EDITION

7

Spreadsheet Modeling & Decision Analysis

A Practical Introduction to Business Analytics

Cliff T. Ragsdale

Spreadsheet Modeling & Decision Analysis 7e

A Practical Introduction to Business Analytics

Cliff Ragsdale

Virginia Polytechnic Institute
and State University

In memory of those
who were killed and injured
in the noble pursuit of education
here at Virginia Tech on April 16, 2007

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Preface

Spreadsheets are one of the most popular and ubiquitous software packages on the planet. Every day, millions of business people use spreadsheet programs to build models of the decision problems they face as a regular part of their work activities. As a result, employers look for experience and ability with spreadsheets in the people they recruit.

Spreadsheets have also become the standard vehicle for introducing undergraduate and graduate students in business and engineering to the concepts and tools covered in the introductory quantitative analysis course. This simultaneously develops students' skills with a standard tool of today's business world and opens their eyes to how a variety of quantitative analysis techniques can be used in this modeling environment. Spreadsheets also capture students' interest and add a new relevance to quantitative analysis, as they see how it can be applied with popular commercial software being used in the business world.

Spreadsheet Modeling & Decision Analysis: A Practical Introduction to Business Analytics provides an introduction to the most commonly used quantitative analysis techniques and shows how these tools can be implemented using Microsoft® Excel. Prior experience with Excel is certainly helpful, but is not a requirement for using this text. In general, a student familiar with computers and the spreadsheet concepts presented in most introductory computer courses should have no trouble using this text. Step-by-step instructions and screen shots are provided for each example, and software tips are included throughout the text as needed.

What's New in the Seventh Edition?

The most significant changes in the seventh edition of *Spreadsheet Modeling & Decision Analysis* are its new focus on business analytics, a new chapter on data mining, and extensive coverage and use of Analytic Solver Platform for Education by Frontline Systems, Inc. Analytic Solver Platform for Education is an add-in for Excel that provides access to analytical tools for performing optimization, simulation, sensitivity analysis, and decision tree analysis, as well as a variety of tools for data mining. Analytic Solver Platform for Education makes it easy to run multiple parameterized optimizations and simulations and apply optimization techniques to simulation models in one integrated, coherent interface. Analytic Solver Platform also offers amazing interactive simulation features in which simulation results are automatically updated in real-time whenever a manual change is made to a spreadsheet. Additionally, when run in its optional "Guided Mode," Analytic Solver Platform provides students with over 100 customized dialogs that provide diagnoses of various model conditions and explain the steps involved in solving problems. Analytic Solver Platform also includes Frontline's XLMiner product that offers easy access to a variety of data mining techniques including discriminant analysis, logistic regression, neural networks, classification and regression trees, k-nearest neighbor classification, cluster analysis, and affinity analysis. Analytic Solver Platform offers numerous other features and, I believe, will transform the way we approach education in quantitative analysis now and in the future.

The most significant changes in the seventh edition of *Spreadsheet Modeling & Decision Analysis* from the sixth edition include:

- Microsoft® Office 2013 is featured throughout.
- Data files and software that accompany the book are now available for download online at www.cengagebrain.com.
- Chapter 1 is re-written from a business analytics perspective and focuses on the use of quantitative analysis to leverage business opportunities. The new business analytics perspective is carried on throughout the text.
- Chapter 4 features new enhancements to Analytic Solver Platform that simplify the creation of spider plots and solver tables.
- Chapter 7 contains new discussion of the triple bottom line perspective as it relates to multi-criteria optimization.
- Chapter 10 (formerly covering discriminant analysis) now provides a full introduction to the topic of data mining including descriptions and examples of the major data mining techniques and the use of XLMiner.
- Several new and revised end-of-chapter problems are incorporated throughout.

Innovative Features

Aside from its strong spreadsheet orientation, the seventh edition of *Spreadsheet Modeling & Decision Analysis* contains several other unique features that distinguish it from traditional quantitative analysis texts.

- Algebraic formulations and spreadsheets are used side-by-side to help develop conceptual thinking skills.
- Step-by-step instructions and numerous annotated screen shots make examples easy to follow and understand.
- Emphasis is placed on model formulation and interpretation rather than on algorithms.
- Realistic examples motivate the discussion of each topic.
- Solutions to example problems are analyzed from a managerial perspective.
- Spreadsheet files for all the examples are provided online.
- A unique and accessible chapter covering data mining is provided.
- Sections entitled "The World of Business Analytics" show how each topic has been applied in a real company.

Organization

The table of contents for *Spreadsheet Modeling & Decision Analysis* is laid out in a fairly traditional format, but topics may be covered in a variety of ways. The text begins with an overview of business analytics in Chapter 1. Chapters 2 through 8 cover various topics in deterministic modeling techniques: linear programming, sensitivity analysis, networks, integer programming, goal programming and multiple objective optimization, and nonlinear and evolutionary programming. Chapters 9 through 11 cover predictive modeling and forecasting techniques: regression analysis, data mining, and time series analysis.

Chapters 12 and 13 cover stochastic modeling techniques: simulation and queuing theory. Chapter 14 covers decision analysis and Chapter 15 (available online) provides an introduction to project management.

After completing Chapter 1, a quick refresher on spreadsheet fundamentals (entering and copying formulas, basic formatting and editing, etc.) is always a good idea. Suggestions for the Excel review may be found at Cengage's Decision Sciences website. Following this, an instructor could cover the material on optimization, forecasting, or simulation, depending on personal preferences. The chapters on queuing and project management make general references to simulation and, therefore, should follow the discussion of that topic.

Ancillary Materials

Several excellent ancillaries for the instructor accompany the revised edition of *Spreadsheet Modeling & Decision Analysis*. All instructor ancillaries are provided online at www.cengagebrain.com. Included in this convenient format are:

- **Instructor's Manual.** The Instructor's Manual, prepared by the author, contains solutions to all the text problems and cases.
- **Test Bank.** The Test Bank, prepared by Tom Bramorski of the University of Wisconsin-Whitewater, includes multiple choice, true/false, and short answer problems for each text chapter. It also includes mini-projects that may be assigned as take-home assignments. The Test Bank is included as Microsoft® Word files. The Test Bank also comes separately in a computerized ExamView™ format that allows instructors to use or modify the questions and create original questions.
- **PowerPoint Presentation Slides.** PowerPoint presentation slides, prepared by the author, provide ready-made lecture material for each chapter in the book.

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My sincere thanks goes to all students and instructors who have used previous editions of this book and provided many valuable comments and suggestions for making it better. I also thank the wonderful SMDA team at Cengage : Aaron Arnsparger, Product Manager; Maggie Kubale, Content Development; Cliff Kallemeyn, Sr. Content Project Manager; and Chris Valentine, Media Developer. I feel very fortunate and privileged to work with each of you.

A very special word of thanks to my friend Dan Fylstra and the crew at Frontline Systems **(http://www.solver.com)** for conceiving and creating Analytic Solver Platform and supporting me so graciously and quickly throughout my revision work on this book. In my opinion, Analytic Solver Platform is the most significant development in OR/MS education since the creation of personal computers and the electronic spreadsheet. (Dan, you get my vote for a lifetime achievement award in analytical modeling and induction in the OR/MS Hall of Fame!)

Once again, I thank my dear wife, Kathy, for her unending patience, support, encouragement, and love. (You will always be the one.) This book is dedicated to our sons, Thomas, Patrick, and Daniel. I am proud of each one of you and will always be so glad that God let me be your daddy and the leader of the Ragsdale ragamuffin band.

Final Thoughts

I hope you enjoy the spreadsheet approach to teaching quantitative analysis as much as I do and that you find this book to be very interesting and helpful. If you find creative ways to use the techniques in this book or need help applying them, I would love to hear from you. Also, any comments, questions, suggestions, or constructive criticism you have concerning this text are always welcome.

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Brief Contents

Contents

Chapter 1

Introduction to Modeling and Decision Analysis

1.0 Introduction

This book is titled *Spreadsheet Modeling and Decision Analysis: A Practical Introduction to Business Analytics*, so let's begin by discussing exactly what this title means. By the very nature of life, all of us must continually make decisions that we hope will solve problems and lead to increased opportunities for ourselves or the organizations for which we work. But making good decisions is rarely an easy task. The problems faced by decision makers in today's competitive, data-intensive, fast-paced business environment are often extremely complex and can be addressed by numerous possible courses of action. Evaluating these alternatives and choosing the best course of action represents the essence of decision analysis.

Since the inception of the electronic spreadsheet in the early 1980s, millions of business people have discovered that one of the most effective ways to analyze and evaluate decision alternatives involves using a spreadsheet package to build computer models of the business opportunities and decision problems they face. A **computer model** is a set of mathematical relationships and logical assumptions implemented in a computer as a representation of some real-world object, decision problem, or phenomenon. Today, electronic spreadsheets provide the most convenient and useful way for business people to implement and analyze computer models. In fact, most business people would probably rate the electronic spreadsheet as their most important analytical tool apart from their brain! Using a **spreadsheet model** (a computer model implemented via a spreadsheet), a businessperson can analyze decision alternatives before having to choose a specific plan for implementation.

This book introduces you to a variety of techniques from the field of business analytics that can be applied in spreadsheet models to assist in the decision-analysis process. For our purposes, we will define **business analytics** as a field of study that uses data, computers, statistics, and mathematics to solve business problems. It involves using the methods and tools of science to drive business decision making. It is the science of making better decisions. Business analytics is also sometimes referred to as operations research, management science, or decision science. See Figure 1.1 for a summary of how business analytics has been applied successfully in a number of real-world situations.

In the not too distant past, business analytics was a highly specialized field that generally could be practiced only by those who had access to mainframe computers and who possessed advanced knowledge of mathematics, computer programming languages, and specialized software packages. However, the proliferation of powerful PCs and the development of easy-to-use electronic spreadsheets have made the tools of business analytics far more practical and available to a much larger audience. Virtually everyone

FIGURE 1.1

Examples of successful business analytics applications

Home Runs in Business Analytics

Over the past decade, scores of business analytics projects saved companies millions of dollars. Each year, the Institute for Operations Research and the Management Sciences (INFORMS) sponsors the Franz Edelman Awards competition to recognize some of the most outstanding business analytics projects during the past year. Here are some of the "home runs" from the 2010 and 2011 Edelman Awards (described in *Interfaces*, Vol. 41, No. 1, January-February, 2010, and Vol. 42, No. 1, January-February 2011).

- New Brunswick Department of Transportation (NBDoT) is charged with maintaining a strong transportation system throughout the province of New Brunswick, Canada—and on a limited budget. As a public entity accountable to its taxpayers, NBDoT must ensure that its strategic plan is defensible to the public it serves. To assist in this process, NBDoT built a linear programming model to help determine how better decisions could be developed. This model incorporates long-term objectives and operational constraints that consider costs, timings, and asset life cycles to produce optimal activity plans. This analysis helped secure a three-year commitment from the Government of New Brunswick for increased funding. NBDoT projects $72 million in annual savings as the return on this $2 million investment.
- In the early 2000s, Procter & Gamble (P&G) needed more advanced inventory tools to allow it to lower inventories while maintaining customer service. It used a multi-echelon inventory planning engine based on the guaranteed service model of safety stock optimization, allowing P&G to capture the multi-echelon nature of its supply chains. Since 2006, multi-echelon inventory optimization has been applied to more than 80% of P&G's global Beauty Care supply chains to address tactical and strategic production-inventory planning problems. Some applications of the multi-echelon decision tool have yielded cost reductions of more than 25%. P&G estimates this technique has reduced investments in inventory by $1.5 billion.
- Industrial and Commercial Bank of China (ICBC) is the world's largest publicly traded bank in terms of profitability, market capitalization, and deposit volume. ICBC has a network of more than 16,000 branch locations and needed to reconfigure them to match its evolving customer distribution. As a result, it required an analytic tool to quickly predict where new branches should be opened to serve promising, new, high-potential markets. The bank partnered with IBM to create a custom branch network optimization system. ICBC has implemented this system in more than 40 major cities in China. ICBC attributes over $1 billion in new deposits to this system in a typical major city.
- Although most consumers of electricity don't give its availability much thought, a lot of work goes into ensuring a constant balance in the real-time demand and generation of power. Midwest Independent Transmission System Operator Inc. (MISO) has transformed this process in the power industry of 13 Midwestern states in the United States through the development of energy and ancillary services markets. MISO uses a mixed-integer optimization model to determine when various power plants should be on or off and has developed other models to predict energy output levels and trading prices. This has increased the efficiency of the existing power plants and transmission lines, improved the reliability of the power grid, and reduced the need for additional investments in infrastructure. It is estimated that the MISO region has saved between $2.1 billion and $3.0 billion from 2007 through 2010 as a result of these efforts.

who uses a spreadsheet today for model building and decision making is a practitioner of business analytics—whether they realize it or not.

1.1 The Modeling Approach to Decision Making

The idea of using models in problem solving and decision analysis is really not new and is certainly not tied to the use of computers. At some point, we all have used a modeling approach to make a decision. For example, if you have ever moved into a dormitory, apartment, or house, you undoubtedly faced a decision about how to arrange the furniture. There were probably a number of different arrangements to consider. One arrangement might give you the most open space but require that you build a loft. Another might give you less space but allow you to avoid the hassle and expense of building a loft. To analyze these different arrangements and make a decision, you did not build the loft. You more likely built a **mental model** of the two arrangements, picturing what each looked like in your mind's eye. Thus, a simple mental model is sometimes all that is required to analyze a problem and make a decision.

For more complex decisions, a mental model might be impossible or insufficient, and other types of models might be required. For example, a set of drawings or blueprints for a house or building provides a **visual model** of the real-world structure. These drawings help illustrate how the various parts of the structure will fit together when it is completed. A road map is another type of visual model because it assists a driver in analyzing the various routes from one location to another.

You have probably also seen car commercials on television showing automotive engineers using **physical**, or **scale**, **models** to study the aerodynamics of various car designs to find the shape that creates the least wind resistance and maximizes fuel economy. Similarly, aeronautical engineers use scale models of airplanes to study the flight characteristics of various fuselage and wing designs. And civil engineers might use scale models of buildings and bridges to study the strengths of different construction techniques.

Another common type of model is a **mathematical model**, which uses mathematical relationships to describe or represent an object or decision problem. Throughout this book, we will study how various mathematical models can be implemented and analyzed on computers using spreadsheet software. But before we move to an in-depth discussion of spreadsheet models, let's look at some of the more general characteristics and benefits of modeling.

1.2 Characteristics and Benefits of Modeling

Although this book focuses on mathematical models implemented in computers via spreadsheets, the examples of nonmathematical models given earlier are worth discussing a bit more because they help illustrate a number of important characteristics and benefits of modeling in general. First, the models mentioned earlier are usually simplified versions of the object or decision problem they represent. To study the aerodynamics of a car design, we do not need to build the entire car complete with engine and stereo because such components have little or no effect on aerodynamics. So, although a model is often a simplified representation of reality, the model is useful as long as it is valid. A **valid model** is one that accurately represents the relevant characteristics of the object or decision problem being studied.

Second, it is often less expensive to analyze decision problems using a model. This is especially easy to understand with respect to scale models of big-ticket items such as cars and planes. Besides the lower financial cost of building a model, the analysis of a model can help avoid costly mistakes that might result from poor decision making. For example, it is far less costly to discover a flawed wing design using a scale model of an aircraft than after the crash of a fully loaded jet liner.

Frank Brock, former executive vice president of the Brock Candy Company, related the following story about blueprints his company prepared for a new production facility. After months of careful design work, he proudly showed the plans to several of his production workers. When he asked for their comments, one worker responded, "It's a fine looking building Mr. Brock, but that sugar valve looks like it's about twenty feet away from the steam valve." "What's wrong with that?" asked Brock. "Well, nothing," said the worker, "except that I have to have my hands on both valves at the same time!"[1] Needless to say, it was far less expensive to discover and correct this "little" problem using a visual model before pouring the concrete and laying the pipes as originally planned.

Third, models often deliver needed information on a timelier basis. Again, it is relatively easy to see that scale models of cars or airplanes can be created and analyzed more quickly than their real-world counterparts. Timeliness is also an issue when vital data will not become available until some later point in time. In these cases, we might create a model to help predict the missing data to assist in current decision making.

Fourth, models are frequently helpful in examining things that would be impossible to do in reality. For example, human models (crash dummies) are used in crash tests to see what might happen to an actual person if a car hits a brick wall at a high speed. Likewise, models of DNA can be used to visualize how molecules fit together. Both of these are difficult, if not impossible, to do without the use of models.

Finally, and probably most importantly, models allow us to gain insight and understanding about the object or decision problem under investigation. The ultimate purpose of using models is to improve decision making. As you will see, the process of building a model can shed important light and understanding on a problem. In some cases, a decision might be made while building the model as a previously misunderstood element of the problem is discovered or eliminated. In other cases, a careful analysis of a completed model might be required to "get a handle" on a problem and gain the insights needed to make a decision. In any event, it is the insight gained from the modeling process that ultimately leads to better decision making.

1.3 Mathematical Models

As mentioned earlier, the modeling techniques in this book differ quite a bit from scale models of cars and planes or visual models of production plants. The models we will build use mathematics to describe a decision problem. We use the term "mathematics" in its broadest sense, encompassing not only the most familiar elements of math, such as algebra, but also the related topic of logic.

Now, let's consider a simple example of a mathematical model:

$$\text{PROFIT} = \text{REVENUE} - \text{EXPENSES} \quad \textbf{1.1}$$

[1] Colson, Charles, and Jack Eckerd, *Why America Doesn't Work* (Denver, Colorado: Word Publishing, 1991), 146–147.

Equation 1.1 describes a simple relationship between revenue, expenses, and profit. This mathematical relationship describes the operation of determining profit—or a mathematical model of profit. Of course, not all models are this simple, but taken piece by piece, the models we will discuss are not much more complex than this one.

Frequently, mathematical models describe functional relationships. For example, the mathematical model in equation 1.1 describes a functional relationship between revenue, expenses, and profit. Using the symbols of mathematics, this functional relationship is represented as follows:

$$\text{PROFIT} = f(\text{REVENUE, EXPENSES}) \qquad \textbf{1.2}$$

In words, the previous expression means "profit is a function of revenue and expenses." We could also say that profit *depends* on (or is *dependent* on) revenue and expenses. Thus, the term PROFIT in equation 1.2 represents a **dependent variable**, whereas REVENUE and EXPENSES are **independent variables**. Frequently, compact symbols (such as A, B, and C) are used to represent variables in an equation such as 1.2. For instance, if we let Y, X_1, and X_2 represent PROFIT, REVENUE, and EXPENSES, respectively, we could rewrite equation 1.2 as follows:

$$Y = f(X_1, X_2) \qquad \textbf{1.3}$$

The notation $f(\cdot)$ represents the function that defines the relationship between the dependent variable Y and the independent variables X_1 and X_2. In the case of determining PROFIT from REVENUE and EXPENSES, the mathematical form of the function $f(\cdot)$ is quite simple because we know that $f(X_1, X_2) = X_1 - X_2$. However, in many other situations we will model, the form of $f(\cdot)$ is quite complex and might involve many independent variables. But regardless of the complexity of $f(\cdot)$ or the number of independent variables involved, many of the decision problems encountered in business can be represented by models that assume the general form,

$$Y = f(X_1, X_2, \ldots, X_k) \qquad \textbf{1.4}$$

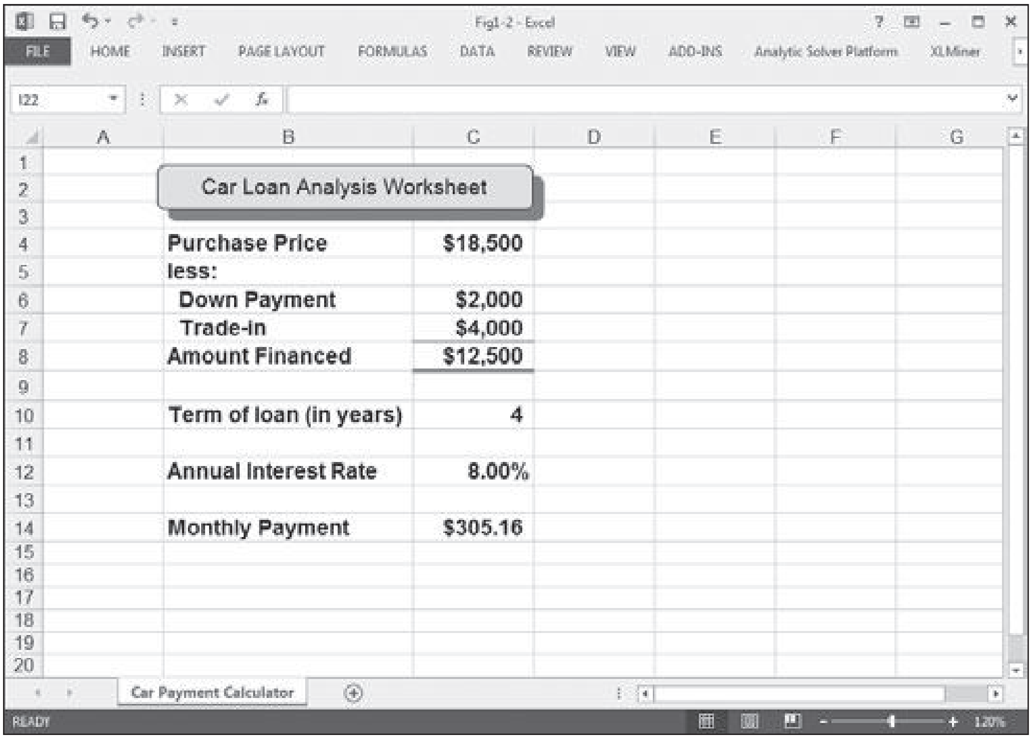
In equation 1.4, the dependent variable Y represents some bottom-line performance measure of the problem we are modeling. The terms $X_1, X_2, \ldots, X_k$ represent the different independent variables that play some role or have some impact in determining the value of Y. Again, $f(\cdot)$ is the function (possibly quite complex) that specifies or describes the relationship between the dependent and independent variables.

The relationship expressed in equation 1.4 is very similar to what occurs in most spreadsheet models. Consider a simple spreadsheet model to calculate the monthly payment for a car loan, as shown in Figure 1.2.

The spreadsheet in Figure 1.2 contains a variety of **input cells** (for example, purchase price, down payment, trade-in, term of loan, annual interest rate) that correspond conceptually to the independent variables $X_1, X_2, \ldots, X_k$ in equation 1.4. Similarly, a variety of mathematical operations are performed using these input cells in a manner analogous to the function $f(\cdot)$ in equation 1.4. The results of these mathematical operations determine the value of some **output cell** in the spreadsheet (for example, monthly payment) that corresponds to the dependent variable Y in equation 1.4. Thus, there is a direct correspondence between equation 1.4 and the spreadsheet in Figure 1.2. This type of correspondence exists for most of the spreadsheet models in this book.

FIGURE 1.2

Example of a simple spreadsheet model



1.4 Categories of Mathematical Models

Not only does equation 1.4 describe the major elements of mathematical or spreadsheet models, but it also provides a convenient means for comparing and contrasting the three categories of modeling techniques presented in this book: Prescriptive Models, Predictive Models, and Descriptive Models. Figure 1.3 summarizes the characteristics and some of the techniques associated with each of these categories.

FIGURE 1.3

Categories and characteristics of business analytics modeling techniques

	Model Characteristics:		
Category	**Form of $f(\cdot)$**	**Values of Independent Variables**	**Business Analytics Techniques**
Prescriptive Models	known, well-defined	known or under decision maker's control	Linear Programming, Networks, Integer Programming, CPM, Goal Programming, EOQ, Nonlinear Programming
Predictive Models	unknown, ill-defined	known or under decision maker's control	Regression Analysis, Time Series Analysis, Discriminant Analysis, Neural Networks, Logistic Regression, Affinity Analysis, Cluster Analysis
Descriptive Models	known, well-defined	unknown or uncertain	Simulation, Queuing, PERT, Inventory Models

In some situations, a manager might face a decision problem involving a very precise, well-defined functional relationship $f(\cdot)$ between the independent variables $X_1, X_2, \ldots, X_k$ and the dependent variable Y. If the values for the independent variables are under the decision maker's control, the decision problem in these types of situations boils down to determining the values of the independent variables $X_1, X_2, \ldots, X_k$ that produce the best possible value for the dependent variable Y. These types of models are called **Prescriptive Models** because their solutions tell the decision maker what actions to take. For example, you might be interested in determining how a given sum of money should be allocated to different investments (represented by the independent variables) to maximize the return on a portfolio without exceeding a certain level of risk.

A second category of decision problems is one in which the objective is to predict or estimate what value the dependent variable Y will take on when the independent variables $X_1, X_2, \ldots, X_k$ take on specific values. If the function $f(\cdot)$ relating the dependent and independent variables is known, this is a very simple task—simply enter the specified values for $X_1, X_2, \ldots, X_k$ into the function $f(\cdot)$ and compute Y. In some cases, however, the functional form of $f(\cdot)$ might be unknown and must be estimated in order for the decision maker to make predictions about the dependent variable Y. These types of models are called **Predictive Models**. For example, a real estate appraiser might know that the value of a commercial property (Y) is influenced by its total square footage (X_1) and age (X_2), among other things. However, the functional relationship $f(\cdot)$ that relates these variables to one another might be unknown. By analyzing the relationship between the selling price, total square footage, and age of other commercial properties, the appraiser might be able to identify a function $f(\cdot)$ that relates these two variables in a reasonably accurate manner.

The third category of models you are likely to encounter in the business world is called **Descriptive Models**. In these situations, a manager might face a decision problem that has a very precise, well-defined functional relationship $f(\cdot)$ between the independent variables $X_1, X_2, \ldots, X_k$ and the dependent variable Y. However, there might be great uncertainty as to the exact values that will be assumed by one or more of the independent variables $X_1, X_2, \ldots, X_k$. In these types of problems, the objective is to describe the outcome or behavior of a given operation or system. For example, suppose a company is building a new manufacturing facility and has several choices about the type of machines to put in the new plant, as well as various options for arranging the machines. Management might be interested in studying how the various plant configurations would affect on-time shipments of orders (Y), given the uncertain number of orders that might be received (X_1) and the uncertain due dates (X_2) that might be required by these orders.

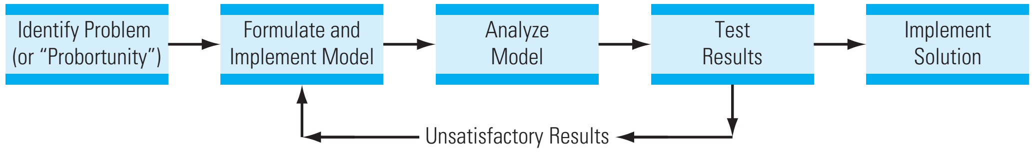
1.5 Business Analytics and the Problem-Solving Process

Business analytics focuses on identifying and leveraging business opportunities. But business **opportunities** can often be viewed or formulated as decision **problems** that need to be solved. As a result, the words "opportunity" and "problem" are used somewhat synonymously throughout this book. Some even use the term "probortunity" to denote that every problem is also an opportunity.

Throughout our discussion, we have said that the ultimate goal in building models is to assist managers in making decisions that solve problems. The modeling techniques we will study represent a small but important part of the total problem-solving process. The problem-solving process discussed here is usually focused on leveraging a business

FIGURE 1.4

A visual model of the problem-solving process



opportunity of one sort or another. To become an effective modeler, it is important to understand how modeling fits into the entire process. Because a model can be used to represent a decision problem or phenomenon, we might be able to create a visual model of the phenomenon that occurs when people solve problems—what we call the problem-solving process. Although a variety of models could be equally valid, the one in Figure 1.4 summarizes the key elements of the problem-solving process and is sufficient for our purposes.

The first step of the problem-solving process, identifying the problem (or "probortunity"), is also the most important. If we do not identify the correct decision problem associated with the business opportunity at hand, all the work that follows will amount to nothing more than wasted effort, time, and money. Unfortunately, identifying the problem to solve is often not as easy as it seems. We know that a problem exists when there is a gap or disparity between the present situation and some desired state. However, we usually are not faced with a neat, well-defined problem. Instead, we often find ourselves facing a "mess"![2] Identifying the real problem involves gathering a lot of information and talking with many people to increase our understanding of the mess. We must then sift through all this information and try to identify the root problem or problems causing the mess. Thus, identifying the real problem (and not just the symptoms of the problem) requires insight, some imagination, time, and a good bit of detective work.

The end result of the problem-identification step is a well-defined statement of the problem. Simply defining a problem well often makes it much easier to solve. There is much truth in the saying, "A problem clearly stated is a problem half solved." Having identified the problem, we turn our attention to creating or formulating a model of the problem. Depending on the nature of the problem, we might use a mental model, a visual model, a scale model, or a mathematical model. Although this book focuses on mathematical models, this does not mean that mathematical models are always applicable or best. In most situations, the best model is the simplest model that accurately reflects the relevant characteristic or essence of the problem being studied.

We will discuss several different business analytics techniques in this book. It is important that you not develop too strong a preference for any one technique. Some people have a tendency to want to formulate every problem they face as something that can be solved by their favorite modeling technique. This simply will not work.

As indicated earlier in Figure 1.3, there are fundamental differences in the types of problems a manager might face. Sometimes, the values of the independent variables affecting a problem are under the manager's control; sometimes they are not. Sometimes, the form of the functional relationship $f(\cdot)$ relating the dependent and independent variables is well-defined, and sometimes it is not. These fundamental characteristics of the problem should guide your selection of an appropriate business analytics modeling technique. Your goal at the model-formulation stage is to select a modeling technique that fits your problem, rather than trying to fit your problem into the required format of a preselected modeling technique.

After you select an appropriate representation or formulation of your problem, the next step is to implement this formulation as a spreadsheet model. We will not dwell on

[2] This characterization is borrowed from Chapter 5, James R. Evans, *Creative Thinking in the Decision and Management Sciences* (Cincinnati, Ohio: South-Western Publishing, 1991), 89–115.

the implementation process now because that is the focus of the remainder of this book. After you verify that your spreadsheet model has been implemented accurately, the next step in the problem-solving process is to use the model to analyze the problem it represents. The main focus of this step is to generate and evaluate alternatives that might lead to a solution of the problem. This often involves playing out a number of scenarios or asking several "What if?" questions. Spreadsheets are particularly helpful in analyzing mathematical models in this manner. In a well-designed spreadsheet model, it should be fairly simple to change some of the assumptions in the model to see what might happen in different situations. As we proceed, we will highlight some techniques for designing spreadsheet models that facilitate this type of "What if" analysis. "What if" analysis is also very appropriate and useful when working with nonmathematical models.

The end result of analyzing a model does not always provide a solution to the actual problem being studied. As we analyze a model by asking various "What if?" questions, it is important to test the feasibility and quality of each potential solution. The blueprints Frank Brock showed to his production employees represented the end result of his analysis of the problem he faced. He wisely tested the feasibility and quality of this alternative before implementing it and discovered an important flaw in his plans. Thus, the testing process can give important new insights into the nature of a problem. The testing process is also important because it provides the opportunity to double-check the validity of the model. At times, we might discover an alternative that appears too good to be true. This could lead us to find that some important assumption has been left out of the model. Testing the results of the model against known results (and simple common sense) helps ensure the structural integrity and validity of the model. After analyzing the model, we might discover that we need to go back and modify it.

The last step of the problem-solving process, implementation, is often the most difficult. Implementation begins by deriving managerial insights from our modeling efforts, framed in the context of the real-world problem we are solving, and communicating those insights to influence actions that affect the business situation. This requires crafting a message that is understood by various stakeholders in an organization and persuading them to take a particular course of action. (See Grossman et al., 2008, for numerous helpful suggestions on this process.) By their very nature, solutions to problems involve people and change. For better or for worse, most people resist change. However, there are ways to minimize the seemingly inevitable resistance to change. For example, it is wise, if possible, to involve anyone who will be affected by the decision in all steps of the problem-solving process. This not only helps develop a sense of ownership and understanding of the ultimate solution, but it also can be the source of important information throughout the problem-solving process. As the Brock Candy story illustrates, even if it is impossible to include those affected by the solution in all steps, their input should be solicited and considered before a solution is accepted for implementation. Resistance to change and new systems can also be eased by creating flexible, user-friendly interfaces for the mathematical models that are often developed in the problem-solving process.

Throughout this book, we focus mostly on the model formulation, implementation, analysis, and testing steps of the problem-solving process, summarized previously in Figure 1.4. Again, this does not imply that these steps are more important than the others. If we do not identify the correct problem, the best we can hope for from our modeling effort is "the right answer to the wrong question," which does not solve the real problem. Similarly, even if we do identify the problem correctly and design a model that leads to a perfect solution, if we cannot implement the solution, then we still have not solved the problem. Developing the interpersonal and investigative skills required to work with people in defining the problem and implementing the solution is as important as the mathematical modeling skills you will develop by working through this book.

1.6 Anchoring and Framing Effects

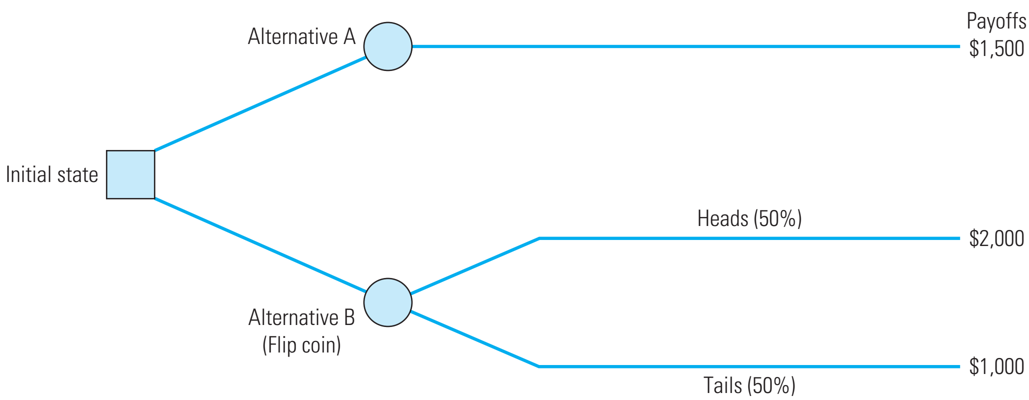
At this point, some of you are probably thinking it is better to rely on subjective judgment and intuition rather than models when making decisions. Most nontrivial decision problems do involve some issues that are difficult or impossible to structure and analyze in the form of a mathematical model. These unstructurable aspects of a decision problem may require the use of judgment and intuition. However, it is important to realize that human cognition is often flawed and can lead to incorrect judgments and irrational decisions. Errors in human judgment often arise because of what psychologists term **anchoring** and **framing** effects associated with decision problems.

Anchoring effects arise when a seemingly trivial factor serves as a starting point (or anchor) for estimations in a decision-making problem. Decision makers adjust their estimates from this anchor but nevertheless remain too close to the anchor and usually under-adjust. In a classic psychological study on this issue, one group of subjects was asked to individually estimate the value of $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8$ (without using a calculator). Another group of subjects were each asked to estimate the value of $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$. The researchers hypothesized that the first number presented (or perhaps the product of the first three or four numbers) would serve as a mental anchor. The results supported the hypothesis. The median estimate of subjects shown the numbers in ascending sequence ($1 \times 2 \times 3 \ldots$) was 512, whereas the median estimate of subjects shown the sequence in descending order ($8 \times 7 \times 6 \ldots$) was 2,250. Of course, the order of multiplication for these numbers is irrelevant, and the product of both series is the same: 40,320.

Framing effects refer to how a decision maker views or perceives the alternatives in a decision problem—often involving a win/loss perspective. The way a problem is framed often influences the choices made by a decision maker and can lead to irrational behavior. For example, suppose you have just been given \$1,000 but must choose one of the following alternatives: (A_1) Receive an additional \$500 with certainty, or (B_1) Flip a fair coin and receive an additional \$1,000 if heads occurs or \$0 additional if tails occurs. Here, alternative A_1 is a "sure win" and is the alternative most people prefer. Now suppose you have been given \$2,000 and must choose one of the following alternatives: (A_2) Give back \$500 immediately, or (B_2) Flip a fair coin and give back \$0 if heads occurs or \$1,000 if tails occurs. When the problem is framed this way, alternative A_2 is a "sure loss," and many people who previously preferred alternative A_1 now opt for alternative B_2 (because it holds a chance of avoiding a loss). However, Figure 1.5 shows a single decision tree for these two scenarios making it clear that, in both cases, the "A" alternative guarantees a total payoff of \$1,500, whereas the "B" alternative offers a 50% chance

FIGURE 1.5

Decision tree for framing effects



of a $2,000 total payoff and a 50% chance of a $1,000 total payoff. (Decision trees will be covered in greater detail in a later chapter.) A purely rational decision maker should focus on the consequences of his or her choices and consistently select the same alternative, regardless of how the problem is framed.

Whether we want to admit it or not, we are all prone to make errors in estimation due to anchoring effects and may exhibit irrationality in decision making due to framing effects. As a result, it is best to use computer models to do what they are best at (that is, modeling structurable portions of a decision problem) and let the human brain do what it is best at (that is, dealing with the unstructurable portion of a decision problem).

1.7 Good Decisions vs. Good Outcomes

The goal of the modeling approach to problem solving is to help individuals make good decisions. But good decisions do not always result in good outcomes. For example, suppose the weather report on the evening news predicts a warm, dry, sunny day tomorrow. When you get up and look out the window tomorrow morning, there is not a cloud in sight. If you decide to leave your umbrella at home and subsequently get soaked in an unexpected afternoon thundershower, did you make a bad decision? Certainly not. Unforeseeable circumstances beyond your control caused you to experience a bad outcome, but it would be unfair to say that you made a bad decision. Good decisions sometimes result in bad outcomes. See Figure 1.6 for the story of another good decision with a bad outcome.

The modeling techniques presented in this book can help you make good decisions but cannot guarantee that good outcomes will always occur as a result of those decisions. Figure 1.7 describes the possible combinations of good and bad decisions and good and bad outcomes. When a good or bad decision is made, luck often plays a role in determining whether a good or bad outcome occurs. However, consistently using a structured, model-based process to make decisions should produce good outcomes (and deserved success) more frequently than making decisions in a more haphazard manner.

FIGURE 1.6

A good decision with a bad outcome

Andre-Francois Raffray thought he had a great deal in 1965 when he agreed to pay a 90-year-old woman named Jeanne Calment $500 a month until she died to acquire her grand apartment in Arles, northwest of Marseilles in the south of France—a town Vincent van Gogh once roamed. Buying apartments "for life" is common in France. The elderly owner gets to enjoy a monthly income from the buyer who gambles on getting a real estate bargain—betting the owner doesn't live too long. Upon the owner's death, the buyer inherits the apartment regardless of how much was paid. But in December of 1995, Raffray died at age 77, having paid more than $180,000 for an apartment he never got to live in.

On the same day, Calment, then the world's oldest living person at 120, dined on *foie gras*, duck thighs, cheese, and chocolate cake at her nursing home near the sought-after apartment. And she does not need to worry about losing her $500 monthly income. Although the amount Raffray already paid is twice the apartment's current market value, his widow is obligated to keep sending the monthly check to Calment. If Calment also outlives her, then the Raffray children will have to pay. "In life, one sometimes makes bad deals," said Calment of the outcome of Raffray's decision. (Source: *The Savannah Morning News*, 12/29/95.)

FIGURE 1.7

Decision quality and outcome quality matrix

		Outcome Quality	
		Good	Bad
Decision Quality	Good	Deserved Success	Bad Luck
	Bad	Dumb Luck	Poetic Justice

Adapted from: J. Russo and P. Shoemaker, *Winning Decisions* (New York, NY: Doubleday, 2002).

1.8 Summary

This book introduces you to a variety of techniques from the field of business analytics that can be applied in spreadsheet models to assist in decision analysis and problem solving. This chapter discussed how spreadsheet models of decision problems can be used to analyze the consequences of possible courses of action before a particular alternative is selected for implementation. It described how models of decision problems differ in a number of important characteristics and how you should select a modeling technique that is most appropriate for the type of problem being faced. The chapter covered how spreadsheet modeling and analysis fit into the problem-solving process. It then discussed how the psychological phenomena of anchoring and framing can influence human judgment and decision making. Finally, it described the importance of distinguishing between the quality of a decision-making process and the quality of decision outcomes.

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THE WORLD OF BUSINESS ANALYTICS

"Business Analysts Trained in Management Science Can Be a Secret Weapon in a CIO's Quest for Bottom-Line Results."

Efficiency nuts. These are the people you see at cocktail parties explaining how the host could disperse that crowd around the popular shrimp dip if he would divide it into three bowls and place them around the room. As she draws the improved traffic flow on a paper napkin, you notice that her favorite word is "optimize"—a tell-tale sign she has studied the field of "operations research" or "management science" (also known as OR/MS or business analytics).

OR/MS professionals are driven to solve logistics problems. This trait may not make them the most popular people at parties, but it is exactly what today's information systems (IS) departments need to deliver more business value. Experts say smart IS executives will learn to exploit the talents of these mathematical wizards in their quest to boost a company's bottom line.

According to Ron J. Ponder, chief information officer (CIO) at Sprint Corp. in Kansas City, Mo., and former CIO at Federal Express Corp., "If IS departments had more participation from operations research analysts, they would be building much better, richer IS solutions." As someone who has a Ph.D. in operations research and who built the renowned package-tracking systems at Federal Express, Ponder is a true believer in OR/MS. Ponder and others say analysts trained in OR/MS can turn ordinary information systems into money-saving, decision-support systems and are ideally suited to be members of the business process reengineering team. "I've always had an operations research department reporting to me, and it's been invaluable. Now I'm building one at Sprint," says Ponder.

The Beginnings

OR/MS got its start in World War II, when the military had to make important decisions about allocating scarce resources to various military operations. One of the first business applications for computers in the 1950s was to solve operations research problems for the petroleum industry. A technique called linear programming was used to figure out how to blend gasoline for the right flash point, viscosity, and octane in the most economical way. Since then, OR/MS has spread throughout business and government, from designing efficient drive-thru window operations for Burger King Corp. to creating ultra-sophisticated computerized stock-trading systems.

A classic OR/MS example is the crew-scheduling problem faced by all major airlines. How do you plan the itineraries of 8,000 pilots and 17,000 flight attendants when there is an astronomical number of combinations of planes, crews, and cities? The OR/MS analysts at United Airlines came up with a scheduling system called Paragon that attempts to minimize the amount of paid time that crews spend waiting for flights. The model factors in constraints such as union rules and Federal Aviation Administration regulations and is projected to save the airline at least $1 million a year.

OR/MS and IS

Somewhere in the 1970s, the OR/MS and IS disciplines went in separate directions. "The IS profession has had less and less contact with the operations research folks . . . and IS lost a powerful intellectual driver," says Peter G. W. Keen, executive director of the International Center for Information Technologies in Washington, D.C. However, many feel that now is an ideal time for the two disciplines to rebuild some bridges.

Today's OR/MS professionals are involved in a variety of IS-related fields, including inventory management, electronic data interchange, computer-integrated manufacturing, network management, and practical applications of artificial intelligence. Furthermore, each side needs something the other side has: OR/MS analysts need corporate data to plug into their models, and the IS folks need to plug the OR/MS models into their strategic information systems. At the same time,

(Continued)

CIOs need intelligent applications that enhance the bottom line and make them heroes with the CEO.

OR/MS analysts can develop a model of how a business process works now and simulate how it could work more efficiently in the future. Therefore, it makes sense to have an OR/MS analyst on the interdisciplinary team that tackles business process reengineering projects. In essence, OR/MS professionals add more value to the IS infrastructure by building "tools that really help decision makers analyze complex situations," says Andrew B. Whinston, director of the Center for Information Systems Management at the University of Texas at Austin.

Although IS departments typically believe their job is done if they deliver accurate and timely information, Thomas M. Cook, president of American Airlines Decision Technologies, Inc., says that adding OR/MS skills to the team can produce intelligent systems that actually recommend solutions to business problems. One of the big success stories at Cook's operations research shop is a "yield management" system that decides how much to overbook and how to set prices for each seat so that a plane is filled up and profits are maximized. The yield management system deals with more than 250 decision variables and accounts for a significant amount of American Airlines' revenue.

Where to Start

So how can the CIO start down the road toward collaboration with OR/MS analysts? If the company already has a group of OR/MS professionals, the IS department can draw on their expertise as internal consultants. Otherwise, the CIO can simply hire a few OR/MS wizards, throw a problem at them, and see what happens. The payback may come surprisingly fast. As one former OR/MS professional put it, "If I couldn't save my employer the equivalent of my own salary in the first month of the year, then I wouldn't feel like I was doing my job."

Adapted from: Mitch Betts, "Efficiency Einsteins," *ComputerWorld*, March 22, 1993, p. 64.

Questions and Problems

1. What is meant by the term *decision analysis*?
2. Define the term *computer model*.
3. What is the difference between a spreadsheet model and a computer model?
4. Define the term *business analytics*.
5. What is the relationship between business analytics and spreadsheet modeling?
6. What kinds of spreadsheet applications would not be considered business analytics?
7. In what ways do spreadsheet models facilitate the decision-making process?
8. What are the benefits of using a modeling approach to decision making?
9. What is a dependent variable?
10. What is an independent variable?
11. Can a model have more than one dependent variable?
12. Can a decision problem have more than one dependent variable?
13. In what ways are Prescriptive Models different from Descriptive Models?
14. In what ways are Prescriptive Models different from Predictive Models?

15. In what ways are Descriptive Models different from Predictive Models?
16. How would you define the words *description*, *prediction*, and *prescription*? Carefully consider what is unique about the meaning of each word.
17. Identify one or more mental models you have used. Can any of them be expressed mathematically? If so, identify the dependent and independent variables in your model.
18. Consider the spreadsheet model shown in Figure 1.2. Is this model Descriptive, Predictive, or Prescriptive in nature, or does it not fall into any of these categories?
19. Discuss the meaning of the term "probortunity."
20. What are the steps in the problem-solving process?
21. Which step in the problem-solving process do you think is most important? Why?
22. Must a model accurately represent every detail of a decision situation to be useful? Why or why not?
23. If you were presented with several different models of a given decision problem, which would you be most inclined to use? Why?
24. Describe an example in which business or political organizations may use anchoring effects to influence decision making.
25. Describe an example in which business or political organizations may use framing effects to influence decision making.
26. Suppose sharks have been spotted along the beach where you are vacationing with a friend. You and your friend have been informed of the shark sightings and are aware of the damage a shark attack can inflict on human flesh. You both decide (individually) to go swimming anyway. You are promptly attacked by a shark while your friend has a nice time body surfing in the waves. Did you make a good or bad decision? Did your friend make a good or bad decision? Explain your answer.
27. Describe an example in which a well-known business, political, or military leader made a good decision that resulted in a bad outcome, or made a bad decision that resulted in a good outcome.

Patrick's Paradox

CASE 1.1

Patrick's luck had changed overnight—but not his skill at mathematical reasoning. The day after graduating from college, he used the $20 that his grandmother had given him as a graduation gift to buy a lottery ticket. He knew his chances of winning the lottery were extremely low and it probably was not a good way to spend this money. But he also remembered from the class he took in management science that bad decisions sometimes result in good outcomes. So he said to himself, "What the heck? Maybe this bad decision will be the one with a good outcome." And with that thought, he bought his lottery ticket.

The next day, Patrick pulled the crumpled lottery ticket out of the back pocket of his jeans and tried to compare his numbers to the winning numbers printed in the paper. When his eyes finally came into focus on the numbers, they also just about popped out of his head. He had a winning ticket! In the ensuing days, he learned that his share of the jackpot would give him a lump sum payout of about $500,000 after taxes. He knew what he was going to do with part of the money, buy a new car, pay off his college loans, and send his grandmother on an all-expenses-paid trip to Hawaii. But he also knew that he couldn't continue to hope for good outcomes to arise from more bad decisions. So he decided to take half of his winnings and invest it for his retirement.

A few days later, Patrick was sitting around with two of his fraternity buddies, Josh and Peyton, trying to figure out how much money his new retirement fund might be worth in 30 years. They were all business majors in college and remembered from their finance class that if you invest p dollars for n years at an annual interest rate of i percent, then in n years you would have $p(1 + i)^n$ dollars. So they figured that if Patrick invested \$250,000 for 30 years in an investment with a 10% annual return, then in 30 years he would have \$4,362,351 (that is, $\$250{,}000(1 + 0.10)^{30}$).

But after thinking about it a little more, they all agreed that it would be unlikely for Patrick to find an investment that would produce a return of exactly 10% each and every year for the next 30 years. If any of this money is invested in stocks, then some years the return might be higher than 10%, and some years it would probably be lower. So to help account for the potential variability in the investment returns, Patrick and his friends came up with a plan. They would assume he could find an investment that would produce a 17.5% annual return 70% of the time and a –7.5% return (or actually a loss) 30% of the time. Such an investment should produce an average annual return of $0.7(17.5\%) + 0.3(-7.5\%) = 10\%$. Josh felt certain that this meant Patrick could still expect his \$250,000 investment to grow to \$4,362,351 in 30 years (because $\$250{,}000(1 + 0.10)^{30} = \$4{,}362{,}351$).

After sitting quietly and thinking about it for a while, Peyton said that he thought Josh was wrong. The way Peyton looked at it, Patrick should see a 17.5% return in 70% of the 30 years (or $0.7(30) = 21$ years) and a –7.5% return in 30% of the 30 years (or $0.3(30) = 9$ years). So, according to Peyton, that would mean Patrick should have $\$250{,}000(1 + 0.175)^{21}(1 - 0.075)^9 = \$3{,}664{,}467$ after 30 years. But that's \$697,884 less than what Josh says Patrick should have.

After listening to Peyton's argument, Josh said he thought Peyton was wrong because his calculation assumes that the "good" return of 17.5% would occur in each of the first 21 years, and the "bad" return of –7.5% would occur in each of the last 9 years. But Peyton countered this argument by saying that the order of good and bad returns does not matter. The commutative law of arithmetic says that when you add or multiply numbers, the order doesn't matter (that is, $X + Y = Y + X$ and $X \times Y = Y \times X$). So Peyton says that because Patrick can expect 21 "good" returns and 9 "bad" returns, and it doesn't matter in what order they occur, then the expected outcome of the investment should be \$3,664,467 after 30 years.

Patrick is now really confused. Both of his friends' arguments seem to make perfect sense logically—but they lead to such different answers, and they can't both be right. What really worries Patrick is that he is starting his new job as a business analyst in a couple of weeks. And if he can't reason his way to the right answer in a relatively simple problem like this, what is he going to do when he encounters the more difficult problems awaiting him in the business world? Now he really wishes he had paid more attention in his business analytics class.

So what do you think? Who is right, Joshua or Peyton? And more importantly, why?

Chapter 2

Introduction to Optimization and Linear Programming

2.0 Introduction

Our world is filled with limited resources. The amount of oil we can pump out of the earth is limited. The amount of land available for garbage dumps and hazardous waste is limited and, in many areas, diminishing rapidly. On a more personal level, each of us has a limited amount of time in which to accomplish or enjoy the activities we schedule each day. Most of us have a limited amount of money to spend while pursuing these activities. Businesses also have limited resources. A manufacturing organization employs a limited number of workers. A restaurant has a limited amount of space available for seating.

Deciding how best to use the limited resources available to an individual or a business is a universal problem. In today's competitive business environment, it is increasingly important to make sure that a company's limited resources are used in the most efficient manner possible. Typically, this involves determining how to allocate the resources in such a way as to maximize profits or minimize costs. **Mathematical programming** (MP) is an area in business analytics that finds the optimal, or most efficient, way of using limited resources to achieve the objectives of an individual or a business. For this reason, mathematical programming is often referred to as **optimization**.

2.1 Applications of Mathematical Optimization

To help you understand the purpose of optimization and the types of problems for which it can be used, let's consider several examples of decision-making situations in which MP techniques have been applied.

Determining Product Mix. Most manufacturing companies can make a variety of products. However, each product usually requires different amounts of raw materials and labor. Similarly, the amount of profit generated by the products varies. The manager of such a company must decide how many of each product to produce in order to maximize profits or to satisfy demand at minimum cost.

Manufacturing. Printed circuit boards, like those used in most computers, often have hundreds or thousands of holes drilled in them to accommodate the different electrical components that must be plugged into them. To manufacture these boards, a computer-controlled drilling machine must be programmed to drill in a given location, move the drill bit to the next location, and then drill again. This process is repeated hundreds or thousands of times to complete all the holes on a circuit board. Manufacturers of these boards would benefit from determining the drilling order that minimizes the total distance the drill bit must be moved.

Routing and Logistics. Many retail companies have warehouses around the country that are responsible for keeping stores supplied with merchandise to sell. The amount of merchandise available at the warehouses and the amount needed at each store tend to fluctuate, as does the cost of shipping or delivering merchandise from the warehouses to the retail locations. Large amounts of money can be saved by determining the least costly method of transferring merchandise from the warehouses to the stores.

Financial Planning. The federal government requires individuals to begin withdrawing money from individual retirement accounts (IRAs) and other tax-sheltered retirement programs no later than age 70.5. Various rules must be followed to avoid paying penalty taxes on these withdrawals. Most individuals want to withdraw their money in a manner that minimizes the amount of taxes they must pay while still obeying the tax laws.

Optimization Is Everywhere

Going to Disney World this summer? Optimization will be your ubiquitous companion—scheduling the crews and planes, pricing the airline tickets and hotel rooms, even helping to set capacities on the theme park rides. If you use Orbitz to book your flights, an optimization engine sifts through millions of options to find the cheapest fares. If you get directions to your hotel from MapQuest, another optimization engine figures out the most direct route. If you ship souvenirs home, an optimization engine tells UPS which truck to put the packages on, exactly where on the truck the packages should go to make them fastest to load and unload, and what route the driver should follow to make his deliveries most efficiently.

(Adapted from: V. Postrel, "Operation Everything," *The Boston Globe*, June 27, 2004.)

2.2 Characteristics of Optimization Problems

These examples represent just a few areas in which MP has been used successfully. We will consider many other examples throughout this book. However, these examples give you some idea of the issues involved in optimization. For instance, each example involves one or more *decisions* that must be made: How many of each product should be produced? Which hole should be drilled next? How much of each product should be shipped from each warehouse to the various retail locations? How much money should an individual withdraw each year from various retirement accounts?

Also, in each example, restrictions, or *constraints*, are likely to be placed on the alternatives available to the decision maker. In the first example, when determining the number of products to manufacture, a production manager is probably faced with a limited amount of raw materials and a limited amount of labor. In the second example, the drill should never return to a position where a hole has already been drilled. In the third example, there is a physical limitation on the amount of merchandise a truck can carry from one warehouse to the stores on its route. In the fourth example, laws determine the minimum and maximum amounts that can be withdrawn from retirement accounts without incurring a penalty. Many other constraints can also be identified for these examples. Indeed, it is not unusual for real-world optimization problems to have hundreds or thousands of constraints.

A final common element in each of the examples is the existence of some goal or *objective* that the decision maker considers when deciding which course of action is best. In the first example, the production manager can decide to produce several different product mixes given the available resources, but the manager will probably choose the mix of products that maximizes profits. In the second example, a large number of possible drilling patterns can be used, but the ideal pattern will probably involve moving the drill bit the shortest total distance. In the third example, merchandise can be shipped in numerous ways from the warehouses to supply the stores, but the company will probably want to identify the routing that minimizes the total transportation cost. Finally, in the fourth example, individuals can withdraw money from their retirement accounts in many ways without violating tax laws, but they probably want to find the method that minimizes their tax liability.

2.3 Expressing Optimization Problems Mathematically

From the preceding discussion, we know that optimization problems involve three elements: decisions, constraints, and an objective. If we intend to build a mathematical model of an optimization problem, we will need mathematical terms or symbols to represent each of these three elements.

2.3.1 DECISIONS

The decisions in an optimization problem are often represented in a mathematical model by the symbols $X_1, X_2, \ldots, X_n$. We will refer to $X_1, X_2, \ldots, X_n$ as the **decision variables** (or simply the variables) in the model. These variables might represent the quantities of different products the production manager can choose to produce. They might represent the amount of different pieces of merchandise to ship from a warehouse to a certain store. They might represent the amount of money to be withdrawn from different retirement accounts.

The exact symbols used to represent the decision variables are not particularly important. You could use $Z_1, Z_2, \ldots, Z_n$ or symbols such as Dog, Cat, and Monkey to represent the decision variables in the model. The choice of which symbols to use is largely a matter of personal preference and might vary from one problem to the next.

2.3.2 CONSTRAINTS

The constraints in an optimization problem can be represented in a mathematical model in a number of ways. Three general ways of expressing the possible constraint relationships in an optimization problem are:

A less than or equal to constraint:	$f(X_1, X_2, \ldots, X_n) \leq b$
A greater than or equal to constraint:	$f(X_1, X_2, \ldots, X_n) \geq b$
An equal to constraint:	$f(X_1, X_2, \ldots, X_n) = b$

In each case, the **constraint** is some function of the decision variables that must be less than or equal to, greater than or equal to, or equal to some specific value (represented by the letter b). We will refer to $f(X_1, X_2, \ldots, X_n)$ as the left-hand-side (LHS) of the constraint and to b as the right-hand-side (RHS) value of the constraint.

For example, we might use a less than or equal to constraint to ensure that the total labor used in producing a given number of products does not exceed the amount of available labor. We might use a greater than or equal to constraint to ensure that the total amount of money withdrawn from a person's retirement accounts is at least the minimum amount required by the IRS. You can use any number of these constraints to represent a given optimization problem depending on the requirements of the situation.

2.3.3 OBJECTIVE

The objective in an optimization problem is represented mathematically by an objective function in the general format:

$$\text{MAX (or MIN):} \quad f(X_1, X_2, \ldots, X_n)$$

The **objective function** identifies some function of the decision variables that the decision maker wants to either MAXimize or MINimize. In our earlier examples, this function might be used to describe the total profit associated with a product mix, the total distance the drill bit must be moved, the total cost of transporting merchandise, or a retiree's total tax liability.

The mathematical formulation of an optimization problem can be described in the general format:

$$\text{MAX (or MIN):} \quad f_0(X_1, X_2, \ldots, X_n) \qquad \textbf{2.1}$$

$$\text{Subject to:} \quad f_1(X_1, X_2, \ldots, X_n) \leq b_1 \qquad \textbf{2.2}$$

$$f_k(X_1, X_2, \ldots, X_n) \geq b_k \qquad \textbf{2.3}$$

$$f_m(X_1, X_2, \ldots, X_n) = b_m \qquad \textbf{2.4}$$

This representation identifies the objective function (equation 2.1) that will be maximized (or minimized) and the constraints that must be satisfied (equations 2.2 through 2.4). Subscripts added to the f and b in each equation emphasize that the functions describing the objective and constraints can all be different. The goal in optimization is to find the values of the decision variables that maximize (or minimize) the objective function without violating any of the constraints.

2.4 Mathematical Programming Techniques

Our general representation of an MP model is just that—general. You can use many kinds of functions to represent the objective function and the constraints in an MP model. Of course, you should always use functions that accurately describe the objective and constraints of the problem you are trying to solve. Sometimes, the functions in a model are linear in nature (that is, form straight lines or flat surfaces); other times, they are nonlinear (that is, form curved lines or curved surfaces). Sometimes, the optimal values of the decision variables in a model must take on integer values (whole numbers); other times, the decision variables can assume fractional values.

Given the diversity of MP problems that can be encountered, many techniques have been developed to solve different types of MP problems. In the next several chapters, we will look at these MP techniques and develop an understanding of how they differ and when each should be used. We will begin by examining a technique called **linear programming** (LP), which involves creating and solving optimization problems with linear objective functions and linear constraints. LP is a very powerful tool that can be applied in many business situations. It also forms a basis for several other techniques discussed later and is, therefore, a good starting point for our investigation into the field of optimization.

2.5 An Example LP Problem

We will begin our study of LP by considering a simple example. You should not interpret this to mean that LP cannot solve more complex or realistic problems. LP has been used to solve extremely complicated problems, saving companies millions of dollars. However, jumping directly into one of these complicated problems would be like starting a marathon without ever having gone out for a jog—you would get winded and could be left behind very quickly. So we'll start with something simple.

> Blue Ridge Hot Tubs manufactures and sells two models of hot tubs: the Aqua-Spa and the Hydro-Lux. Howie Jones, the owner and manager of the company, needs to decide how many of each type of hot tub to produce during his next production cycle. Howie buys prefabricated fiberglass hot tub shells from a local supplier and adds the pump and tubing to the shells to create his hot tubs. (This supplier has the capacity to deliver as many hot tub shells as Howie needs.) Howie installs the same type of pump into both hot tubs. He will have only 200 pumps available during his next production cycle. From a manufacturing standpoint, the main difference between the two models of hot tubs is the amount of tubing and labor required. Each Aqua-Spa requires 9 hours of labor and 12 feet of tubing. Each Hydro-Lux requires 6 hours of labor and 16 feet of tubing. Howie expects to have 1,566 production labor hours and 2,880 feet of tubing available during the next production cycle. Howie earns a profit of $350 on each Aqua-Spa he sells and $300 on each Hydro-Lux he sells. He is confident that he can sell all the hot tubs he produces. The question is, how many Aqua-Spas and Hydro-Luxes should Howie produce if he wants to maximize his profits during the next production cycle?

2.6 Formulating LP Models

The process of taking a practical problem—such as determining how many Aqua-Spas and Hydro-Luxes Howie should produce—and expressing it algebraically in the form of an LP model is known as **formulating** the model. Throughout the next several chapters, you will see that formulating an LP model is as much an art as a science.

2.6.1 STEPS IN FORMULATING AN LP MODEL

There are some general steps you can follow to help make sure your formulation of a particular problem is accurate. We will walk through these steps using the hot tub example.

1. **Understand the problem.** This step appears to be so obvious that it hardly seems worth mentioning. However, many people have a tendency to jump into a problem and start writing the objective function and constraints before they really understand

the problem. If you do not fully understand the problem you face, it is unlikely that your formulation of the problem will be correct.

The problem in our example is fairly easy to understand: How many Aqua-Spas and Hydro-Luxes should Howie produce to maximize his profit, while using no more than 200 pumps, 1,566 labor hours, and 2,880 feet of tubing?

2. **Identify the decision variables.** After you are sure you understand the problem, you need to identify the decision variables. That is, what are the fundamental decisions that must be made in order to solve the problem? The answers to this question often will help you identify appropriate decision variables for your model. Identifying the decision variables means determining what the symbols $X_1, X_2, \ldots, X_n$ represent in your model.

   In our example, the fundamental decision Howie faces is this: How many Aqua-Spas and Hydro-Luxes should be produced? In this problem, we will let X_1 represent the number of Aqua-Spas to produce and X_2 represent the number of Hydro-Luxes to produce.

3. **State the objective function as a linear combination of the decision variables.** After determining the decision variables you will use, the next step is to create the objective function for the model. This function expresses the mathematical relationship between the decision variables in the model to be maximized or minimized.

   In our example, Howie earns a profit of \$350 on each Aqua-Spa (X_1) he sells and \$300 on each Hydro-Lux (X_2) he sells. Thus, Howie's objective of maximizing the profit he earns is stated mathematically as:

$$\text{MAX:} \qquad 350X_1 + 300X_2$$

   For whatever values might be assigned to X_1 and X_2, the previous function calculates the associated total profit that Howie would earn. Obviously, Howie wants to maximize this value.

4. **State the constraints as linear combinations of the decision variables.** As mentioned earlier, there are usually some limitations on the values that can be assumed by the decision variables in an LP model. These restrictions must be identified and stated in the form of constraints.

   In our example, Howie faces three major constraints. Because only 200 pumps are available and each hot tub requires 1 pump, Howie cannot produce more than a total of 200 hot tubs. This restriction is stated mathematically as:

$$1X_1 + 1X_2 \leq 200$$

   This constraint indicates that each unit of X_1 produced (that is, each Aqua-Spa built) will use 1 of the 200 pumps available—as will each unit of X_2 produced (that is, each Hydro-Lux built). The total number of pumps used (represented by $1X_1 + 1X_2$) must be less than or equal to 200. Another restriction Howie faces is that he has only 1,566 labor hours available during the next production cycle. Because each Aqua-Spa he builds (each unit of X_1) requires 9 labor hours, and each Hydro-Lux (each unit of X_2) requires 6 labor hours, the constraint on the number of labor hours is stated as:

$$9X_1 + 6X_2 \leq 1{,}566$$

   The total number of labor hours used (represented by $9X_1 + 6X_2$) must be less than or equal to the total labor hours available, which is 1,566.

   The final constraint specifies that only 2,880 feet of tubing is available for the next production cycle. Each Aqua-Spa produced (each unit of X_1) requires 12 feet of tubing, and each Hydro-Lux produced (each unit of X_2) requires 16 feet of tubing.

The following constraint is necessary to ensure that Howie's production plan does not use more tubing than is available:

$$12X_1 + 16X_2 \leq 2{,}880$$

The total number of feet of tubing used (represented by $12X_1 + 16X_2$) must be less than or equal to the total number of feet of tubing available, which is 2,880.

5. **Identify any upper or lower bounds on the decision variables.** Often, simple upper or lower bounds apply to the decision variables. You can view upper and lower bounds as additional constraints in the problem.

   In our example, there are simple lower bounds of zero on the variables X_1 and X_2 because it is impossible to produce a negative number of hot tubs. Therefore, the following two constraints also apply to this problem:

$$X_1 \geq 0$$
$$X_2 \geq 0$$

   Constraints like these are often referred to as nonnegativity conditions and are quite common in LP problems.

2.7 Summary of the LP Model for the Example Problem

The complete LP model for Howie's decision problem can be stated as:

$$\begin{array}{lrcrcrl} \text{MAX:} & 350X_1 & + & 300X_2 & & & \quad \textbf{2.5} \\ \text{Subject to:} & 1X_1 & + & 1X_2 & \leq & 200 & \quad \textbf{2.6} \\ & 9X_1 & + & 6X_2 & \leq & 1{,}566 & \quad \textbf{2.7} \\ & 12X_1 & + & 16X_2 & \leq & 2{,}880 & \quad \textbf{2.8} \\ & 1X_1 & & & \geq & 0 & \quad \textbf{2.9} \\ & & & 1X_2 & \geq & 0 & \quad \textbf{2.10} \end{array}$$

In this model, the decision variables X_1 and X_2 represent the number of Aqua-Spas and Hydro-Luxes to produce, respectively. Our goal is to determine the values for X_1 and X_2 that maximize the objective in equation 2.5 while simultaneously satisfying all the constraints in equations 2.6 through 2.10.

2.8 The General Form of an LP Model

The technique of linear programming is so-named because the MP problems to which it applies are linear in nature. That is, it must be possible to express all the functions in an LP model as some weighted sum (or linear combination) of the decision variables. So, an LP model takes on the general form:

$$\begin{array}{ll} \text{MAX (or MIN):} & c_1X_1 + c_2X_2 + \cdots + c_nX_n \quad \textbf{2.11} \\ \text{Subject to:} & a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n \leq b_1 \quad \textbf{2.12} \\ & a_{k1}X_1 + a_{k2}X_2 + \cdots + a_{kn}X_n \geq b_k \quad \textbf{2.13} \\ & a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n = b_m \quad \textbf{2.14} \end{array}$$

Up to this point, we have suggested that the constraints in an LP model represent some type of limited resource. Although this is frequently the case, in later chapters, you will see examples of LP models in which the constraints represent things other than limited resources. The important point here is that *any* problem that can be formulated in the preceding fashion is an LP problem.

The symbols $c_1, c_2, \ldots, c_n$ in equation 2.11 are called **objective function coefficients** and might represent the marginal profits (or costs) associated with the decision variables $X_1, X_2, \ldots, X_n$, respectively. The symbol a_{ij} found throughout equations 2.12 through 2.14 represents the numeric coefficient in the i^{th} constraint for variable X_j. The objective function and constraints of an LP problem represent different weighted sums of the decision variables. The b_i symbols in the constraints, once again, represent values that the corresponding linear combination of the decision variables must be less than or equal to, greater than or equal to, or equal to.

You should now see a direct connection between the LP model we formulated for Blue Ridge Hot Tubs in equations 2.5 through 2.10 and the general definition of an LP model given in equations 2.11 through 2.14. In particular, note that the various symbols used in equations 2.11 through 2.14 to represent numeric constants (that is, the c_j, a_{ij}, and b_i) were replaced by actual numeric values in equations 2.5 through 2.10. Also, note that our formulation of the LP model for Blue Ridge Hot Tubs did not require the use of equal to constraints. Different problems require different types of constraints, and you should use whatever types of constraints are necessary for the problem at hand.

2.9 Solving LP Problems: An Intuitive Approach

After an LP model has been formulated, our interest naturally turns to solving it. But before we actually solve our example problem for Blue Ridge Hot Tubs, what do you think is the optimal solution to the problem? Just by looking at the model, what values for X_1 and X_2 do you think would give Howie the largest profit?

Following one line of reasoning, it might seem that Howie should produce as many units of X_1 (Aqua-Spas) as possible because each of these generates a profit of \$350, whereas each unit of X_2 (Hydro-Luxes) generates a profit of only \$300. But what is the maximum number of Aqua-Spas that Howie could produce?

Howie can produce the maximum number of units of X_1 by making no units of X_2 and devoting all his resources to the production of X_1. Suppose we let $X_2 = 0$ in the model in equations 2.5 through 2.10 to indicate that no Hydro-Luxes will be produced. What then is the largest possible value of X_1? If $X_2 = 0$, then the inequality in equation 2.6 tells us:

$$X_1 \le 200 \qquad \textbf{2.15}$$

So we know that X_1 cannot be any greater than 200 if $X_2 = 0$. However, we also have to consider the constraints in equations 2.7 and 2.8. If $X_2 = 0$, then the inequality in equation 2.7 reduces to:

$$9X_1 \le 1{,}566 \qquad \textbf{2.16}$$

If we divide both sides of this inequality by 9, we find that the previous constraint is equivalent to:

$$X_1 \le 174 \qquad \textbf{2.17}$$

Now consider the constraint in equation 2.8. If $X_2 = 0$, then the inequality in equation 2.8 reduces to:

$$12X_1 \leq 2{,}880 \qquad \textbf{2.18}$$

Again, if we divide both sides of this inequality by 12, we find that the previous constraint is equivalent to:

$$X_1 \leq 240 \qquad \textbf{2.19}$$

So, if $X_2 = 0$, the three constraints in our model imposing upper limits on the value of X_1 reduce to the values shown in equations 2.15, 2.17, and 2.19. The most restrictive of these constraints is equation 2.17. Therefore, the maximum number of units of X_1 that can be produced is 174. In other words, 174 is the largest value X_1 can take on and still satisfy all the constraints in the model.

If Howie builds 174 units of X_1 (Aqua-Spas) and 0 units of X_2 (Hydro-Luxes), he will have used all of the labor that is available for production ($9X_1 = 1{,}566$ if $X_1 = 174$). However, he will have 26 pumps remaining ($200 - X_1 = 26$ if $X_1 = 174$) and 792 feet of tubing remaining ($2{,}880 - 12X_1 = 792$ if $X_1 = 174$). Also, notice that the objective function value (or total profit) associated with this solution is:

$$\$350X_1 + \$300X_2 = \$350 \times 174 + \$300 \times 0 = \$60{,}900$$

From this analysis, we see that the solution $X_1 = 174$, $X_2 = 0$ is a *feasible solution* to the problem because it satisfies all the constraints of the model. But is it the *optimal solution*? In other words, is there any other possible set of values for X_1 and X_2 that also satisfies all the constraints *and* results in a higher objective function value? As you will see, the intuitive approach to solving LP problems that we have taken here cannot be trusted because there actually is a *better* solution to Howie's problem.

2.10 Solving LP Problems: A Graphical Approach

The constraints of an LP model define the set of feasible solutions—or the **feasible region**—for the problem. The difficulty in LP is determining which point or points in the feasible region correspond to the best possible value of the objective function. For simple problems with only two decision variables, it is fairly easy to sketch the feasible region for the LP model and locate the optimal feasible point graphically. Because the graphical approach can be used only if there are two decision variables, it has limited practical use. However, it is an extremely good way to develop a basic understanding of the strategy involved in solving LP problems. Therefore, we will use the graphical approach to solve the simple problem faced by Blue Ridge Hot Tubs. Chapter 3 shows how to solve this and other LP problems using a spreadsheet.

To solve an LP problem graphically, you first must plot the constraints for the problem and identify its feasible region. This is done by plotting the *boundary lines* of the constraints and identifying the points that will satisfy all the constraints. So, how do we do this for our example problem (repeated here)?

MAX:	$350X_1 + 300X_2$	**2.20**
Subject to:	$1X_1 + 1X_2 \leq 200$	**2.21**
	$9X_1 + 6X_2 \leq 1{,}566$	**2.22**
	$12X_1 + 16X_2 \leq 2{,}880$	**2.23**
	$1X_1 \geq 0$	**2.24**
	$1X_2 \geq 0$	**2.25**

2.10.1 PLOTTING THE FIRST CONSTRAINT

The boundary of the first constraint in our model, which specifies that no more than 200 pumps can be used, is represented by the straight line defined by the equation:

$$X_1 + X_2 = 200 \qquad \textbf{2.26}$$

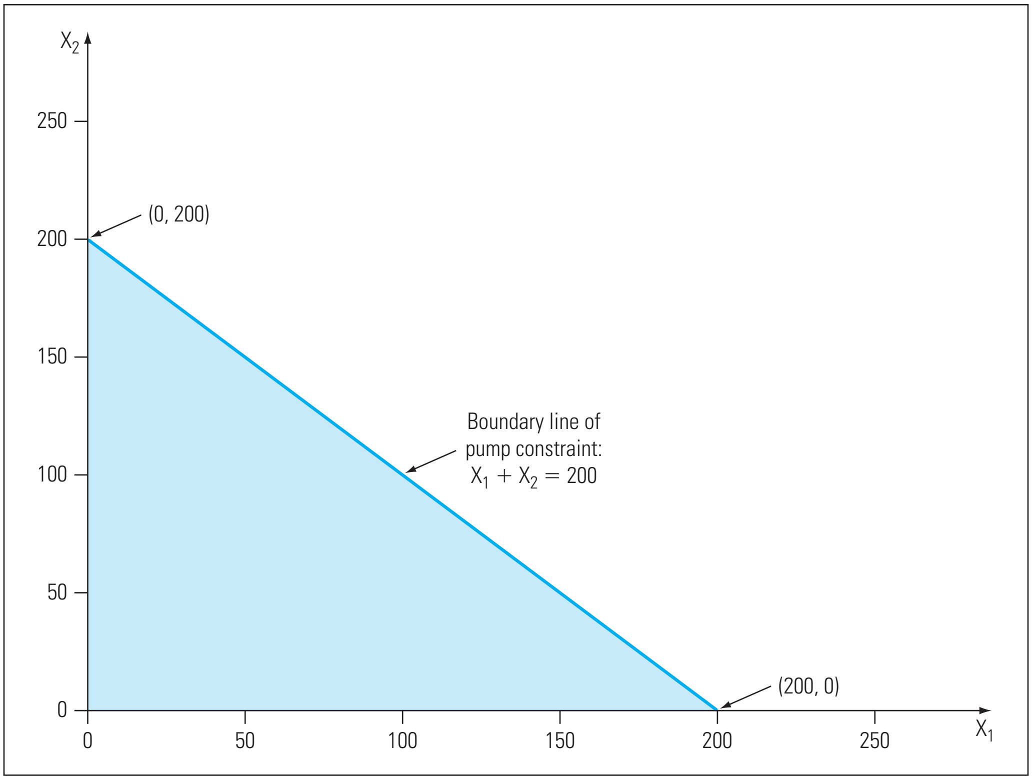
If we can find any two points on this line, the entire line can be plotted easily by drawing a straight line through these points. If $X_2 = 0$, we can see from equation 2.26 that $X_1 = 200$. Thus, the point $(X_1, X_2) = (200, 0)$ must fall on this line. If we let $X_1 = 0$, from equation 2.26, it is easy to see that $X_2 = 200$. So, the point $(X_1, X_2) = (0, 200)$ must also fall on this line. These two points are plotted on the graph in Figure 2.1 and connected to form the straight line representing equation 2.26.

Note that the graph of the line associated with equation 2.26 actually extends beyond the X_1 and X_2 axes shown in Figure 2.1. However, we can disregard the points beyond these axes because the values assumed by X_1 and X_2 cannot be negative (because we also have the constraints given by $X_1 \geq 0$ and $X_2 \geq 0$).

The line connecting the points (0, 200) and (200, 0) in Figure 2.1 identifies the points (X_1, X_2) that satisfy the equality $X_1 + X_2 = 200$. But recall that the first constraint in the LP model is the inequality $X_1 + X_2 \leq 200$. Thus, after plotting the boundary line of a constraint, we must determine which area on the graph corresponds to feasible solutions for the original constraint. This can be done easily by picking an arbitrary point on either side of the boundary line and checking whether it satisfies the original constraint. For example, if we test the point $(X_1, X_2) = (0, 0)$, we see that this point satisfies the first constraint. Therefore, the area of the graph on the same side of the boundary line as the point (0, 0) corresponds to the feasible solutions of our first constraint. This area of feasible solutions is shaded in Figure 2.1.

FIGURE 2.1

Graphical representation of the pump constraint



2.10.2 PLOTTING THE SECOND CONSTRAINT

Some of the feasible solutions to one constraint in an LP model usually will not satisfy one or more of the other constraints in the model. For example, the point $(X_1, X_2) = (200, 0)$ satisfies the first constraint in our model, but it does not satisfy the second constraint, which requires that no more than 1,566 labor hours be used (because $9 \times 200 + 6 \times 0 = 1{,}800$). So, what values for X_1 and X_2 will simultaneously satisfy both of these constraints? To answer this question, we need to plot the second constraint on the graph as well. This is done in the same manner as before—by locating two points on the boundary line of the constraint and connecting these points with a straight line.

The boundary line for the second constraint in our model is given by:

$$9X_1 + 6X_2 = 1{,}566 \qquad \textbf{2.27}$$

If $X_1 = 0$ in equation 2.27, then $X_2 = 1{,}566/6 = 261$. So, the point (0, 261) must fall on the line defined by equation 2.27. Similarly, if $X_2 = 0$ in equation 2.27, then $X_1 = 1{,}566/9 = 174$. So, the point (174, 0) must also fall on this line. These two points are plotted on the graph and connected with a straight line representing equation 2.27, as shown in Figure 2.2.

The line drawn in Figure 2.2 representing equation 2.27 is the boundary line for our second constraint. To determine the area on the graph that corresponds to feasible solutions to the second constraint, we again need to test a point on either side of this line to see if it is feasible. The point $(X_1, X_2) = (0, 0)$ satisfies $9X_1 + 6X_2 \leq 1{,}566$. Therefore, all points on the same side of the boundary line satisfy this constraint.

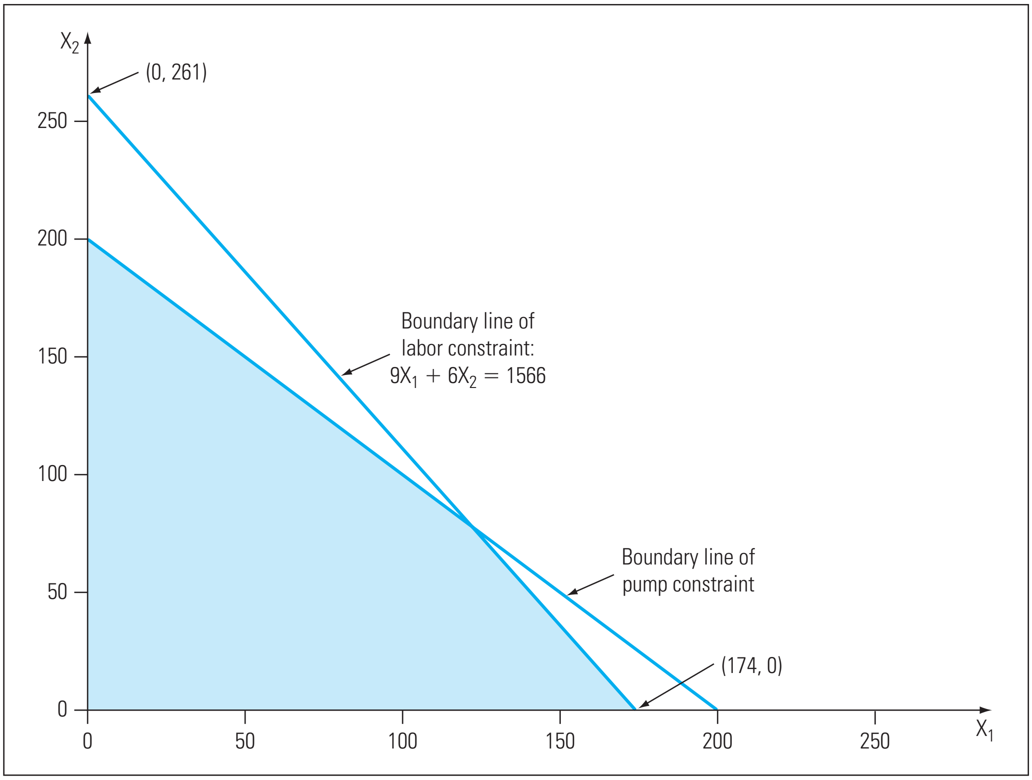
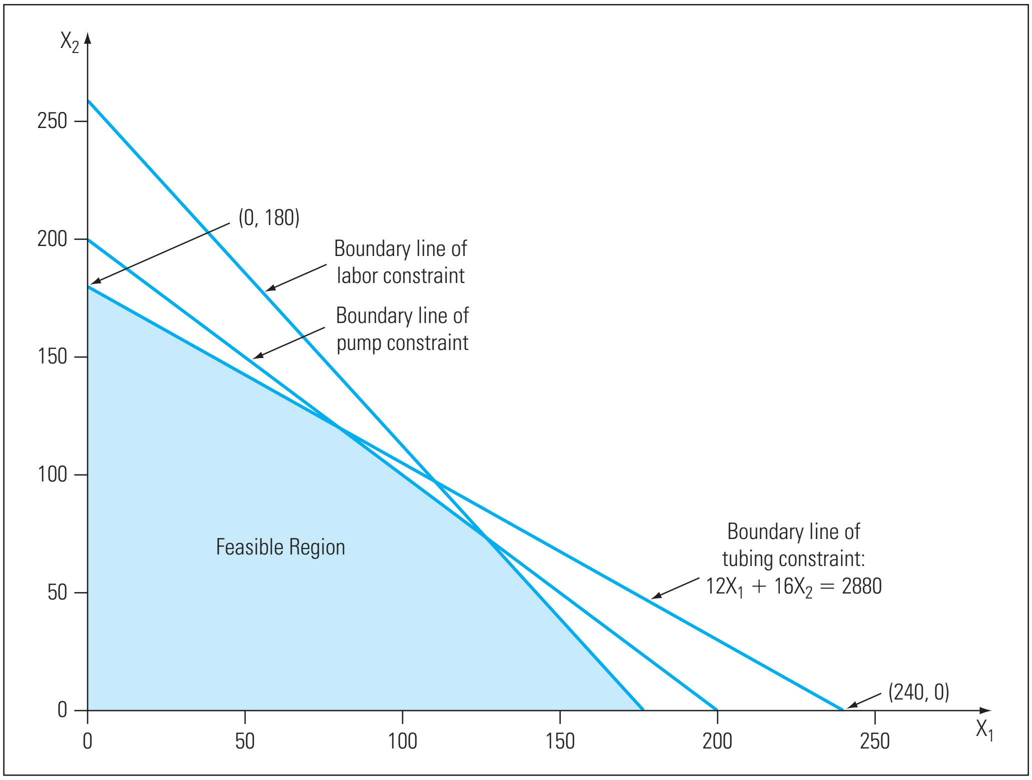


FIGURE 2.2

Graphical representation of the pump and labor constraints

FIGURE 2.3

Graphical representation of the feasible region



2.10.3 PLOTTING THE THIRD CONSTRAINT

To find the set of values for X_1 and X_2 that satisfies all the constraints in the model, we need to plot the third constraint. This constraint requires that no more than 2,880 feet of tubing be used in producing the hot tubs. Again, we will find two points on the graph that fall on the boundary line for this constraint and connect them with a straight line.

The boundary line for the third constraint in our model is:

$$12X_1 + 16X_2 = 2{,}880 \qquad \textbf{2.28}$$

If $X_1 = 0$ in equation 2.28, then $X_2 = 2{,}880/16 = 180$. So, the point (0, 180) must fall on the line defined by equation 2.28. Similarly, if $X_2 = 0$ in equation 2.28, then $X_1 = 2{,}880/12 = 240$. So, the point (240, 0) must also fall on this line. These two points are plotted on the graph and connected with a straight line representing equation 2.28, as shown in Figure 2.3.

Again, the line drawn in Figure 2.3 representing equation 2.28 is the boundary line for our third constraint. To determine the area on the graph that corresponds to feasible solutions to this constraint, we need to test a point on either side of this line to see if it is feasible. The point $(X_1, X_2) = (0, 0)$ satisfies $12X_1 + 16X_2 \leq 2{,}880$. Therefore, all points on the same side of the boundary line satisfy this constraint.

2.10.4 THE FEASIBLE REGION

It is now easy to see which points satisfy all the constraints in our model. These points correspond to the shaded area in Figure 2.3, labeled "Feasible Region." The **feasible region** is the set of points or values that the decision variables can assume and simultaneously satisfy all the constraints in the problem. Take a moment now to carefully compare the graphs in Figures 2.1, 2.2, and 2.3. In particular, notice that when we added the second constraint in Figure 2.2, some of the feasible solutions associated with the first constraint were eliminated because these solutions did not satisfy the second constraint. Similarly, when we added the third constraint in Figure 2.3, another portion of the feasible solutions for the first constraint was eliminated.

2.10.5 PLOTTING THE OBJECTIVE FUNCTION

Now that we have isolated the set of feasible solutions to our LP problem, we need to determine which of these solutions is best. That is, we must determine which point in the feasible region will maximize the value of the objective function in our model. At first glance, it might seem that trying to locate this point is like searching for a needle in a haystack. After all, as shown by the shaded region in Figure 2.3, there are an *infinite* number of feasible solutions to this problem. Fortunately, we can easily eliminate most of the feasible solutions in an LP problem from consideration. It can be shown that if an LP problem has an optimal solution with a finite objective function value, this solution will always occur at a point in the feasible region where two or more of the boundary lines of the constraints intersect. These points of intersection are sometimes called **corner points** or **extreme points** of the feasible region.

To see why the finite optimal solution to an LP problem occurs at an extreme point of the feasible region, consider the relationship between the objective function and the feasible region of our example LP model. Suppose we are interested in finding the values of X_1 and X_2 associated with a given level of profit, such as \$35,000. Then, mathematically, we are interested in finding the points (X_1, X_2) for which our objective function equals \$35,000, or where:

$$\$350X_1 + \$300X_2 = \$35{,}000 \tag{2.29}$$

This equation defines a straight line, which we can plot on our graph. Specifically, if $X_1 = 0$ then, from equation 2.29, $X_2 = 116.67$. Similarly, if $X_2 = 0$ in equation 2.29, then $X_1 = 100$. So, the points $(X_1, X_2) = (0, 116.67)$ and $(X_1, X_2) = (100, 0)$ both fall on the line defining a profit level of \$35,000. (Note that all the points on this line produce a profit level of \$35,000.) This line is shown in Figure 2.4.

Now, suppose we are interested in finding the values of X_1 and X_2 that produce some higher level of profit, such as \$52,500. Then, mathematically, we are interested in finding the points (X_1, X_2) for which our objective function equals \$52,500, or where:

$$\$350X_1 + \$300X_2 = \$52{,}500 \tag{2.30}$$

This equation also defines a straight line, which we could plot on our graph. If we do this, we'll find that the points $(X_1, X_2) = (0, 175)$ and $(X_1, X_2) = (150, 0)$ both fall on this line, as shown in Figure 2.5.

2.10.6 FINDING THE OPTIMAL SOLUTION USING LEVEL CURVES

The lines in Figure 2.5 representing the two objective function values are sometimes referred to as **level curves** because they represent different levels or values of the objective. Note that the two level curves in Figure 2.5 are *parallel* to one another. If we repeat this process of drawing lines associated with larger and larger values of our objective function, we will continue to observe a series of parallel lines shifting away from the

FIGURE 2.4

Graph showing values of X_1 and X_2 that produce an objective function value of \$35,000

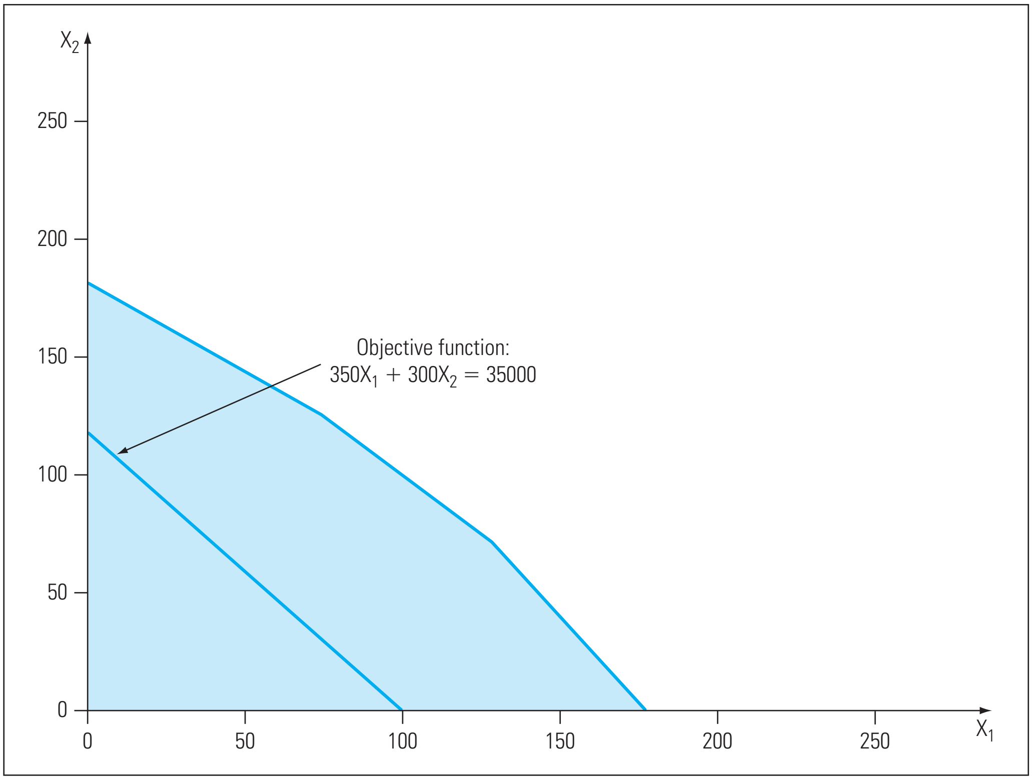
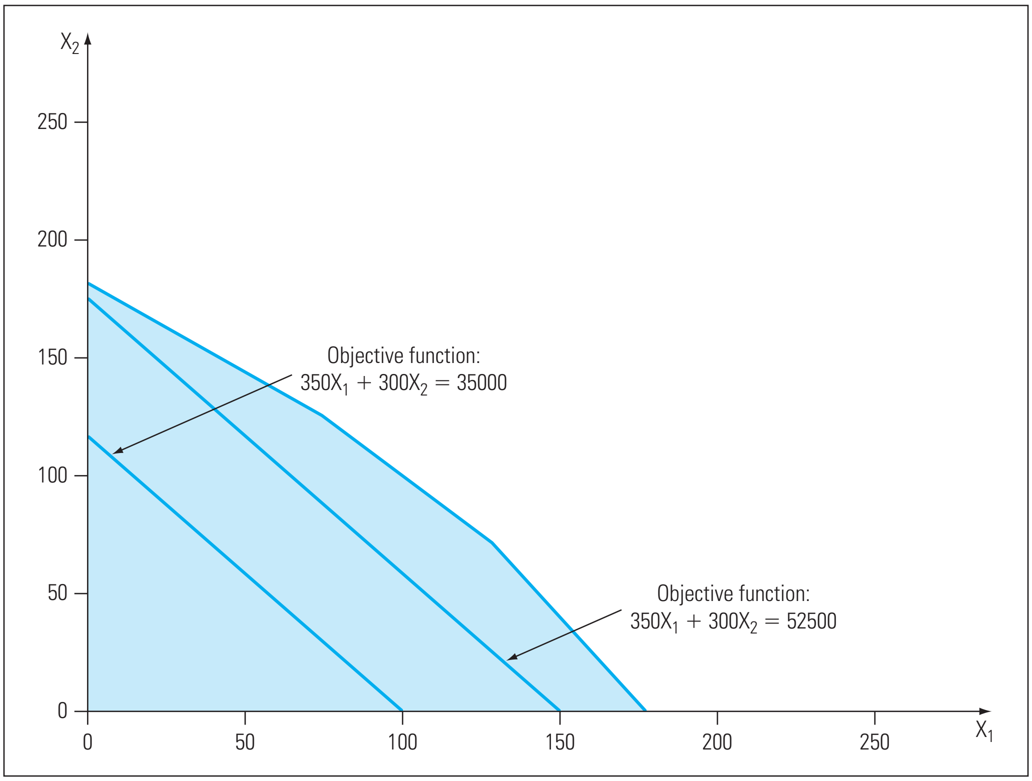


FIGURE 2.5

Parallel level curves for two different objective function values



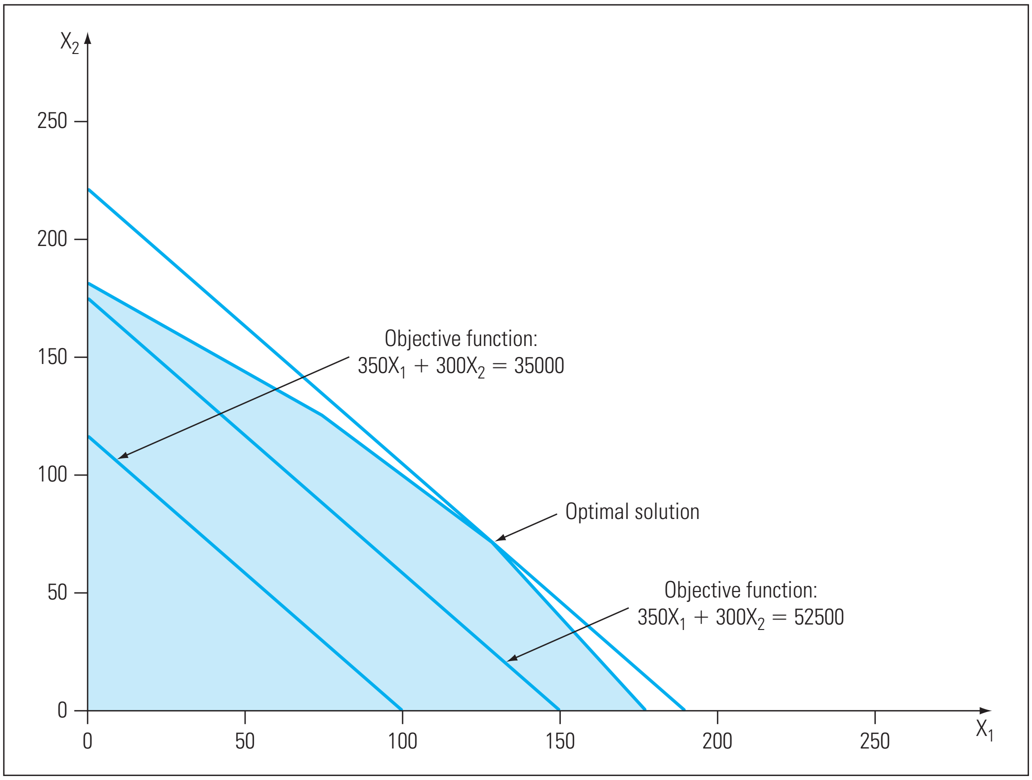


FIGURE 2.6

Graph showing optimal solution where the level curve is tangent to the feasible region

origin—that is, away from the point (0, 0). The very last level curve we can draw that still intersects the feasible region determines the maximum profit we can achieve. This point of intersection, shown in Figure 2.6, represents the optimal feasible solution to the problem.

As shown in Figure 2.6, the optimal solution to our example problem occurs at the point where the largest possible level curve intersects the feasible region at a single point. This is the feasible point that produces the largest profit for Blue Ridge Hot Tubs. But how do we figure out exactly what point this is and how much profit it provides?

If you compare Figure 2.6 to Figure 2.3, you see that the optimal solution occurs where the boundary lines of the pump and labor constraints intersect (or are equal). Thus, the optimal solution is defined by the point (X_1, X_2) that simultaneously satisfies equations 2.26 and 2.27, which are repeated here:

$$X_1 + X_2 = 200$$
$$9X_1 + 6X_2 = 1{,}566$$

From the first equation, we easily conclude that $X_2 = 200 - X_1$. If we substitute this definition of X_2 into the second equation we get:

$$9X_1 + 6(200 - X_1) = 1{,}566$$

Using simple algebra, we can solve this equation to find that $X_1 = 122$. And because $X_2 = 200 - X_1$, we can conclude that $X_2 = 78$. Therefore, we have determined that the optimal solution to our example problem occurs at the point $(X_1, X_2) = (122, 78)$. This point satisfies all the constraints in our model and corresponds to the point in Figure 2.6 identified as the optimal solution.

The total profit associated with this solution is found by substituting the optimal values of $X_1 = 122$ and $X_2 = 78$ into the objective function. Thus, Blue Ridge Hot Tubs can realize a profit of $66,100 if it produces 122 Aqua-Spas and 78 Hydro-Luxes ($\$350 \times 122 + \$300 \times 78 = \$66{,}100$). Any other production plan results in a lower total profit. In particular, note that the solution we found earlier using the intuitive approach (which produced a total profit of $60,900) is inferior to the optimal solution identified here.

2.10.7 FINDING THE OPTIMAL SOLUTION BY ENUMERATING THE CORNER POINTS

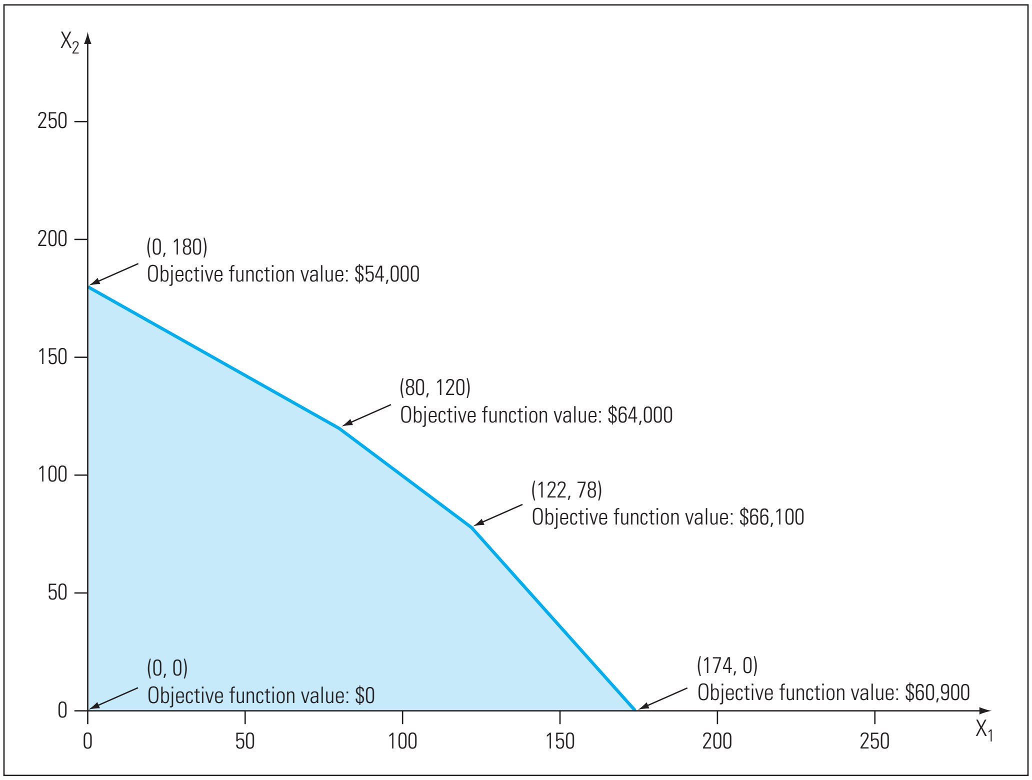
Earlier, we indicated that if an LP problem has a finite optimal solution, this solution will always occur at some corner point of the feasible region. So, another way of solving an LP problem is to identify all the corner points, or extreme points, of the feasible region and calculate the value of the objective function at each of these points. The corner point with the largest objective function value is the optimal solution to the problem.

This approach is illustrated in Figure 2.7, where the X_1 and X_2 coordinates for each of the extreme points are identified along with the associated objective function values. As expected, this analysis also indicates that the point $(X_1, X_2) = (122, 78)$ is optimal.

Enumerating the corner points to identify the optimal solution is often more difficult than the level curve approach because it requires that you identify the coordinates for *all* the extreme points of the feasible region. If there are many intersecting constraints, the number of extreme points can become rather large, making this procedure very tedious. Also, a special condition exists for which this procedure will not work. This condition, known as an unbounded solution, is described shortly.

FIGURE 2.7

Objective function values at each extreme point of the feasible region



2.10.8 SUMMARY OF GRAPHICAL SOLUTION TO LP PROBLEMS

To summarize this section, a two-variable LP problem is solved graphically by performing these steps:

1. Plot the boundary line of each constraint in the model.
2. Identify the feasible region, that is, the set of points on the graph that simultaneously satisfies all the constraints.
3. Locate the optimal solution by one of the following methods:
   a. Plot one or more level curves for the objective function and determine the direction in which parallel shifts in this line produce improved objective function values. Shift the level curve in a parallel manner in the improving direction until it intersects the feasible region at a single point. Then find the coordinates for this point. This is the optimal solution.
   b. Identify the coordinates of all the extreme points of the feasible region, and calculate the objective function values associated with each point. If the feasible region is bounded, the point with the best objective function value is the optimal solution.

2.10.9 UNDERSTANDING HOW THINGS CHANGE

It is important to realize that if changes occur in any of the coefficients in the objective function or constraints of this problem, then the level curve, feasible region, and optimal solution to this problem might also change. To be an effective LP modeler, it is important for you to develop some intuition about how changes in various coefficients in the model will impact the solution to the problem. We will study this in greater detail in Chapter 4 when discussing sensitivity analysis. However, the spreadsheet shown in Figure 2.8 (and the file named Fig2-8.xlsm that accompanies this book) allows you

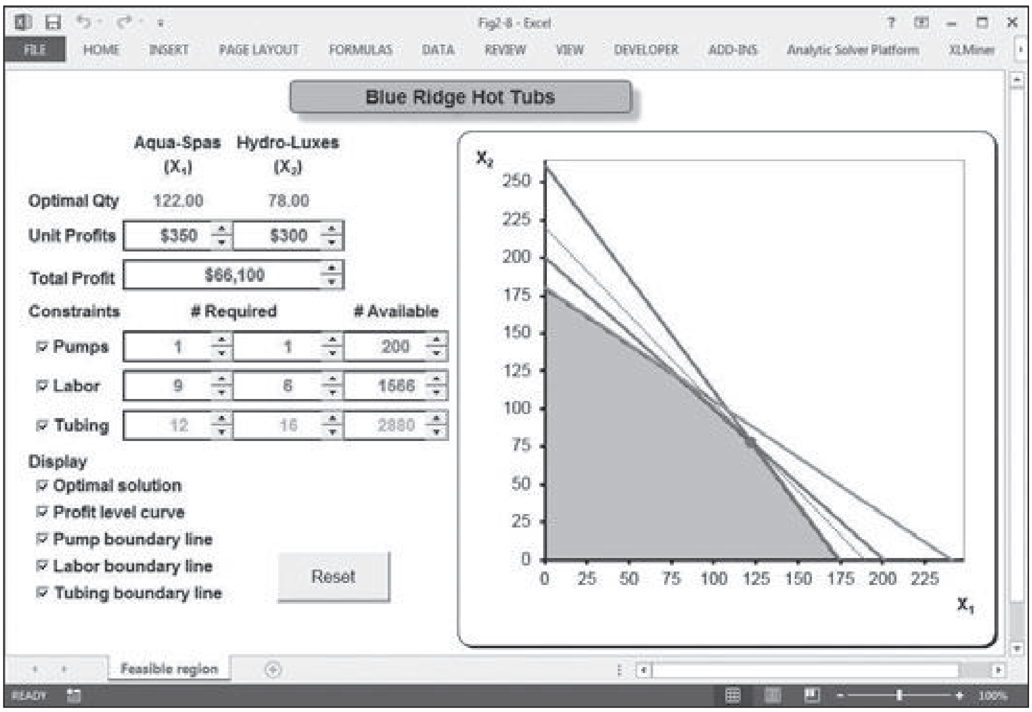


FIGURE 2.8

Interactive spreadsheet for the Blue Ridge Hot Tubs LP problem