

EXPONENTS AND RADICALS

$$x^{m}x^{n} = x^{m+n}$$

$$(x^{m})^{n} = x^{mn}$$

$$(xy)^{n} = x^{n}y^{n}$$

$$x^{1/n} = \sqrt[n]{x}$$

$$\sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}$$

$$\sqrt[n]{x^{1/n}} = \sqrt[n]{x}\sqrt[n]{y}$$

$$\sqrt[n]{x^{1/n}} = \sqrt[n]{x}\sqrt[n]{x}$$

$$x^{m/n} = \sqrt[n]{x}\sqrt[n]{x}$$

SPECIAL PRODUCTS

$$(x + y)(x - y) = x^{2} - y^{2}$$

$$(x + y)^{2} = x^{2} + 2xy + y^{2}$$

$$(x - y)^{2} = x^{2} - 2xy + y^{2}$$

$$(x + y)^{3} = x^{3} + 3x^{2}y + 3xy^{2} + y^{3}$$

$$(x - y)^{3} = x^{3} - 3x^{2}y + 3xy^{2} - y^{3}$$

FACTORING FORMULAS

$$x^{2} - y^{2} = (x + y)(x - y)$$

$$x^{2} + 2xy + y^{2} = (x + y)^{2}$$

$$x^{2} - 2xy + y^{2} = (x - y)^{2}$$

$$x^{3} + y^{3} = (x + y)(x^{2} - xy + y^{2})$$

$$x^{3} - y^{3} = (x - y)(x^{2} + xy + y^{2})$$

QUADRATIC FORMULA

If $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

INEQUALITIES AND ABSOLUTE VALUE

If a < b and b < c, then a < c.

If a < b, then a + c < b + c.

If a < b and c > 0, then ca < cb.

If a < b and c < 0, then ca > cb.

If a > 0, then

$$|x| = a$$
 means $x = a$ or $x = -a$.

|x| < a means -a < x < a.

|x| > a means x > a or x < -a.

DISTANCE AND MIDPOINT FORMULAS

Distance between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint of
$$P_1P_2$$
: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

LINES

 $m = \frac{y_2 - y_1}{x_2 - x_1}$ Slope of line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$

Point-slope equation of line $y - y_1 = m(x - x_1)$ through $P_1(x_1, y_1)$ with slope m

Slope-intercept equation of y = mx + b

line with slope m and y-intercept b

Two-intercept equation of line $\frac{x}{a} + \frac{y}{b} = 1$ with x-intercept a and y-intercept b

The lines $y = m_1 x + b_1$ and $y = m_2 x + b_2$ are

Parallel if the slopes are the same

Perpendicular if the slopes are $m_1 = \frac{-1}{m_2}$ negative reciprocals

LOGARITHMS

 $y = \log_a x$ means $a^y = x$

 $a^{\log_a x} = x$ $\log_a a^x = x$

 $\log_a 1 = 0$ $\log_a a = 1$

Common and natural logarithms

 $\log x = \log_{10} x$ $\ln x = \log_e x$

Laws of logarithms

 $\log_a xy = \log_a x + \log_a y$

 $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

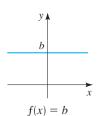
 $\log_a x^b = b \log_a x$

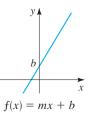
Change of base formula

 $\log_b x = \frac{\log_a x}{\log_a h}$

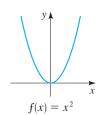
GRAPHS OF FUNCTIONS

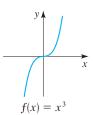
Linear functions: f(x) = mx + b

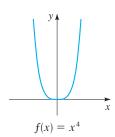


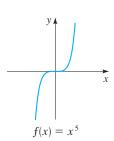


Power functions: $f(x) = x^n$

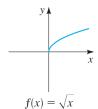


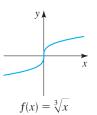




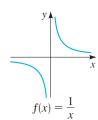


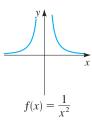
Root functions: $f(x) = \sqrt[n]{x}$



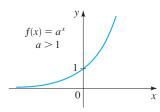


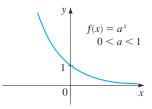
Reciprocal functions: $f(x) = 1/x^n$



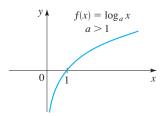


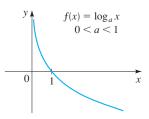
Exponential functions: $f(x) = a^x$





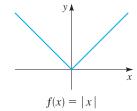
Logarithmic functions: $f(x) = \log_a x$

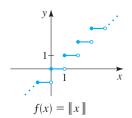




Absolute value function

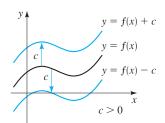




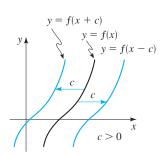


SHIFTING OF FUNCTIONS

Vertical shifting



Horizontal shifting



SEVENTH EDITION

COLLEGE ALGEBRA

ABOUT THE AUTHORS

JAMES STEWART received his MS from Stanford University and his PhD from the University of Toronto. He did research at the University of London and was influenced by the famous mathematician George Polya at Stanford University. Stewart is Professor Emeritus at McMaster University and is currently Professor of Mathematics at the University of Toronto. His research field is harmonic analysis and the connections between mathematics and music. James Stewart is the author of a bestselling calculus textbook series published by Cengage Learning, including Calculus, Calculus: Early Transcendentals, and Calculus: Concepts and Contexts; a series of precalculus texts; and a series of highschool mathematics textbooks.

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Stewart, Redlin, and Watson have also published *Precalculus, Trigonometry, Algebra and Trigonometry,* and (with Phyllis Panman) *College Algebra: Concepts and Contexts*.

ABOUT THE COVER

The cover photograph shows the Telefónica Tower in Barcelona, Spain, designed by the Spanish architect Santiago Calatrava. The tower, popularly known as Torre Telefónica or Torre Calatrava, is a telecommunications tower in the Muntjuïc neighborhood of Barcelona. Calatrava has always been very interested in how mathematics can help him realize the buildings he imagines. As a young student, he taught himself descriptive geometry from

books in order to represent three-dimensional objects in two dimensions. Trained as both an engineer and an architect, he wrote a doctoral thesis in 1981 entitled "On the Foldability of Space Frames," which is filled with mathematics, especially geometric transformations. His strength as an engineer enables him to be daring in his architecture.

SEVENTH EDITION

COLLEGE ALGEBRA

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With the assistance of Phyllis Panman



Australia • Brazil • Japan • Korea • Mexico • Singapore • Spain • United Kingdom • United States





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CONTENTS

	FOCUS ON MODELING Fitting Lines to Data 174	
	Chapter 1 Test 172	
1.10	Chapter 1 Review 167	
1.10	Modeling Variation 159	
1.8	Solving Absolute Value Equations and Inequalities 150 Solving Equations and Inequalities Graphically 154	
1.7 1.8	Solving Absolute Value Equations and Inequalities, 150	
1.6	Solving Other Types of Equations 132	
1.5	Complex Numbers 126	
1.4	Solving Quadratic Equations 115	
1.3	Lines 104	
1.2	Graphs of Equations in Two Variables; Circles 94	
1.1	Chapter Overview 87 The Coordinate Plane 88	
CHAPTER 1	EQUATIONS AND GRAPHS	87
	FOCUS ON MODELING Making the Best Decisions 81	
	Chapter P Test 79	
	Chapter P Review 74	
P.9	Modeling with Equations 61	
P.8	Solving Basic Equations 53	
P.7	Rational Expressions 44	
P.6	Factoring 37	
P.5	Algebraic Expressions 32	
P.4	Rational Exponents and Radicals 25	
P.2 P.3	Integer Exponents and Scientific Notation 18	
P.1 P.2	Modeling the Real World with Algebra 2 Real Numbers 6	
	Chapter Overview 1	
CHAPTER P	Prerequisites	1
	PROLOGUE: PRINCIPLES OF PROBLEM SOLVING P1	
	ARE YOU READY FOR COLLEGE ALGEBRA? xvii	
	TO THE STUDENT xvi	
	PREFACE ix	

CHAPTER 2	Functions	183
	Chapter Overview 183	
2.1	Functions 184	
2.2	Graphs of Functions 195	
2.3	Getting Information from the Graph of a Function 206	
2.4	Average Rate of Change of a Function 219	
2.5	Linear Functions and Models 226	
2.6	Transformations of Functions 234	
2.7	Combining Functions 246	
2.8	One-to-One Functions and Their Inverses 255	
	Chapter 2 Review 265	
	Chapter 2 Test 271	
	FOCUS ON MODELING Modeling with Functions 273	
CHAPTER 3	POLYNOMIAL AND RATIONAL FUNCTIONS	281
	Chapter Overview 281	
3.1	Quadratic Functions and Models 282	
3.2	Polynomial Functions and Their Graphs 290	
3.3	Dividing Polynomials 305	
3.4	Real Zeros of Polynomials 311	
3.5	Complex Zeros and the Fundamental Theorem of Algebra 323	
3.6	Rational Functions 331	
3.7	Polynomial and Rational Inequalities 347	
	Chapter 3 Review 353	
	Chapter 3 Test 359	
	FOCUS ON MODELING Fitting Polynomial Curves to Data 361	
CHAPTER 4	EXPONENTIAL AND LOGARITHMIC FUNCTIONS	365
	Chapter Overview 365	
4.1	Exponential Functions 366	
4.2	The Natural Exponential Function 374	
4.3	Logarithmic Functions 380	
4.4	Laws of Logarithms 390	
4.5	Exponential and Logarithmic Equations 396	
4.6	Modeling with Exponential Functions 406	
4.7	Logarithmic Scales 417	
	Chapter 4 Review 422	
	Chapter 4 Test 427	
	FOCUS ON MODELING Fitting Exponential and Power Curves to Data	/2 <u>Q</u>
-		720
	Cumulative Review Test: Chapters 2, 3, and 4 (Website)	

CHAPTER 5	Systems of Equations and Inequalities	437
	Chapter Overview 437	
5.1	Systems of Linear Equations in Two Variables 438	
5.2	Systems of Linear Equations in Several Variables 448	
5.3	Partial Fractions 457	
5.4	Systems of Nonlinear Equations 462	
5.5		
	Chapter 5 Review 477	
	Chapter 5 Test 481	
	FOCUS ON MODELING Linear Programming 482	
CHAPTER 6	Matrices and Determinants	489
	Chapter Overview 489	
6.1	Matrices and Systems of Linear Equations 490	
6.2	The Algebra of Matrices 503	
6.3	Inverses of Matrices and Matrix Equations 515	
6.4	Determinants and Cramer's Rule 525	
	Chapter 6 Review 536	
	Chapter 6 Test 541	
	FOCUS ON MODELING Computer Graphics 542	
CHAPTER 7	CONIC SECTIONS	547
	Chapter Overview 547	
7.1	Parabolas 548	
7.2	Ellipses 556	
7.3	Hyperbolas 565	
7.4	Shifted Conics 573	
	Chapter 7 Review 582	
	Chapter 7 Test 586	
	FOCUS ON MODELING Conics in Architecture 587	
_	Cumulative Review Test: Chapters 5, 6, and 7 (Website)	
	Cumulative neview lest. Chapters 3, 0, and 7 (Website)	
CHAPTER 8	SEQUENCES AND SERIES	591
	Chapter Overview 591	
8.1	Sequences and Summation Notation 592	
8.2	Arithmetic Sequences 603	
8.3	Geometric Sequences 608	
8.4	Mathematics of Finance 617	
8.5	Mathematical Induction 623	
8.6	The Binomial Theorem 629	
	Chapter 8 Review 637	
	Chapter 8 Test 642	
	FOCUS ON MODELING Modeling with Recursive Sequences 643	

CHAPTER 9 COUNTING AND PROBABILITY

647

Chapter Overview 647

- **9.1** Counting 648
- 9.2 Probability 660
- 9.3 Binomial Probability 672
- 9.4 Expected Value 677 Chapter 9 Review 681 Chapter 9 Test 686
 - FOCUS ON MODELING The Monte Carlo Method 687 Cumulative Review Test: Chapters 8 and 9 (Website)

APPENDIX A Calculations and Significant Figures 691 APPENDIX B Graphing with a Graphing Calculator 693 APPENDIX C Using the TI-83/84 Graphing Calculator 699

ANSWERS A1 INDEX 11

PREFACE

For many students a College Algebra course represents the first opportunity to discover the beauty and practical power of mathematics. Thus instructors are faced with the challenge of teaching the concepts and skills of algebra while at the same time imparting a sense of its utility in the real world. In this edition, as in the previous editions, our aim is to provide instructors and students with tools they can use to meet this challenge.

The emphasis is on understanding concepts. To help instructors foster conceptual understanding in their students, we use the rule of four: "Topics should be presented geometrically, numerically, algebraically, and verbally." A major part of conceptual understanding is an appreciation for the logical structure of algebra. For this reason every statement about algebra in this book is supported by a proof, an explanation, or an intuitive argument. Students also need to achieve a certain level of technical skill. Indeed, *conceptual understanding* and *technical skill* go hand in hand, each reinforcing the other. Above all, we present algebra as a problem-solving art with numerous applications to modeling and solving real-world problems.

In this Seventh Edition our objective is to further enhance the effectiveness of the book as an instructional tool for teachers and as a learning tool for students. Many of the changes in this edition are a result of suggestions we received from instructors and students who are using the current edition; others are a result of insights we have gained from our own teaching. Some chapters have been reorganized and rewritten, new sections have been added (as described below), the review material at the end of each chapter has been substantially expanded, and exercise sets have been enhanced to further focus on the main concepts of college algebra. In all these changes and numerous others (small and large) we have retained the main features that have contributed to the success of this book.

New to the Seventh Edition

- Exercises More than 20% of the exercises are new, and groups of exercises now have headings that identify the type of exercise. New Skills Plus exercises in most sections contain more challenging exercises that require students to extend and synthesize concepts.
- Review Material The review material at the end of each chapter now includes a new *Concept Check*. Each *Concept Check* provides a step-by-step review of all the main concepts and applications of the chapter. Answers to the *Concept Check* questions are on tear-out sheets at the back of the book.
- **Discovery Projects** References to *Discovery Projects*, including brief descriptions of the content of each project, are located in boxes where appropriate in each chapter. These boxes highlight the applications of college algebra in many different real-world contexts. (The projects are located at the book companion website: www.stewartmath.com.)
- CHAPTER P Prerequisites This chapter begins with a preview of how algebra can model real-world situations. The chapter now concludes with the new Section P.9, "Modeling with Equations." This section shows how the properties and equations studied in the preceding sections can be used to solve real-world problems.
- CHAPTER 1 Equations and Graphs This chapter has been reorganized so that the material on graphs of equations (Sections 1.1–1.3) is followed by several sections on solving different types of equations and inequalities algebraically (Sections 1.4–1.8), which is then followed by a section on solving equations and inequalities graphically (Section 1.9). Finally, we use the methods of the preceding sections to model variation (Section 1.10).

- CHAPTER 2 Functions This chapter now includes the new Section 2.5, "Linear Functions and Models." This section highlights the connection between the slope of a line and the rate of change of a linear function. These two interpretations of slope help prepare students for the concept of the derivative in calculus.
- CHAPTER 3 Polynomial and Rational Functions This chapter now includes the new Section 3.7, "Polynomial and Rational Inequalities." Section 3.6, "Rational Functions," has a new subsection on rational functions with "holes." The sections on complex numbers and on variation have been moved to Chapter 1.
- CHAPTER 4 Exponential and Logarithmic Functions This chapter now includes two sections on the applications of these functions. Section 4.6, "Modeling with Exponential Functions," focuses on modeling growth and decay, Newton's Law of Cooling, and other such applications. Section 4.7, "Logarithmic Scales," covers the concept of a logarithmic scale with applications involving the pH, Richter, and decibel scales.
- CHAPTER 5 Systems of Equations and Inequalities The material on systems of inequalities has been rewritten to emphasize the steps used in graphing the solution of a system of inequalities.

Teaching with the Help of This Book

We are keenly aware that good teaching comes in many forms and that there are many different approaches to teaching and learning the concepts and skills of college algebra. The organization and exposition of the topics in this book are designed to accommodate different teaching and learning styles. In particular, each topic is presented algebraically, graphically, numerically, and verbally, with emphasis on the relationships between these different representations. The following are some special features that can be used to complement different teaching and learning styles:

Diagnostic Test For a student to achieve success in any mathematics course, it is important that he or she has the necessary prerequisite knowledge. For this reason we have included four *Diagnostic Tests* at the beginning of the book (pages xvii–xx) to test preparedness for college algebra.

Exercise Sets The most important way to foster conceptual understanding and hone technical skill is through the problems that the instructor assigns. To that end we have provided a wide selection of exercises.

- Concept Exercises These exercises ask students to use mathematical language to state fundamental facts about the topics of each section.
- Skills Exercises These exercises reinforce and provide practice with all the learning objectives of each section. They comprise the core of each exercise set.
- Skills Plus Exercises The Skills Plus exercises contain challenging problems that often require the synthesis of previously learned material with new concepts.
- Applications Exercises We have included substantial applied problems from many different real-world contexts. We believe that these exercises will capture students' interest.
- Discovery, Writing, and Group Learning Each exercise set ends with a block of exercises labeled *Discuss* Discover Prove Write. These exercises are designed to encourage students to experiment, preferably in groups, with the concepts developed in the section and then to write about what they have learned rather than simply looking for the answer. New *Prove* exercises highlight the importance of deriving a formula.
- Now Try Exercise ... At the end of each example in the text the student is directed to one or more similar exercises in the section that help to reinforce the concepts and skills developed in that example.

• Check Your Answer Students are encouraged to check whether an answer they obtained is reasonable. This is emphasized throughout the text in numerous Check Your Answer sidebars that accompany the examples (see, for instance, pages 55, 69, and 135).

Graphing Calculators and Computers We make use of graphing calculators and computers in examples and exercises throughout the book. Our calculator-oriented examples are always preceded by examples in which students must graph or calculate by hand so that they can understand precisely what the calculator is doing when they later use it to simplify the routine, mechanical part of their work. The graphing calculator sections, subsections, examples, and exercises, all marked with the special symbol are optional and may be omitted without loss of continuity.

- Using a Graphing Calculator Appendices B and C contain general guidelines on using graphing calculators and a quick reference guide to using TI-83/84 calculators.
- Graphing, Regression, Matrix Algebra Graphing calculators are used throughout
 the text to graph and analyze functions, families of functions, and sequences; to
 calculate and graph regression curves; to perform matrix algebra; to graph linear
 inequalities; and other powerful uses.
- Simple Programs We exploit the programming capabilities of a graphing calculator to simulate real-life situations, to sum series, or to compute the terms of a recursive sequence (see, for instance, pages 544, 595, 599, and 689).

Focus on Modeling The theme of modeling has been used throughout to unify and clarify the many applications of college algebra. We have made a special effort to clarify the essential process of translating problems from English into the language of mathematics (see pages 274 and 444).

- Constructing Models There are many applied problems throughout the book in which students are given a model to analyze (see, for instance, page 286). But the material on modeling, in which students are required to *construct* mathematical models, has been organized into clearly defined sections and subsections (see, for instance, pages 273, 406, and 443).
- Focus on Modeling Each chapter concludes with a *Focus on Modeling* section. For example, the *Focus on Modeling* after Chapter 1 introduces the basic idea of modeling a real-life situation by fitting lines to data (linear regression). Other sections present ways in which polynomial, exponential, logarithmic, and trigonometric functions, and systems of inequalities can all be used to model familiar phenomena from the sciences and from everyday life (see, for instance, pages 361, 428, 482, and 643).

Review Sections and Chapter Tests Each chapter ends with an extensive review section that includes the following.

- **Properties and Formulas** The *Properties and Formulas* at the end of each chapter contains a summary of the main formulas and procedures of the chapter (see, for instance, pages 353 and 422).
- Concept Check and Concept Check Answers The Concept Check at the end of each chapter is designed to get the students to think about and explain each concept presented in the chapter and then to use the concept in a given problem. This provides a step-by-step review of all the main concepts in a chapter (see, for instance, pages 266, 355, and 538). Answers to the Concept Check questions are on tear-out sheets at the back of the book.
- Review Exercises The Review Exercises at the end of each chapter recapitulate the basic concepts and skills of the chapter and include exercises that combine the different ideas learned in the chapter.

- Chapter Test Each review section concludes with a Chapter Test designed to help students gauge their progress.
- Cumulative Review Tests Cumulative Review Tests following selected chapters are available at the book companion website. These tests contain problems that combine skills and concepts from the preceding chapters. The problems are designed to highlight the connections between the topics in these related chapters.
- Answers Brief answers to odd-numbered exercises in each section (including the review exercises) and to all questions in the Concepts exercises and Chapter Tests, are given in the back of the book.

Mathematical Vignettes Throughout the book we make use of the margins to provide historical notes, key insights, or applications of mathematics in the modern world. These serve to enliven the material and show that mathematics is an important, vital activity and that even at this elementary level it is fundamental to everyday life.

- Mathematical Vignettes These vignettes include biographies of interesting mathematicians and often include a key insight that the mathematician discovered (see, for instance, the vignettes on Viète, page 119; Salt Lake City, page 89; and radiocarbon dating, page 403).
- Mathematics in the Modern World This is a series of vignettes that emphasize the central role of mathematics in current advances in technology and the sciences (see, for instance, pages 136, 464, 550, and 604).

Book Companion Website A website that accompanies this book is located at www.stewartmath.com. The site includes many useful resources for teaching college algebra, including the following.

- Discovery Projects Discovery Projects for each chapter are available at the book companion website. The projects are referenced in the text in the appropriate sections. Each project provides a challenging yet accessible set of activities that enable students (perhaps working in groups) to explore in greater depth an interesting aspect of the topic they have just learned (see, for instance, the Discovery Projects Visualizing a Formula, Relations and Functions, and Will the Species Survive?, referenced on pages 34, 199, and 510).
- Focus on Problem Solving Several Focus on Problem Solving sections are available on the website. Each such section highlights one of the problem-solving principles introduced in the Prologue and includes several challenging problems (see for instance Recognizing Patterns, Using Analogy, Introducing Something Extra, Taking Cases, and Working Backward).
- Cumulative Review Tests Cumulative Review Tests following Chapters 4, 7, and 9 are available on the website.

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Ancillaries

Instructor Resources

Instructor Companion Site

Everything you need for your course in one place! This collection of book-specific lecture and class tools is available online via **www.cengage.com/login**. Access and download PowerPoint presentations, images, instructor's manual, and more.

Complete Solutions Manual

This manual contains solutions to all exercises from the text, including Chapter Review Exercises, and Chapter Tests. This manual can be found on the instructor companion website.

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This textbook was written for you to use as a guide to mastering college algebra. Here are some suggestions to help you get the most out of your course.

First of all, you should read the appropriate section of text *before* you attempt your homework problems. Reading a mathematics text is quite different from reading a novel, a newspaper, or even another textbook. You may find that you have to reread a passage several times before you understand it. Pay special attention to the examples, and work them out yourself with pencil and paper as you read. Then do the linked exercises referred to in "*Now Try Exercise*..." at the end of each example. With this kind of preparation you will be able to do your homework much more quickly and with more understanding.

Don't make the mistake of trying to memorize every single rule or fact you may come across. Mathematics doesn't consist simply of memorization. Mathematics is a *problem-solving art*, not just a collection of facts. To master the subject you must solve problems—lots of problems. Do as many of the exercises as you can. Be sure to write your solutions in a logical, step-by-step fashion. Don't give up on a problem if you can't solve it right away. Try to understand the problem more clearly—reread it thoughtfully and relate it to what you have learned from your teacher and from the examples in the text. Struggle with it until you solve it. Once you have done this a few times you will begin to understand what mathematics is really all about.

Answers to the odd-numbered exercises, as well as all the answers to the concept exercises and to each chapter test, appear at the back of the book. Solutions to all the *Concept Check* questions can be found on cut-out cards at the back of the book. If your answer differs from the one given, don't immediately assume that you are wrong. There may be a calculation that connects the two answers and makes both correct. For example, if you get $1/(\sqrt{2}-1)$ but the answer given is $1+\sqrt{2}$, your answer is correct, because you can multiply both numerator and denominator of your answer by $\sqrt{2}+1$ to change it to the given answer. In rounding approximate answers, follow the guidelines in Appendix A: *Calculations and Significant Figures*.

The symbol is used to warn against committing an error. We have placed this symbol in the margin to point out situations where we have found that many of our students make the same mistake.

Abbreviations

The following abbreviations are used throughout the text.

cm	centimeter	kPa	kilopascal	N	Newton
dB	decibel	L	liter	qt	quart
F	farad	lb	pound	OZ	ounce
ft	foot	lm	lumen	S	second
g	gram	\mathbf{M}	mole of solute	Ω	ohm
gal	gallon		per liter of	\mathbf{V}	volt
h	hour		solution	\mathbf{W}	watt
H	henry	m	meter	yd	yard
Hz	Hertz	mg	milligram	yr	year
in.	inch	MHz	megahertz	$^{\circ}\mathbf{C}$	degree Celsius
J	Joule	mi	mile	$^{\circ}\mathbf{F}$	degree Fahrenheit
kcal	kilocalorie	min	minute	K	Kelvin
kg	kilogram	mL	milliliter	\Rightarrow	implies
km	kilometer	mm	millimeter	\Leftrightarrow	is equivalent to

To succeed in your College Algebra course you need to use some of the skills that you learned in your previous mathematics classes. In particular, you need to be familiar with the real number system, algebraic expressions, solving basic equations, and graphing. The following diagnostic tests are designed to assess your knowledge of these topics. After taking each test you can check your answers using the answer key on page xx. If you have difficulty with any topic, you can refresh your skills by studying the review materials from Chapters P and 1 that are referenced after each test.

A DIAGNOSTIC TEST: Real Numbers and Exponents

1.	Perform the indicated operations.	Write your	final	answer	as	an	integer	or	as	a
	fraction in lowest terms.									

(a)
$$\frac{1}{3} + \frac{1}{2}$$
 (b) $2 - \frac{2}{3} + \frac{1}{4}$ (c) $4(2 - \frac{2}{3})$ (d) $\frac{12}{\frac{4}{3} + \frac{1}{6}}$

2. Determine whether the given number is an integer, rational, or irrational.

(a) 10 (b)
$$\frac{16}{3}$$
 (c) 5^2 (d) $\sqrt{5}$ 3. Is the inequality true or false?

(a)
$$-2 < 0$$
 (b) $5 \ge 5$ (c) $5 > 5$
(d) $3 \le -10$ (e) $-2 > -6$

(a)
$$-1 < x \le 5$$
 (b) $x < 3$ (c) $x \ge 4$

5. Express the interval using inequalities.
(a)
$$(2, \infty)$$
 (b) $[-3, -1]$ **(c)** $[0, 9)$

(a)
$$(-3)^4$$
 (b) -3^4 (c) 3^{-4} (d) $\frac{5^{12}}{5^{10}}$ (e) $\left(\frac{3}{4}\right)^{-2}$ (f) $16^{3/4}$

7. Simplify the expression. Write your final answer without negative exponents.

(a)
$$(4x^2y^3)(2xy^2)$$
 (b) $\left(\frac{5a^{1/2}}{a^2}\right)^2$ (c) $(x^{-2}y^{-3})(xy^2)^2$

Answers to Test A are on page xx. If you had difficulty with any of the questions on Test A, you should review the material covered in Sections P.2, P.3, and P.4.

B DIAGNOSTIC TEST: Algebraic Expressions

1. Expand and simplify.

(a)
$$4(x+3) + 5(2x-1)$$
 (b) $(x+3)(x-5)$ (c) $(2x-1)(3x+2)$

(d)
$$(a-2b)(a+2b)$$
 (e) $(y-3)^2$ (f) $(2x+5)^2$

xvii

- **2.** Factor the expression.

- (a) $4x^2 + 2x$ (b) $3xy^2 6x^2y$ (c) $x^2 + 8x + 15$ (d) $x^2 x 2$ (e) $2x^2 + 5x 12$ (f) $x^2 16$
- **3.** Simplify the rational expression.

 - (a) $\frac{x^2 + 4x + 3}{x^2 2x 3}$ (b) $\frac{2x^2 3x 2}{x^2 1} \cdot \frac{x + 1}{2x + 1}$
 - (c) $\frac{x^2 x}{x^2 9} \frac{x + 1}{x + 3}$ (d) $\frac{\frac{1}{x} \frac{1}{y}}{\frac{2}{x^2}}$
- 4. Rationalize the denominator and simplify.

 - (a) $\frac{\sqrt{3}}{\sqrt{7}}$ (b) $\frac{12}{3-\sqrt{5}}$

Answers to Test B are on page xx. If you had difficulty with any of the questions on Test B, you should review the material covered in Sections P.5, P.6, and P.7.

C DIAGNOSTIC TEST: Equations

- 1. Solve the linear equation.
 - (a) 3x 1 = 5 (b) 2x + 3 = 8

 - (c) 2x = 5x + 6 (d) x + 11 = 6 4x
- **2.** Solve the equation.

 - (a) $\frac{1}{3}x = 6$ (b) $\frac{1}{2}x \frac{3}{2} = \frac{7}{2}$
- 3. Find all real solutions of the equation.

 - **(a)** $x^2 7 = 0$ **(b)** $x^3 + 8 = 0$
 - (c) $2x^3 54 = 0$ (d) $x^4 16 = 0$
- 4. Solve the equation for the indicated variable.
 - (a) 4x + y = 108, for x (b) $8 = \frac{mn}{t^2}$, for m

Answers to Test C are on page xx. If you had difficulty with any of the questions on Test C, you should review the material covered in Section P.8.

DIAGNOSTIC TEST: The Coordinate Plane

- 1. Graph the following points in a coordinate plane.
 - (a) (2, 4)
- (b) (-1,3) (c) (3,-1)(e) (5,0) (f) (0,-1)
- **(d)** (0, 0)

- 2. Find the distance between the given pair of points.
 - (a) (1, 3), (5, 6)
- **(b)** (-2,0), (3,12) **(c)** (0,-4), (4,0)
- **3.** Find the midpoint of the segment PQ.
 - (a) P(3,7), Q(5,13)
- **(b)** P(-2,3), Q(8,-7)
- 4. Graph the equation in a coordinate plane by plotting points.
 - (a) y = x + 2
- **(b)** $y = 4 x^2$

Answers to Test D are on page xx. If you had difficulty with any of the questions on Test D, you should review the material covered in Sections 1.1 and 1.2.

A Answers

- **1.** (a) $\frac{5}{6}$ (b) $\frac{19}{12}$ (c) $\frac{16}{3}$ (d) 8 **2.** (a) Integer and rational (b) Rational
- (c) Integer and rational (d) Irrational 3. (a) True (b) True (c) False
- (d) False (e) True 4. (a) (-1, 5] (b) $(-\infty, 3)$ (c) $[4, \infty)$

ANSWERS TO DIAGNOSTIC TESTS

- **5.** (a) x > 2 (b) $-3 \le x \le -1$ (c) $0 \le x < 9$ **6.** (a) 81 (b) -81
- (c) $\frac{1}{81}$ (d) 25 (e) $\frac{16}{9}$ (f) 8 7. (a) $8x^3y^5$ (b) $\frac{25}{a^3}$ (c) y

B Answers

- **1.** (a) 14x + 7 (b) $x^2 2x 15$ (c) $6x^2 + x 2$ (d) $a^2 4b^2$
- (e) $y^2 6y + 9$ (f) $4x^2 + 20x + 25$ 2. (a) 2x(2x + 1)
- **(b)** 3xy(y-2x) **(c)** (x+3)(x+5) **(d)** (x-2)(x+1)
- (e) (2x-3)(x+4) (f) (x-4)(x+4) 3. (a) $\frac{x+3}{x-3}$ (b) $\frac{x-2}{x-1}$
- (c) $\frac{1}{x-3}$ (d) $\frac{y-x}{2}$ 4. (a) $\frac{\sqrt{21}}{7}$ (b) $9+3\sqrt{5}$

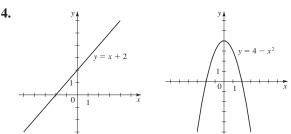
C Answers

- **1.** (a) 2 (b) $\frac{5}{2}$ (c) -2 (d) -1 **2.** (a) 18 (b) 10 **3.** (a) $-\sqrt{7}$, $\sqrt{7}$
- **(b)** -2 **(c)** 3 **(d)** -2, 2 **4. (a)** $x = 27 \frac{1}{4}y$ **(b)** $m = \frac{8k^2}{n}$

D Answers

1. • (2, 4)

2. (a) 5 (b) 13 (c) $4\sqrt{2} \approx 5.66$ **3.** (a) (4, 10) (b) (3, -2)



PROLOGUE

PRINCIPLES OF PROBLEM SOLVING



GEORGE POLYA (1887-1985) is famous among mathematicians for his ideas on problem solving. His lectures on problem solving at Stanford University attracted overflow crowds whom he held on the edges of their seats, leading them to discover solutions for themselves. He was able to do this because of his deep insight into the psychology of problem solving. His well-known book How To Solve It has been translated into 15 languages. He said that Euler (see page 63) was unique among great mathematicians because he explained how he found his results. Polya often said to his students and colleagues, "Yes, I see that your proof is correct, but how did you discover it?" In the preface to How To Solve It, Polya writes, "A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery."

The ability to solve problems is a highly prized skill in many aspects of our lives; it is certainly an important part of any mathematics course. There are no hard and fast rules that will ensure success in solving problems. However, in this Prologue we outline some general steps in the problem-solving process and we give principles that are useful in solving certain types of problems. These steps and principles are just common sense made explicit. They have been adapted from George Polya's insightful book *How To Solve It*.

1. Understand the Problem

The first step is to read the problem and make sure that you understand it. Ask yourself the following questions:

What is the unknown?
What are the given quantities?
What are the given conditions?

For many problems it is useful to

draw a diagram

and identify the given and required quantities on the diagram. Usually, it is necessary to

introduce suitable notation

In choosing symbols for the unknown quantities, we often use letters such as a, b, c, m, n, x, and y, but in some cases it helps to use initials as suggestive symbols, for instance, V for volume or t for time.

2. Think of a Plan

Find a connection between the given information and the unknown that enables you to calculate the unknown. It often helps to ask yourself explicitly: "How can I relate the given to the unknown?" If you don't see a connection immediately, the following ideas may be helpful in devising a plan.

■ Try to Recognize Something Familiar

Relate the given situation to previous knowledge. Look at the unknown and try to recall a more familiar problem that has a similar unknown.

■ Try to Recognize Patterns

Certain problems are solved by recognizing that some kind of pattern is occurring. The pattern could be geometric, numerical, or algebraic. If you can see regularity or repetition in a problem, then you might be able to guess what the pattern is and then prove it.

■ Use Analogy

Try to think of an analogous problem, that is, a similar or related problem but one that is easier than the original. If you can solve the similar, simpler problem, then it might give you the clues you need to solve the original, more difficult one. For instance, if a problem involves very large numbers, you could first try a similar problem with smaller numbers. Or if the problem is in three-dimensional geometry, you could look for something similar in two-dimensional geometry. Or if the problem you start with is a general one, you could first try a special case.

■ Introduce Something Extra

You might sometimes need to introduce something new—an auxiliary aid—to make the connection between the given and the unknown. For instance, in a problem for which a diagram is useful, the auxiliary aid could be a new line drawn in the diagram. In a more algebraic problem the aid could be a new unknown that relates to the original unknown.

Take Cases

You might sometimes have to split a problem into several cases and give a different argument for each case. For instance, we often have to use this strategy in dealing with absolute value.

■ Work Backward

Sometimes it is useful to imagine that your problem is solved and work backward, step by step, until you arrive at the given data. Then you might be able to reverse your steps and thereby construct a solution to the original problem. This procedure is commonly used in solving equations. For instance, in solving the equation 3x - 5 = 7, we suppose that x is a number that satisfies 3x - 5 = 7 and work backward. We add 5 to each side of the equation and then divide each side by 3 to get x = 4. Since each of these steps can be reversed, we have solved the problem.

■ Establish Subgoals

In a complex problem it is often useful to set subgoals (in which the desired situation is only partially fulfilled). If you can attain or accomplish these subgoals, then you might be able to build on them to reach your final goal.

Indirect Reasoning

Sometimes it is appropriate to attack a problem indirectly. In using **proof by contradiction** to prove that P implies Q, we assume that P is true and Q is false and try to see why this cannot happen. Somehow we have to use this information and arrive at a contradiction to what we absolutely know is true.

Mathematical Induction

In proving statements that involve a positive integer n, it is frequently helpful to use the Principle of Mathematical Induction, which is discussed in Section 8.5.

3. Carry Out the Plan

In Step 2, a plan was devised. In carrying out that plan, you must check each stage of the plan and write the details that prove that each stage is correct.

4. Look Back

Having completed your solution, it is wise to look back over it, partly to see whether any errors have been made and partly to see whether you can discover an easier way to solve the problem. Looking back also familiarizes you with the method of solution, which may be useful for solving a future problem. Descartes said, "Every problem that I solved became a rule which served afterwards to solve other problems."

We illustrate some of these principles of problem solving with an example.

PROBLEM ■ Average Speed

A driver sets out on a journey. For the first half of the distance, she drives at the leisurely pace of 30 mi/h; during the second half she drives 60 mi/h. What is her average speed on this trip?

It is tempting to take the average of the speeds and say that the average speed for the entire trip is

$$\frac{30 + 60}{2} = 45 \text{ mi/h}$$

But is this simple-minded approach really correct?

Let's look at an easily calculated special case. Suppose that the total distance traveled is 120 mi. Since the first 60 mi is traveled at 30 mi/h, it takes 2 h. The second 60 mi is traveled at 60 mi/h, so it takes one hour. Thus, the total time is 2 + 1 = 3 hours and the average speed is

$$\frac{120}{3} = 40 \text{ mi/h}$$

So our guess of 45 mi/h was wrong.

SOLUTION

Understand the problem. We need to look more carefully at the meaning of average speed. It is defined as

average speed =
$$\frac{\text{distance traveled}}{\text{time elapsed}}$$

Introduce notation. Let d be the distance traveled on each half of the trip. Let t_1 and t_2 be the times taken

for the first and second halves of the trip. Now we can write down the information we have been given. For the first half of the trip we have

$$30 = \frac{d}{t_1}$$

and for the second half we have

$$60 = \frac{d}{t_2}$$

Identify the unknown. Now we identify the quantity that we are asked to find:

average speed for entire trip =
$$\frac{\text{total distance}}{\text{total time}} = \frac{2d}{t_1 + t_2}$$

To calculate this quantity, we need to know t_1 and t_2 , so we solve the above equations Connect the given for these times: with the unknown.

$$t_1 = \frac{d}{30}$$
 $t_2 = \frac{d}{60}$

Now we have the ingredients needed to calculate the desired quantity:

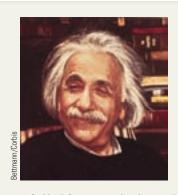
average speed =
$$\frac{2d}{t_1 + t_2} = \frac{2d}{\frac{d}{30} + \frac{d}{60}}$$

$$= \frac{60(2d)}{60\left(\frac{d}{30} + \frac{d}{60}\right)}$$
Multiply numerator and denominator by 60
$$= \frac{120d}{2d + d} = \frac{120d}{3d} = 40$$

So the average speed for the entire trip is 40 mi/h.

State what is given.

Try a special case.



Don't feel bad if you can't solve these problems right away. Problems 1 and 4 were sent to Albert Einstein by his friend Wertheimer. Einstein (and his friend Bucky) enjoyed the problems and wrote back to Wertheimer. Here is part of his reply:

Your letter gave us a lot of amusement. The first intelligence test fooled both of us (Bucky and me). Only on working it out did I notice that no time is available for the downhill run! Mr. Bucky was also taken in by the second example, but I was not. Such drolleries show us how stupid we are!

(See *Mathematical Intelligencer*, Spring 1990, page 41.)



PROBLEMS

- 1. Distance, Time, and Speed An old car has to travel a 2-mile route, uphill and down. Because it is so old, the car can climb the first mile—the ascent—no faster than an average speed of 15 mi/h. How fast does the car have to travel the second mile—on the descent it can go faster, of course—to achieve an average speed of 30 mi/h for the trip?
- **2. Comparing Discounts** Which price is better for the buyer, a 40% discount or two successive discounts of 20%?
- **3. Cutting up a Wire** A piece of wire is bent as shown in the figure. You can see that one cut through the wire produces four pieces and two parallel cuts produce seven pieces. How many pieces will be produced by 142 parallel cuts? Write a formula for the number of pieces produced by *n* parallel cuts.



- **4. Amoeba Propagation** An amoeba propagates by simple division; each split takes 3 minutes to complete. When such an amoeba is put into a glass container with a nutrient fluid, the container is full of amoebas in one hour. How long would it take for the container to be filled if we start with not one amoeba, but two?
- **5. Batting Averages** Player A has a higher batting average than player B for the first half of the baseball season. Player A also has a higher batting average than player B for the second half of the season. Is it necessarily true that player A has a higher batting average than player B for the entire season?
- **6. Coffee and Cream** A spoonful of cream is taken from a pitcher of cream and put into a cup of coffee. The coffee is stirred. Then a spoonful of this mixture is put into the pitcher of cream. Is there now more cream in the coffee cup or more coffee in the pitcher of cream?
- **7. Wrapping the World** A ribbon is tied tightly around the earth at the equator. How much more ribbon would you need if you raised the ribbon 1 ft above the equator everywhere? (You don't need to know the radius of the earth to solve this problem.)
- **8. Ending Up Where You Started** A woman starts at a point *P* on the earth's surface and walks 1 mi south, then 1 mi east, then 1 mi north, and finds herself back at *P*, the starting point. Describe all points *P* for which this is possible. [*Hint:* There are infinitely many such points, all but one of which lie in Antarctica.]

Many more problems and examples that highlight different problem-solving principles are available at the book companion website: www.stewartmath.com. You can try them as you progress through the book.



Andersen Ross/Blend Images/Alamy

Prerequisites

- P.1 Modeling the Real World with Algebra
- P.2 Real Numbers
- P.3 Integer Exponents and Scientific Notation
- P.4 Rational Exponents and Radicals
- P.5 Algebraic Expressions
- P.6 Factoring
- P.7 Rational Expressions
- P.8 Solving Basic Equations
- P.9 Modeling with Equations

FOCUS ON MODELING

Making the Best Decisions

In this chapter we begin by taking a look at the central reason for studying algebra: its usefulness in describing (or modeling) real-world situations.

In algebra we use letters to stand for numbers. This allows us to write equations and solve problems. Of course, the letters in our equations must obey the same rules that numbers do. So in this chapter we review properties of numbers and algebraic expressions. You are probably already familiar with many of these properties, but it is helpful to get a fresh look at how these properties work together to solve real-world problems.

In the *Focus on Modeling* at the end of the chapter we see how equations can help us make the best decisions in some everyday situations. This theme of using algebra to model real-world situations is further developed throughout the textbook.

MODELING THE REAL WORLD WITH ALGEBRA

Using Algebra Models Making Algebra Models

In algebra we use letters to stand for numbers. This allows us to describe patterns that we see in the real world.

For example, if we let N stand for the number of hours you work and let W stand for your hourly wage, then the formula

$$P = NW$$

gives your pay P. The formula P = NW is a description or model for pay. We can also call this formula an algebra model. We summarize the situation as follows:

Real World Algebra Model

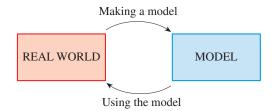
You work for an hourly wage. You would like to know your pay for any number of hours worked.

P = NW

The model P = NW gives the pattern for finding the pay for any worker, with any hourly wage, working any number of hours. That's the power of algebra: By using letters to stand for numbers, we can write a single formula that describes many different situations.

We can now use the model P = NW to answer questions such as "I make \$10 an hour, and I worked 35 hours; how much do I get paid?" or "I make \$8 an hour; how many hours do I need to work to get paid \$1000?"

In general, a **model** is a mathematical representation (such as a formula) of a realworld situation. **Modeling** is the process of making mathematical models. Once a model has been made, it can be used to answer questions about the thing being modeled.



The examples we study in this section are simple, but the methods are far reaching. This will become more apparent as we explore the applications of algebra in subsequent Focus on Modeling sections that follow each chapter.

Using Algebra Models

We begin our study of modeling by using models that are given to us. In the next subsection we learn how to make our own models.

EXAMPLE 1 Using a Model for Pay

Aaron makes \$9 an hour at his part-time job. Use the model P = NW to answer the following questions:

- (a) Aaron worked 35 hours last week. How much did he get paid?
- (b) Aaron wants to earn enough money to buy a calculus text that costs \$126. How many hours does he need to work to earn this amount?

SOLUTION

(a) We know that N=35 h and W=\$9. To find P, we substitute these values into the model.

$$P = NW$$
 Model
= 35×9 Substitute $N = 35$, $W = 9$
= 315 Calculate

So Aaron was paid \$315.

(b) Aaron's hourly wage is W = \$9, and the amount of pay he needs to buy the book is P = \$126. To find N, we substitute these values into the model.

$$P = NW$$
 Model
 $126 = 9N$ Substitute $P = 126$, $W = 9$
 $\frac{126}{9} = N$ Divide by 9
 $N = 14$ Calculate

So Aaron must work 14 hours to buy this book.



EXAMPLE 2 Using an Elevation-Temperature Model

A mountain climber uses the model

$$T = 20 - 10h$$

to estimate the temperature T (in $^{\circ}$ C) at elevation h (in kilometers, km).

- (a) Make a table that gives the temperature for each 1-km change in elevation, from elevation 0 km to elevation 5 km. How does temperature change as elevation increases?
- **(b)** If the temperature is 5°C, what is the elevation?

SOLUTION

(a) Let's use the model to find the temperature at elevation h = 3 km.

$$T = 20 - 10h$$
 Model
= 20 - 10(3) Substitute $h = 3$
= -10 Calculate

So at an elevation of 3 km the temperature is -10° C. The other entries in the following table are calculated similarly.

Temperature (°C)
20°
10°
0°
-10°
-20°
-30°

We see that temperature decreases as elevation increases.



$$T = 20 - 10h$$
 Model
 $5 = 20 - 10h$ Substitute $T = 5$
 $-15 = -10h$ Subtract 20
 $\frac{-15}{-10} = h$ Divide by -10
 $1.5 = h$ Calculator

The elevation is 1.5 km.



Making Algebra Models

In the next example we explore the process of making an algebra model for a real-life situation.

12 mi/gal

40 mi/gal

EXAMPLE 3 Making a Model for Gas Mileage

The gas mileage of a car is the number of miles it can travel on one gallon of gas.

- (a) Find a formula that models gas mileage in terms of the number of miles driven and the number of gallons of gasoline used.
- (b) Henry's car used 10.5 gal to drive 230 mi. Find its gas mileage.

THINKING ABOUT THE PROBLEM

Let's try a simple case. If a car uses 2 gal to drive 100 mi, we easily see that

gas mileage =
$$\frac{100}{2}$$
 = 50 mi/gal

So gas mileage is the number of miles driven divided by the number of gallons used.

SOLUTION

(a) To find the formula we want, we need to assign symbols to the quantities involved.

In Words	In Algebra
Number of miles driven	N
Number of gallons used	G
Gas mileage (mi/gal)	M

We can express the model as follows:

gas mileage =
$$\frac{\text{number of miles driven}}{\text{number of gallons used}}$$
$$M = \frac{N}{G} \qquad \text{Model}$$

(b) To get the gas mileage, we substitute N = 230 and G = 10.5 in the formula.

$$M = \frac{N}{G}$$
 Model
 $= \frac{230}{10.5}$ Substitute $N = 230$, $G = 10.5$
 ≈ 21.9 Calculator

The gas mileage for Henry's car is about 21.9 mi/gal.

Now Try Exercise 19

P.1 EXERCISES

CONCEPTS

1. The model L = 4S gives the total number of legs that S sheep have. Using this model, we find that 12 sheep have

L =_____legs.

2. Suppose gas costs \$3.50 a gallon. We make a model for the cost C of buying x gallons of gas by writing the formula

 $C = \underline{\hspace{1cm}}$.

SKILLS

3–12 ■ Using Models Use the model given to answer the questions about the object or process being modeled.

3. The sales tax T in a certain county is modeled by the formula T = 0.06x. Find the sales tax on an item whose price is \$120.

- **4.** Mintonville School District residents pay a wage tax T that is modeled by the formula T = 0.005x. Find the wage tax paid by a resident who earns \$62,000 per year.
- 5. The distance d (in mi) driven by a car traveling at a speed of v miles per hour for t hours is given by

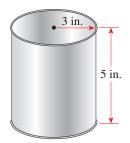
$$d = vt$$

If the car is driven at 70 mi/h for 3.5 h, how far has it traveled?

6. The volume *V* of a cylindrical can is modeled by the formula

$$V = \pi r^2 h$$

where r is the radius and h is the height of the can. Find the volume of a can with radius 3 in. and height 5 in.



 \sim 7. The gas mileage M (in mi/gal) of a car is modeled by M = N/G, where N is the number of miles driven and G is the number of gallons of gas used.

- (a) Find the gas mileage M for a car that drove 240 mi on 8 gal of gas.
- **(b)** A car with a gas mileage M = 25 mi/gal is driven 175 mi. How many gallons of gas are used?
- **8.** A mountain climber models the temperature T (in $^{\circ}$ F) at elevation h (in ft) by

$$T = 70 - 0.003h$$

- (a) Find the temperature T at an elevation of 1500 ft.
- **(b)** If the temperature is 64°F, what is the elevation?
- 9. The portion of a floating iceberg that is below the water surface is much larger than the portion above the surface. The total volume V of an iceberg is modeled by

$$V = 9.5S$$

where *S* is the volume showing above the surface.

- (a) Find the total volume of an iceberg if the volume showing above the surface is 4 km³.
- (b) Find the volume showing above the surface for an iceberg with total volume 19 km³.
- 10. The power P measured in horsepower (hp) needed to drive a certain ship at a speed of s knots is modeled by

$$P = 0.06s^3$$

- (a) Find the power needed to drive the ship at 12 knots.
- **(b)** At what speed will a 7.5-hp engine drive the ship?
- 11. An ocean diver models the pressure P (in lb/in²) at depth d (in ft) by

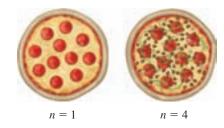
$$P = 14.7 + 0.45d$$

- (a) Make a table that gives the pressure for each 10-ft change in depth, from a depth of 0 ft to 60 ft.
- **(b)** If the pressure is 30 lb/in², what is the depth?
- 12. Arizonans use an average of 40 gal of water per person each day. The number of gallons W of water used by x Arizonans each day is modeled by W = 40x.
 - (a) Make a table that gives the number of gallons of water used for each 1000-person change in population, from 0 to 5000.
 - (b) What is the population of an Arizona town whose water usage is 120,000 gal per day?

- 13. The number N of cents in q quarters
- **14.** The average *A* of two numbers *a* and *b*
- **15.** The cost C of purchasing x gallons of gas at \$3.50 a gallon
- **16.** The amount T of a 15% tip on a restaurant bill of x dollars
- 17. The distance d in miles that a car travels in t hours at 60 mi/h
- **18.** The speed r of a boat that travels d miles in 3 h

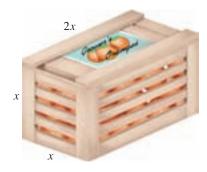
APPLICATIONS

- 19. Cost of a Pizza A pizza parlor charges \$12 for a cheese pizza and \$1 for each topping.
 - (a) How much does a 3-topping pizza cost?
 - **(b)** Find a formula that models the cost *C* of a pizza with *n* toppings.
 - (c) If a pizza costs \$16, how many toppings does it have?



- 20. Renting a Car At a certain car rental agency a compact car rents for \$30 a day and 10¢ a mile.
 - (a) How much does it cost to rent a car for 3 days if the car is driven 280 mi?
 - **(b)** Find a formula that models the cost *C* of renting this car for *n* days if it is driven *m* miles.
 - (c) If the cost for a 3-day rental was \$140, how many miles was the car driven?
- **21. Energy Cost for a Car** The cost of the electricity needed to drive an all-electric car is about 4 cents per mile. The cost of the gasoline needed to drive the average gasoline-powered car is about 12 cents per mile.

- (a) Find a formula that models the energy cost *C* of driving *x* miles for (i) the all-electric car and (ii) the average gasoline-powered car.
- **(b)** Find the cost of driving 10,000 mi with each type of car.
- 22. Volume of Fruit Crate A fruit crate has square ends and is twice as long as it is wide.
 - (a) Find the volume of the crate if its width is 20 in.
 - **(b)** Find a formula for the volume *V* of the crate in terms of its width *x*.



23. Grade Point Average In many universities students are given grade points for each credit unit according to the following scale:

A 4 points
B 3 points
C 2 points
D 1 point
F 0 point

For example, a grade of A in a 3-unit course earns $4 \times 3 = 12$ grade points and a grade of B in a 5-unit course earns $3 \times 5 = 15$ grade points. A student's grade point average (GPA) for these two courses is the total number of grade points earned divided by the number of units; in this case the GPA is (12 + 15)/8 = 3.375.

- (a) Find a formula for the GPA of a student who earns a grade of A in a units of course work, B in b units, C in c units, D in d units, and F in f units.
- (b) Find the GPA of a student who has earned a grade of A in two 3-unit courses, B in one 4-unit course, and C in three 3-unit courses.

P.2 REAL NUMBERS

Real Numbers Properties of Real Numbers Addition and Subtraction Multiplication and Division The Real Line Sets and Intervals Absolute Value and Distance

In the real world we use numbers to measure and compare different quantities. For example, we measure temperature, length, height, weight, blood pressure, distance, speed, acceleration, energy, force, angles, age, cost, and so on. Figure 1 illustrates some situations in which numbers are used. Numbers also allow us to express relationships between different quantities—for example, relationships between the radius and volume of a ball, between miles driven and gas used, or between education level and starting salary.

Count

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Length





Weight

FIGURE 1 Measuring with real numbers

■ Real Numbers

Let's review the types of numbers that make up the real number system. We start with the **natural numbers**:

The integers consist of the natural numbers together with their negatives and 0:

$$\dots$$
, -3 , -2 , -1 , 0 , 1 , 2 , 3 , 4 , \dots

We construct the **rational numbers** by taking ratios of integers. Thus any rational number r can be expressed as

$$r = \frac{m}{n}$$

where m and n are integers and $n \neq 0$. Examples are

$$\frac{1}{2}$$
 $-\frac{3}{7}$ $46 = \frac{46}{1}$ $0.17 = \frac{17}{100}$

(Recall that division by 0 is always ruled out, so expressions like $\frac{3}{0}$ and $\frac{0}{0}$ are undefined.) There are also real numbers, such as $\sqrt{2}$, that cannot be expressed as a ratio of integers and are therefore called **irrational numbers**. It can be shown, with varying degrees of difficulty, that these numbers are also irrational:

$$\sqrt{3}$$
 $\sqrt{5}$ $\sqrt[3]{2}$ π $\frac{3}{\pi^2}$

The set of all real numbers is usually denoted by the symbol \mathbb{R} . When we use the word *number* without qualification, we will mean "real number." Figure 2 is a diagram of the types of real numbers that we work with in this book.

Every real number has a decimal representation. If the number is rational, then its corresponding decimal is repeating. For example,

$$\frac{1}{2} = 0.5000... = 0.5\overline{0}$$
 $\frac{2}{3} = 0.66666... = 0.\overline{6}$ $\frac{157}{495} = 0.3171717... = 0.3\overline{17}$ $\frac{9}{7} = 1.285714285714... = 1.\overline{285714}$

(The bar indicates that the sequence of digits repeats forever.) If the number is irrational, the decimal representation is nonrepeating:

$$\sqrt{2} = 1.414213562373095...$$
 $\pi = 3.141592653589793...$

If we stop the decimal expansion of any number at a certain place, we get an approximation to the number. For instance, we can write

$$\pi \approx 3.14159265$$

where the symbol \approx is read "is approximately equal to." The more decimal places we retain, the better our approximation.

The different types of real numbers were invented to meet specific needs. For example, natural numbers are needed for counting, negative numbers for describing debt or below-zero temperatures, rational numbers for concepts like "half a gallon of milk," and irrational numbers for measuring certain distances, like the diagonal of a square.

Rational numbers

$$\frac{1}{2}$$
, $-\frac{3}{7}$, 0.17, 0.6, 0.317

Integers

Natural numbers

$$\dots, -3, -2, -1, 0, 1, 2, \dots 46, \dots$$

Irrational numbers

$$\sqrt{3}$$
, $\sqrt{5}$, $\sqrt[3]{2}$, π , $\frac{3}{\pi^2}$

FIGURE 2 The real number system

A repeating decimal such as

$$x = 3.5474747...$$

is a rational number. To convert it to a ratio of two integers, we write

$$1000x = 3547.47474747...$$

$$10x = 35.47474747...$$

$$990x = 3512.0$$

Thus $x = \frac{3512}{990}$. (The idea is to multiply x by appropriate powers of 10 and then subtract to eliminate the repeating part.)

Properties of Real Numbers

We all know that 2 + 3 = 3 + 2, and 5 + 7 = 7 + 5, and 513 + 87 = 87 + 513, and so on. In algebra we express all these (infinitely many) facts by writing

$$a + b = b + a$$

where a and b stand for any two numbers. In other words, "a + b = b + a" is a concise way of saying that "when we add two numbers, the order of addition doesn't matter." This fact is called the *Commutative Property* of addition. From our experience with numbers we know that the properties in the following box are also valid.

PROPERTIES OF REAL NUMBERS						
Property	Example	Description				
Commutative Properties						
a+b=b+a	7 + 3 = 3 + 7	When we add two numbers, order doesn't matter.				
ab = ba	$3 \cdot 5 = 5 \cdot 3$	When we multiply two numbers, order doesn't matter.				
Associative Properties						
(a + b) + c = a + (b + c)	(2+4)+7=2+(4+7)	When we add three numbers, it doesn't matter which two we add first.				
(ab)c = a(bc)	$(3\cdot7)\cdot5=3\cdot(7\cdot5)$	When we multiply three numbers, it doesn't matter which two we multiply first.				
Distributive Property						
a(b+c) = ab + ac	$2 \cdot (3+5) = 2 \cdot 3 + 2 \cdot 5$	When we multiply a number by a sum of two				
(b+c)a = ab + ac	$(3+5)\cdot 2 = 2\cdot 3 + 2\cdot 5$	numbers, we get the same result as we get if we multiply the number by each of the terms and then add the results.				

The Distributive Property applies whenever we multiply a number by a sum. Figure 3 explains why this property works for the case in which all the numbers are positive integers, but the property is true for any real numbers a, b, and c.

The Distributive Property is crucial because it describes the way addition and multiplication interact with each other.

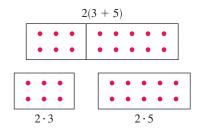


FIGURE 3 The Distributive Property

EXAMPLE 1 Using the Distributive Property

(a)
$$2(x + 3) = 2 \cdot x + 2 \cdot 3$$
 Distributive Property
= $2x + 6$ Simplify

In the last step we removed the parentheses because, according to the Associative Property, the order of addition doesn't matter.

Now Try Exercise 15

On't assume that -a is a negative

positive depends on the value of a. For

number. Whether -a is negative or

example, if a = 5, then -a = -5, a

negative number, but if a = -5, then

itive number.

-a = -(-5) = 5 (Property 2), a pos-

Addition and Subtraction

The number 0 is special for addition; it is called the additive identity because a + 0 = a for any real number a. Every real number a has a **negative**, -a, that satisfies a + (-a) = 0. Subtraction is the operation that undoes addition; to subtract a number from another, we simply add the negative of that number. By definition

$$a - b = a + (-b)$$

To combine real numbers involving negatives, we use the following properties.

PROPERTIES OF NEGATIVES

Property

1. (-1)a = -a

2.
$$-(-a) = a$$

3.
$$(-a)b = a(-b) = -(ab)$$

4.
$$(-a)(-b) = ab$$

5.
$$-(a+b) = -a-b$$

6. $-(a-b) = b-a$

Example

$$(-1)5 = -5$$

$$-(-5) = 5$$

3.
$$(-a)b = a(-b) = -(ab)$$
 $(-5)7 = 5(-7) = -(5 \cdot 7)$

$$(-4)(-3) = 4 \cdot 3$$

 $-(3+5) = -3-5$

$$-(5-8)=8-5$$

Property 6 states the intuitive fact that a - b and b - a are negatives of each other. Property 5 is often used with more than two terms:

$$-(a+b+c) = -a-b-c$$

EXAMPLE 2 Using Properties of Negatives

Let x, y, and z be real numbers.

(a)
$$-(x+2) = -x-2$$

Property 5:
$$-(a + b) = -a - b$$

(b)
$$-(x + y - z) = -x - y - (-z)$$
 Property 5: $-(a + b) = -a - b$

Property 5:
$$-(a + b) = -a - b$$

$$=-x-y+z$$

$$= -x - y + z$$
 Property 2: $-(-a) = a$

Now Try Exercise 27

Multiplication and Division

The number 1 is special for multiplication; it is called the multiplicative identity because $a \cdot 1 = a$ for any real number a. Every nonzero real number a has an **inverse**, 1/a, that satisfies $a \cdot (1/a) = 1$. **Division** is the operation that undoes multiplication;

$$a \div b = a \cdot \frac{1}{b}$$

We write $a \cdot (1/b)$ as simply a/b. We refer to a/b as the **quotient** of a and b or as the **fraction** a over b; a is the **numerator** and b is the **denominator** (or **divisor**). To combine real numbers using the operation of division, we use the following properties.

PROPERTIES OF FRACTIONS

Property

$$1. \ \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

$$2. \ \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

$$3. \ \frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

$$4. \ \frac{a}{b} + \frac{c}{d} = \frac{ad + bd}{bd}$$

$$5. \ \frac{ac}{bc} = \frac{a}{b}$$

6. If
$$\frac{a}{b} = \frac{c}{d}$$
, then $ad = be$

Example

$$\frac{2}{3} \cdot \frac{5}{7} = \frac{2 \cdot 5}{3 \cdot 7} = \frac{10}{21}$$

$$\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \cdot \frac{7}{5} = \frac{14}{15}$$

3.
$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$
 $\frac{2}{5} + \frac{7}{5} = \frac{2+7}{5} = \frac{9}{5}$

4.
$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$
 $\frac{2}{5} + \frac{3}{7} = \frac{2 \cdot 7 + 3 \cdot 5}{35} = \frac{29}{35}$

$$\frac{2\cdot 5}{3\cdot 5} = \frac{2}{3}$$

6. If
$$\frac{a}{b} = \frac{c}{d}$$
, then $ad = bc$ $\frac{2}{3} = \frac{6}{9}$, so $2 \cdot 9 = 3 \cdot 6$

Description

When **multiplying fractions**, multiply numerators and denominators.

When dividing fractions, invert the divisor and multiply.

When adding fractions with the same denominator, add the numerators.

When adding fractions with different denominators, find a common denominator. Then add the numerators.

Cancel numbers that are common factors in numerator and denominator.

Cross-multiply.

When adding fractions with different denominators, we don't usually use Property 4. Instead we rewrite the fractions so that they have the smallest possible common denominator (often smaller than the product of the denominators), and then we use Property 3. This denominator is the Least Common Denominator (LCD) described in the next example.

EXAMPLE 3 Using the LCD to Add Fractions

Evaluate: $\frac{5}{36} + \frac{7}{120}$

SOLUTION Factoring each denominator into prime factors gives

$$36 = 2^2 \cdot 3^2$$
 and $120 = 2^3 \cdot 3 \cdot 5$

We find the least common denominator (LCD) by forming the product of all the prime factors that occur in these factorizations, using the highest power of each prime factor. Thus the LCD is $2^3 \cdot 3^2 \cdot 5 = 360$. So

$$\frac{5}{36} + \frac{7}{120} = \frac{5 \cdot 10}{36 \cdot 10} + \frac{7 \cdot 3}{120 \cdot 3}$$
 Use common denominator
$$= \frac{50}{360} + \frac{21}{360} = \frac{71}{360}$$
 Property 3: Adding fractions with the same denominator

The Real Line

The real numbers can be represented by points on a line, as shown in Figure 4. The positive direction (toward the right) is indicated by an arrow. We choose an arbitrary reference point O, called the **origin**, which corresponds to the real number 0. Given any convenient unit of measurement, each positive number x is represented by the point on the line a distance of x units to the right of the origin, and each negative number -x is represented by the point x units to the left of the origin. The number associated with the point P is called the coordinate of P, and the line is then called a **coordinate line**, or a real number line, or simply a real line. Often we identify the point with its coordinate and think of a number as being a point on the real line.

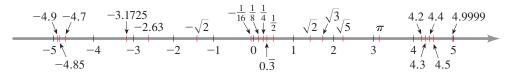


FIGURE 4 The real line

The real numbers are *ordered*. We say that **a** is less than **b** and write a < b if b-a is a positive number. Geometrically, this means that a lies to the left of b on the number line. Equivalently, we can say that **b** is greater than **a** and write b > a. The symbol $a \le b$ (or $b \ge a$) means that either a < b or a = b and is read "a is less than or equal to b." For instance, the following are true inequalities (see Figure 5):

Sets and Intervals

A set is a collection of objects, and these objects are called the elements of the set. If S is a set, the notation $a \in S$ means that a is an element of S, and $b \notin S$ means that b is not an element of S. For example, if Z represents the set of integers, then $-3 \in Z$ but $\pi \notin Z$.

Some sets can be described by listing their elements within braces. For instance, the set A that consists of all positive integers less than 7 can be written as

$$A = \{1, 2, 3, 4, 5, 6\}$$

We could also write A in set-builder notation as

$$A = \{x \mid x \text{ is an integer and } 0 < x < 7\}$$

which is read "A is the set of all x such that x is an integer and 0 < x < 7."

DISCOVERY PROJECT

Real Numbers in the Real World

Real-world measurements always involve units. For example, we usually measure distance in feet, miles, centimeters, or kilometers. Some measurements involve different types of units. For example, speed is measured in miles per hour or meters per second. We often need to convert a measurement from one type of unit to another. In this project we explore different types of units used for different purposes and how to convert from one type of unit to another. You can find the project at www.stewartmath.com.

If S and T are sets, then their union $S \cup T$ is the set that consists of all elements that are in S or T (or in both). The **intersection** of S and T is the set $S \cap T$ consisting of all elements that are in both S and T. In other words, $S \cap T$ is the common part of S and T. The **empty set**, denoted by \emptyset , is the set that contains no element.

EXAMPLE 4 Union and Intersection of Sets

If $S = \{1, 2, 3, 4, 5\}$, $T = \{4, 5, 6, 7\}$, and $V = \{6, 7, 8\}$, find the sets $S \cup T$, $S \cap T$, and $S \cap V$.

SOLUTION

$$S \cup T = \{1, 2, 3, 4, 5, 6, 7\}$$
 All elements in S or T
$$S \cap T = \{4, 5\}$$
 Elements common to both S and T
$$S \cap V = \emptyset$$
 S and V have no element in common





FIGURE 6 The open interval (a, b)



FIGURE 7 The closed interval [a, b]

Certain sets of real numbers, called **intervals**, occur frequently in calculus and correspond geometrically to line segments. If a < b, then the **open interval** from a to b consists of all numbers between a and b and is denoted (a, b). The **closed interval** from a to b includes the endpoints and is denoted [a, b]. Using set-builder notation, we can write

$$(a,b) = \{x \mid a < x < b\}$$
 $[a,b] = \{x \mid a \le x \le b\}$

Note that parentheses () in the interval notation and open circles on the graph in Figure 6 indicate that endpoints are excluded from the interval, whereas square brackets and solid circles in Figure 7 indicate that the endpoints are *included*. Intervals may also include one endpoint but not the other, or they may extend infinitely far in one direction or both. The following table lists the possible types of intervals.

Notation	Set description	Graph
(a,b)	$\{x \mid a < x < b\}$	
[<i>a</i> , <i>b</i>]	$\{x \mid a \le x \le b\}$	<i>a b</i>
[a,b)	$\{x \mid a \le x < b\}$	<i>a b</i>
(a,b]	$\{x \mid a < x \le b\}$	<i>a b</i>
(a, ∞)	$\{x \mid a < x\}$	<i>a b</i> →
$[a, \infty)$	$\{x \mid a \le x\}$	<i>a</i>
$(-\infty,b)$	$\{x \mid x < b\}$	<i>a</i>
$(-\infty, b]$	$\{x \mid x \le b\}$	<i>b b</i>
$(-\infty,\infty)$	\mathbb{R} (set of all real numbers)	<i>→</i>

The symbol ∞ ("infinity") does not stand for a number. The notation (a, ∞) , for instance, simply indicates that the interval has no endpoint on the right but extends infinitely far in the positive direction.

EXAMPLE 5 Graphing Intervals

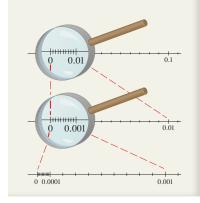
Express each interval in terms of inequalities, and then graph the interval.

(a)
$$[-1,2) = \{x \mid -1 \le x < 2\}$$

(b) $[1.5,4] = \{x \mid 1.5 \le x \le 4\}$
(c) $(-3,\infty) = \{x \mid -3 < x\}$

No Smallest or Largest Number in an Open Interval

Any interval contains infinitely many numbers—every point on the graph of an interval corresponds to a real number. In the closed interval [0, 1], the smallest number is 0 and the largest is 1, but the open interval (0, 1) contains no smallest or largest number. To see this, note that 0.01 is close to zero, but 0.001 is closer, 0.0001 is closer yet, and so on. We can always find a number in the interval (0, 1) closer to zero than any given number. Since 0 itself is not in the interval, the interval contains no smallest number. Similarly, 0.99 is close to 1, but 0.999 is closer, 0.9999 closer yet, and so on. Since 1 itself is not in the interval, the interval has no largest number.



| -3 | = 3 | 5 | = 5 |-3 | 0 | 5

FIGURE 10

EXAMPLE 6 Finding Unions and Intersections of Intervals

Graph each set.

(a)
$$(1,3) \cap [2,7]$$

(b)
$$(1,3) \cup [2,7]$$

SOLUTION

(a) The intersection of two intervals consists of the numbers that are in both intervals. Therefore

$$(1,3) \cap [2,7] = \{x \mid 1 < x < 3 \text{ and } 2 \le x \le 7\}$$

= $\{x \mid 2 \le x < 3\} = [2,3)$

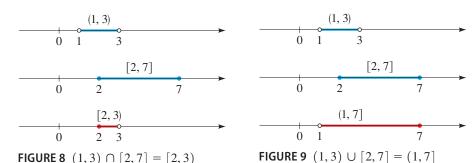
This set is illustrated in Figure 8.

(b) The union of two intervals consists of the numbers that are in either one interval or the other (or both). Therefore

$$(1,3) \cup [2,7] = \{x \mid 1 < x < 3 \text{ or } 2 \le x \le 7\}$$

= $\{x \mid 1 < x \le 7\} = (1,7]$

This set is illustrated in Figure 9.



Now Try Exercise 61

Absolute Value and Distance

The **absolute value** of a number a, denoted by |a|, is the distance from a to 0 on the real number line (see Figure 10). Distance is always positive or zero, so we have $|a| \ge 0$ for every number a. Remembering that -a is positive when a is negative, we have the following definition.

DEFINITION OF ABSOLUTE VALUE

If a is a real number, then the **absolute value** of a is

$$|a| = \begin{cases} a & \text{if } a \ge 0 \\ -a & \text{if } a < 0 \end{cases}$$

EXAMPLE 7 Evaluating Absolute Values of Numbers

(a)
$$|3| = 3$$

(b)
$$|-3| = -(-3) = 3$$

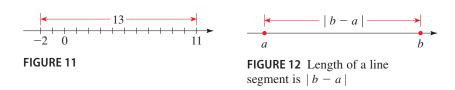
(c)
$$|0| = 0$$

(d)
$$|3 - \pi| = -(3 - \pi) = \pi - 3$$
 (since $3 < \pi \implies 3 - \pi < 0$)

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PROPERTIES OF ABSOLUTE VALUE							
Property	Example	Description					
1. $ a \ge 0$	$ -3 =3\geq 0$	The absolute value of a number is always positive or zero.					
2. $ a = -a $	5 = -5	A number and its negative have the same absolute value.					
3. $ ab = a b $	$ -2\cdot 5 = -2 5 $	The absolute value of a product is the product of the absolute values.					
4.	$\left \frac{12}{-3} \right = \frac{\mid 12 \mid}{\mid -3 \mid}$	The absolute value of a quotient is the quotient of the absolute values.					
5. $ a+b \le a + b $	$ -3+5 \le -3 + 5 $	Triangle Inequality					

What is the distance on the real line between the numbers -2 and 11? From Figure 11 we see that the distance is 13. We arrive at this by finding either |11-(-2)|=13 or |(-2)-11|=13. From this observation we make the following definition (see Figure 12).



DISTANCE BETWEEN POINTS ON THE REAL LINE

If a and b are real numbers, then the **distance** between the points a and b on the real line is

$$d(a,b) = |b-a|$$

From Property 6 of negatives it follows that

$$|b-a| = |a-b|$$

This confirms that, as we would expect, the distance from a to b is the same as the distance from b to a.

EXAMPLE 8 Distance Between Points on the Real Line

The distance between the numbers -8 and 2 is

$$d(a,b) = |2 - (-8)| = |-10| = 10$$

We can check this calculation geometrically, as shown in Figure 13.

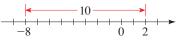


FIGURE 13

P.2 EXERCISES

CONCEPTS

- 1. Give an example of each of the following:
 - (a) A natural number
 - (b) An integer that is not a natural number
 - (c) A rational number that is not an integer
 - (d) An irrational number
- 2. Complete each statement and name the property of real numbers you have used.
 - (a) $ab = \underline{\hspace{1cm}}$; Property
 - **(b)** a + (b + c) = ______; _____ Property
 - (c) a(b + c) = Property
- **3.** Express the set of real numbers between but not including 2 and 7 as follows.
 - (a) In set-builder notation: ____
 - (b) In interval notation: ___
- **4.** The symbol |x| stands for the _____ of the number x. If x is not 0, then the sign of |x| is always ____
- 5. The distance between a and b on the real line is d(a, b) =- So the distance between -5 and 2 is -.
- **6–8** *Yes or No*? If *No*, give a reason. Assume that a and b are nonzero real numbers.
- **6.** (a) Is the sum of two rational numbers always a rational
 - (b) Is the sum of two irrational numbers always an irrational number?
- 7. (a) Is a-b equal to b-a?
 - **(b)** Is -2(a-5) equal to -2a-10?
- 8. (a) Is the distance between any two different real numbers always positive?
 - (b) Is the distance between a and b the same as the distance between b and a?

SKILLS

- **9–10 Real Numbers** List the elements of the given set that are
 - (a) natural numbers
 - (b) integers
 - (c) rational numbers
 - (d) irrational numbers
- **9.** $\{-1.5, 0, \frac{5}{2}, \sqrt{7}, 2.71, -\pi, 3.1\overline{4}, 100, -8\}$
- **10.** $\{1.3, 1.3333..., \sqrt{5}, 5.34, -500, 1\frac{2}{3}, \sqrt{16}, \frac{246}{579}, -\frac{20}{5}\}$
- 11–18 Properties of Real Numbers State the property of real numbers being used.
- **11.** 3 + 7 = 7 + 3
- **12.** 4(2+3) = (2+3)4
- **13.** (x + 2y) + 3z = x + (2y + 3z)

- **14.** 2(A + B) = 2A + 2B
- **15.** (5x + 1)3 = 15x + 3
 - **16.** (x + a)(x + b) = (x + a)x + (x + a)b
 - **17.** 2x(3 + y) = (3 + y)2x
 - **18.** 7(a + b + c) = 7(a + b) + 7c
 - **19–22 Properties of Real Numbers** Rewrite the expression using the given property of real numbers.
 - **19.** Commutative Property of Addition, x + 3 =
 - **20.** Associative Property of Multiplication, 7(3x) =
 - **21.** Distributive Property, 4(A + B) =
 - **22.** Distributive Property, 5x + 5y =
 - 23–28 Properties of Real Numbers Use properties of real numbers to write the expression without parentheses.
 - **23.** 3(x + y)
- **24.** (a b)8

- **25.** 4(2*m*)
- **26.** $\frac{4}{3}(-6y)$
- **27.** $-\frac{5}{2}(2x-4y)$
- **28.** (3a)(b+c-2d)
- **29–32 Arithmetic Operations** Perform the indicated operations.
- **29.** (a) $\frac{3}{10} + \frac{4}{15}$
- **(b)** $\frac{1}{4} + \frac{1}{5}$
- 30. (a) $\frac{2}{3} \frac{3}{5}$
- **(b)** $1 + \frac{5}{8} \frac{1}{6}$
- 31. (a) $\frac{2}{3}(6-\frac{3}{2})$
- **(b)** $(3 + \frac{1}{4})(1 \frac{4}{5})$
- 32. (a) $\frac{2}{\frac{2}{3}} \frac{\frac{2}{3}}{2}$
- **(b)** $\frac{\frac{2}{5} + \frac{1}{2}}{\frac{1}{2} + \frac{3}{2}}$
- 33–34 Inequalities Place the correct symbol (<, >, or =) in the space.
- **33.** (a) $3 \frac{7}{2}$
- **(b)** -3 $-\frac{7}{2}$ **(c)** 3.5 $\frac{7}{2}$

- **34.** (a) $\frac{2}{3}$ 0.67
- **(b)** $\frac{2}{3}$ -0.67
- (c) | 0.67 | | -0.67 |
- **35–38** Inequalities State whether each inequality is true or
- 35. (a) -3 < -4
- **(b)** 3 < 4
- **36.** (a) $\sqrt{3} > 1.7325$
- **(b)** $1.732 \ge \sqrt{3}$
- 37. (a) $\frac{10}{2} \ge 5$
- **(b)** $\frac{6}{10} \ge \frac{5}{6}$
- 38. (a) $\frac{7}{11} \ge \frac{8}{13}$
- **(b)** $-\frac{3}{5} > -\frac{3}{4}$
- **39–40** Inequalities Write each statement in terms of inequalities.
- **39.** (a) x is positive.
 - **(b)** *t* is less than 4.
 - (c) a is greater than or equal to π .
 - (d) x is less than $\frac{1}{3}$ and is greater than -5.
 - (e) The distance from p to 3 is at most 5.

- **40.** (a) y is negative.
 - (b) z is greater than 1.
 - (c) *b* is at most 8.
 - (d) w is positive and is less than or equal to 17.
 - (e) y is at least 2 units from π .
- **41–44 Sets** Find the indicated set if

$$A = \{1, 2, 3, 4, 5, 6, 7\}$$

$$B = \{2, 4, 6, 8\}$$

$$C = \{7, 8, 9, 10\}$$

- \sim 41. (a) $A \cup B$
- **(b)** $A \cap B$
- **42.** (a) $B \cup C$
- (b) $B \cap C$
- **43.** (a) $A \cup C$
- (b) $A \cap C$
- **44.** (a) $A \cup B \cup C$
- (b) $A \cap B \cap C$
- **45–46 Sets** Find the indicated set if

$$A = \{x \mid x \ge -2\} \qquad B = \{x \mid x < 4\}$$
$$C = \{x \mid -1 < x \le 5\}$$

- **45.** (a) $B \cup C$
- (b) $B \cap C$
- **46.** (a) $A \cap C$
- (b) $A \cap B$
- **47–52** Intervals Express the interval in terms of inequalities, and then graph the interval.
- **47.** (−3, 0)
- **48.** (2, 8]

49. [2, 8)

- **50.** $\left[-6, -\frac{1}{2}\right]$
- **51.** [2, ∞)

- **52.** $(-\infty, 1)$
- **53–58** Intervals Express the inequality in interval notation, and then graph the corresponding interval.
- **53.** $x \le 1$

- **54.** $1 \le x \le 2$
- **55.** $-2 < x \le 1$
- **56.** $x \ge -5$
- **57.** x > -1
- **58.** -5 < x < 2
- **59–60** Intervals Express each set in interval notation.



60. (a)
$$0 2$$

- **61–66** Intervals Graph the set.
- **61.** (-2,0) ∪ (-1,1)
- **62.** $(-2,0) \cap (-1,1)$
- **63.** $[-4,6] \cap [0,8)$
- **64.** $[-4,6) \cup [0,8)$
- **65.** $(-\infty, -4) \cup (4, \infty)$
- **66.** $(-\infty, 6] \cap (2, 10)$
- **67–72 Absolute Value** Evaluate each expression.
- **67.** (a) | 100 |
- **(b)** |-73|
- **68.** (a) $|\sqrt{5} 5|$
- **(b)** $|10 \pi|$

- **69.** (a) ||-6|-|-4|| (b) $\frac{-1}{|-1|}$
- **70.** (a) |2 |-12|
 - **(b)** -1 |1 |-1|
- 71. (a) $|(-2) \cdot 6|$
- **(b)** $|(-\frac{1}{3})(-15)|$
- 72. (a) $\left| \frac{-6}{24} \right|$ (b) $\left| \frac{7-12}{12-7} \right|$
- **73–76 Distance** Find the distance between the given numbers.
- **73.** -3 -2 -1 0 1 2 3
- **74.** -3 -2 -1 0 1 2 3
- **75.** (a) 2 and 17 (b) -3 and 21 (c) $\frac{11}{8}$ and $-\frac{3}{10}$

- **76.** (a) $\frac{7}{15}$ and $-\frac{1}{21}$ (b) -38 and -57 (c) -2.6 and -1.8

SKILLS Plus

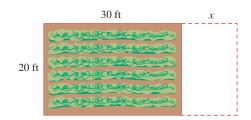
- 77–78 Repeating Decimal Express each repeating decimal as a fraction. (See the margin note on page 7.)
- 77. (a) $0.\overline{7}$
- **(b)** $0.2\overline{8}$
- (c) $0.\overline{57}$

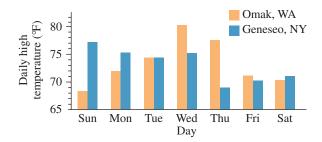
- **78.** (a) $5.\overline{23}$
- **(b)** $1.3\overline{7}$
- (c) $2.1\overline{35}$
- **79–82** Simplifying Absolute Value Express the quantity without using absolute value.
- **79.** $|\pi 3|$
- **80.** $|1-\sqrt{2}|$
- **81.** |a b|, where a < b
- **82.** a + b + |a b|, where a < b
- **83–84** Signs of Numbers Let a, b, and c be real numbers such that a > 0, b < 0, and c < 0. Find the sign of each expression.
- **83.** (a) −*a*

- **(b)** bc **(c)** a b **(d)** ab + ac
- **84.** (a) -b
- **(b)** a + bc
- - (c) c a (d) ab^2

APPLICATIONS

85. Area of a Garden Mary's backyard vegetable garden measures 20 ft by 30 ft, so its area is $20 \times 30 = 600$ ft². She decides to make it longer, as shown in the figure, so that the area increases to A = 20(30 + x). Which property of real numbers tells us that the new area can also be written A = 600 + 20x?

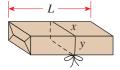


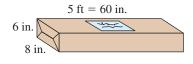


87. Mailing a Package The post office will accept only packages for which the length plus the "girth" (distance around) is no more than 108 in. Thus for the package in the figure, we must have

$$L + 2(x + y) \le 108$$

- (a) Will the post office accept a package that is 6 in. wide, 8 in. deep, and 5 ft long? What about a package that measures 2 ft by 2 ft by 4 ft?
- **(b)** What is the greatest acceptable length for a package that has a square base measuring 9 in. by 9 in.?





DISCUSS DISCOVER PROVE WRITE

- 88. DISCUSS: Sums and Products of Rational and Irrational Numbers Explain why the sum, the difference, and the product of two rational numbers are rational numbers. Is the product of two irrational numbers necessarily irrational? What about the sum?
- 89. DISCOVER PROVE: Combining Rational and Irrational Numbers Is $\frac{1}{2} + \sqrt{2}$ rational or irrational? Is $\frac{1}{2} \cdot \sqrt{2}$ rational or irrational? Experiment with sums and products of other rational and irrational numbers. Prove the following.
 - (a) The sum of a rational number *r* and an irrational number *t* is irrational.
 - **(b)** The product of a nonzero rational number *r* and an irrational number *t* is irrational.

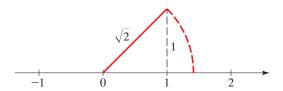
[*Hint:* For part (a), suppose that r + t is a rational number q, that is, r + t = q. Show that this leads to a contradiction. Use similar reasoning for part (b).]

90. DISCOVER: Limiting Behavior of Reciprocals Complete the tables. What happens to the size of the fraction 1/x as x gets large? As x gets small?

x	1/x
1	
2	
10	
100	
1000	

x	1/ <i>x</i>
1.0	
0.5	
0.1	
0.01	
0.001	

91. DISCOVER: Irrational Numbers and Geometry Using the following figure, explain how to locate the point $\sqrt{2}$ on a number line. Can you locate $\sqrt{5}$ by a similar method? What about $\sqrt{6}$? List some other irrational numbers that can be located this way.



92. DISCUSS: Commutative and Noncommutative OperationsWe have seen that addition and multiplication are both commutative operations.

- (a) Is subtraction commutative?
- **(b)** Is division of nonzero real numbers commutative?
- (c) Are the actions of putting on your socks and putting on your shoes commutative?
- (d) Are the actions of putting on your hat and putting on your coat commutative?
- (e) Are the actions of washing laundry and drying it commutative?
- (f) Give an example of a pair of actions that are commutative.
- (g) Give an example of a pair of actions that are not commutative.
- **93. WRITE: Real Numbers in the Real World** Write a paragraph describing different real-world situations in which you would use natural numbers, integers, rational numbers, and irrational numbers. Give examples for each type of situation.
- **94. PROVE: Triangle Inequality** We prove Property 5 of absolute values, the Triangle Inequality:

$$|x+y| \le |x| + |y|$$

- (a) Verify that the Triangle Inequality holds for x = 2 and y = 3, for x = -2 and y = -3, and for x = -2 and y = 3.
- **(b)** Prove that the Triangle Inequality is true for all real numbers *x* and *y*. [*Hint:* Take cases.]

P.3 INTEGER EXPONENTS AND SCIENTIFIC NOTATION

■ Exponential Notation ■ Rules for Working with Exponents ■ Scientific Notation

In this section we review the rules for working with exponential notation. We also see how exponents can be used to represent very large and very small numbers.

Exponential Notation

A product of identical numbers is usually written in exponential notation. For example, $5 \cdot 5 \cdot 5$ is written as 5^3 . In general, we have the following definition.

EXPONENTIAL NOTATION

If a is any real number and n is a positive integer, then the **nth power** of a is

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

The number a is called the **base**, and n is called the **exponent**.

EXAMPLE 1 Exponential Notation

(a)
$$\left(\frac{1}{2}\right)^5 = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{32}$$

(b)
$$(-3)^4 = (-3) \cdot (-3) \cdot (-3) \cdot (-3) = 81$$

(c)
$$-3^4 = -(3 \cdot 3 \cdot 3 \cdot 3) = -81$$

Now Try Exercise 11

We can state several useful rules for working with exponential notation. To discover the rule for multiplication, we multiply 5^4 by 5^2 :

$$5^4 \cdot 5^2 = (5 \cdot 5 \cdot 5 \cdot 5)(5 \cdot 5) = 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 5^6 = 5^{4+2}$$
4 factors 2 factors 6 factors

It appears that to multiply two powers of the same base, we add their exponents. In general, for any real number a and any positive integers m and n, we have

$$a^{m}a^{n} = \underbrace{(a \cdot a \cdot \cdots \cdot a)}_{m \text{ factors}} \underbrace{(a \cdot a \cdot \cdots \cdot a)}_{n \text{ factors}} = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{m + n \text{ factors}} = a^{m+n}$$

Thus $a^m a^n = a^{m+n}$.

We would like this rule to be true even when m and n are 0 or negative integers. For instance, we must have

$$2^0 \cdot 2^3 = 2^{0+3} = 2^3$$

But this can happen only if $2^0 = 1$. Likewise, we want to have

$$5^4 \cdot 5^{-4} = 5^{4+(-4)} = 5^{4-4} = 5^0 = 1$$

and this will be true if $5^{-4} = 1/5^4$. These observations lead to the following definition.

Note the distinction between $(-3)^4$ and -3^4 . In $(-3)^4$ the exponent applies to -3, but in -3^4 the exponent applies only to 3.

ZERO AND NEGATIVE EXPONENTS

If $a \neq 0$ is a real number and n is a positive integer, then

$$a^0 = 1 \qquad \text{and} \qquad a^{-n} = \frac{1}{a^n}$$

EXAMPLE 2 Zero and Negative Exponents

(a)
$$\left(\frac{4}{7}\right)^0 = 1$$

(b)
$$x^{-1} = \frac{1}{x^1} = \frac{1}{x}$$

(c)
$$(-2)^{-3} = \frac{1}{(-2)^3} = \frac{1}{-8} = -\frac{1}{8}$$

Now Try Exercise 13

Rules for Working with Exponents

Familiarity with the following rules is essential for our work with exponents and bases. In the table the bases a and b are real numbers, and the exponents m and n are integers.

LAWS OF EXPONENTS

Description

1. $a^m a^n = a^{m+n}$ $3^2 \cdot 3^5 = 3^{2+5} = 3^7$ To multiply two powers of the same number, add the exponents.

2. $\frac{a^m}{a^n} = a^{m-n}$ $\frac{3^5}{3^2} = 3^{5-2} = 3^3$ To divide two powers of the same number, subtract the exponents.

3. $(a^m)^n = a^{mn}$ $(3^2)^5 = 3^{2 \cdot 5} = 3^{10}$ To raise a power to a new power, multiply the exponents.

4. $(ab)^n = a^n b^n$ $(3 \cdot 4)^2 = 3^2 \cdot 4^2$ To raise a product to a power, raise each factor to the power.

5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ $\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2}$ To raise a quotient to a power, raise both numerator and denominator to the power.

6. $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n \quad \left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2$ To raise a fraction to a negative power, invert the fraction and change the sign of the exponent.

7. $\frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}$ $\frac{3^{-2}}{4^{-5}} = \frac{4^5}{3^2}$ To move a number raised to a power from numerator to denominator or from denominator to numerator, change the sign of the exponent.

Proof of Law 3 If m and n are positive integers, we have

$$(a^{m})^{n} = \underbrace{(a \cdot a \cdot \cdots \cdot a)^{n}}_{m \text{ factors}}$$

$$= \underbrace{(a \cdot a \cdot \cdots \cdot a)(a \cdot a \cdot \cdots \cdot a) \cdot \cdots \cdot (a \cdot a \cdot \cdots \cdot a)}_{m \text{ factors}}$$

$$= \underbrace{(a \cdot a \cdot \cdots \cdot a)(a \cdot a \cdot \cdots \cdot a) \cdot \cdots \cdot (a \cdot a \cdot \cdots \cdot a)}_{m \text{ factors}}$$

$$= \underbrace{a \cdot a \cdot \cdots \cdot a}_{mn \text{ factors}} = a^{mn}$$

The cases for which $m \le 0$ or $n \le 0$ can be proved by using the definition of negative exponents.

20

$$(ab)^n = (\underline{ab})(\underline{ab}) \cdot \cdot \cdot \cdot (\underline{ab}) = (\underline{a \cdot a \cdot \cdot \cdot \cdot \cdot a}) \cdot (\underline{b \cdot b \cdot \cdot \cdot \cdot \cdot b}) = a^n b^n$$

$$n \text{ factors}$$

$$n \text{ factors}$$

Here we have used the Commutative and Associative Properties repeatedly. If $n \le 0$, Law 4 can be proved by using the definition of negative exponents.

You are asked to prove Laws 2, 5, 6, and 7 in Exercises 58 and 59.

EXAMPLE 3 Using Laws of Exponents

(a)
$$x^4x^7 = x^{4+7} = x^{11}$$
 Law 1: $a^m a^n = a^{m+n}$

(b)
$$y^4y^{-7} = y^{4-7} = y^{-3} = \frac{1}{y^3}$$
 Law 1: $a^ma^n = a^{m+n}$

(c)
$$\frac{c^9}{c^5} = c^{9-5} = c^4$$
 Law 2: $\frac{a^m}{a^n} = a^{m-n}$
(d) $(b^4)^5 = b^{4\cdot 5} = b^{20}$ Law 3: $(a^m)^n = a^m$

(d)
$$(b^4)^5 = b^{4\cdot 5} = b^{20}$$
 Law 3: $(a^m)^n = a^{mn}$

(e)
$$(3x)^3 = 3^3x^3 = 27x^3$$
 Law 4: $(ab)^n = a^nb^n$

(f)
$$\left(\frac{x}{2}\right)^5 = \frac{x^5}{2^5} = \frac{x^5}{32}$$
 Law 5: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Now Try Exercises 19 and 21

EXAMPLE 4 Simplifying Expressions with Exponents

Simplify:

(a)
$$(2a^3b^2)(3ab^4)^3$$
 (b) $\left(\frac{x}{y}\right)^3 \left(\frac{y^2x}{z}\right)^4$

SOLUTION

(a)
$$(2a^3b^2)(3ab^4)^3 = (2a^3b^2)[3^3a^3(b^4)^3]$$
 Law 4: $(ab)^n = a^nb^n$
 $= (2a^3b^2)(27a^3b^{12})$ Law 3: $(a^m)^n = a^{mn}$
 $= (2)(27)a^3a^3b^2b^{12}$ Group factors with the same base
 $= 54a^6b^{14}$ Law 1: $a^ma^n = a^{m+n}$

(b)
$$\left(\frac{x}{y}\right)^3 \left(\frac{y^2 x}{z}\right)^4 = \frac{x^3}{y^3} \frac{(y^2)^4 x^4}{z^4}$$
 Laws 5 and 4
$$= \frac{x^3}{y^3} \frac{y^8 x^4}{z^4}$$
 Law 3
$$= (x^3 x^4) \left(\frac{y^8}{y^3}\right) \frac{1}{z^4}$$
 Group factors with the same base
$$= \frac{x^7 y^5}{4}$$
 Laws 1 and 2

Now Try Exercises 25 and 29

When simplifying an expression, you will find that many different methods will lead to the same result; you should feel free to use any of the rules of exponents to arrive at your own method. In the next example we see how to simplify expressions with negative exponents.

Mathematics in the Modern World

Although we are often unaware of its presence, mathematics permeates nearly every aspect of life in the modern world. With the advent of modern technology, mathematics plays an ever greater role in our lives. Today you were probably awakened by a digital alarm clock, sent a text, surfed the Internet, watched HDTV or a streaming video, listened to music on your cell phone, drove a car with digitally controlled fuel injection, then fell asleep in a room whose temperature is controlled by a digital thermostat. In each of these activities mathematics is crucially involved. In general, a property such as the intensity or frequency of sound, the oxygen level in the exhaust emission from a car, the colors in an image, or the temperature in your bedroom is transformed into sequences of numbers by sophisticated mathematical algorithms. These numerical data, which usually consist of many millions of bits (the digits 0 and 1), are then transmitted and reinterpreted. Dealing with such huge amounts of data was not feasible until the invention of computers, machines whose logical processes were invented by mathematicians.

The contributions of mathematics in the modern world are not limited to technological advances. The logical processes of mathematics are now used to analyze complex problems in the social, political, and life sciences in new and surprising ways. Advances in mathematics continue to be made, some of the most exciting of these just within the past decade.

In other Mathematics in the Modern World, we will describe in more detail how mathematics affects all of us in our everyday activities.

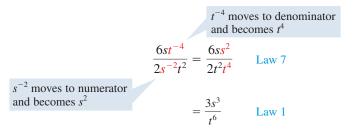
EXAMPLE 5 Simplifying Expressions with Negative Exponents

Eliminate negative exponents, and simplify each expression.

(a)
$$\frac{6st^{-4}}{2s^{-2}t^2}$$
 (b) $\left(\frac{y}{3z^3}\right)^{-2}$

SOLUTION

(a) We use Law 7, which allows us to move a number raised to a power from the numerator to the denominator (or vice versa) by changing the sign of the exponent.



(b) We use Law 6, which allows us to change the sign of the exponent of a fraction by inverting the fraction.

$$\left(\frac{y}{3z^3}\right)^{-2} = \left(\frac{3z^3}{y}\right)^2 \qquad \text{Law 6}$$

$$= \frac{9z^6}{y^2} \qquad \text{Laws 5 and 4}$$

Now Try Exercise 31

Scientific Notation

Exponential notation is used by scientists as a compact way of writing very large numbers and very small numbers. For example, the nearest star beyond the sun, Proxima Centauri, is approximately 40,000,000,000,000 km away. The mass of a hydrogen atom is about 0.00000000000000000000000166 g. Such numbers are difficult to read and to write, so scientists usually express them in *scientific notation*.

SCIENTIFIC NOTATION

A positive number x is said to be written in **scientific notation** if it is expressed as follows:

$$x = a \times 10^n$$
 where $1 \le a < 10$ and n is an integer

For instance, when we state that the distance to the star Proxima Centauri is 4×10^{13} km, the positive exponent 13 indicates that the decimal point should be moved 13 places to the *right*:

$$4 \times 10^{13} = 40,000,000,000,000$$

Move decimal point 13 places to the right

When we state that the mass of a hydrogen atom is 1.66×10^{-24} g, the exponent -24indicates that the decimal point should be moved 24 places to the left:

Move decimal point 24 places to the left

Write each number in scientific notation.

- (a) 56,920
- **(b)** 0.000093

SOLUTION

- (a) $56,920 = 5.692 \times 10^4$ 4 places
- **(b)** $0.000093 = 9.3 \times 10^{-5}$ 5 places

Now Try Exercise 35

EXAMPLE 7 Changing from Scientific Notation to Decimal Notation

Write each number in decimal notation.

- (a) 6.97×10^9
- **(b)** 4.6271×10^{-6}

SOLUTION

- (a) $6.97 \times 10^9 = 6,970,000,000$
- Move decimal 9 places to the right

- **(b)** $4.6271 \times 10^{-6} = 0.0000046271$
- Move decimal 6 places to the left

6 places

Now Try Exercise 37

To use scientific notation on a calculator, press the key labeled **EE** or EXP or EEX to enter the exponent. For example, to enter the number 3.629×10^{15} on a TI-83 or TI-84 calculator, we enter

$$3.629$$
 2ND EE 15

and the display reads

3.629E15

Scientific notation is often used on a calculator to display a very large or very small number. For instance, if we use a calculator to square the number 1,111,111, the display panel may show (depending on the calculator model) the approximation

Here the final digits indicate the power of 10, and we interpret the result as

$$1.234568 \times 10^{12}$$

EXAMPLE 8 Calculating with Scientific Notation

If $a \approx 0.00046$, $b \approx 1.697 \times 10^{22}$, and $c \approx 2.91 \times 10^{-18}$, use a calculator to approximate the quotient ab/c.

SOLUTION We could enter the data using scientific notation, or we could use laws of exponents as follows:

$$\frac{ab}{c} \approx \frac{(4.6 \times 10^{-4})(1.697 \times 10^{22})}{2.91 \times 10^{-18}}$$
$$= \frac{(4.6)(1.697)}{2.91} \times 10^{-4+22+18}$$
$$\approx 2.7 \times 10^{36}$$

For guidelines on working with significant figures, see Appendix A, Calculations and Significant Figures.

We state the answer rounded to two significant figures because the least accurate of the given numbers is stated to two significant figures.

CONCEPTS

- 1. Using exponential notation, we can write the product $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$ as ______
- 2. Is there a difference between $(-5)^4$ and -5^4 ?
- **3.** In the expression 3⁴ the number 3 is called the ____ and the number 4 is called the ____
- 4. When we multiply two powers with the same base, we _____ the exponents. So $3^4 \cdot 3^5 =$ _____.
- 5. When we divide two powers with the same base, we _ the exponents. So $\frac{3^5}{3^2} = \underline{\hspace{1cm}}$.
- **6.** When we raise a power to a new power, we _____ the exponents. So $(3^4)^2 =$ _____
- 7. Express the following numbers without using exponents.
 - (a) $2^{-1} =$ (b) $2^{-3} =$

 - (c) $\left(\frac{1}{2}\right)^{-1} =$ _____ (d) $\frac{1}{2^{-3}} =$ _____
- 8. Scientists express very large or very small numbers using __ notation. In scientific notation 8,300,000 is _____, and 0.0000327 is _____
- **9–10** *Yes or No?* If *No*, give a reason.
- **9.** (a) Is the expression $\left(\frac{2}{3}\right)^{-2}$ equal to $\frac{3}{4}$?
 - **(b)** Is there a difference between $(-5)^4$ and -5^4 ?
- **10.** (a) Is the expression $(x^2)^3$ equal to x^5 ?
 - **(b)** Is the expression $(2x^4)^3$ equal to $2x^{12}$?

SKILLS

11–18 ■ **Exponential Notation** Evaluate each expression.

- -2^6
- **(b)** $(-2)^6$
- (c) $\left(\frac{1}{5}\right)^2 \cdot (-3)^3$

- 12. (a) $(-5)^3$
- **(b)** -5^3
- (c) $(-5)^2 \cdot (\frac{2}{5})^2$
- **13.** (a) $\left(\frac{5}{3}\right)^0 \cdot 2^{-1}$ (b) $\frac{2^{-3}}{2^0}$
- (c) $(\frac{1}{4})^{-2}$
- **14.** (a) $-2^{-3} \cdot (-2)^0$ (b) $-2^3 \cdot (-2)^0$ (c) $\left(\frac{-2}{3}\right)^{-3}$

- 15. (a) $5^3 \cdot 5$
- **(b)** $3^2 \cdot 3^0$
- (c) $(2^2)^3$

- 16. (a) $3^8 \cdot 3^5$
- **(b)** $6^0 \cdot 6$
- (c) $(5^4)^2$
- 17. (a) $5^4 \cdot 5^{-2}$ (b) $\frac{10^7}{10^4}$ (c) $\frac{3^2}{3^4}$
- **18.** (a) $3^{-3} \cdot 3^{-1}$ (b) $\frac{5^4}{5}$
- (c) $\frac{7^2}{7^5}$

19–24 ■ Expressions with Exponents Simplify each expression.

- **19.** (a) x^2x^3
- **(b)** $(-x^2)^3$
- (c) $t^{-3}t^5$

- **20.** (a) $v^5 \cdot v^2$
- **(b)** $(8x)^2$
- (c) x^4x^{-3}
- **21.** (a) $x^{-5} \cdot x^3$ (b) $w^{-2}w^{-4}w^5$ (c) $\frac{y^{10}y^0}{y^7}$
- **22.** (a) $y^2 \cdot y^{-5}$ (b) $z^5 z^{-3} z^{-4}$ (c) $\frac{x^6}{1.10}$
- **23.** (a) $\frac{a^9a^{-2}}{a}$ (b) $(a^2a^4)^3$ (c) $(2x)^2(5x^6)$

- **24.** (a) $\frac{z^2z^4}{z^3z^{-1}}$
- **(b)** $(2a^3a^2)^4$
- (c) $(-3z^2)^3(2z^3)$

25–34 ■ Simplifying Expressions with Exponents Simplify each expression, and eliminate any negative exponent(s).

- **25.** (a) $(3x^2y)(2x^3)$
- **(b)** $(2a^2b^{-1})(3a^{-2}b^2)$
- (c) $(4y^2)(x^4y)^2$
- **26.** (a) $(4x^3y^2)(7y^5)$
- **(b)** $(9y^{-2}z^2)(3y^3z)$
- (c) $(8x^7y^2)(\frac{1}{2}x^3y)^{-2}$
- **27.** (a) $(2x^2y^3)^2(3y)$
- **(b)** $\frac{x^2y^{-1}}{-5}$
- (c) $\left(\frac{x^2y}{3}\right)^3$
- **28.** (a) $(5x^{-4}y^3)(8x^3)^2$
- **(b)** $\frac{y^{-2}z^{-3}}{y^{-1}}$
- (c) $\left(\frac{a^3b^{-2}}{b^3}\right)^2$
- **29.** (a) $(x^3y^3)^{-1}$
- **(b)** $(a^2b^{-2})^{-3}(a^3)^{-2}$
- (c) $\left(\frac{x^2}{v^{-2}}\right)^{-2} \left(\frac{2y^{-3}}{r^2}\right)^3$
- **30.** (a) $(x^{-2}y^4)^{-3}$
- **(b)** $(y^2)^{-1}(2x^{-3}y^4)^{-3}$
- (c) $\left(\frac{2a^{-1}}{b^{-2}}\right)^{-3} \left(\frac{b^{-1}}{2a^2}\right)^2$
- 31. (a) $\frac{3x^{-2}y^5}{9x^{-3}y^2}$
- **(b)** $\left(\frac{2x^3y^{-1}}{y^2}\right)^{-2}$
- (c) $\left(\frac{y^{-1}}{x^{-2}}\right)^{-1} \left(\frac{3x^{-3}}{v^2}\right)^{-2}$
- 32. (a) $\frac{\frac{1}{2}a^{-3}b^{-4}}{2a^{-5}b^{-1}}$
- **(b)** $\left(\frac{x^2y}{5x^4}\right)^{-2}$
- (c) $\left(\frac{2y^{-1}z}{z^2}\right)^{-1} \left(\frac{y}{3z^2}\right)^2$
- **33.** (a) $\left(\frac{3a}{b^3}\right)^{-1}$
- **(b)** $\left(\frac{q^{-1}r^{-1}s^{-2}}{r^{-5}sa^{-8}}\right)^{-1}$
- **34.** (a) $\left(\frac{s^2t^{-4}}{5c^{-1}t}\right)^{-2}$
- **(b)** $\left(\frac{xy^{-2}z^{-3}}{x^2y^3z^{-4}}\right)^{-3}$

35–36 ■ **Scientific Notation** Write each number in scientific notation.

- **35.** (a) 69,300,000
- **(b)** 7,200,000,000,000
- (c) 0.000028536
- (d) 0.0001213
- **36.** (a) 129,540,000
- **(b)** 7,259,000,000
- (c) 0.0000000014
- (d) 0.0007029

37–38 ■ **Decimal Notation** Write each number in decimal notation.

- **37.** (a) 3.19×10^5
- **(b)** 2.721×10^8
- (c) 2.670×10^{-8}
- (d) 9.999×10^{-9}
- **38.** (a) 7.1×10^{14}
- **(b)** 6×10^{12}
- (c) 8.55×10^{-3}
- (d) 6.257×10^{-10}

39–40 ■ Scientific Notation Write the number indicated in each statement in scientific notation.

- **39.** (a) A light-year, the distance that light travels in one year, is about 5,900,000,000,000 mi.
 - **(b)** The diameter of an electron is about 0.0000000000000 cm.
 - (c) A drop of water contains more than 33 billion billion molecules.
- **40.** (a) The distance from the earth to the sun is about 93 million miles.
 - (b) The mass of an oxygen molecule is about
 - (c) The mass of the earth is about 5,970,000,000,000,000,000,000,000 kg.
- **41–46 Scientific Notation** Use scientific notation, the Laws of Exponents, and a calculator to perform the indicated operations. State your answer rounded to the number of significant digits indicated by the given data.
- **41.** $(7.2 \times 10^{-9})(1.806 \times 10^{-12})$
 - **42.** $(1.062 \times 10^{24})(8.61 \times 10^{19})$
 - $\frac{1.295643 \times 10^9}{(3.610 \times 10^{-17})(2.511 \times 10^6)}$
 - **44.** $(73.1)(1.6341 \times 10^{28})$ 0.0000000019
 - - $\frac{(0.0000162)(0.01582)}{(594,621,000)(0.0058)}$ **46.** $\frac{(3.542 \times 10^{-6})^9}{(5.05 \times 10^4)^{12}}$

SKILLS Plus

- **47. Distances Between Powers** Which pair of numbers is closer together?
 - 10^{10} and 10^{50}
- 10¹⁰⁰ and 10¹⁰¹
- **48.** Signs of Numbers Let a, b, and c be real numbers with a > 0, b < 0, and c < 0. Determine the sign of each expression.
 - (a) b^5
- (c) ab^2c^3

- (a) b^3 (b) b^{10} (c) ab^2c^2 (d) $(b-a)^3$ (e) $(b-a)^4$ (f) $\frac{a^3c^3}{b^6c^6}$

APPLICATIONS

- **49. Distance to the Nearest Star** Proxima Centauri, the star nearest to our solar system, is 4.3 light-years away. Use the information in Exercise 39(a) to express this distance in miles.
- **50. Speed of Light** The speed of light is about 186,000 mi/s. Use the information in Exercise 40(a) to find how long it takes for a light ray from the sun to reach the earth.
- **51. Volume of the Oceans** The average ocean depth is 3.7×10^3 m, and the area of the oceans is 3.6×10^{14} m². What is the total volume of the ocean in liters? (One cubic meter contains 1000 liters.)



52. National Debt As of July 2013, the population of the United States was 3.164×10^8 , and the national debt was 1.674×10^{13} dollars. How much was each person's share of the debt?

[Source: U.S. Census Bureau and U.S. Department of Treasury]

- **53.** Number of Molecules A sealed room in a hospital, measuring 5 m wide, 10 m long, and 3 m high, is filled with pure oxygen. One cubic meter contains 1000 L, and 22.4 L of any gas contains 6.02×10^{23} molecules (Avogadro's number). How many molecules of oxygen are there in the room?
- **54.** Body-Mass Index The body-mass index is a measure that medical researchers use to determine whether a person is overweight, underweight, or of normal weight. For a person who weighs W pounds and who is H inches tall, the bodymass index B is given by

$$B = 703 \frac{W}{H^2}$$

A body-mass index is considered "normal" if it satisfies $18.5 \le B \le 24.9$, while a person with body-mass index $B \ge 30$ is considered obese.

(a) Calculate the body-mass index for each person listed in the table, then determine whether he or she is of normal weight, underweight, overweight, or obese.

Person	Weight	Height				
Brian Linda Larry	295 lb 105 lb 220 lb	5 ft 10 in. 5 ft 6 in. 6 ft 4 in.				
Helen	110 lb	5 ft 2 in.				

(b) Determine your own body-mass index.

55. Interest on a CD A sum of \$5000 is invested in a 5-year certificate of deposit paying 3% interest per year, compounded monthly. After n years the amount of interest I that has accumulated is given by

$$I = 5000[(1.0025)^{12n} - 1]$$

Complete the following table, which gives the amount of interest accumulated after the given number of years.

Year	Total interest
1	\$152.08
2	308.79
3	
4	
5	

DISCUSS **WRITE** DISCOVER **PROVE**

56. DISCUSS: How Big Is a Billion? If you had a million (10^6) dollars in a suitcase and you spent a thousand (10³) dollars

- each day, how many years would it take you to use all the money? If you spent at the same rate, how many years would it take you to empty a suitcase filled with a billion (10⁹) dollars?
- 57. DISCUSS: Easy Powers That Look Hard Calculate these expressions in your head. Use the Laws of Exponents to help
 - (a) $\frac{18^5}{9^5}$
- **(b)** $20^6 \cdot (0.5)^6$
- **58. PROVE:** Laws of Exponents Prove the following laws of exponents for the case in which m and n are positive integers

 - (a) Law 2: $\frac{a^m}{a^n} = a^{m-n}$ (b) Law 5: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
- **59. PROVE:** Laws of Exponents Prove the following laws of exponents.
 - (a) Law 6: $\left(\frac{a}{b}\right)^{-n} = \frac{b^n}{a^n}$ (b) Law 7: $\frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}$

RATIONAL EXPONENTS AND RADICALS

Radicals Rational Exponents Rationalizing the Denominator; Standard Form

In this section we learn to work with expressions that contain radicals or rational exponents.

Radicals

We know what 2^n means whenever n is an integer. To give meaning to a power, such as $2^{4/5}$, whose exponent is a rational number, we need to discuss radicals.

The symbol $\sqrt{\ }$ means "the positive square root of." Thus

$$\sqrt{a} = b$$
 means $b^2 = a$ and $b \ge 0$

Since $a = b^2 \ge 0$, the symbol \sqrt{a} makes sense only when $a \ge 0$. For instance,

$$\sqrt{9} = 3$$
 because $3^2 = 9$ and $3 \ge 6$

Square roots are special cases of *n*th roots. The *n*th root of *x* is the number that, when raised to the *n*th power, gives x.

DEFINITION OF nth ROOT

If n is any positive integer, then the **principal** nth root of a is defined as follows:

$$\sqrt[n]{a} = b$$
 means $b^n = a$

If *n* is even, we must have $a \ge 0$ and $b \ge 0$.

It is true that the number 9 has two square roots, 3 and -3, but the notation $\sqrt{9}$ is reserved for the *positive* square root of 9 (sometimes called the principal square root of 9). If we want the negative root, we must write $-\sqrt{9}$, which is -3.

$$\sqrt[4]{81} = 3$$
 because $3^4 = 81$ and $3 \ge 0$
 $\sqrt[3]{-8} = -2$ because $(-2)^3 = -8$

But $\sqrt{-8}$, $\sqrt[4]{-8}$, and $\sqrt[6]{-8}$ are not defined. (For instance, $\sqrt{-8}$ is not defined because the square of every real number is nonnegative.)

Notice that

$$\sqrt{4^2} = \sqrt{16} = 4$$
 but $\sqrt{(-4)^2} = \sqrt{16} = 4 = |-4|$

So the equation $\sqrt{a^2} = a$ is not always true; it is true only when $a \ge 0$. However, we can always write $\sqrt{a^2} = |a|$. This last equation is true not only for square roots, but for any even root. This and other rules used in working with nth roots are listed in the following box. In each property we assume that all the given roots exist.

PROPERTIES OF nth ROOTS

Property

1. $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$

$$2. \ \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

3.
$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$$

4.
$$\sqrt[n]{a^n} = a$$
 if n is odd

5.
$$\sqrt[n]{a^n} = |a|$$
 if *n* is even $\sqrt[4]{(-3)^4} = |-3| = 3$

Example

$$\sqrt[3]{-8 \cdot 27} = \sqrt[3]{-8} \sqrt[3]{27} = (-2)(3) = -6$$

$$\sqrt[4]{\frac{16}{81}} = \frac{\sqrt[4]{16}}{\sqrt[4]{81}} = \frac{2}{3}$$

$$\sqrt[3]{729} = \sqrt[6]{729} = 3$$

4.
$$\sqrt[n]{a^n} = a$$
 if *n* is odd $\sqrt[3]{(-5)^3} = -5$, $\sqrt[5]{2^5} = 2$

$$\sqrt[4]{(-3)^4} = |-3| = 3$$

EXAMPLE 1 Simplifying Expressions Involving *n*th Roots

(a)
$$\sqrt[3]{x^4} = \sqrt[3]{x^3}x$$

= $\sqrt[3]{x^3}\sqrt[3]{x}$
= $x\sqrt[3]{x}$

Factor out the largest cube

Property 1: $\sqrt[3]{ab} = \sqrt[3]{a}\sqrt[3]{b}$

Property 4: $\sqrt[3]{a^3} = a$

(b)
$$\sqrt[4]{81x^8y^4} = \sqrt[4]{81}\sqrt[4]{x^8}\sqrt[4]{y^4}$$

= $3\sqrt[4]{(x^2)^4}|y|$
= $3x^2|y|$

Property 1:
$$\sqrt[4]{abc} = \sqrt[4]{a}\sqrt[4]{b}\sqrt[4]{c}$$

Property 5:
$$\sqrt[4]{a^4} = |a|$$

Property 5:
$$\sqrt[4]{a^4} = |a|, |x^2| = x^2$$

Now Try Exercises 27 and 35

It is frequently useful to combine like radicals in an expression such as $2\sqrt{3} + 5\sqrt{3}$. This can be done by using the Distributive Property. For example,

$$2\sqrt{3} + 5\sqrt{3} = (2+5)\sqrt{3} = 7\sqrt{3}$$

The next example further illustrates this process.

EXAMPLE 2 Combining Radicals

(a)
$$\sqrt{32} + \sqrt{200} = \sqrt{16 \cdot 2} + \sqrt{100 \cdot 2}$$
 Factor out the largest squares
$$= \sqrt{16}\sqrt{2} + \sqrt{100}\sqrt{2}$$
 Property 1
$$= 4\sqrt{2} + 10\sqrt{2} = 14\sqrt{2}$$
 Distributive Property

For instance, if we let a = 9 and b = 16, then we see the error:

$$\sqrt{9+16} \stackrel{?}{=} \sqrt{9} + \sqrt{16}$$
$$\sqrt{25} \stackrel{?}{=} 3+4$$
$$5 \stackrel{?}{=} 7 \quad \text{Wrong!}$$

(b) If
$$b > 0$$
, then

$$\sqrt{25b} - \sqrt{b^3} = \sqrt{25}\sqrt{b} - \sqrt{b^2}\sqrt{b}$$
 Property 1: $\sqrt{xy} = \sqrt{x}\sqrt{y}$
$$= 5\sqrt{b} - b\sqrt{b}$$
 Property 5, $b > 0$
$$= (5 - b)\sqrt{b}$$
 Distributive Property

(c)
$$\sqrt{49x^2 + 49} = \sqrt{49(x^2 + 1)}$$
 Factor out 49
= $7\sqrt{x^2 + 1}$ Property 1: $\sqrt{ab} = \sqrt{a}\sqrt{b}$

Now Try Exercises 39, 43, and 47

Rational Exponents

To define what is meant by a *rational exponent* or, equivalently, a *fractional exponent* such as $a^{1/3}$, we need to use radicals. To give meaning to the symbol $a^{1/n}$ in a way that is consistent with the Laws of Exponents, we would have to have

$$(a^{1/n})^n = a^{(1/n)n} = a^1 = a$$

So by the definition of *n*th root,

$$a^{1/n} = \sqrt[n]{a}$$

In general, we define rational exponents as follows.

DEFINITION OF RATIONAL EXPONENTS

For any rational exponent m/n in lowest terms, where m and n are integers and n > 0, we define

$$a^{m/n} = (\sqrt[n]{a})^m$$
 or equivalently $a^{m/n} = \sqrt[n]{a^m}$

If *n* is even, then we require that $a \ge 0$.

With this definition it can be proved that the Laws of Exponents also hold for rational exponents (see page 19).

EXAMPLE 3 Using the Definition of Rational Exponents

(a)
$$4^{1/2} = \sqrt{4} = 2$$

(b)
$$8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4$$
 Alternative solution: $8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$

(c)
$$125^{-1/3} = \frac{1}{125^{1/3}} = \frac{1}{\sqrt[3]{125}} = \frac{1}{5}$$

Now Try Exercises 49 and 51

EXAMPLE 4 Using the Laws of Exponents with Rational Exponents

(a)
$$a^{1/3}a^{7/3} = a^{8/3}$$
 Law 1: $a^m a^n = a^{m+n}$

(b)
$$\frac{a^{2/5}a^{7/5}}{a^{3/5}} = a^{2/5+7/5-3/5} = a^{6/5}$$
 Law 1, Law 2: $\frac{a^m}{a^n} = a^{m-n}$