you must plan to attend class regularly. Pay attention to what your teacher says and does, and take notes. If you must miss class, write an outline of the text corresponding to the missed material, including working out each text example on your notebook paper.

You must make a commitment to daily work. Do not expect to save up and do your mathematics work once or twice a week. It will take a daily commitment on your part, and you will find mathematics difficult if you try to "get it done" in spurts. You could not expect to become proficient in tennis, soccer, or playing the piano by practicing once a week, and the same is true of mathematics. Try to schedule a regular time to study mathematics each day.

Read the text carefully. Many students expect to get through a mathematics course by beginning with the homework problems, then reading some examples, and reading the text only as a desperate attempt to find an answer. This procedure is backward; do your homework only *after* reading the text.

Writing Mathematics

The fundamental objective of education has always been to prepare students for life. A measure of your success with this text is a measure of its usefulness to you in your life. What are the basics for your knowledge "in life"? In this information age with access to a world of knowledge on the Internet, we still would respond by saying that the basics remain "reading, 'riting, and 'rithmetic." As you progress through the material in this text, we will give you opportunities to read mathematics and to consider some of the great ideas in the history of civilization, to develop your problem-solving skills ('rithmetic), and to communicate mathematical ideas to others ('riting). Perhaps you think of mathematics as "working problems" and "getting answers," but it is so much more. Mathematics is a way of thought that includes all three Rs, and to strengthen your skills you will be asked to communicate your knowledge in written form.

To begin building your skills in writing mathematics, you might keep a journal summarizing each day's work. Keep a record of your feelings and perceptions about what happened in class. Ask yourself, "How long did the homework take?" "What time of the day or night did I spend working and studying mathematics?" "What is the most important idea that I should remember from the day's lesson?" To help you with your journals or writing of mathematics, you will find problems in this text designated "**IN YOUR OWN WORDS**." (For example, look at Problems 1–6 of the problem set at the end of this section.) There are no right answers or wrong answers to this type of problem, but you are encouraged to look at these for ideas of what you might write in your journal.

Journal Ideas

Write in your journal every day.
Include important ideas.
Include new words, ideas, formulas, or concepts.
Include questions that you want to ask later.
If possible, carry your journal with you so you can write in it anytime you get an idea.

Reasons for Keeping a Journal

It will record ideas you might otherwise forget.
It will keep a record of your progress.
If you have trouble later, it may help you diagnose areas for change or improvement.
It will build your writing skills.

Guidelines for Problem Solving

We begin this study of **problem solving** by looking at the *process* of problem solving. As a mathematics teacher, I often hear the comment, "I can do mathematics, but I can't solve word problems." There ***is*** a great fear and avoidance of "real-life" problems because they do not fit into the same mold as the "examples in the book." Few practical problems from everyday life come in the same form as those you study in school.

To compound the difficulty, learning to solve problems takes time. All too often, the mathematics curriculum is so packed with content that the real process of problem solving is slighted and, because of time limitations, becomes an exercise in mimicking the instructor's steps instead of developing into an approach that can be used long after the final examination is over.

Before we build problem-solving skills, it is necessary to build certain prerequisite skills necessary for problem solving. It is my goal to develop your skills in the mechanics of mathematics, in understanding the important concepts, and finally in applying those skills to solve a new type of problem. I have segregated the problems in this text to help you build these different skills:

IN YOUR OWN WORDS	This type of problem asks you to discuss or rephrase main ideas or procedures using your own words.
Level 1 Problems	These are mechanical and drill problems and are directly related to an example in the text.
Level 2 Problems	These problems require an understanding of the concepts and are loosely related to an example in the text.
Level 3 Problems	These problems are extensions of the examples but generally do not have corresponding examples.
Problem Solving	These require problem-solving skills or original thinking and generally do not have direct examples in the text. These should be considered Level 3 problems.
Research Problems	These problems require Internet research or library work. Most are intended for individual research but a few are group research projects. You will find these problems for research in the chapter summary and at the Web address for this text:

www.mathnature.com

The model for problem solving that we will use was first published in 1945 by the great, charismatic mathematician George Pólya. His book *How to Solve It* (Princeton University Press, 1973) has become a classic. In Pólya's book, you will find this

Historical Note


Karl Smith Library

George Pólya
(1887–1985)

Born in Hungary, Pólya attended the universities of Budapest, Vienna, Göttingen, and Paris. He was a professor of mathematics at Stanford University. Pólya's research and winning personality earned him a place of honor not only among mathematicians, but among students and teachers as well. His discoveries spanned an impressive range of mathematics, real and complex analysis, probability, combinatorics, number theory, and geometry. Pólya's *How to Solve It* has been translated into 15 languages. His books have a clarity and elegance seldom seen in mathematics, making them a joy to read. For example, here is his explanation of why he became a mathematician: "It is a little shortened but not quite wrong to say: I thought, I am not good enough for physics and I am too good for philosophy. Mathematics is in between."

problem-solving model as well as a treasure trove of strategy, know-how, rules of thumb, good advice, anecdotes, history, and problems at all levels of mathematics. We will refer to his problem-solving model as the **problem-solving procedure,** summarized as follows.

Guidelines for Problem Solving

In order to solve a problem for which there is no immediate solution or no known procedure, use the following steps:

Step 1 *Understand the problem.* Ask questions, experiment, or otherwise rephrase the question in your own words.

Step 2 *Devise a plan.* Find the connection between the data and the unknown. Look for patterns, relate to a previously solved problem or a known formula, or simplify the given information to give you an easier problem.

Step 3 *Carry out the plan.* Check the steps as you go.

Step 4 *Look back.* Examine the solution obtained. In other words, check your answer.

Pay attention to boxes that look like this—they are used to tell you about important procedures that are used throughout the text.

Pólya's original statement of this procedure is reprinted in the following box.*

UNDERSTANDING THE PROBLEM

First

You have to *understand* the problem.

What is the unknown? What are the data? What is the condition? Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?

Draw a figure. Introduce a suitable notation.

Separate the various parts of the condition. Can you write them down?

DEVISING A PLAN

Second

Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found.

Have you seen it before? Or have you seen the same problem in a slightly different form?

Do you know a related problem? Do you know a theorem that could be useful?

Look at the unknown! And try to think of a familiar problem having the same or a similar unknown.

Is the problem related to one you have solved before? Could you use it?

Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible?

Could you restate the problem? Could you restate it still differently? Go back to definitions.

If you cannot solve the proposed problem try to solve first some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or the data, or both if necessary, so that the new unknown and the new data are nearer to each other? Did you use all the data? Did you see the whole condition? Have you taken into account all essential notions involved in the problem?

CARRYING OUT THE PLAN

Third

Carry out your plan.

Carrying out your plan of the solution, *check each step*. Can you see clearly that the step is correct? Can you prove that it is correct?

LOOKING BACK

Fourth

Examine the solution.

Can you *check the result*? Can you check the argument?

Can you derive the result differently? Can you see it at a glance?

---

*This is taken word for word as it was written by Pólya in 1941. It was printed in *How to Solve It* (Princeton, NJ: Princeton University Press, 1973).

Let's apply this procedure for problem solving to the map shown in Figure 1.1; we refer to this problem as the **street problem.** Melissa lives at the YWCA (point *A*) and works at Macy's (point *B*). She usually walks to work. How many different routes can Melissa take?

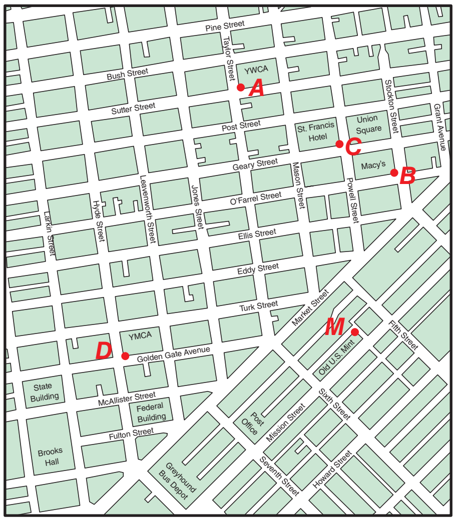


FIGURE 1.1 **Portion of a map of San Francisco**

Where would you begin with this problem?

Step 1. ***Understand the Problem.*** Can you restate it in your own words? Can you trace out one or two possible paths? What assumptions are reasonable? We assume that Melissa will not do any backtracking—that is, she always travels toward her destination. We also assume that she travels along the city streets—she cannot cut diagonally across a lot or a block.

Step 2. ***Devise a Plan.*** Simplify the question asked. Consider the simplified drawing shown in Figure 1.2.

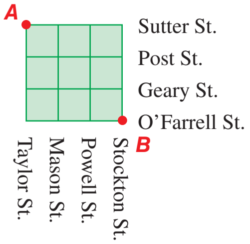


FIGURE 1.2 **Simplified portion of Figure 1.1**

Step 3. ***Carry Out the Plan.*** Count the number of ways it is possible to arrive at each point, or, as it is sometimes called, a *vertex*.

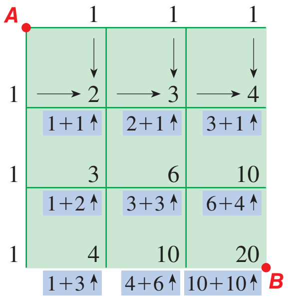
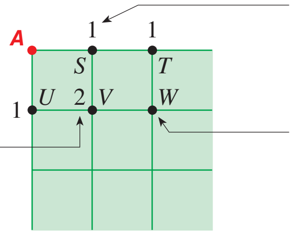


FIGURE 1.3 **Map with solution**



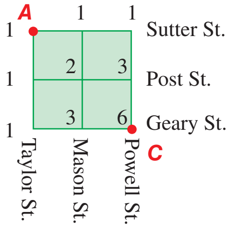
This means one way to arrive at this point.

This means two ways of arriving at this point (from point *S* or point *U*).

How many ways to this point? Do you see why the answer is 2 + 1 = 3 ways? Remember no backtracking is allowed, so you must get here from point *V* or from point *T*.

Now fill in all the possibilities on Figure 1.3, as shown by the above procedure.

Step 4. ***Look Back.*** Does the answer "20 different routes" make sense? Do you think you could fill in all of them?



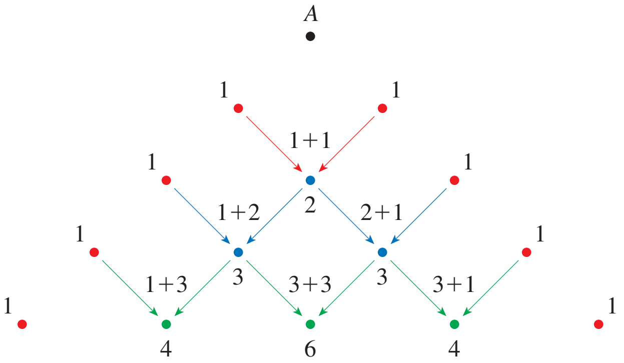
Example 1 Problem solving—from here to there

In how many different ways could Melissa get from the YWCA (point *A*) to the St. Francis Hotel (point *C* in Figure 1.1), using the method of Figure 1.3?

Solution Draw a simplified version of Figure 1.3, as shown in the margin. There are 6 different paths.

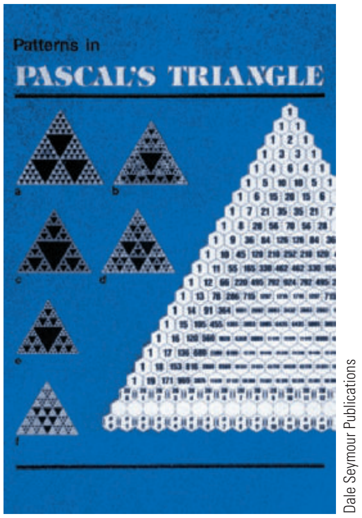
Problem Solving by Patterns

Let's formulate a general solution. Consider a map with a starting point *A*:



In the Sherlock Holmes mystery *The Final Solution*, Moriarty is a mathematician who wrote a treatise on Pascal's triangle.

Do you see the pattern for building this figure? Each new row is found by adding the two previous numbers, as shown by the arrows. This pattern is known as **Pascal's triangle.** In Figure 1.4, the rows and diagonals are numbered for easy reference.



Dale Seymour Publications

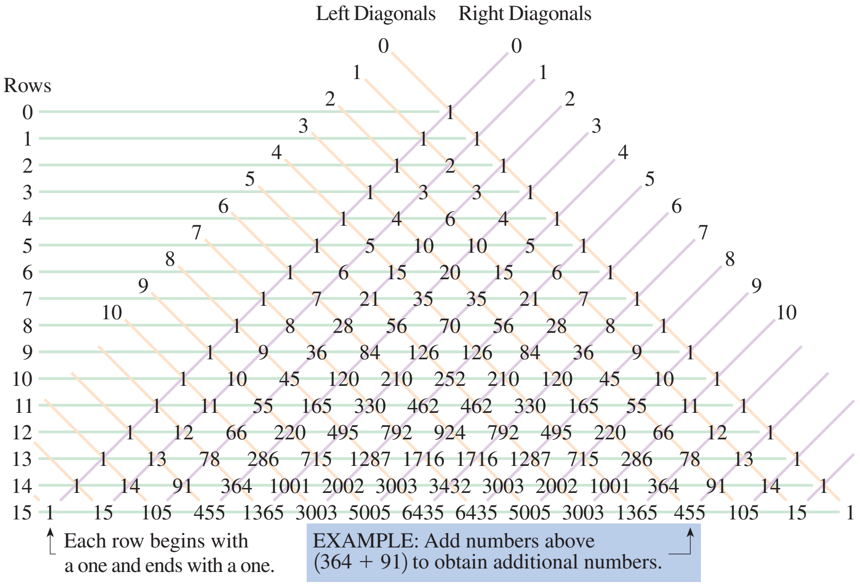


FIGURE 1.4 **Pascal's triangle**

www.mathnature.com
There is an online interactive version of Pascal's triangle.

How does this pattern apply to Melissa's trip from the YWCA to Macy's? It is 3 blocks down and 3 blocks over. Look at Figure 1.4 and count out these blocks, as shown in Figure 1.5.

Historical Note


Karl Smith Library

Blaise Pascal
(1623–1662)

Described as "the greatest 'might-have-been' in the history of mathematics," Pascal was a person of frail health, and because he needed to conserve his energy, he was forbidden to study mathematics. This aroused his curiosity and forced him to acquire most of his knowledge of the subject by himself. At 18, he had invented one of the first calculating machines. However, at 27, because of his health, he promised God that he would abandon mathematics and spend his time in religious study. Three years later he broke this promise and wrote *Traite du triangle arithmétique*, in which he investigated what we today call Pascal's triangle. The very next year he was almost killed when his runaway horse jumped an embankment. He took this to be a sign of God's displeasure with him and again gave up mathematics—this time permanently.

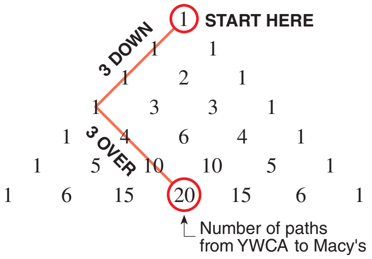
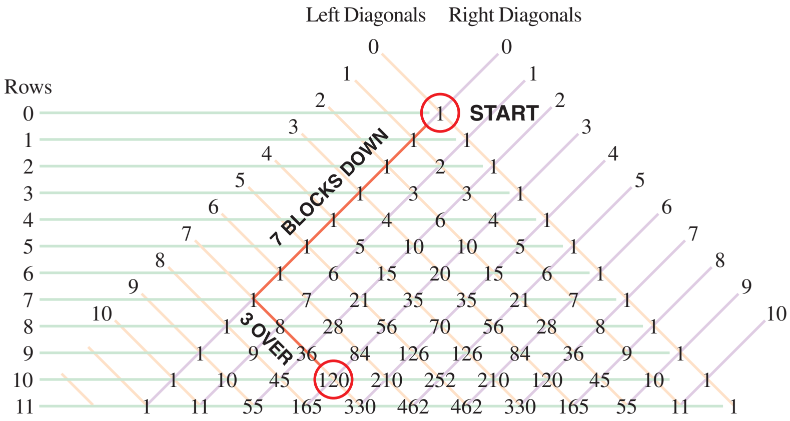


FIGURE 1.5 **Using Pascal's triangle to solve the street problem**

Example 2 Pascal's triangle to track paths

In how many different ways could Melissa get from the YWCA (point *A* in Figure 1.1) to the YMCA (point *D*)?

Solution Look at Figure 1.1; from point *A* to point *D* is 7 blocks down and 3 blocks left. Use Figure 1.4 as follows:



We see that there are 120 paths.

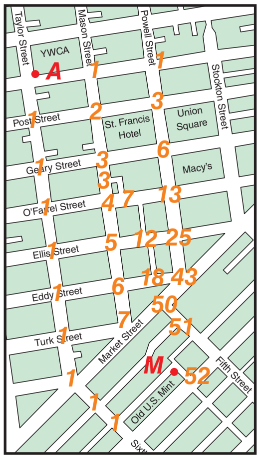
Pascal's triangle applies to the street problem only if the streets are rectangular. If the map shows irregularities (for example, diagonal streets or obstructions), then you must revert back to numbering the vertices.

Example 3 Travel with irregular paths

In how many different ways could Melissa get from the YWCA (point *A*) to the Old U.S. Mint (point *M*)?

Solution If the streets are irregular or if there are obstructions, you cannot use Pascal's triangle, but you can still count the blocks in the same fashion, as shown in the figure in the margin.

There are 52 paths from point *A* to point *M* (if, as usual, we do not allow backtracking).



Problem solving is a difficult task to master, and you are not expected to master it after one section of this text (or even after several chapters of this text). However, you must make building your problem-solving skills an ongoing process. One of the most important aspects of problem solving is to relate new problems to old problems. The problem-solving techniques outlined here should be applied when you are faced with a new problem. When you are faced with a problem similar to one you have already worked, you can apply previously developed techniques (as we did in Examples 1–3). Now, because Example 4 seems to be a new type of problem, we again apply the guidelines for problem solving.

Historical Note


Eyn Newe
Vnnd wolgegründte
vnderweyſung aller Kauffmanß Rech
nung in dreyen Büchern / mit ſchönen Re
geln vñ fragſtucken begriffen. Sunder
lich was fortl vnnd behendigkait in der
Welſchẽ Practica vñ Tolletn gebraucht
wirdt / des gleychen fürmalß weder in
Teutſcher noch in Welſcher ſprach nie
gedrückt. durch Petrum Apianũ
von Leyßnick / d Aſtronomei
zů Ingolſtat Ordina
riũ / verfertiget.

Bettmann/Corbis

The title page of an arithmetic book by Petrus Apianus in 1527 is reproduced above. It was the first time Pascal's triangle appeared in print.

Example 4 Cows and chickens

Pólya's Method

A jokester tells you that he has a group of cows and chickens and that he counted 13 heads and 36 feet. How many cows and chickens does he have?

Solution Let's use Pólya's problem-solving guidelines.

Understand the Problem. A good way to make sure you understand a problem is to attempt to phrase it in a simpler setting:

one chicken and one cow:	2 heads and 6 feet (chickens have 2 feet; cows have 4)
two chickens and one cow:	3 heads and 8 feet
one chicken and two cows:	3 heads and 10 feet

Devise a Plan. How you organize the material is often important in problem solving. Let's organize the information into a table:

No. of chickens	*No. of cows*	*No. of heads*	*No. of feet*
0	13	13	52

Do you see why we started here? The problem says we must have 13 heads. There are other possible starting places (13 chickens and 0 cows, for example), but an important aspect of problem solving is to start with *some* plan.

No. of chickens	*No. of cows*	*No. of heads*	*No. of feet*
1	12	13	50
2	11	13	48
3	10	13	46
4	9	13	44

Carry Out the Plan. Now, look for patterns. Do you see that as the number of cows decreases by one and the number of chickens increases by one, the number of feet must decrease by two? Does this make sense to you? Remember, step 1 requires that you not just push numbers around, but that you understand what you are doing. Since we need 36 feet for the solution to this problem, we see

$$44 - 36 = 8$$

so the number of chickens must increase by an additional four. The answer is 8 chickens and 5 cows.

Look Back.

No. of chickens	*No. of cows*	*No. of heads*	*No. of feet*
8	5	13	36

Check: 8 chickens have 16 feet and 5 cows have 20 feet, so the total number of heads is $8 + 5 = 13$, and the number of feet is 36.

Example 5 Number of birth orders

Pólya's Method

If a family has 5 children, in how many different birth orders could the parents have a 3-boy, 2-girl family?

Solution

Understand the Problem. Part of understanding the problem might involve estimation. For example, if a family has 1 child, there are 2 possible orders (B or G). If a family has 2 children, there are 4 orders (BB, BG, GB, GG); for 3 children, 8 orders; for 4 children, 16 orders; and for 5 children, a total of 32 orders. This means, for example, that an answer of 140 possible orders is an unreasonable answer.

Devise a Plan. You might begin by enumeration:

BBBGG, BBGBG, BBGGB, · · ·

This would seem to be too tedious. Instead, rewrite this as a simpler problem and look for a pattern.

1 child:
B ← one way
G ← one way

2 children:
BB ← one way
BG, GB ← two ways
GG ← one way

3 children:
BBB ← one way
BBG, BGB, GBB ← three ways
BGG, GBG, GGB ← three ways
GGG ← one way

Look at the possibilities:

1 child 1B 1G
2 children 1BB 2 1GG
$\uparrow$ ways for 1 boy and 1 girl

3 children 1BBB 3 3 1GGG *Look familiar?*
$\uparrow$ ways for 2 boys and 1 girl
$\uparrow$ ways for 1 boy and 2 girls

Look at Pascal's triangle in Figure 1.4; for 5 children, look at row 5.

Carry Out the Plan.

1	5	10	10	5	1
↑	↑	↑	↑	↑	↑
5 boys	4 boys and 1 girl	3 boys and 2 girls	2 boys and 3 girls	1 boy and 4 girls	5 girls

The family could have 3 boys and 2 girls in a total of 10 ways.

Look Back. We predicted that there are a total of 32 ways a family could have 5 children; let's sum the number of possibilities we found in carrying out the plan to see if it totals 32:

$$1 + 5 + 10 + 10 + 5 + 1 = 32$$

In this text, we are concerned about the thought process and not just typical "manipulative" mathematics. Using common sense is part of that thought process, and the following example illustrates how you can use common sense to analyze a given situation.

Example 6 Birthday dilemma

Pólya's Method

"I'm nine years old," says Adam.
"I'm ten years old," says Eve.
"My tenth birthday is tomorrow," says Adam.
"My tenth birthday was yesterday," says Eve.
"But I'm older than Eve," says Adam.

How is this possible if both children are telling the truth?

Solution We use Pólya's problem-solving guidelines for this example.

Understand the Problem. What do we mean by the words of the problem? A birthday is the celebration of the day of one's birth. Is a person's age always the same as the number of birthdays?

Devise a Plan. The only time the number of birthdays is different from the person's age is when we are dealing with a leap year. Let's suppose that the day of this conversation is in a leap year on February 29.

Carry Out the Plan. Eve was born ten years ago (a nonleap year) on February 28 and Adam was born ten years ago on March 1. But if someone is born on March 1, then that person is younger than someone born on February 28, right? Not necessarily! Suppose that Adam was born in New York City just after midnight on March 1 and that Eve was born before 9:00 P.M. in Los Angeles on February 28.

Look Back. Since Adam was born before 9:00 P.M. on February 28, he is older than Eve, even though his birthday is on March 1.

The following example illustrates the necessity of carefully reading the question.

Example 7 Meet for dinner

Pólya's Method

Nick and Marsha are driving from Santa Rosa, CA, to Los Angeles, a distance of 460 miles. They leave at 11:00 A.M. and average 50 mph. On the other hand, Mary and Dan leave at 1:00 P.M. in Dan's sports car. Who is closer to Los Angeles when they meet for dinner in San Luis Obispo at 5:00 P.M.?

Solution

Understand the Problem. If they are sitting in the same restaurant, then they are all the same distance from Los Angeles.

The last example of this section illustrates that problem solving may require that you change the conceptual mode.

Example 8 Pascal's triangle—first time in print

If you have been reading the historical notes in this chapter, you may have noticed that Blaise Pascal was born in 1623 and died in 1662. You may also have noticed that the first time Pascal's triangle appeared in print was in 1527. How can this be?

Solution It was a reviewer of this text who brought this apparent discrepancy to my attention. However, the facts are all correct. How could Pascal's triangle have been in print almost 100 years before he was born? The fact is, the number pattern we call Pascal's triangle is *named after* Pascal, but was not *discovered* by Pascal. This number pattern seems to have been discovered several times, by Johann Scheubel in the 16th century and by the Chinese mathematician Nakone Genjun; and recent research has traced the triangle pattern as far back as Omar Khayyám (1048–1122).

"Wait!" you exclaim. "How was I to answer the question in Example 8—I don't know all those facts about the triangle." You are not expected to know these facts, but you are expected to begin to think critically about the information you are given and the assumptions you are making. It was never stated that Blaise Pascal was the first to think of or publish Pascal's triangle!

Complete solutions are found in an Instructor's Manual.
Answers that vary are generally not shown here.

Problem Set 1.1

Level 1

1. **IN YOUR OWN WORDS** In the text, it was stated that "the most important prerequisite for this course is an openness to try out new ideas—a willingness to experience the suggested activities rather than to sit on the sideline as a spectator." Do you agree or disagree that this is the *most* important prerequisite? Discuss.
2. **IN YOUR OWN WORDS** What do you think the primary goal of mathematics education should be? What do you think it is in the United States? Discuss the differences between what it is and what you think it should be.
3. **IN YOUR OWN WORDS** In the chapter overview (did you read it?), it was pointed out that this text was written for people who think they don't like mathematics, or people who think they can't work math problems, or people who think they are never going to use math. Do any of those descriptions apply to you or to someone you know? Discuss.
4. **IN YOUR OWN WORDS** At the beginning of this section, three hints for success were listed. Discuss each of these from your perspective. Are there any other hints that you might add to this list?
5. **IN YOUR OWN WORDS** In Example 1, we concluded that there were 6 different paths. List each of those paths.
(1) down 2, over 2; (2) down 1, over 2, down 1; (3) down 1, over 1, down 1, over 1; (4) over 2, down 2; (5) over 1, down 2, over 1; (6) over 1, down 1, over 1, down 1

6. **IN YOUR OWN WORDS** In Example 2, the solution was found by going 7 blocks down and 3 blocks over. Could the solution also have been obtained by going 3 blocks over and 7 blocks down? Would this solution end up in a different location in Pascal's triangle? Describe a property of Pascal's triangle that is relevant to an answer for this question.

yes yes

The triangle is symmetric.

7. Describe the location of the numbers 1, 2, 3, 4, 5, · · · in Pascal's triangle. first diagonal

8. Describe the location of the numbers 1, 4, 10, 20, 35, · · · in Pascal's triangle. third diagonal

9.

Tony Freeman / PhotoEdit

a. If a family has 5 children, in how many ways could the parents have 2 boys and 3 girls as children? 10

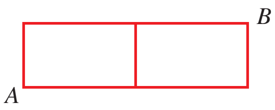
b. If a family has 6 children, in how many ways could the parents have 3 boys and 3 girls as children? 20

10. a. If a family has 7 children, in how many ways could the parents have 4 boys and 3 girls as children? 35

b. If a family has 8 children, in how many ways could the parents have 3 boys and 5 girls as children? 56

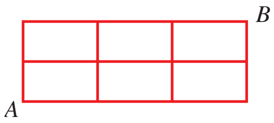
In Problems 11–14, *what is the number of direct routes from point A to point B?*

11.



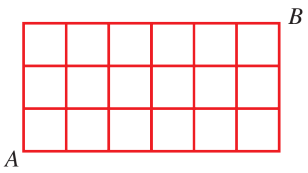
3 paths

12.



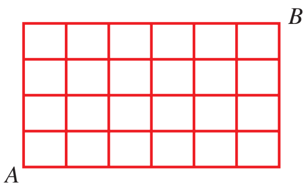
10 paths

13.



84 paths

14.



210 paths

Use the map in Figure 1.6 *to determine the number of different paths from point A to the point indicated in Problems* 15–18. *Remember, no backtracking is allowed.*

FIGURE 1.6 **Map of a portion of San Francisco**

15. *E* 10 paths

16. *F* 35 paths

17. *G* 330 paths

18. *H* 15 paths

Level 2

19. A car pulls onto the USS *Nimitz*, which is now a car ferry.

U.S. Navy photo by Mass Communication Specialist 3rd Class Casey J. Amdahl/Released

USS *Nimitz*

As a car enters the ferry, there are four rows of traffic directors arranged in a triangular pattern, such that one director is in the first row, two in the second, three in the third, and four in the fourth row. They are directing traffic into five parking spots, labeled *A* through *E* in Figure 1.7.

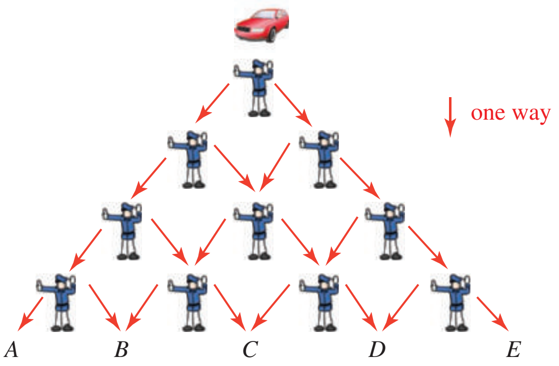


FIGURE 1.7 **Traffic directors lead cars to spaces.**

Given that cars must proceed according to the traffic arrows, how many different paths are there into each of the parking spots?
1 4 6 4 1

20. The ferry portion on the USS *Nimitz* houses 10 rows of parking spaces. Repeat Problem 19 for 10 rows instead of four. 1 10 45 120 210 252 210 120 45 10 1

21. Ten full crates of walnuts weigh 410 pounds, whereas an empty crate weighs 10 pounds. How much do the walnuts alone weigh? 310 lb

22. There are three separate, equal-size boxes, and inside each box there are two separate small boxes, and inside each of the small boxes there are three even smaller boxes. How many boxes are there all together? 27 boxes

23. Jerry's mother has three children. The oldest is a boy named Harry, who has brown eyes. Everyone says he is a math whiz. The next younger is a girl named Henrietta. Everyone calls her Mary because she hates her name. The youngest child has green eyes and can wiggle his ears. What is his first name? Jerry

24. A deaf-mute walks into a stationery store and wants to purchase a pencil sharpener. To communicate this need, the customer pantomimes by sticking a finger in one ear and rotating the other hand around the other ear. The very next customer is a blind person who needs a pair of scissors. How should this customer communicate this idea to the clerk? The blind person says, "I need a pair of scissors."

25. **a.** What is the sum of the numbers in row 1 of Pascal's triangle? 2
b. What is the sum of the numbers in row 2 of Pascal's triangle? 4
c. What is the sum of the numbers in row 3 of Pascal's triangle? 8
d. What is the sum of the numbers in row 4 of Pascal's triangle? 16

26. What is the sum of the numbers in row n of Pascal's triangle? 2^n

Use the map in Figure 1.6 *to determine the number of different paths from point A to the point indicated in Problems* 27–30. *Remember, no backtracking is allowed.*

27. *J*	**28.** *I*	**29.** *L*	**30.** *K*
49 paths	37 paths	284 paths	11 paths

Problems 31–44 *are not typical math problems but are problems that require only common sense (and sometimes creative thinking).*

31. How many 3-cent stamps are there in a dozen? 12

32. Which weighs more—a ton of coal or a ton of feathers? same

33. If you take 7 cards from a deck of 52 cards, how many cards do you have? 7

34. Oak Park Cemetery in Oak Park, New Jersey, will not bury anyone living west of the Mississippi. Why? They don't bury the living.

35. If posts are spaced 10 feet apart, how many posts are needed for 100 feet of straight-line fence? 11

36. At six o'clock, the grandfather clock struck 6 times. If it was 30 seconds between the first and last strokes, how long will it take the same clock to strike noon? 66 sec

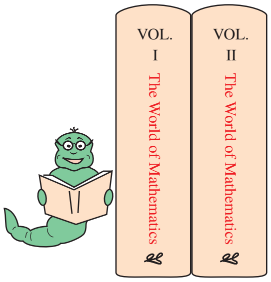
37. A person arrives at home just in time to hear one chime from the grandfather clock. A half-hour later it strikes once. Another half-hour later it chimes once. Still another half-hour later it chimes once. What time did the person arrive home? 12:00

38. Two girls were born on the same day of the same month of the same year to the same parents, but they are not twins. Explain how this is possible. triplets

39. How many outs are there in a baseball game that lasts the full 9 innings? 54

40. Two U.S. coins total $0.30, yet one of these coins is not a nickel. What are the coins? quarter and a nickel

41. Two volumes of Newman's *The World of Mathematics* stand side by side, in order, on a shelf. A bookworm starts at page i of Volume I and bores its way in a straight line to the last page of Volume II. Each cover is 2 mm thick, and the first volume is $\frac{17}{19}$ as thick as the second volume. The first volume is 38 mm thick without its cover. How far does the bookworm travel? 4 mm



42. A farmer has to get a fox, a goose, and a bag of corn across a river in a boat that is large enough only for him and one of these three items. If he leaves the fox alone with the goose, the fox will eat the goose. If he leaves the goose alone with the corn, the goose will eat the corn. How does he get all the items across the river?

Take goose across; return and pick up the fox. Deliver the fox and take the goose back to the first side. Drop off the goose, pick up the corn, and leave it with the fox. Return one last time to fetch the goose.

1 lump in first cup; 4 lumps in second cup; 5 lumps in third cup. Now put the first cup (containing the lump) inside the second cup.

43. Can you place ten lumps of sugar in three empty cups so that there is an odd number of lumps in each cup?

44. Six glasses are standing in a row. The first three are empty, and the last three are full of water. By handling and moving only one glass, it is possible to change this arrangement so that no empty glass is next to another empty one and no full glass is next to another full glass. How can this be done? Pour the contents of glass 5 into glass 2.

Level 3

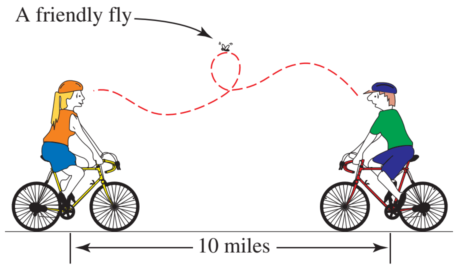
45. IN YOUR OWN WORDS Suppose you have a long list of numbers to add, and you have misplaced your calculator. Discuss the different approaches that could be used for adding this column of numbers.

46. IN YOUR OWN WORDS You are faced with a long division problem, and you have misplaced your calculator. You do not remember how to do long division. Discuss your alternatives to come up with the answer to your problem.

47. IN YOUR OWN WORDS You have 10 items in your grocery cart. Six people are waiting in the express lane (10 items or less); one person is waiting in the first checkout stand and two people are waiting in another checkout stand. The other checkout stands are closed. What additional information do you need in order to decide which lane to enter?

48. IN YOUR OWN WORDS You drive up to your bank and see five cars in front of you waiting for two lanes of the drive-through banking services. What additional information do you need in order to decide whether to drive through or park your car and enter the bank to do your banking?

49. A boy cyclist and a girl cyclist are 10 miles apart and pedaling toward each other. The boy's rate is 6 miles per hour, and the girl's rate is 4 miles per hour. There is also a friendly fly zooming continuously back and forth from one bike to the other. If the fly's rate is 20 miles per hour, by the time the cyclists reach each other, how far does the fly fly? 20 mi



50. Two race cars face each other on a single 30-mile track, and each is moving at 60 mph. A fly on the front of one car flies back and forth on a zigzagging path between the cars until they meet. If the fly travels at 25 mph, how far will it have traveled when the cars collide? 15 miles; since the cars travel faster than the fly, the fly can never get off the front of the car.

51. Alex, Beverly, and Cal live on the same straight road. Alex lives 10 miles from Beverly, and Cal lives 2 miles from Beverly. How far does Alex live from Cal? 8 mi or 12 mi

52. In a different language, *liro cas* means "red tomato." The meaning of *dum cas dan* is "big red barn" and *xer dan* means "big horse." What are the words for "red barn" in this language? *dum cas*

53. Assume that the first "gh" sound in *ghghgh* is pronounced as in *hiccough,* the second "gh" as in *Edinburgh*, and the third "gh" as in *laugh*. How should the word *ghghgh* be pronounced? puff

54. Write down a three-digit number. Write the number in reverse order. Subtract the smaller of the two numbers from the larger to obtain a new number. Write down the new number. Reverse the digits again, but add the numbers this time. Complete this process for another three-digit number. Do you notice a pattern, and does your pattern work for all three-digit numbers?
1,089; does not work for palindromes. It also does not work if the last digit is one less than the first digit and the middle digit is a zero. In this case, the result is 198.

55. Start with a common fraction between 0 and 1. Form a new fraction, using the following rules: *New denominator*: Add the numerator and denominator of the original fraction. *New numerator*: Add the new denominator to the original numerator. Write the new fraction and use a calculator to find a decimal equivalent to four decimal places. Repeat these steps again, this time calling the new fraction the original. Continue the process until a pattern appears about the decimal equivalent. What is the decimal equivalent (correct to two decimal places)? 1.62

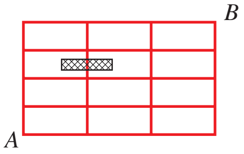
56. The number 6 has four divisors—namely, 1, 2, 3, and 6. List all numbers less than 20 that have exactly four divisors. 6, 8, 10, 14, and 15

Problem Solving Level 3

Each section of the text has one or more problems designated by "Problem Solving." These problems may require additional insight, information, or effort to solve. True problem-solving ability comes from solving problems that "are not like the examples" but rather require independent thinking. I hope you will make it a habit to read these problems and attempt to work those that interest you, even though they may not be part of your regular class assignment.

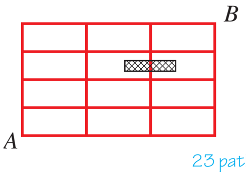
57. Consider the routes from A to B and notice that there is now a barricade blocking the path. Work out a general solution for the number of paths with a blockade, and then illustrate your general solution by giving the number of paths for each of the following street patterns.

a.



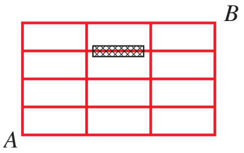
26 paths

b.



23 paths

c.



27 paths

58. HISTORICAL QUEST Thoth, an ancient Egyptian god of wisdom and learning, has abducted Ahmes, a famous Egyptian scribe, in order to assess his intellectual prowess. Thoth places Ahmes before a large funnel set in the ground (see Figure 1.8 on the next page). It has a circular opening 1,000 ft in diameter, and its walls are quite slippery. If Ahmes attempts to enter the funnel, he will slip down the wall. At the bottom of the funnel is a sleep-inducing liquid that will instantly put Ahmes to sleep for eight hours if he touches it.* Thoth hands Ahmes two objects: a rope 1,006.28 ft in length and the skull of a chicken. Thoth says to Ahmes, "If you are able to get to the central tower and touch it, we will live in harmony for the next millennium. If not, I will detain you for further testing. Please note that with each passing hour, I will decrease the rope's length by a foot." How can Ahmes reach the central ankh tower and touch it?
Tie one end of the rope onto the ankh, and then walk around the outer perimeter to form a rope bridge to the center.

---

*From "The Thoth Maneuver," by Clifford A. Pickover, *Discover*, March 1996, p. 108. Clifford Pickover © 1996. Reprinted with permission of *Discover* Magazine. Nenad Jakesevic and Sonja Lamut © 1996. Reprinted with permission of *Discover* Magazine.

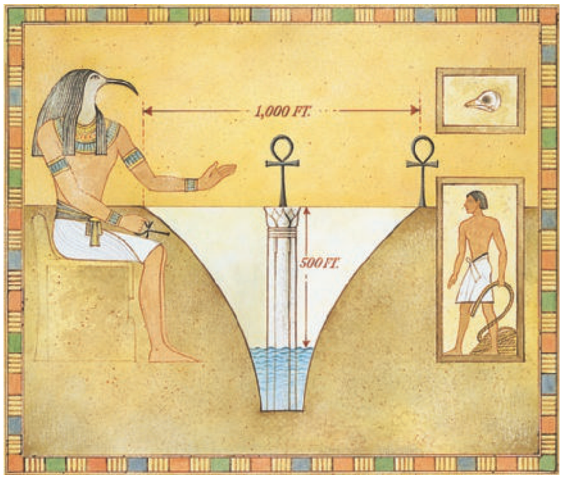
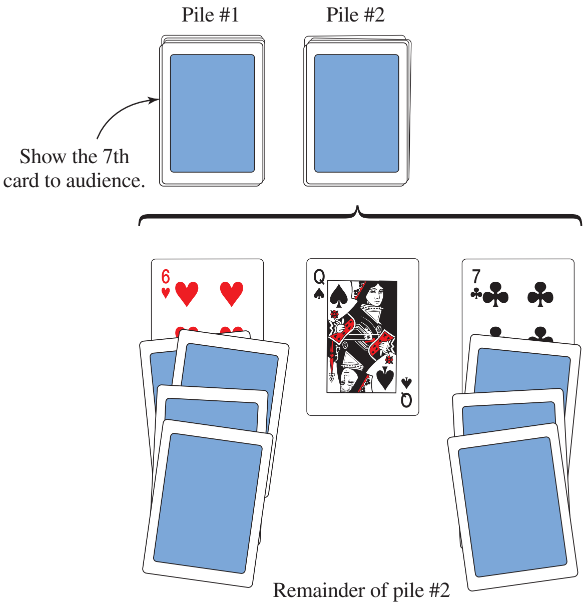


FIGURE 1.8 **Ahmes's dilemma.**
Note that there are two ankh-shaped towers. One stands on a cylindrical platform in the center of the funnel. The platform's surface is at ground level. The distance from the surface to the liquid is 500 ft. The other ankh tower is on land, at the edge of the funnel.

"The Thoth Maneuver," by Clifford A. Pickover, Discover, March 1996, p. 108. Clifford Pickover/© 1996. Reprinted with permission of Discover Magazine. Nenad Jakesevic and Sonja Lamut/© 1996. Reprinted with permission of Discover Magazine.



Remainder of pile #2 goes on top of pile #1.

59. A magician divides a deck of cards into two equal piles, counts down from the top of the first pile to the seventh card, and shows it to the audience without looking at it herself. These seven cards are replaced face down in the same order on top of the first pile. She then picks up the other pile and deals the top three cards up in a row in front of her. If the first card is a six, then she starts counting with "six" and counts to ten, thus placing four more cards on this pile as shown. In turn, the magician does the same for the next two cards. If the card is a ten or a face card, then no additional cards are added. The remainder of this pile is placed on top of the first pile. Next, the magician adds the values of the three face-up cards ($6 + 10 + 7$ for this illustration) and counts down in the first deck this number of cards. That card is the card that was originally shown to the audience. Explain why this trick works.

Begin with 26 cards in each pile. The hidden card is 7 cards down. Three cards are removed, leaving 23 cards. See IM for necessary algebraic steps.

60. A very magical teacher had a student select a two-digit number between 50 and 100 and write it on the board out of view of the instructor. Next, the student was asked to add 76 to the number, producing a three-digit sum. If the digit in the hundreds place is added to the remaining two-digit number and this result is subtracted from the original number, the answer is 23, which was predicted by the instructor. How did the instructor know the answer would be 23? *Note:* This problem is dedicated to my friend Bill Leonard of Cal State, Fullerton. His favorite number is 23.

Let the given number be $10t + u$. See IM for the necessary algebraic steps.

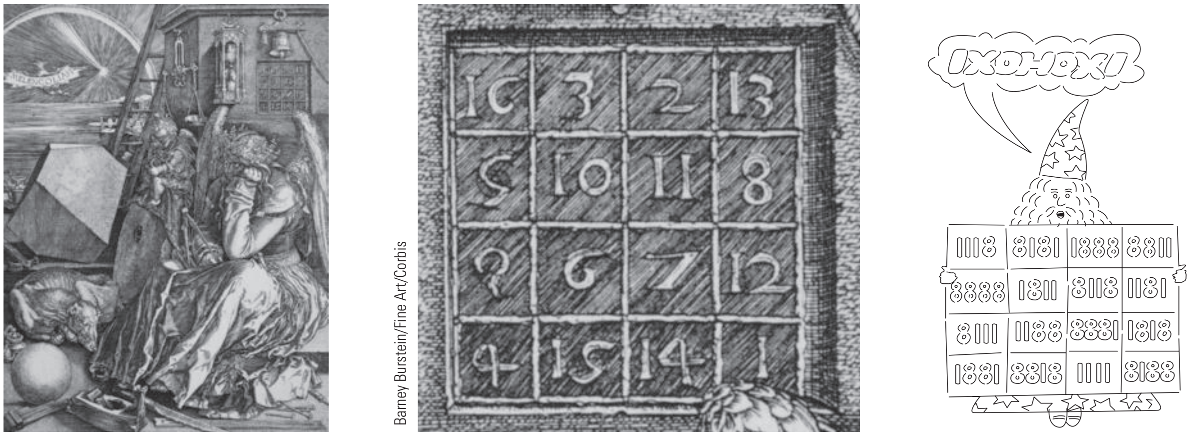
1.2 Inductive and Deductive Reasoning

Studying numerical patterns is one frequently used technique of problem solving.

Magic Squares

A magic square is an arrangement of numbers in the shape of a square with the sums of each vertical column, each horizontal row, and each diagonal all equal. One of the most famous ones appeared in a 1514 engraving, *Melancholia* by Dürer, as shown in Figure 1.9. (Notice that the date appears in the magic square.)

Historical Note



Barney Burstein/Fine Art/Corbis

Barney Burstein/Fine Art/Corbis

Melancholia by Albrecht Dürer

Detail of *Melancholia*

IXOHOXI

FIGURE 1.9 Early magic squares

Magic squares can be constructed using the first 9, 16, 25, 36, 49, 64, and 81 consecutive numbers. You may want to try some of them. One with the first 25 numbers is shown here.

23	12	1	20	9
4	18	7	21	15
10	24	13	2	16
11	5	19	8	22
17	6	25	14	3

There are formal methods for finding magic squares, but we will not describe them here. Figure 1.9 shows a rather interesting magic square called IXOHOXI because it is a magic square when it is turned upside down and also when it is reflected in a mirror.

Let's consider some other simple patterns.

A Pattern of Nines

A very familiar pattern is found in the ordinary "times tables." By pointing out patterns, teachers can make it easier for children to learn some of their multiplication tables. For example, consider the multiplication table for 9s:

$$\begin{aligned}
1 \times 9 &= 9\\
2 \times 9 &= 18\\
3 \times 9 &= 27\\
4 \times 9 &= 36\\
5 \times 9 &= 45\\
6 \times 9 &= 54\\
7 \times 9 &= 63\\
8 \times 9 &= 72\\
9 \times 9 &= 81\\
10 \times 9 &= 90
\end{aligned}$$

What patterns do you notice? You should be able to see many number relationships by looking at the totals. For example, notice that the sum of the digits to the right of the equality is 9 in all the examples ($1 + 8 = 9$, $2 + 7 = 9$, $3 + 6 = 9$, and so on). Will this always be the case for multiplication by 9? (Consider $11 \times 9 = 99$. The sum of the digits is 18. However, notice the result if you add the digits of 18.) Do you see any other patterns? Can you explain why they "work"? This pattern of adding after multiplying by 9 generates a sequence of numbers: 9, 9, 9, · · · . We call this the *nine pattern*. The two number tricks described in the news clip at the left use this nine pattern.

Nine is one of the most fascinating of all numbers. Here are two interesting tricks that involve the number nine. You need a calculator for these.

Mix up the serial number on a dollar bill. You now have two numbers, the original serial number and the mixed-up one. Subtract the smaller from the larger. If you add the digits of the answer, you will obtain a 9 or a number larger than 9; if it is larger than 9, add the digits of this answer again. Repeat the process until you obtain a single digit as with the *nine pattern*. That digit will *always* be 9.

Here is another trick. Using a calculator keyboard or push-button phone, choose any three-digit column, row, or diagonal, and arrange these digits in any order. Multiply this number by another row, column, or diagonal. If you repeatedly add the digits the answer will *always* be nine.

Example 1 Eight pattern

Pólya's Method

Find the *eight pattern.*

Solution We use Pólya's problem-solving guidelines for this example.

Understand the Problem. What do we mean by the *eight pattern*? Do you understand the example for the *nine pattern*?

Devise a Plan. We will carry out the multiplications of successive counting numbers by 8 and if there is more than a single-digit answer, we add the digits.* What we are looking for is a pattern for these single-digit numerals (shown in blue).

Carry Out the Plan.

$$\begin{array}{ccccc} 1 \times 8 = 8, & 2 \times 8 = \underbrace{16}, & 3 \times 8 = \underbrace{24}, & 4 \times 8 = \underbrace{32}, & 5 \times 8 = \underbrace{40}, \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 8 & 1 + 6 = 7 & 2 + 4 = 6 & 3 + 2 = 5 & 4 + 0 = 4 \end{array}$$

Continue with some additional terms:

$$\begin{array}{rclcrclcrcl} 6 \times 8 &=& 48 & \text{and} & 4 + 8 &=& 12 & \text{and} & 1 + 2 &=& 3 \\ 7 \times 8 &=& 56 & \text{and} & 5 + 6 &=& 11 & \text{and} & 1 + 1 &=& 2 \\ 8 \times 8 &=& 64 & \text{and} & 6 + 4 &=& 10 & \text{and} & 1 + 0 &=& 1 \\ 9 \times 8 &=& 72 & \text{and} & 7 + 2 &=& 9 & & & & \\ 10 \times 8 &=& 80 & \text{and} & 8 + 0 &=& 8 & & & & \end{array}$$

We now see the eight pattern: 8, 7, 6, 5, 4, 3, 2, 1, 9, 8, 7, 6, · · · .

Look Back. Let's do more than check the arithmetic, since this pattern seems clear. The problem seems to be asking whether we understand the concept of a *nine pattern* or an *eight pattern*. Verify that the *seven pattern* is 7, 5, 3, 1, 8, 6, 4, 2, 9, 7, 5, 3, 1, · · · .

Order of Operations

Complicated arithmetic problems can sometimes be solved by using patterns. Given a difficult problem, a problem solver will often try to *solve a simpler, but similar, problem.* A **numerical expression** is one or more numbers connected by valid mathematical operations, and to **simplify** a numerical expression means to write the numerical expression as a single number.

Imagination is a sort of faint perception.
—Aristotle

The second suggestion for solving using Pólya's problem-solving procedure stated, "If you cannot solve the proposed problem try to solve some related problem." The appropriate related problem is a simpler one, if possible. For example, suppose we wish to compute the following number:

$$10 + 123{,}456{,}789 \times 9$$

Instead of doing a lot of arithmetic, let's study the following pattern:

$$\begin{array}{c} 2 + 1 \times 9 \\ 3 + 12 \times 9 \\ 4 + 123 \times 9 \end{array}$$

---

*__Counting numbers__ are the numbers we use for counting—namely, 1, 2, 3, 4, · · · . Sometimes they are also called **natural numbers.** The integers are the counting numbers, their opposites, and 0, namely, · · ·, −3, −2, −1, 0, 1, 2, 3, · · · . We assume a knowledge of these numbers.

Do you see the next entry in this pattern? Do you see that if we continue the pattern we will eventually reach the desired expression of $10 + 123{,}456{,}789 \times 9$? Using Pólya's strategy, we begin by working these easier problems. Thus, we begin with $2 + 1 \times 9$. There is a possibility of ambiguity in calculating this number:

Left to right	*Multiplication first*
$2 + 1 \times 9 = 3 \times 9 = 27$	$2 + 1 \times 9 = 2 + 9 = 11$

Although either of these might be acceptable in certain situations, it is not acceptable to get two different answers to the same problem. We therefore agree to do a problem like this by multiplying first. If we wish to change this order, we use parentheses, as in $(2 + 1) \times 9 = 27$. We summarize with a procedure known as the **order-of-operations agreement.**

This is important! Take time looking at what this says.

Order of Operations

To simplify a numerical expression containing only the operations of addition, subtraction, multiplication, and division, follow these steps (in order):

Step 1 Perform any operations enclosed in parentheses.

Step 2 Perform multiplications and divisions as they occur by working from left to right.

Step 3 Perform additions and subtractions as they occur by working from left to right.

Thus, the correct result for $2 + 1 \times 9$ is 11. Also,

$$3 + 12 \times 9 = 3 + 108 = 111$$
$$4 + 123 \times 9 = 4 + 1{,}107 = 1{,}111$$
$$5 + 1{,}234 \times 9 = 5 + 11{,}106 = 11{,}111$$

Do you see a pattern? If so, then make a prediction about the desired result. If you do not see a pattern, continue with this pattern to see more terms, or go back and try another pattern. For this example, we predict

$$10 + 123{,}456{,}789 \times 9 = 1{,}111{,}111{,}111$$

The most difficult part of this type of problem solving is coming up with a correct pattern. For this example, you might guess that

$$\begin{array}{l} 2 + 1 \times 1 \\ 3 + 12 \times 2 \\ 4 + 123 \times 3 \\ 5 + 1{,}234 \times 4 \\ \quad\vdots \end{array}$$

leads to $10 + (123{,}456{,}789 \times 9)$. Calculating, we find

$$2 + 1 \times 1 = 2 + 1 = 3$$
$$3 + 12 \times 2 = 3 + 24 = 27$$
$$4 + 123 \times 3 = 4 + 369 = 373$$
$$5 + 1{,}234 \times 4 = 5 + 4{,}936 = 4{,}941$$

If you begin a pattern and it does not lead to a pattern of answers, then you need to remember that part of Pólya's problem-solving procedure is to work both backward and forward. Be willing to give up one pattern and begin another.

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We also point out that the patterns you find are not necessarily unique. One last time, we try a pattern for $10 + 123{,}456{,}789 \times 9$:

$$
\begin{aligned}
10 + 1 \times 9 &= 10 + 9 = 19\\
10 + 12 \times 9 &= 10 + 108 = 118\\
10 + 123 \times 9 &= 10 + 1{,}107 = 1{,}117\\
10 + 1{,}234 \times 9 &= 10 + 11{,}106 = 11{,}116\\
&\vdots
\end{aligned}
$$

We do see a pattern here (although not quite as easily as the one we found with the first pattern for this example):

$$10 + 123{,}456{,}789 \times 9 = 1{,}111{,}111{,}111$$

Inductive Reasoning

The type of reasoning used here and in Section 1.1—first observing patterns and then predicting answers for more complicated problems—is called **inductive reasoning.** It is a very important method of thought and is sometimes called the *scientific method*. It involves reasoning from particular facts or individual cases to a general **conjecture**—a statement you think may be true. That is, a generalization is made on the basis of some observed occurrences. The more individual occurrences we observe, the better able we are to make a correct generalization. Peter in the *B.C.* cartoon makes the mistake of generalizing on the basis of a single observation.

Example 2 Sum of 100 odd numbers

Pólya's Method

What is the sum of the first 100 consecutive odd numbers?

Solution We use Pólya's problem-solving guidelines for this example.

Understand the Problem. Do you know what the terms mean? Odd numbers are $1, 3, 5, \cdots$, and *sum* indicates addition:

$$1 + 3 + 5 + \cdots + \underset{\uparrow}{?}$$

The first thing you need to understand is what the last term will be, so you will know when you have reached 100 consecutive odd numbers.

$1 + 3$ is two terms.
$1 + 3 + 5$ is three terms.
$1 + 3 + 5 + 7$ is four terms.

It seems as if the last term is always one less than twice the number of terms. Thus, the sum of the first 100 consecutive odd numbers is

$$1 + 3 + 5 + \cdots + 195 + 197 + \underset{\uparrow}{199}$$

This is one less than 2(100).

Devise a Plan. The plan we will use is to look for a pattern:

$$
\begin{aligned}
1 &= 1 && \text{One term}\\
1 + 3 &= 4 && \text{Sum of two terms}\\
1 + 3 + 5 &= 9 && \text{Sum of three terms}
\end{aligned}
$$

Do you see a pattern yet? If not, continue:

$$
\begin{aligned}
1 + 3 + 5 + 7 &= 16\\
1 + 3 + 5 + 7 + 9 &= 25
\end{aligned}
$$

Carry Out the Plan. It appears that the sum of two terms is $2 \cdot 2$; of three terms, $3 \cdot 3$; of four terms, $4 \cdot 4$; and so on. The sum of the first 100 consecutive odd numbers is therefore $100 \cdot 100$.

Look Back. Does $100 \cdot 100 = 10{,}000$ seem correct?

The numbers $2 \cdot 2$, $3 \cdot 3$, $4 \cdot 4$, and $100 \cdot 100$ from Example 2 are usually written as 2^2, 3^2, 4^2, and 100^2. The number b^2 means $b \cdot b$ and is pronounced ***b* squared,** and the number b^3 means $b \cdot b \cdot b$ and is pronounced ***b* cubed.** The process of repeated multiplication is called **exponentiation.** Numbers that are multiplied are called **factors,** so we note that b^2 means we have two factors and one multiplication, whereas b^3 indicates three factors (two multiplications).

Deductive Reasoning

Another method of reasoning used in mathematics is called **deductive reasoning.** This method of reasoning produces results that are *certain* within the logical system being developed. That is, deductive reasoning involves reaching a conclusion by using a formal structure based on a set of **undefined terms** and a set of accepted unproved **axioms** or **premises.** For example, consider the following argument:

1. If you read the *Times*, then you are well informed.
2. You read the *Times*.
3. Therefore, you are well informed.

Statements 1 and 2 are the *premises* of the argument; statement 3 is called the **conclusion.** If you accept statements 1 and 2 as true, then you *must* accept statement 3 as true. Such reasoning is called *deductive reasoning*; and if the conclusion follows from the premises, the reasoning is said to be **valid.**

Deductive Reasoning

Deductive reasoning consists of reaching a conclusion by using a formal structure based on a set of *undefined terms* and on a set of accepted unproved *axioms* or *premises*. The conclusions are said to be *proved* and are called **theorems.**

Reasoning that is not valid is called **invalid** reasoning. Logic accepts no conclusions except those that are inescapable. This is possible because of the strict way in which concepts are defined. Difficulty in simplifying arguments may arise because of their length, the vagueness of the words used, the literary style, or the possible emotional impact of the words used.

Consider the following two arguments:

1. If George Washington was assassinated, then he is dead. Therefore, if he is dead, he was assassinated.
2. If you use heroin, then you first used marijuana. Therefore, if you use marijuana, then you will use heroin.

Logically, these two arguments are exactly the same, and both are *invalid* forms of reasoning. Nearly everyone would agree that the first is invalid, but many people see the second as valid. The reason lies in the emotional appeal of the words used.

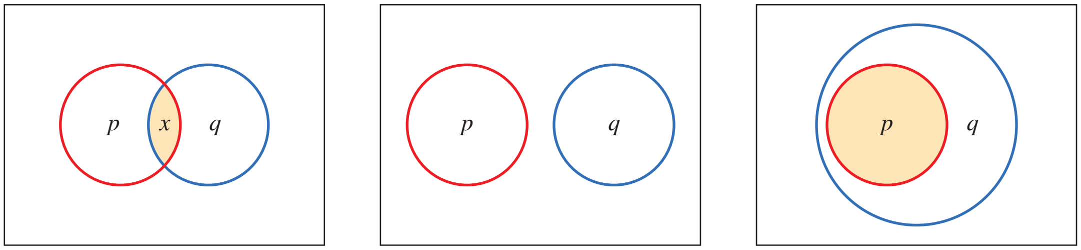
To avoid these difficulties, we look at the *form* of the arguments and not at the independent truth or falsity of the statements. One type of logic problem is called a **syllogism.** A syllogism has three parts: two *premises*, or hypotheses, and a *conclusion*. The premises give us information from which we form a conclusion. With the syllogism, we are interested in knowing whether the conclusion *necessarily follows* from the premises.

If it does, it is called a *valid syllogism*; if not, it is called *invalid*. Consider the following examples:

Valid Forms of Reasoning		*Invalid Forms of Reasoning*
All Chevrolets are automobiles.	***Premise***	Some people are nice.
All automobiles have four wheels.	***Premise***	Some people are broke.
All Chevrolets have four wheels.	***Conclusion***	There are some nice broke people.
All teachers are crazy.	***Premise***	All dodos are extinct
Karl Smith is a teacher.	***Premise***	No dinosaurs are dodos.
Karl Smith is crazy.	***Conclusion***	Therefore, all dinosaurs are extinct.

To analyze such arguments, we need to have a systematic method of approach. We will use **Euler circles,** named after one of the most famous mathematicians in all of mathematics, Leonhard Euler.

For two sets p and q, we make the interpretations shown in Figure 1.10.



Some p are q.

(x means that intersection is not empty)

No p are q.

Disjoint sets

All p are q.

Subsets

FIGURE 1.10 Euler circles for syllogisms

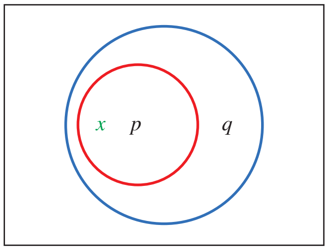
Example 3 Testing for valid arguments

Test the validity of the following arguments.

a. All dictionaries are books.
This is a dictionary.
Therefore, this is a book.

b. If you like potato chips, then you will like Krinkles.
You do not like potato chips.
Therefore, you do not like Krinkles.

Solution

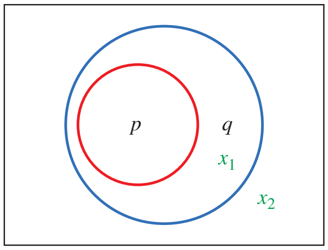


a. Begin by drawing Euler circles showing the first premise:

All dictionaries are books.

Let p: dictionaries
q: books

For the second premise, we place x (this object) inside the circle of dictionaries (labeled p). The conclusion, "This object is a book," cannot be avoided (since x *must* be in q), so it is valid.



b. Again, begin by using Euler circles:

If you like potato chips, then you will like Krinkles.

The first premise is the same as

All people who like potato chips like Krinkles.

Let p: people who like potato chips
q: people who like Krinkles

For the second premise, you will place the x (you) outside the circle labeled p. Notice that you are not forced to place x into a single region; it could be placed in either of two places—those labeled x_1 and x_2. Since the stated conclusion is not forced, the argument is not valid.

Historical Note

Karl Smith Library

Leonhard Euler
(1707–1783)

Euler's name is attached to every branch of mathematics, and we will visit his work many times in this text. His name is pronounced "Oiler" and it is sometimes joked that if you want to give a student a one-word mathematics test, just ask the student to pronounce Leonhard's last name. He was the most prolific writer on the subject of mathematics, and his mathematical textbooks were masterfully written. His writing was not at all slowed down by his total blindness for the last 17 years of his life. He possessed a phenomenal memory, had almost total recall, and could mentally calculate long and complicated problems.

Problem Set 1.2

Level 1

1. IN YOUR OWN WORDS Discuss the nature of *inductive* and *deductive reasoning*.

2. IN YOUR OWN WORDS Explain what is meant by the *seven pattern*.

3. IN YOUR OWN WORDS What do we mean by *order of operations*? See page 19.

4. IN YOUR OWN WORDS What is the scientific method? See page 20.

5. IN YOUR OWN WORDS Explain inductive reasoning. Give an original example of an occasion when you have used inductive reasoning or heard it being used. Answers vary.

6. IN YOUR OWN WORDS Explain deductive reasoning. Give an original example of an occasion when you have used deductive reasoning or heard it being used. Answers vary.

Perform the operations in Problems 7–18.

7. a. $5 + 2 \times 6$ 17 **b.** $7 + 3 \times 2$ 13

8. a. $14 + 6 \times 3$ 32 **b.** $30 \div 5 \times 2$ 12

9. a. $3 \times 8 + 3 \times 7$ 45 **b.** $3(8 + 7)$ 45

10. a. $(8 + 6) \div 2$ 7 **b.** $8 + 6 \div 2$ 11

11. a. $12 + 6/3$ 14 **b.** $(12 + 6)/3$ 6

12. a. $450 + 550/10$ 505 **b.** $\dfrac{450 + 550}{10}$ 100

13. a. $20/2 \cdot 5$ 50 **b.** $20/(2 \cdot 5)$ 2

14. a. $1 + 3 \times 2 + 4 + 3 \times 6$ 29
b. $3 + 6 \times 2 + 8 + 4 \times 3$ 35

15. a. $10 + 5 \times 2 + 6 \times 3$ 38
b. $4 + 3 \times 8 + 6 + 4 \times 5$ 54

16. a. $8 + 2(3 + 12) - 5 \times 3$ 23
b. $25 - 4(12 - 2 \times 6) + 3$ 28

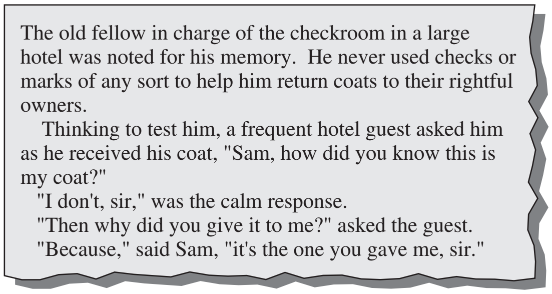
17. a. $3 + 9 \div 3 \times 2 + 2 \times 6 \div 3$ 13
b. $[(3 + 9) \div 3] \times 2 + [(2 \times 6) \div 3]$ 12

18. a. $3 + [(9 \div 3) \times 2] + [(2 \times 6) \div 3]$ 13
b. $[(3 + 9) \div (3 \times 2)] + [(2 \times 6) \div 3]$ 6

19. Does the *B.C.* cartoon illustrate inductive or deductive reasoning? Explain your answer. inductive

By permission John L. Hart FLP and Creators Syndicate, Inc.

20. Does the news clip below illustrate inductive or deductive reasoning? Explain your answer. deductive



The old fellow in charge of the checkroom in a large hotel was noted for his memory. He never used checks or marks of any sort to help him return coats to their rightful owners.

Thinking to test him, a frequent hotel guest asked him as he received his coat, "Sam, how did you know this is my coat?"

"I don't, sir," was the calm response.

"Then why did you give it to me?" asked the guest.

"Because," said Sam, "it's the one you gave me, sir."

Lucille J. Goodyear

Problems 21–24 *are modeled after Example* 1. *Find the requested pattern.*

21. three pattern 3, 6, 9, ⋯

22. four pattern 4, 8, 3, 7, 2, 6, 1, 5, 9, ⋯

23. five pattern 5, 1, 6, 2, 7, 3, 8, 4, 9, ⋯

24. six pattern 6, 3, 9, ⋯

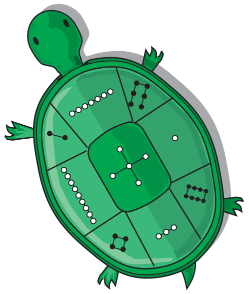
25. a. What is the sum of the first 25 consecutive odd numbers? 625

b. What is the sum of the first 250 consecutive odd numbers? 62,500

26. a. What is the sum of the first 50 consecutive odd numbers? 2,500

b. What is the sum of the first 1,000 consecutive odd numbers? 1,000,000

27. HISTORICAL QUEST The first known example of a magic square comes from China. Legend tells us that around the year 200 B.C. the emperor Yu of the Shang dynasty received the following magic square etched on the back of a tortoise's shell:



2	7	6
9	5	1
4	3	8

The incident supposedly took place along the Lo River, so this magic square has come to be known as the Lo-shu magic square.

The even numbers are black (female numbers) and the odd numbers are white (male numbers). Translate this magic square into modern symbols.

This same magic square (called *wafq* in Arabic) appears in Islamic literature from the 10th century A.D. and is attributed to Jabir ibn Hayyan.

28. HISTORICAL QUEST The Lo-shu magic square in Problem 27 has the even numbers in black (yin numbers) and the odd numbers in white (yang). What is the relationship between the yin and the yang numbers in this magic square? Do you think this is a coincidence, a special property of the Lo-shu square, or is it something else? The yin and the yang sums are equal if we do not use the middle number. This is not a coincidence.

29. Consider the square shown in Figure 1.11.

10	7	8	11
14	11	12	15
13	10	11	14
15	12	13	16

FIGURE 1.11 Magic square?

Is this a magic square? no

30. Circle any number of the magic square in Figure 1.11. Cross out all the numbers in the same row and column. Then circle any remaining number and cross out all the numbers in the same row and column. Circle the remaining number. The sum of the circled numbers is 48. Why? Answers vary.

Level 2

HISTORICAL QUEST Magic squares remind us of Sudoku number puzzles. *Sudoku* (or su doku) is a number puzzle, usually consisting of a 9 × 9 grid divided into nine 3 × 3 boxes, into which numbers already appear in a few cells. You may have seen such a puzzle online or in your newspaper, but would you believe that Sudoku was invented by a 74-year-old Indianapolis man who never received credit for his invention of this type of puzzle problem? In May 1979, a puzzle created by Howard Garns was published, but few paid attention.* Then, in April 1984, Japan's puzzle group Nikoli discovered Garns's puzzle, invented the word Sudoku—(Su = number, Doku = single)—trademarked the name, and published new versions of the puzzle. At that time, and with this new name, the puzzle became a worldwide phenomenon.

The object of the puzzle is to complete all the remaining cells with the numbers from 1 to 9, so that:

- each row contains all the numbers from 1 to 9
- each column contains all the numbers from 1 to 9
- each 3 × 3 box contains all the numbers from 1 to 9

Complete the Sudoku puzzles in Problems 31–32.

31.

	7				6	4	1	
		4			2	8		
	6		4					5
8			3			7		
			7	9	8			
		6			5			3
9					3		4	
		2	1			5		
	4	7	6				2	

32.

		4		2			1	9
			6					5
				8	1		6	
	4	7		1				6
			3		7			
3				6		5	9	
	3		2	5				
8					9			
2	9			3		1		

33. The following square of numbers consists of nine square numbers. Is this a magic square with nine distinct square numbers?

127^2	46^2	58^2
2^2	113^2	94^2
74^2	82^2	97^2

This is not a magic square; one diagonal is 38,307 and the other sums are 21,609.

34. The following square of numbers consists of nine square numbers. Is this a magic square with nine distinct square numbers?

35^2	$3,495^2$	$2,958^2$
$3,642^2$	$2,125^2$	$1,785^2$
$2,775^2$	$2,058^2$	$3,005^2$

This is not a magic square; one diagonal is 13,546,875; other sums are 20,966,014.

Use Euler circles to check the validity of the arguments in Problems 35–46.

35. All mathematicians are eccentrics.
All eccentrics are rich.
Therefore, all mathematicians are rich. valid

36. All snarks are fribbles.
All fribbles are ugly.
Therefore, all snarks are ugly. valid

37. All cats are animals.
This is not an animal.
Therefore, this is not a cat. valid

38. All bachelors are handsome.
Some bachelors do not drink lemonade.
Therefore, some handsome men do not drink lemonade. valid

39. No students are enthusiastic.
You are enthusiastic.
Therefore, you are not a student. valid

40. No politicians are honest.
Some dishonest people are found out.
Therefore, some politicians are found out. not valid

41. All candy is fattening.
All candy is delicious.
Therefore, all fattening food is delicious. not valid

42. All parallelograms are rectangles.
All rectangles are polygons.
Therefore, all parallelograms are polygons. valid

43. No professors are ignorant.
All ignorant people are vain.
Therefore, no professors are vain. not valid

44. No monkeys are soldiers.
All monkeys are mischievous.
Therefore, some mischievous creatures are not soldiers. valid

45. All lions are fierce.
Some lions do not drink coffee.
Therefore, some fierce creatures do not drink coffee. valid

46. All red hair is pretty.
No pretty things are valuable.
Therefore, no red hair is valuable. valid

Level 3

47. Refer to the lyrics of "By the Time I Get to Phoenix" on the next page. Tell whether each answer you give is arrived at inductively or deductively.

a. In what basic direction (north, south, east, or west) is the person traveling? east; deductive reasoning

b. What method of transportation or travel is the person using? car; deductive reasoning

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Dell Pencil Puzzles & Word Games (May 1979, issue #16, p. 6).