EIGHTH EDITION

Algebra & Trigonometry

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EIGHTH EDITION

ALGEBRA AND TRIGONOMETRY

Richard N. Aufmann
Richard D. Nation

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***Algebra and Trigonometry*, Eighth Edition**
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Preface

We are proud to offer the eighth edition of *Algebra and Trigonometry*. Your success in algebra and trigonometry is important to us. To guide you to that success, we have created a textbook with features that promote learning and support various learning styles. These features are highlighted below. We encourage you to examine the features and use them to help you successfully complete this course.

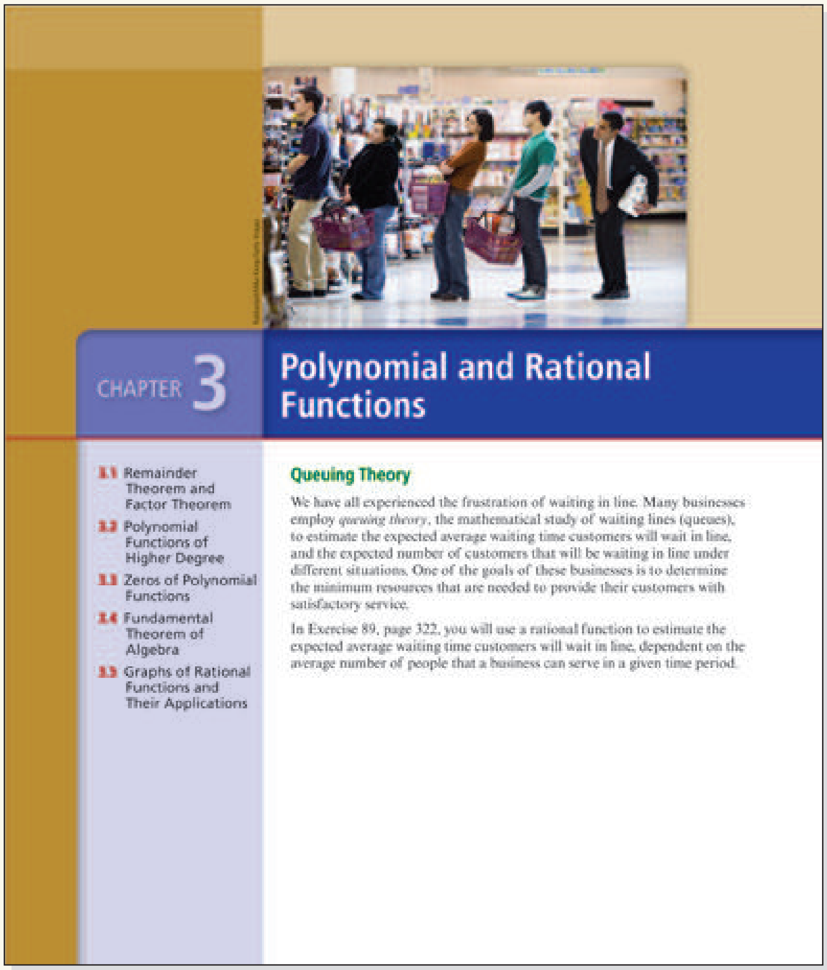
Features

Chapter Openers

Each Chapter Opener demonstrates a contemporary application of a mathematical concept developed in that chapter.

Related Exercise References

Each Chapter Opener cites a particular exercise within the chapter that is related to the chapter opener topic.



CHAPTER 3

Polynomial and Rational Functions

3.1 Remainder Theorem and Factor Theorem
3.2 Polynomial Functions of Higher Degree
3.3 Zeros of Polynomial Functions
3.4 Fundamental Theorem of Algebra
3.5 Graphs of Rational Functions and Their Applications

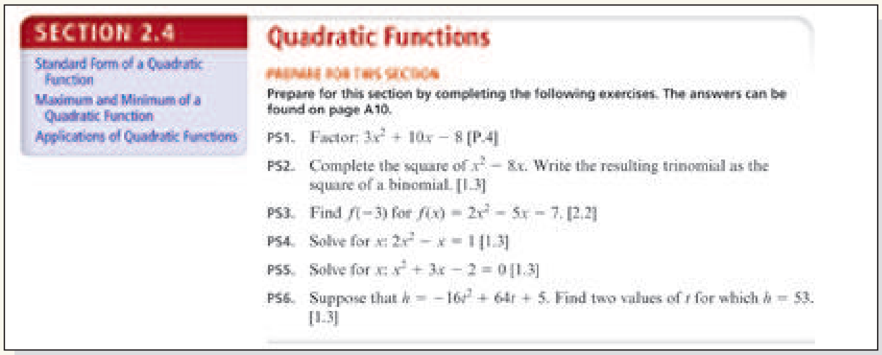
Queuing Theory

We have all experienced the frustration of waiting in line. Many businesses employ *queuing theory*, the mathematical study of waiting lines (queues), to estimate the expected average waiting time customers will wait in line, and the expected number of customers that will be waiting in line under different situations. One of the goals of these businesses is to determine the minimum resources that are needed to provide their customers with satisfactory service.

In Exercise 89, page 322, you will use a rational function to estimate the expected average waiting time customers will wait in line, dependent on the average number of people that a business can serve in a given time period.

Listing of Major Concepts

A list of major concepts in each section is provided in the margin of the first page of each section.



SECTION 2.4

Standard Form of a Quadratic Function
Maximum and Minimum of a Quadratic Function
Applications of Quadratic Functions

Quadratic Functions

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A10.

PS1. Factor: $3x^2 + 10x - 8$ [P.4]

PS2. Complete the square of $x^2 - 8x$. Write the resulting trinomial as the square of a binomial. [1.3]

PS3. Find $f(-3)$ for $f(x) = 2x^2 - 5x - 7$. [2.2]

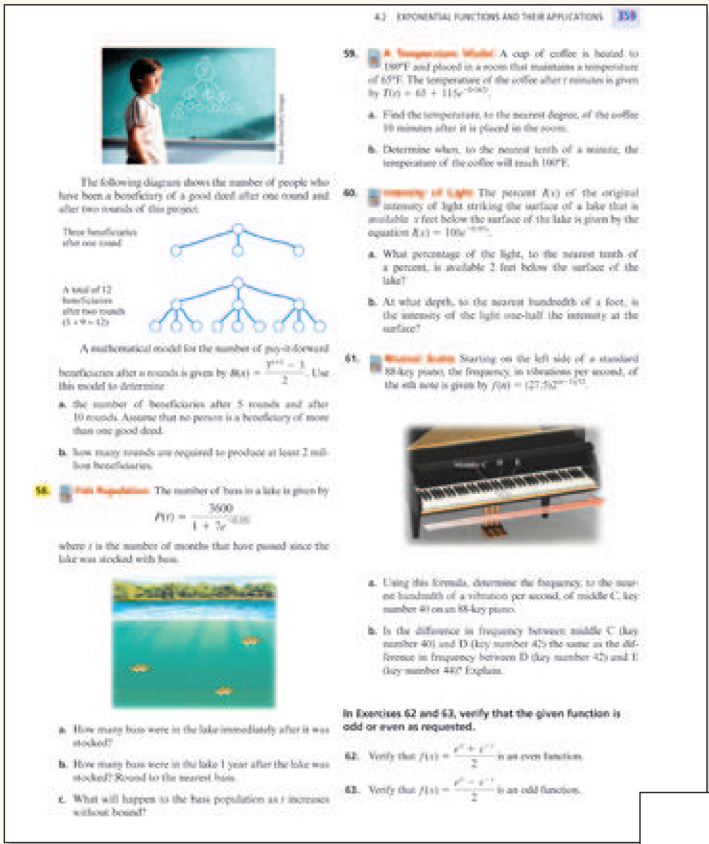
PS4. Solve for x: $2x^2 - x = 1$ [1.3]

PS5. Solve for x: $x^2 + 3x - 2 = 0$ [1.3]

PS6. Suppose that $h = -16t^2 + 64t + 5$. Find two values of t for which $h = 53$. [1.3]

Prepare for This Section

Each section (after the first section) of a chapter opens with review exercises titled Prepare for This Section. These exercises give you a chance to test your understanding of prerequisite skills and concepts before proceeding to the new topics presented in the section.

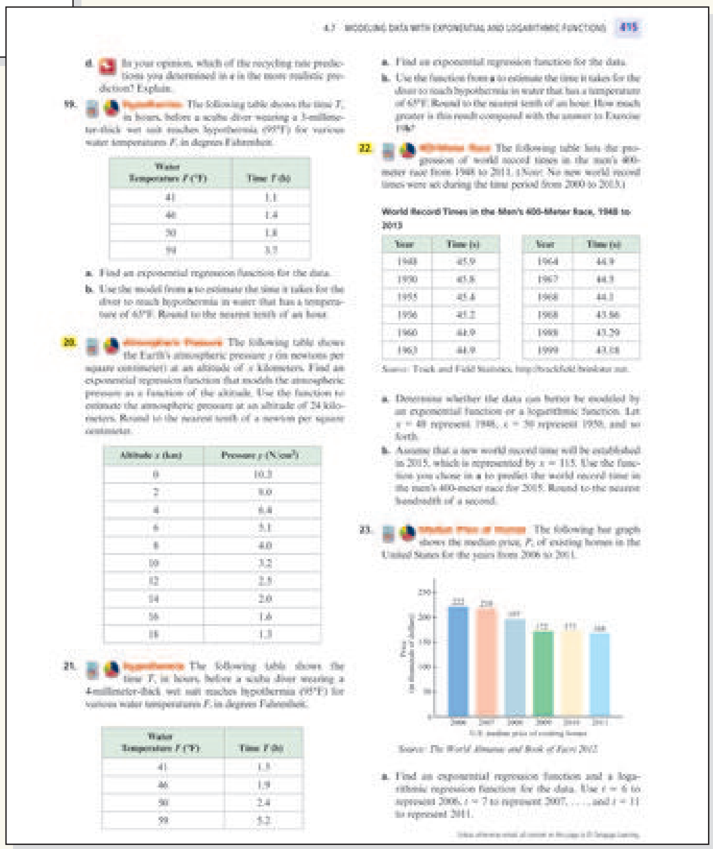


Thoughtfully Designed Exercise Sets

We have thoroughly reviewed each exercise set to ensure a smooth progression from routine exercises to exercises that are more challenging. The exercises illustrate the many facets of topics discussed in the text. The exercise sets emphasize skill building, skill maintenance, conceptual understanding, and, as appropriate, applications. Each chapter includes a Chapter Review Exercise set and each chapter, except Chapter P, includes a Cumulative Review Exercise set.

Contemporary Applications

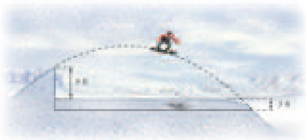
Carefully developed mathematics is complemented by abundant, relevant, and contemporary applications. Many of these feature real data, tables, graphs, and charts. Applications demonstrate the value of algebra and cover topics from a wide variety of disciplines. Besides providing motivation to study mathematics, the applications will help you develop good problem-solving skills.



By incorporating many interactive learning techniques, including the key features outlined below, *Algebra and Trigonometry* uses the proven Aufmann Interactive Methods (AIM) to help you understand concepts and obtain greater mathematical fluency. The AIM consists of **Annotated Examples** followed by **Try Exercises** (and solutions) and a conceptual **Question/Answer** follow-up. See the samples below:

EXAMPLE 5 Calculate the Airtime for a Snowboarder's Jump

The height $h(t)$, in feet, of a snowboarder t seconds after beginning a certain jump can be approximated by $h(t) = -16t^2 + 22.9t + 9$. If the snowboarder lands at a point that is 3 feet below the base of the jump, determine the *airtime* (the time the snowboarder is in the air) for this jump. Round to the nearest tenth of a second.



Solution

Because $h(t)$ represents the height of the snowboarder t seconds after the beginning of the jump, the snowboarder lands when $h(t) = -3$, 3 feet *below* the base of the jump.

$$h(t) = -16t^2 + 22.9t + 9$$
$$-3 = -16t^2 + 22.9t + 9 \quad \text{• Replace } h(t) \text{ with } -3.$$
$$0 = -16t^2 + 22.9t + 12$$
$$t = \frac{-22.9 \pm \sqrt{22.9^2 - 4(-16)(12)}}{2(-16)} \quad \text{• Use the quadratic formula.}$$
$$= \frac{-22.9 \pm \sqrt{1292.41}}{-32}$$
$$\approx -0.4 \quad \text{or} \quad 1.8 \quad \text{• Use a calculator.}$$

Because a negative time is not possible, the airtime for this jump is approximately 1.8 seconds.

Try Exercise 48, page 207

Engaging Examples

Examples are designed to capture your attention and help you master important concepts.

Annotated Examples

Step-by-step solutions are provided for most numbered examples.

Try Exercises

A reference to an exercise follows all worked examples. This exercise provides you with an opportunity to test your understanding of concepts by working an exercise related to the worked example.

Solutions to Try Exercises

Complete solutions to the Try Exercises can be found in the Solutions to the Try Exercises appendix.

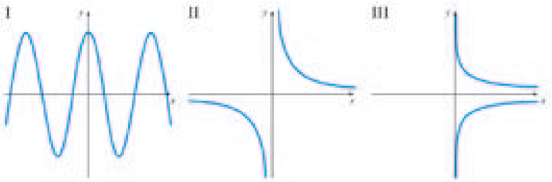
Exercise Set 2.4, page 206

48. The soccer ball hits the ground when $h(t) = 0$.

$$h(t) = -4.9t^2 + 12.8t$$
$$0 = -4.9t^2 + 12.8t \quad \text{• Replace } h(t) \text{ with } 0.$$
$$0 = t(-4.9t + 12.8) \quad \text{• Solve for } t.$$
$$t = 0 \quad \text{or} \quad t = \frac{-12.8}{-4.9} \approx 2.6$$

The soccer ball hits the ground in approximately 2.6 seconds.

Question • Which of the graphs below, I, II, or III, is **a.** symmetric with respect to the x-axis? **b.** symmetric with respect to the y-axis?



Answer • **a.** III is symmetric with respect to the x-axis. **b.** I is symmetric with respect to the y-axis.

Question/Answer

In each section, we have posed at least one question that encourages you to pause and think about the concepts presented in the current discussion. To ensure that you do not miss this important information, the answer is provided as a footnote on the same page.

Zero of a Function

A value *a* in the domain of a function *f* for which $f(a) = 0$ is called a **zero** of *f*.

EXAMPLES

- Let $f(x) = 2x - 4$. When $x = 2$, we have

$$f(x) = 2x - 4$$
$$f(2) = 2(2) - 4$$
$$= 0$$

Because $f(2) = 0$, 2 is a zero of *f*.

Immediate Examples of Definitions and Concepts

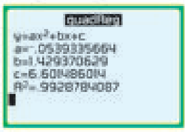
Immediate examples of many definitions and concepts are provided to enhance your understanding of new topics.

Margin Notes

Margin Notes alert you to a point requiring special attention or are used to provide study tips.

Note
For nonlinear regression calculations, the value of *r* is not shown on a TI-83/TI-83 Plus/TI-84 Plus graphing calculator. In such cases, the coefficient of determination is used to determine how well the data fit the model.

2. Find the regression equation using QuadReg in the STAT CALC menu. For a TI-83/TI-83 Plus/TI-84 Plus calculator, press STAT ▶ 5 ENTER. Scroll to Calculate and press ENTER.



TO REVIEW
Pythagorean Theorem
See pages 102–103.

The distance between the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in Figure 2.5 is the length of the hypotenuse of a right triangle whose sides are horizontal and vertical line segments that measure $|x_2 - x_1|$ and $|y_2 - y_1|$, respectively. Applying the Pythagorean Theorem to this triangle produces

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

To Review Notes

To Review Notes in the margin will help you recognize the prerequisite skills needed to understand new concepts. These notes direct you to the appropriate page or section for review.

Calculus Connection Icons

Calculus Connection Icons identify topics that will be revisited in a subsequent calculus course.

CALCULUS CONNECTION

Difference Quotient

The expression

$$\frac{f(x+h) - f(x)}{h}, \quad h \neq 0$$

is called the **difference quotient** of *f*. It enables us to study the manner in which a function changes in value as the independent variable changes.

EXAMPLE 7 Solve a Logarithmic Equation

Solve: $\ln(3x + 8) = \ln(2x + 2) + \ln(x - 2)$

Algebraic Solution

$\ln(3x + 8) = \ln(2x + 2) + \ln(x - 2)$

$\ln(3x + 8) = \ln[(2x + 2)(x - 2)]$ • Product property

$\ln(3x + 8) = \ln(2x^2 - 2x - 4)$

$3x + 8 = 2x^2 - 2x - 4$ • One-to-one property of logarithms

$0 = 2x^2 - 5x - 12$ • Subtract $3x + 8$ from each side.

$0 = (2x + 3)(x - 4)$ • Factor.

$x = -\frac{3}{2}$ or $x = 4$ • Solve for *x*.

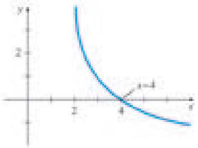
A check will show that 4 is a solution but that $-\frac{3}{2}$ is not a solution.

Visualize the Solution

The graph of

$y = \ln(3x + 8) - \ln(2x + 2) - \ln(x - 2)$

has only one *x*-intercept. Thus there is only one real solution.



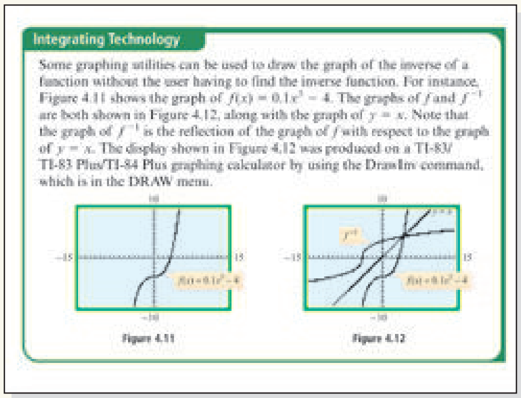
▶ Try Exercise 40, page 388

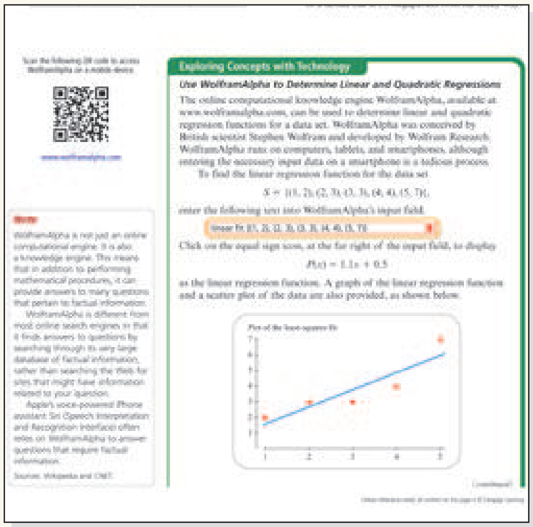
Visualize the Solution

When appropriate, both algebraic and graphical solutions are provided to help you visualize the mathematics of an example and to create a link between the algebraic and visual components of a solution.

▶ Integrating Technology

Integrating Technology boxes show how technology can be used to illustrate concepts and solve many mathematical problems.



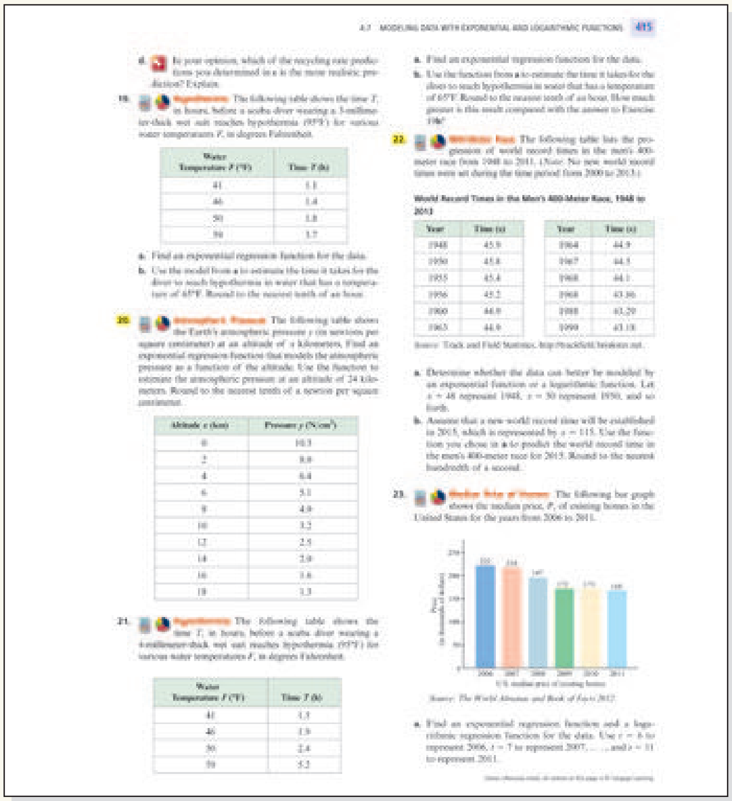


◀ Exploring Concepts with Technology

The optional Exploring Concepts with Technology feature appears after the last section in each chapter and provides you the opportunity to use a calculator, a mobile device, or a computer to solve computationally difficult problems.

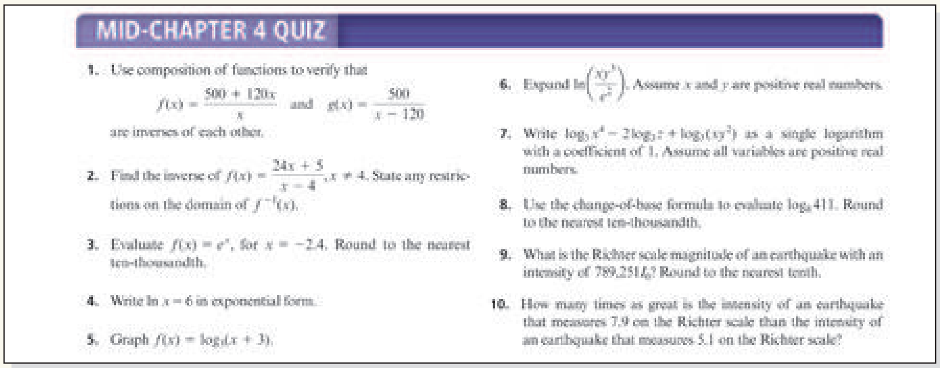
▶ Modeling

Modeling sections and exercises rely on the use of a graphing calculator, a mobile device, or a computer. These optional sections and exercises introduce the idea of a mathematical model and help you see the relevance of mathematical concepts.



▶ Mid-Chapter Quizzes

The Mid-Chapter Quizzes help you assess your understanding of the concepts studied earlier in the chapter. The answers for all exercises in the Mid-Chapter Quizzes, along with a reference to the section in which a particular concept was presented, are provided in the Answers to Selected Exercises appendix.

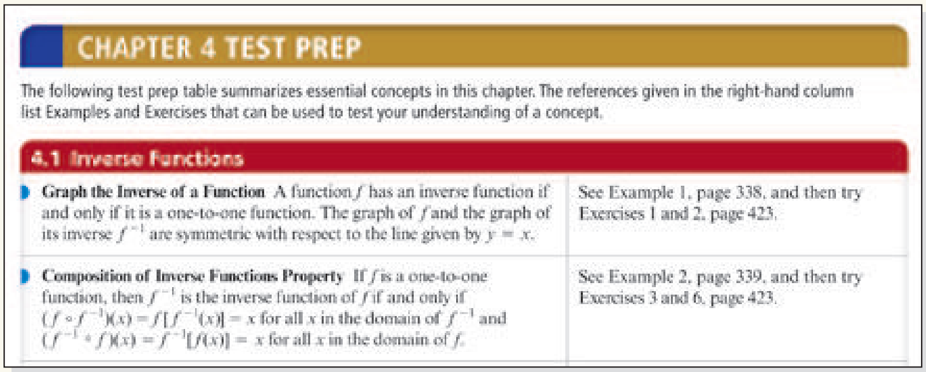


MID-CHAPTER 4 QUIZ

1. Use composition of functions to verify that $f(x) = \frac{500 + 120x}{x}$ and $g(x) = \frac{500}{x - 120}$ are inverses of each other.
2. Find the inverse of $f(x) = \frac{24x + 5}{x - 4}$, $x \neq 4$. State any restrictions on the domain of $f^{-1}(x)$.
3. Evaluate $f(x) = e^x$, for $x = -2.4$. Round to the nearest ten-thousandth.
4. Write $\ln x = 6$ in exponential form.
5. Graph $f(x) = \log_3(x + 3)$.
6. Expand $\ln\left(\frac{xy^3}{e^2}\right)$. Assume x and y are positive real numbers.
7. Write $\log_3 x^4 - 2\log_3 z + \log_3(xy^2)$ as a single logarithm with a coefficient of 1. Assume all variables are positive real numbers.
8. Use the change-of-base formula to evaluate $\log_6 411$. Round to the nearest ten-thousandth.
9. What is the Richter scale magnitude of an earthquake with an intensity of $789{,}251I_0$? Round to the nearest tenth.
10. How many times as great is the intensity of an earthquake that measures 7.9 on the Richter scale than the intensity of an earthquake that measures 5.1 on the Richter scale?

◀ Chapter Test Preps

The Chapter Test Preps summarize the major concepts discussed in each chapter. These Test Preps help you prepare for a chapter test. For each concept, there is a reference to a worked example illustrating the concept and at least one exercise in the Chapter Review Exercise set relating to that concept.



CHAPTER 4 TEST PREP

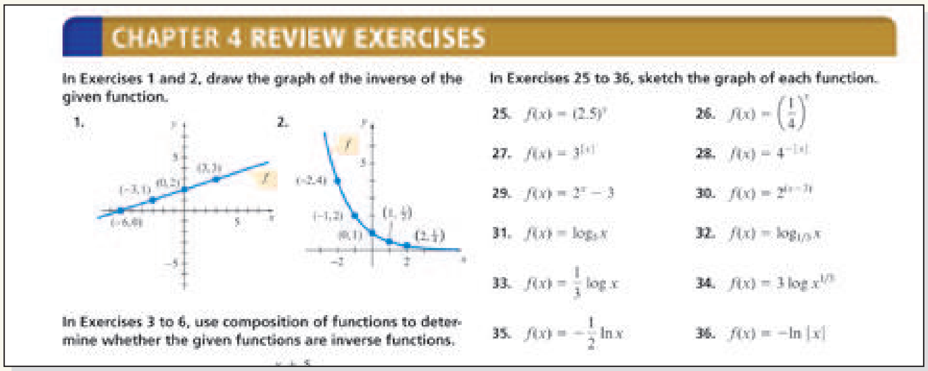
The following test prep table summarizes essential concepts in this chapter. The references given in the right-hand column list Examples and Exercises that can be used to test your understanding of a concept.

4.1 Inverse Functions

Graph the Inverse of a Function A function f has an inverse function if and only if it is a one-to-one function. The graph of f and the graph of its inverse f^{-1} are symmetric with respect to the line given by $y = x$.	See Example 1, page 338, and then try Exercises 1 and 2, page 423.
Composition of Inverse Functions Property If f is a one-to-one function, then f^{-1} is the inverse function of f if and only if $(f \circ f^{-1})(x) = f[f^{-1}(x)] = x$ for all x in the domain of f^{-1} and $(f^{-1} \circ f)(x) = f^{-1}[f(x)] = x$ for all x in the domain of f.	See Example 2, page 339, and then try Exercises 3 and 6, page 423.

▼ Chapter Review Exercise Sets and Chapter Tests

The Chapter Review Exercise sets and the Chapter Tests at the end of each chapter are designed to provide you with another opportunity to assess your understanding of the concepts presented in a chapter. The answers for all exercises in the Chapter Review Exercise sets and the Chapter Tests, along with a reference to the section in which the concept was presented, are provided in the Answers to Selected Exercises appendix.



CHAPTER 4 REVIEW EXERCISES

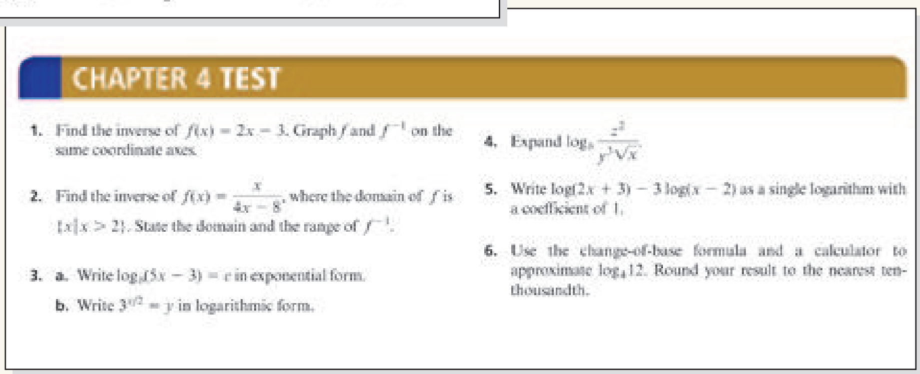
In Exercises 1 and 2, draw the graph of the inverse of the given function.

1. 2.

In Exercises 3 to 6, use composition of functions to determine whether the given functions are inverse functions.

In Exercises 25 to 36, sketch the graph of each function.

25. $f(x) = (2.5)^x$
26. $f(x) = \left(\frac{1}{4}\right)^x$
27. $f(x) = 3^{|x|}$
28. $f(x) = 4^{-|x|}$
29. $f(x) = 2^x - 3$
30. $f(x) = 2^{(x-3)}$
31. $f(x) = \log_5 x$
32. $f(x) = \log_{1/3} x$
33. $f(x) = \frac{1}{3}\log x$
34. $f(x) = 3\log x^{1/3}$
35. $f(x) = -\frac{1}{2}\ln x$
36. $f(x) = -\ln |x|$



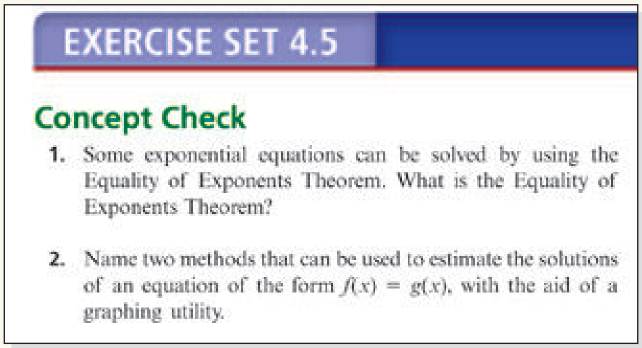
CHAPTER 4 TEST

1. Find the inverse of $f(x) = 2x - 3$. Graph f and f^{-1} on the same coordinate axes.
2. Find the inverse of $f(x) = \frac{x}{4x - 8}$, where the domain of f is $\{x \mid x > 2\}$. State the domain and the range of f^{-1}.
3. a. Write $\log_b(5x - 3) = c$ in exponential form.
   b. Write $3^{x/2} = y$ in logarithmic form.
4. Expand $\log_b \frac{z^2}{y^3\sqrt{x}}$.
5. Write $\log(2x + 3) - 3\log(x - 2)$ as a single logarithm with a coefficient of 1.
6. Use the change-of-base formula and a calculator to approximate $\log_4 12$. Round your result to the nearest ten-thousandth.

New Features in This Eighth Edition!

NEW Concept Check Exercises

Each exercise set starts with exercises that are designed to test your understanding of new concepts.


EXERCISE SET 4.5

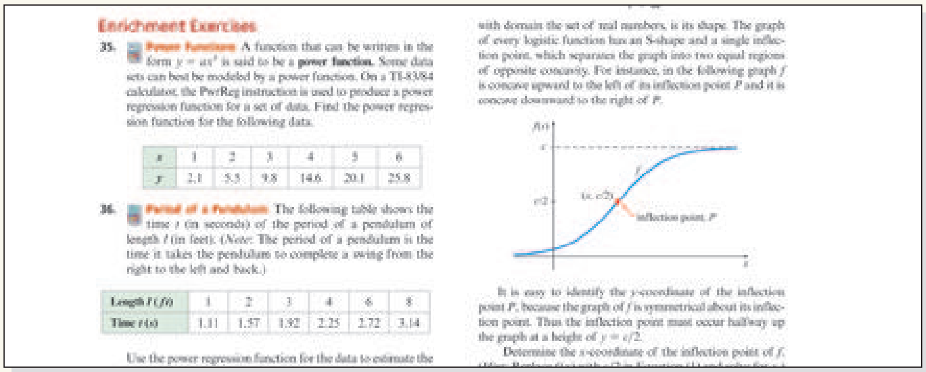
Concept Check

1. Some exponential equations can be solved by using the Equality of Exponents Theorem. What is the Equality of Exponents Theorem?

2. Name two methods that can be used to estimate the solutions of an equation of the form $f(x) = g(x)$, with the aid of a graphing utility.

NEW Enrichment Exercises

Each exercise set concludes with an exercise or exercises designed to extend the concepts presented in the section or to provide exercises that challenge your problem-solving abilities.


Enrichment Exercises

35. **Power Functions** A function that can be written in the form $y = ax^b$ is said to be a **power function.** Some data sets can best be modeled by a power function. On a TI-83/84 calculator, the PwrReg instruction is used to produce a power regression function for a set of data. Find the power regression function for the following data.

x	1	2	3	4	5	6
y	2.1	5.5	9.8	14.6	20.1	25.8

36. **Period of a Pendulum** The following table shows the time t (in seconds) of the period of a pendulum of length l (in feet). (*Note:* The period of a pendulum is the time it takes the pendulum to complete a swing from the right to the left and back.)

Length l (ft)	1	2	3	4	6	8
Time t (s)	1.11	1.57	1.92	2.25	2.72	3.14

Use the power regression function for the data to estimate the

with domain the set of real numbers, is its shape. The graph of every logistic function has an S-shape and a single inflection point, which separates the graph into two equal regions of opposite concavity. For instance, in the following graph f is concave upward to the left of its inflection point P and it is concave downward to the right of P.

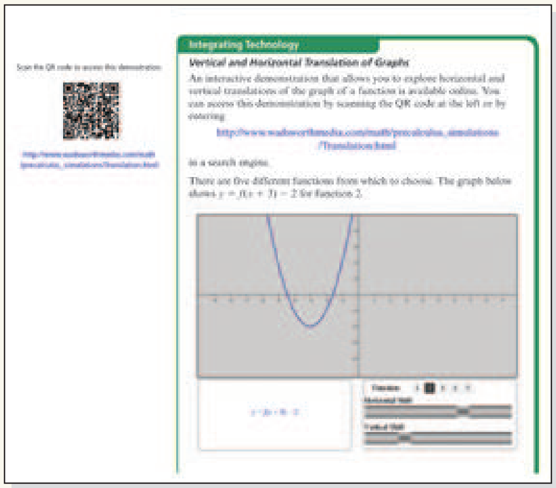
It is easy to identify the y-coordinate of the inflection point P, because the graph of f is symmetrical about its inflection point. Thus the inflection point must occur halfway up the graph at a height of $y = c/2$.

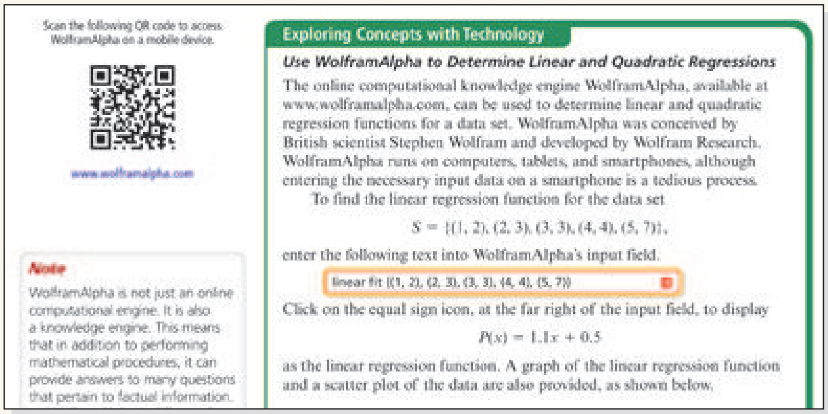
Determine the x-coordinate of the inflection point of f.

NEW Interactive Demonstrations

Our new Interactive Demonstrations allow you to adjust parameters and immediately see the change produced by your adjustment. These Interactive Demonstrations run on computers and on mobile devices. Instructions for using each of the demonstrations are provided.

You can access these Interactive Demonstrations by scanning a QR code with a QR reader app or by using the Web address listed below the QR code.

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www.wolframalpha.com

Exploring Concepts with Technology

Use WolframAlpha to Determine Linear and Quadratic Regressions

The online computational knowledge engine WolframAlpha, available at www.wolframalpha.com, can be used to determine linear and quadratic regression functions for a data set. WolframAlpha was conceived by British scientist Stephen Wolfram and developed by Wolfram Research. WolframAlpha runs on computers, tablets, and smartphones, although entering the necessary input data on a smartphone is a tedious process.

To find the linear regression function for the data set

$$S = \{(1, 2), (2, 3), (3, 3), (4, 4), (5, 7)\},$$

enter the following text into WolframAlpha's input field.

linear fit {(1, 2), (2, 3), (3, 3), (4, 4), (5, 7)}

Click on the equal sign icon, at the far right of the input field, to display

$$P(x) = 1.1x + 0.5$$

as the linear regression function. A graph of the linear regression function and a scatter plot of the data are also provided, as shown below.

Note

WolframAlpha is not just an online computational engine. It is also a knowledge engine. This means that in addition to performing mathematical procedures, it can provide answers to many questions that pertain to factual information.

NEW An Alternative Technology Approach via WolframAlpha

Several Integrating Technology features and some Exploring Concepts with Technology features show how WolframAlpha can be used to perform computations, solve equations, graph functions, and find regression functions. WolframAlpha, which runs on computers and mobile devices, often provides an alternative to a graphing calculator. WolframAlpha can be accessed by scanning a QR code or by using the address www.wolframalpha.com.

In addition to the new features, the following changes appear in this eighth edition of *Algebra and Trigonometry*.

Chapter P **Preliminary Concepts**

- A new chapter opener introduces some of the concepts in this chapter.
- **P.1** New application exercises and Enrichment Exercises have been added.
- **P.2** One example has been revised and Enrichment Exercises have been added.
- **P.3** One example has been revised.
- **P.4** Enrichment Exercises have been added.
- **P.5** A continued fraction exercise has been added to the Enrichment Exercises.
- **P.6** Enrichment Exercises have been added.

Chapter 1 **Equations and Inequalities**

- A new chapter opener introduces some of the concepts in this chapter.
- **1.1** New application exercises and Enrichment Exercises have been added.
- **1.2** Three examples have been updated and a new example has been added. Several exercises have been revised and Enrichment Exercises have been added.
- **1.3** New application exercises and Enrichment Exercises have been added.
- **1.4 and 1.5** An Enrichment Exercise has been added.

Chapter 2 **Functions and Graphs**

- **2.2** The introduction to piecewise-defined functions has been expanded and the example has been revised. New exercises involving piecewise-defined functions have been added.
- **2.3** A new example has been added and several exercises have been updated.
- **2.4** The business application example has been revised and application exercises have been added.
- **2.5** Three new Interactive Demonstrations have been added. They illustrate translations, reflections, and compressing and stretching of graphs. Enrichment Exercises have been added.
- **2.7** Two exercises have been updated and a new application exercise has been added.
- The Exploring Concepts with Technology feature has been revised to illustrate how WolframAlpha can be used to find linear and quadratic regression functions.

Chapter 3 **Polynomial and Rational Functions**

- A new chapter opener introduces some of the concepts in this chapter.
- **3.2** The Integrating Technology feature has been expanded to include instructions on how to use WolframAlpha to graph, find extrema, and evaluate polynomial functions. The cubic regression example has been updated and new application exercises have been added.
- **3.3 and 3.4** One example has been revised.
- **3.5** An application exercise and an exercise that involves the parabolic asymptote of a rational function have been added.
- The Exploring Concepts with Technology feature has been revised to illustrate how WolframAlpha can be used to find zeros of a polynomial function and to find cubic and quartic regression functions.
- Several exercises in the Review Exercises and the Chapter Test have been updated.

Chapter 4 **Exponential and Logarithmic Functions**

- A new chapter opener introduces some of the concepts in this chapter.
- **4.2** A new Integrating Technology feature illustrates how to use WolframAlpha to solve exponential equations. A new exercise that demonstrates the rapid growth of an exponential function has been included in the Enrichment Exercises.
- **4.3** Several exercises have been revised and two application exercises have been added.
- **4.4** A new Integrating Technology feature illustrates how to use WolframAlpha to evaluate logarithms with various bases. Several examples and exercises have been updated.
- **4.6** Several population growth and compound interest exercises have been updated.

- **4.7** Two examples and several exercises have been updated.
- The Exploring Concepts with Technology feature has been revised to illustrate how WolframAlpha can be used to find exponential and logarithmic regression functions.
- Several exercises in the Review Exercises and the Chapter Test have been updated.

Chapter 5 **Trigonometric Functions**

- A new chapter opener introduces some of the concepts in this chapter.
- **5.1** Two examples have been revised and a new application example has been added. Application exercises involving arc length and angular speed have been added.
- **5.2** A new example and several application exercises have been added.
- **5.3** A new example has been added and several changes have been made to the exercise set.
- **5.4** The introduction to the wrapping function has been enhanced with a graphical representation. An Interactive Demonstration that involves the wrapping function has been added. Exercises that allow the student to graphically estimate the value of the wrapping function, as well as new application exercises, have been added.
- **5.5** Definition boxes that allow the student to easily find important concepts have been added. The new definition boxes now include simple examples of the definitions. Many examples have been revised so there is a consistent approach to graphing trigonometric functions.
- **5.6** Definition boxes that allow the student to easily find important concepts have been added. The new definition boxes now include simple examples of the definitions. Many examples have been revised so there is a consistent approach to graphing trigonometric functions. Application exercises have been added.
- **5.7** An Interactive Demonstration that involves the graphs of trigonometric functions has been added. An exercise involving the phenomenon of beats and other application exercises have been added.
- **5.8** New application exercises have been added.

Chapter 6 **Trigonometric Identities, Inverse Functions, and Equations**

- A new chapter opener introduces some of the concepts in this chapter.
- **6.3** A visual insight exercise that illustrates a half-angle identity has been added to the Enrichment Exercises.
- **6.6** The example that uses a sine regression to model the illumination of the moon has been updated. All sine regression application exercises have been updated.
- The Exploring Concepts with Technology feature has been revised to illustrate how WolframAlpha can be used to play musical tones and beats.
- The sine regression application exercises in the Chapter Review and the Chapter Test have been updated.

Chapter 7 **Applications of Trigonometry**

- **7.1** Additional introductory information concerning the ASA, AAS, and the SSA cases has been added.
- **7.3** The dot product example has been revised.
- The Exploring Concepts with Technology feature has been revised to illustrate how WolframAlpha can be used to perform vector operations and solve vector application problems.

Chapter 8 **Topics in Analytic Geometry**

- A new chapter opener introduces some of the concepts in this chapter.
- **8.1** The Integrating Technology feature of this section has been expanded to include instructions on how to use WolframAlpha to graph parabolas and to find the focus and vertex of a parabola. A new Interactive Demonstration illustrates the relationships between the standard form of the equation of a parabola and its graph. A 3-D optical illusion exercise has been added to the Enrichment Exercises.

- **8.2** The new Interactive Demonstration illustrates the relationships between the standard form of the equation of an ellipse and its graph. The Integrating Technology feature has been expanded to include instructions on how to use WolframAlpha to graph ellipses and to determine the foci, vertices, and center of an ellipse. Several application exercises have been added.
- **8.3** The new Interactive Demonstration illustrates the relationships between the standard form of the equation of a hyperbola and its graph. The Integrating Technology feature has been expanded to include instructions on how to use WolframAlpha to graph a hyperbola and to determine the foci, vertices, and center of a hyperbola. Several application exercises have been added.
- **8.4** A new example involving the rotation-of-axes formulas has been added. A new example illustrates how to use WolframAlpha to graph a conic.
- **8.5** Two new Integrating Technology features and two Enrichment Exercises have been added.
- **8.6** The figue that illustrates the focus-directrix definitions of the conics has been expanded to include an ellipse and a hyperbola.
- **8.7** Exercises that involve parametric equations in an *xyz*-coordinate system have been added.
- The Exploring Concepts with Technology feature now provides instructions on how to use our new Interactive Demonstration that involves conics and polar equations.
- Application exercises have been added to the Review Exercises and to the Chapter Test.

Chapter 9 Systems of Equations and Inequalities

- **9.1** The Integrating Technology feature has been expanded to include instructions on how to use WolframAlpha to solve systems of linear equations. Application exercises and Enrichment Exercises have been added.
- **9.2** Application exercises have been added.
- **9.3** The Integrating Technology feature has been expanded to include instructions on how to use WolframAlpha to solve nonlinear systems of equations. Enrichment Exercises have been added.
- The new Exploring Concepts with Technology feature illustrates how to use WolframAlpha to solve linear programming problems.

Chapter 10 Matrices

- **10.1** The Echelon Form Procedure and the Gaussian Elimination example have been revised. A new example and new exercises have been added.
- **10.2** A social network graph introduces the concept of adjacency matrices. A social networks exercise has been added to the Enrichment Exercises.
- **10.3** The Integrating Technology feature has been expanded to include instructions on how to use WolframAlpha to find the inverse of a matrix.
- **10.4** Cramer's Rule is now included in this section. Three examples have been added.
- New exercises have been added to the Review Exercises and to the Chapter Test.

Chapter 11 Sequences, Series, and Probability

- A new chapter opener introduces some of the concepts in this chapter.
- **11.1** New examples have been added.
- **11.2** A new example and new application exercises have been added.
- **11.5** A new example has been added.
- **11.6** Three examples have been revised and several application exercises have been added.
- **11.7** Two examples have been revised and several application exercises have been added.

Supplements

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Jennifer Byall—*Globe University*
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CHAPTER P

Preliminary Concepts

P.1 The Real Number System

P.2 Integer and Rational Number Exponents

P.3 Polynomials

P.4 Factoring

P.5 Rational Expressions

P.6 Complex Numbers

Sabermetrics

The film *Moneyball* was based on the book *Moneyball: The Art of Winning an Unfair Game* by Michael Lewis. It recounts the true story of how the Oakland Athletics baseball team used mathematics to select players for its team. They used what has become known as **sabermetrics,** introduced by Bill James, to objectively evaluate a player's performance using mathematics.

> Bill James defined sabermetrics as "the search for objective knowledge about baseball." Thus, sabermetrics attempts to answer objective questions about baseball, such as "which player on the Red Sox contributed the most to the team's offense?" or "How many home runs will Miguel Cabrera hit next year?" It cannot deal with the subjective judgments which are also important to the game, such as "Who is your favorite player?" or "That was a great game."[1]

In sabermetrics, a SLOB is not a bad thing. Instead, a SLOB is one of the measures of a player's performance. SLOB stands for "**sl**ugging times **o**n **b**ase average." A SLOB value of 0.3 is considered very good. For instance, Lou Gehrig had a SLOB value of 0.283. Many of the sabermetric measures are based on ratios such as the expressions given in Exercises 129 and 130 on page 16.

[1] David J. Grabiner, "The Sabermetric Manifesto," *The Baseball Archive*. Available online at http://remarque.org/~grabiner/manifesto.txt

SECTION P.1

Sets
Union and Intersection of Sets
Interval Notation
Absolute Value and Distance
Exponential Expressions
Order of Operations Agreement
Simplifying Variable Expressions

The Real Number System

Sets

Human beings share the desire to organize and classify. Astronomers classify stars by such characteristics as color, mass, size, temperature, and distance from Earth. Mathematicians likewise place objects with similar properties in *sets*. A **set** is a collection of objects. The objects are called **elements** of the set. Sets are denoted by placing braces around the elements in the set.

The numbers that we use to count things, such as the number of books in a library or the number of songs in a music collection, are called the **natural numbers.**

$$\textbf{Natural numbers} = \{1, 2, 3, 4, 5, 6, \ldots\}$$

Each natural number greater than 1 is a *prime* number or a *composite* number. A **prime number** is a natural number greater than 1 that is divisible (evenly) only by itself and 1. For example, 2, 3, 5, 7, 11, and 13 are the first six prime numbers. A natural number, other than 1, that is not a prime number is a **composite number.** The numbers 4, 6, 8, and 9 are the first four composite numbers. Note that each of these numbers is divisible by a number other than itself and 1. For instance, 8 is divisible by 1, 2, 4, and 8.

The whole numbers include zero and the natural numbers.

$$\textbf{Whole numbers} = \{0, 1, 2, 3, 4, 5, 6, \ldots\}$$

We also need numbers to measure temperature below zero or, in accounting, when a company incurs a loss.

$$\textbf{Integers} = \{\ldots, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, \ldots\}$$

The integers $\ldots, -6, -5, -4, -3, -2, -1$ are **negative integers.** The integers 1, 2, 3, 4, 5, 6, . . . are **positive integers** (or natural numbers). The integer 0 is neither a positive nor a negative integer.

Still other numbers are needed to designate part of a whole, such as a screw that is three-fourths inch long.

$$\textbf{Rational numbers} = \left\{\frac{p}{q}, \text{ where } p \text{ and } q \text{ are integers and } q \neq 0\right\}$$

The numbers $\frac{3}{4}$, $-\frac{9}{2}$, and $\frac{7}{1}$ are examples of rational numbers. Note that $\frac{7}{1} = 7$. Because any integer n can be written with a denominator of $1\left(n = \frac{n}{1}\right)$, **all integers are rational numbers.**

A rational number written as a fraction can be written as a decimal by dividing the numerator by the denominator. As shown below, the result is either a **terminating decimal** such as 0.45 or a **repeating decimal** such as 0.218181818 . . . , where the digits 18 are continually repeated. In this case, we frequently place a bar over the repeating digits and write $0.2181818\ldots = 0.2\overline{18}$.

$$\begin{array}{r} 0.45 \\ 20\,\overline{)\;9.00} \\ \underline{-8\,0} \\ 1\,00 \\ \underline{-1\,00} \\ 0 \end{array}$$

This is a terminating decimal. The remainder is zero.

$$\frac{9}{20} = 0.45$$

$$\begin{array}{r} 0.21818 \\ 55\,\overline{)\;12.00000} \\ \underline{-11\,0} \\ 1\,00 \\ \underline{-55} \\ 450 \\ \underline{-440} \\ 100 \\ \underline{-55} \\ 450 \\ \underline{-440} \\ 10 \end{array}$$

This is a repeating decimal. Note that the remainders 10 and 45 are repeating. The remainder is never zero.

$$\frac{12}{55} = 0.2\overline{18}$$

Math Matters

Archimedes (c. 287–212 B.C.E.) was the first to calculate π with any degree of precision. He was able to show that

$$3\frac{10}{71} < \pi < 3\frac{1}{7}$$

from which we get the approximation

$$3\frac{1}{7} = \frac{22}{7} \approx \pi$$

The use of the symbol π for this quantity was introduced by Leonhard Euler (1707–1783) in 1739, approximately 2000 years after Archimedes.

Numbers that are not rational numbers are called **irrational numbers.** In decimal form, an irrational number has a decimal representation that never terminates nor repeats. One of the best known irrational numbers is pi, denoted by the Greek letter π. An approximate value of π is 3.14592654. . . . Other examples of irrational numbers are 2.13113111311113. . . and the square root of any prime number such as $\sqrt{11} \approx 3.31662479\ldots$. The rational numbers and irrational numbers taken together are the **real numbers.**

The relationships among the various sets of numbers are shown in Figure P.1.

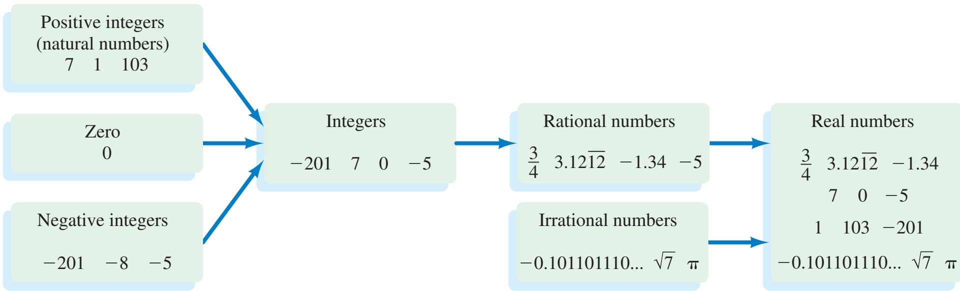


Figure P.1

EXAMPLE 1 Classify Real Numbers

Determine which of the following numbers are

a. integers **b.** rational numbers **c.** irrational numbers

d. real numbers **e.** prime numbers **f.** composite numbers

$$-0.2,\quad 0,\quad 0.\overline{3},\quad 0.71771777177771\ldots,\quad \pi,\quad 6,\quad 7,\quad 41,\quad 51$$

Solution

a. Integers: 0, 6, 7, 41, 51

b. Rational numbers: $-0.2, 0, 0.\overline{3}, 6, 7, 41, 51$

c. Irrational numbers: $0.71771777177771\ldots, \pi$

d. Real numbers: $-0.2, 0, 0.\overline{3}, 0.71771777177771\ldots, \pi, 6, 7, 41, 51$

e. Prime numbers: 7, 41

f. Composite numbers: 6, 51

Try Exercise 8, page 14

Each member of a set is called an **element** of the set. For instance, if $C = \{2, 3, 5\}$, then the elements of C are 2, 3, and 5. The notation $2 \in C$ is read "2 is an element of C." Set A is a **subset** of set B if every element of A is also an element of B, and we write $A \subseteq B$. For instance, the set of negative integers $\{-1, -2, -3, -4, \ldots\}$ is a subset of the set of integers. The set of positive integers $\{1, 2, 3, 4, \ldots\}$ (the natural numbers) is also a subset of the set of integers.

Question • Are the integers a subset of the rational numbers?

The **empty set**, or **null set**, is the set that contains no elements. The symbol $\varnothing$ is used to represent the empty set. The set of people who have run a 2-minute mile is the empty set.

Answer • Yes.

Math Matters

Sophie Germain (1776–1831) was born in Paris, France. Because enrollment in the university she wanted to attend was available only to men, Germain attended under the name of Antoine-August Le Blanc. Eventually her ruse was discovered, but not before she came to the attention of Pierre Lagrange, one of the best mathematicians of the time. He encouraged her work and became a mentor to her. A certain type of prime number is named after her, called a *Germain prime number.* It is a number p such that p and $2p + 1$ are both prime. For instance, 11 is a Germain prime because $2(11) + 1 = 23$, and 11 and 23 are both prime numbers. Germain primes are used in public key cryptography, a method used to send secure communications over the Internet.

Note

The order of the elements of a set is not important. For instance, the set of natural numbers less than 6 given at the right could have been written {3, 5, 2, 1, 4}. It is customary, however, to list elements of a set in numerical order.

The set of natural numbers less than 6 is $\{1, 2, 3, 4, 5\}$. This is an example of a **finite set**; all the elements of the set can be listed. The set of all natural numbers is an example of an **infinite set**. There is no largest natural number, so all the elements of the set of natural numbers cannot be listed.

Sets are often written using **set-builder notation**. Set-builder notation can be used to describe almost any set, but it is especially useful when writing infinite sets. For instance, the set

$$\{2n \mid n \in \text{natural numbers}\}$$

is read as "the set of elements $2n$ such that n is a natural number." By replacing n with each of the natural numbers, we obtain the set of positive even integers: $\{2, 4, 6, 8, \ldots\}$.

The set of real numbers greater than 2 is written

$$\{x \mid x > 2, x \in \text{real numbers}\}$$

and is read "the set of x such that x is greater than 2 and x is an element of the real numbers."

Much of the work we do in this text uses the real numbers. With this in mind, we will frequently write, for instance, $\{x \mid x > 2, x \in \text{real numbers}\}$ in a shortened form as $\{x \mid x > 2\}$, where we assume that x is a real number.

Math Matters

A **fuzzy set** is one in which each element is given a "degree" of membership. The concepts behind fuzzy sets are used in a wide variety of applications such as traffic lights, washing machines, and computer speech recognition programs.

EXAMPLE 2 Use Set-Builder Notation

List the four smallest elements in $\{n^3 \mid n \in \text{natural numbers}\}$.

Solution

Because we want the four *smallest* elements, we choose the four smallest natural numbers. Thus $n = 1, 2, 3,$ and 4. Therefore, the four smallest elements of the set $\{n^3 \mid n \in \text{natural numbers}\}$ are 1, 8, 27, and 64.

Try Exercise 12, page 14

Union and Intersection of Sets

Just as operations such as addition and multiplication are performed on real numbers, operations are performed on sets. Two operations performed on sets are union and intersection. The union of two sets A and B is the set of elements that belong to A or to B, or to both A and B.

Definition of the Union of Two Sets

The **union** of two sets, written $A \cup B$, is the set of all elements that belong to either A or B. In set-builder notation, this is written

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

EXAMPLE

Given $A = \{2, 3, 4, 5\}$ and $B = \{0, 1, 2, 3, 4\}$, find $A \cup B$.

$$A \cup B = \{0, 1, 2, 3, 4, 5\}$$

• Note that an element that belongs to both sets is listed only once.

The intersection of the two sets A and B is the set of elements that belong to both A and B.

Definition of the Intersection of Two Sets

The **intersection** of two sets, written $A \cap B$, is the set of all elements that are common to both A and B. In set-builder notation, this is written

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

EXAMPLE

Given $A = \{2, 3, 4, 5\}$ and $B = \{0, 1, 2, 3, 4\}$, find $A \cap B$.

$A \cap B = \{2, 3, 4\}$ • The intersection of two sets contains the elements common to both sets.

If the intersection of two sets is the empty set, the two sets are said to be **disjoint.** For example, if $A = \{2, 3, 4\}$ and $B = \{7, 8\}$, then $A \cap B = \varnothing$ and A and B are disjoint sets.

EXAMPLE 3 Find the Union and Intersection of Sets

Find each intersection or union given $A = \{0, 2, 4, 6, 10, 12\}$, $B = \{0, 3, 6, 12, 15\}$, and $C = \{1, 2, 3, 4, 5, 6, 7\}$.

a. $A \cup C$ **b.** $B \cap C$ **c.** $A \cap (B \cup C)$ **d.** $B \cup (A \cap C)$

Solution

a. $A \cup C = \{0, 1, 2, 3, 4, 5, 6, 7, 10, 12\}$ • The elements that belong to A or C

b. $B \cap C = \{3, 6\}$ • The elements that belong to B and C

c. First, determine $B \cup C = \{0, 1, 2, 3, 4, 5, 6, 7, 12, 15\}$. Then
$A \cap (B \cup C) = \{0, 2, 4, 6, 12\}$ • The elements that belong to A and $(B \cup C)$

d. First, determine $A \cap C = \{2, 4, 6\}$. Then
$B \cup (A \cap C) = \{0, 2, 3, 4, 6, 12, 15\}$ • The elements that belong to B or $(A \cap C)$

▸ Try Exercise 22, page 14

Interval Notation

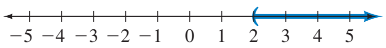


Figure P.2

The graph of $\{x \mid x > 2\}$ is shown in Figure P.2. The set is the real numbers greater than 2. The parenthesis at 2 indicates that 2 is not included in the set. Rather than write this set of real numbers using set-builder notation, we can write the set in **interval notation** as $(2, \infty)$.

In general, the interval notation

(a, b) represents all real numbers between a and b, not including a and b. This is an **open interval.** In set-builder notation, we write $\{x \mid a < x < b\}$. The graph of $(-4, 2)$ is shown in Figure P.3.

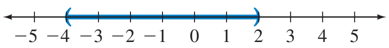


Figure P.3

$[a, b]$ represents all real numbers between a and b, including a and b. This is a **closed interval.** In set-builder notation, we write $\{x \mid a \le x \le b\}$. The graph of $[0, 4]$ is shown in Figure P.4. The brackets at 0 and 4 indicate that those numbers are included in the graph.

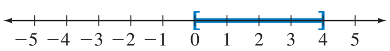


Figure P.4

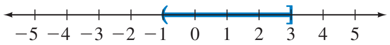


Figure P.5

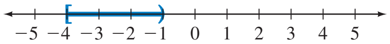
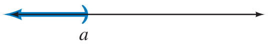


Figure P.6

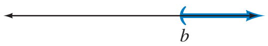
$(a, b]$ represents all real numbers between a and b, not including a but including b. This is a **half-open interval.** In set-builder notation, we write $\{x \mid a < x \le b\}$. The graph of $(-1, 3]$ is shown in Figure P.5.

$[a, b)$ represents all real numbers between a and b, including a but not including b. This is a **half-open interval.** In set-builder notation, we write $\{x \mid a \le x < b\}$. The graph of $[-4, -1)$ is shown in Figure P.6.

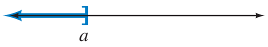
Subsets of the real numbers whose graphs extend forever in one or both directions can be represented by interval notation using the **infinity symbol** ∞ or the **negative infinity symbol** $-\infty$.



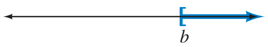
$(-\infty, a)$ represents all real numbers less than a.



(b, ∞) represents all real numbers greater than b.



$(-\infty, a]$ represents all real numbers less than or equal to a.



$[b, \infty)$ represents all real numbers greater than or equal to b.

$(-\infty, \infty)$ represents all real numbers.

EXAMPLE 4 Graph a Set Given in Interval Notation

Graph $(-\infty, 3]$. Write the interval in set-builder notation.

Solution

The set is the real numbers less than or equal to 3. In set-builder notation, this is the set $\{x \mid x \le 3\}$. Draw a right bracket at 3, and darken the number line to the left of 3, as shown in Figure P.7.

Figure P.7

▸ Try Exercise 40, page 14

> **Caution**
>
> It is *never* correct to use a bracket when using the infinity symbol. For instance, $[-\infty, 3]$ is not correct. Nor is $[2, \infty]$ correct. Neither negative infinity nor positive infinity is a real number and therefore cannot be contained in a closed interval.

The set $\{x \mid x \le -2\} \cup \{x \mid x > 3\}$ is the set of real numbers that are either less than or equal to -2 *or* greater than 3. We also could write this in interval notation as $(-\infty, -2] \cup (3, \infty)$. The graph of the set is shown in Figure P.8.

The set $\{x \mid x > -4\} \cap \{x \mid x < 1\}$ is the set of real numbers that are greater than -4 *and* less than 1. Note from Figure P.9 that this set is the interval $(-4, 1)$, which can be written in set-builder notation as $\{x \mid -4 < x < 1\}$.

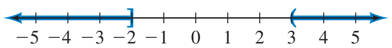


Figure P.8

Figure P.9

EXAMPLE 5 Graph Intervals

Graph the following. Write **a.** and **b.** using interval notation. Write **c.** and **d.** using set-builder notation.

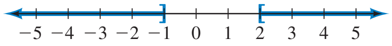
a. $\{x \mid x \le -1\} \cup \{x \mid x \ge 2\}$

b. $\{x \mid x \ge -1\} \cap \{x \mid x < 5\}$

c. $(-\infty, 0) \cup [1, 3]$

d. $[-1, 3] \cap (1, 5)$

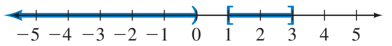
Solution

a. 

$(-\infty, -1] \cup [2, \infty)$

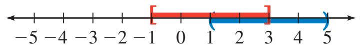
b. 

$[-1, 5)$

c. 

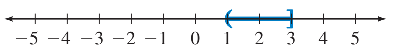
$\{x \mid x < 0\} \cup \{x \mid 1 \le x \le 3\}$

d. The graphs of $[-1, 3]$, in red, and $(1, 5)$, in blue, are shown below.



Note that the intersection of the sets occurs where the graphs intersect. Although $1 \in [-1, 3]$, $1 \notin (1, 5)$. Therefore, 1 does not belong to the intersection of the sets. On the other hand, $3 \in [-1, 3]$ and $3 \in (1, 5)$. Therefore, 3 belongs to the intersection of the sets.

Thus we have the following.



$\{x \mid 1 < x \le 3\}$

Try Exercise 50, page 14

Absolute Value and Distance

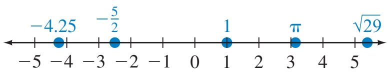


Figure P.10

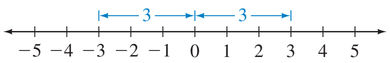


Figure P.11

The real numbers can be represented geometrically by a **coordinate axis** called a **real number line**. Figure P.10 shows a portion of a real number line. The number associated with a point on a real number line is called the **coordinate** of the point. The point corresponding to zero is called the **origin**. Every real number corresponds to a point on the number line, and every point on the number line corresponds to a real number.

The *absolute value* of a real number a, denoted $|a|$, is the distance between a and 0 on the number line. For instance, $|3| = 3$ and $|-3| = 3$ because both 3 and -3 are 3 units from zero. See Figure P.11.

In general, if $a \ge 0$, then $|a| = a$; however, if $a < 0$, then $|a| = -a$ because $-a$ is positive when $a < 0$. This leads to the following definition.

Note

The second part of the definition of absolute value states that if $a < 0$, then $|a| = -a$. For instance, if $a = -4$, then $|a| = |-4| = -(-4) = 4$.

Definition of Absolute Value

The **absolute value** of the real number a is defined by

$$|a| = \begin{cases} a & \text{if } a \ge 0 \\ -a & \text{if } a < 0 \end{cases}$$

EXAMPLE

$|5| = 5$ $\qquad |-4| = 4$ $\qquad |0| = 0$

EXAMPLE 6 Simplify an Absolute Value Expression

Simplify $|x + 4| - |2x - 6|$ given that $-3 \le x \le 2$.

Solution

Recall that $|a| = -a$ when $a < 0$ and $|a| = a$ when $a \ge 0$.

(continued)

When $-3 \le x \le 2$, $x + 4 > 0$ and $2x - 6 < 0$. Therefore, $|x + 4| = x + 4$ and

$|2x - 6| = -(2x - 6)$. Thus

$$\begin{aligned} |x + 4| - |2x - 6| &= (x + 4) - [-(2x - 6)] \\ &= (x + 4) + (2x - 6) \\ &= 3x - 2 \end{aligned}$$

▶ Try Exercise 60, page 14

The definition of *distance* between two points on a real number line makes use of absolute value.

Definition of the Distance Between Points on a Real Number Line

If a and b are the coordinates of two points on a real number line, the **distance** between the graph of a and the graph of b, denoted by $d(a, b)$, is given by $d(a, b) = |a - b|$.

EXAMPLE

Find the distance between the point whose coordinate on the real number line is -2 and the point whose coordinate is 5.

$$d(-2, 5) = |-2 - 5| = |-7| = 7$$

Note in Figure P.12 that there are 7 units between -2 and 5. Also note that the order of the coordinates in the formula does not matter.

$$d(5, -2) = |5 - (-2)| = |7| = 7$$

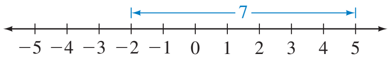


Figure P.12

EXAMPLE 7 Use Absolute Value to Express the Distance Between Two Points

Express the distance between a and -3 on the number line using absolute value notation.

Solution

$$d(a, -3) = |a - (-3)| = |a + 3|$$

▶ Try Exercise 70, page 15

Math Matters

The expression 10^{100} is called a *googol*. The term was coined by the 9-year-old nephew of the American mathematician Edward Kasner. Many calculators do not provide for numbers of this magnitude, but it is no serious loss. To appreciate the magnitude of a googol, consider that if all the atoms in the known universe were counted, the number would not even be close to a googol. But if a googol is too small for you, try 10^{googol}, which is called a *googolplex.* As a final note, the name of the Internet site Google.com is a takeoff on the word *googol*.

Exponential Expressions

A compact method of writing $5 \cdot 5 \cdot 5 \cdot 5$ is 5^4. The expression 5^4 is written in **exponential notation.** Similarly, we can write

$$\frac{2x}{3} \cdot \frac{2x}{3} \cdot \frac{2x}{3} \quad \text{as} \quad \left(\frac{2x}{3}\right)^3$$

Exponential notation can be used to express the product of any expression that is used repeatedly as a factor.

Definition of Natural Number Exponents

If b is any real number and n is a natural number, then

$$b^n = \overbrace{b \cdot b \cdot b \cdot \cdots \cdot b}^{b \text{ is a factor } n \text{ times}}$$

where b is the **base** and n is the **exponent**.

EXAMPLE

$$\left(\frac{3}{4}\right)^3 = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{27}{64}$$

$$-5^4 = -(5 \cdot 5 \cdot 5 \cdot 5) = -625$$

$$(-5)^4 = (-5)(-5)(-5)(-5) = 625$$

Pay close attention to the difference between -5^4 (the base is 5) and $(-5)^4$ (the base is -5).

EXAMPLE 8 Evaluate an Exponential Expression

Evaluate.

a. $(-3^4)(-4)^2$ **b.** $\dfrac{-4^4}{(-4)^4}$

Solution

a. $(-3^4)(-4)^2 = -(3 \cdot 3 \cdot 3 \cdot 3) \cdot (-4)(-4) = -81 \cdot 16 = -1296$

b. $\dfrac{-4^4}{(-4)^4} = \dfrac{-(4 \cdot 4 \cdot 4 \cdot 4)}{(-4)(-4)(-4)(-4)} = \dfrac{-256}{256} = -1$

Try Exercise 76, page 15

Order of Operations Agreement

The approximate pressure p, in pounds per square inch, on a scuba diver x feet below the water's surface is given by

$$p = 15 + 0.5x$$

The pressure on the diver at various depths is given below.

10 feet	$15 + 0.5(10) = 15 + 5 = 20$ pounds
20 feet	$15 + 0.5(20) = 15 + 10 = 25$ pounds
40 feet	$15 + 0.5(40) = 15 + 20 = 35$ pounds
70 feet	$15 + 0.5(70) = 15 + 35 = 50$ pounds

Note that the expression $15 + 0.5(70)$ has two operations, addition and multiplication. When an expression contains more than one operation, the operations must be performed in a specified order, as given by the Order of Operations Agreement.

The Order of Operations Agreement

If grouping symbols are present, evaluate by first performing the operations within the grouping symbols, innermost grouping symbols first, while observing the order given in steps 1 to 3.

Step 1 Evaluate exponential expressions.
Step 2 Do multiplication and division as they occur from left to right.
Step 3 Do addition and subtraction as they occur from left to right.

EXAMPLE

$5 - 7(23 - 5^2) - 16 \div 2^3$

$= 5 - 7(23 - 25) - 16 \div 2^3$ • Begin inside the parentheses and evaluate $5^2 = 25$.

$= 5 - 7(-2) - 16 \div 2^3$ • Continue inside the parentheses and evaluate $23 - 25 = -2$.

$= 5 - 7(-2) - 16 \div 8$ • Evaluate $2^3 = 8$.

$= 5 - (-14) - 2$ • Perform multiplication and division from left to right.

$= 17$ • Perform addition and subtraction from left to right.

EXAMPLE 9 Use the Order of Operations Agreement

Evaluate: $3 \cdot 5^2 - 6(-3^2 - 4^2) \div (-15)$

Solution

$3 \cdot 5^2 - 6(-3^2 - 4^2) \div (-15)$

$= 3 \cdot 5^2 - 6(-9 - 16) \div (-15)$ • Begin inside the parentheses.

$= 3 \cdot 5^2 - 6(-25) \div (-15)$ • Simplify $-9 - 16$.

$= 3 \cdot 25 - 6(-25) \div (-15)$ • Evaluate 5^2.

$= 75 + 150 \div (-15)$ • Do muliplication and division from left to right.

$= 75 + (-10)$

$= 65$ • Do addition.

▶ **Try Exercise 80, page 15**

Recall

Subtraction can be rewritten as addition of the opposite. Therefore,
$3x^2 - 4xy + 5x - y - 7$
$= 3x^2 + (-4xy) + 5x + (-y) + (-7)$
In this form, we can see that the terms (addends) are $3x^2$, $-4xy$, $5x$, $-y$, and -7.

$$3x^2 - 4xy + 5x - y - 7$$

One of the ways in which the Order of Operations Agreement is used is to evaluate variable expressions. The addends of a variable expression are called **terms**. The terms for the expression at the right are $3x^2$, $-4xy$, $5x$, $-y$, and -7. Observe that the sign of a term is the sign that immediately precedes it.

The terms $3x^2$, $-4xy$, $5x$, and $-y$ are **variable terms**. The term -7 is a **constant term**. Each variable term has a **numerical coefficient** and a **variable part**. The numerical coefficient for the term $3x^2$ is 3; the numerical coefficient for the term $-4xy$ is -4; the numerical coefficient for the term $5x$ is 5; and the numerical coefficient for the term $-y$ is -1. When the numerical coefficient is 1 or -1 (as in x and $-x$), the 1 is usually not written.

To **evaluate** a variable expression, replace the variables by their given values and then use the Order of Operations Agreement to simplify the result.

EXAMPLE 10 Evaluate a Variable Expression

a. Evaluate $\dfrac{x^3 - y^3}{x^2 + xy + y^2}$ when $x = 2$ and $y = -3$.

b. Evaluate $(x + 2y)^2 - 4z$ when $x = 3$, $y = -2$, and $z = -4$.

Solution

a. $\dfrac{x^3 - y^3}{x^2 + xy + y^2}$

$$\frac{2^3 - (-3)^3}{2^2 + 2(-3) + (-3)^2} = \frac{8 - (-27)}{4 - 6 + 9} = \frac{35}{7} = 5$$

b. $(x + 2y)^2 - 4z$

$$\begin{aligned}[3 + 2(-2)]^2 - 4(-4) &= [3 + (-4)]^2 - 4(-4)\\ &= (-1)^2 - 4(-4)\\ &= 1 - 4(-4)\\ &= 1 + 16 = 17\end{aligned}$$

Try Exercise 90, page 15

Simplifying Variable Expressions

Addition, multiplication, subtraction, and *division* are the operations of arithmetic. **Addition** of the two real numbers a and b is designated by $a + b$. If $a + b = c$, then c is the **sum** and the real numbers a and b are called **terms.**

Multiplication of the real numbers a and b is designated by ab or $a \cdot b$. If $ab = c$, then c is the **product** and the real numbers a and b are called **factors** of c.

The number $-b$ is referred to as the **additive inverse** of b. **Subtraction** of the real numbers a and b is designated by $a - b$ and is defined as the sum of a and the additive inverse of b. That is,

$$a - b = a + (-b)$$

If $a - b = c$, then c is called the **difference** of a and b.

The **multiplicative inverse** or **reciprocal** of the nonzero number b is $1/b$. The **division** of a and b, designated by $a \div b$ with $b \neq 0$, is defined as the product of a and the reciprocal of b. That is,

$$a \div b = a\left(\frac{1}{b}\right) \quad \text{provided that } b \neq 0$$

If $a \div b = c$, then c is called the **quotient** of a and b.

The notation $a \div b$ is often represented by the fractional notation a/b or $\dfrac{a}{b}$. The real number a is the **numerator,** and the nonzero real number b is the **denominator** of the fraction.

Properties of Real Numbers

Let a, b, and c be real numbers.

	Addition Properties	Multiplication Properties
Closure	$a + b$ is a unique real number.	ab is a unique real number.
Commutative	$a + b = b + a$	$ab = ba$

(continued)

	Addition Properties	Multiplication Properties
Associative	$(a + b) + c = a + (b + c)$	$(ab)c = a(bc)$
Identity	There exists a unique real number 0 such that $a + 0 = 0 + a = a$.	There exists a unique real number 1 such that $a \cdot 1 = 1 \cdot a = a$.
Inverse	For each real number a, there is a unique real number $-a$ such that $a + (-a) = (-a) + a = 0$.	For each *nonzero* real number a, there is a unique real number $1/a$ such that $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$.
Distributive	$a(b + c) = ab + ac$	

EXAMPLE 11 Identify Properties of Real Numbers

Identify the property of real numbers illustrated in each statement.

a. $(2a)b = 2(ab)$

b. $\left(\frac{1}{5}\right)11$ is a real number.

c. $4(x + 3) = 4x + 12$

d. $(a + 5b) + 7c = (5b + a) + 7c$

e. $\left(\frac{1}{2} \cdot 2\right)a = 1 \cdot a$

f. $1 \cdot a = a$

Solution

a. Associative property of multiplication

b. Closure property of multiplication

c. Distributive property

d. Commutative property of addition

e. Inverse property of multiplication

f. Identity property of multiplication

▶ **Try Exercise 102, page 15**

Note

Normally, we will not show, as we did at the right, all the steps involved in the simplification of a variable expression. For instance, we will just write $(6x)2 = 12x$, $3(4p + 5) = 12p + 15$, and $3x^2 + 9x^2 = 12x^2$. It is important to know, however, that every step in the simplification process depends on one of the properties of real numbers.

We can identify which properties of real numbers have been used to rewrite an expression by closely comparing the original and final expressions and noting any changes. For instance, to simplify $(6x)2$, both the commutative property and associative property of multiplication are used.

$$(6x)2 = 2(6x) \quad \text{• Commutative property of multiplication}$$
$$= (2 \cdot 6)x \quad \text{• Associative property of multiplication}$$
$$= 12x$$

To simplify $3(4p + 5)$, use the distributive property.

$$3(4p + 5) = 3(4p) + 3(5) \quad \text{• Distributive property}$$
$$= 12p + 15$$

Terms that have the same variable part are called **like terms**. The distributive property is also used to simplify an expression with like terms such as $3x^2 + 9x^2$.

$$3x^2 + 9x^2 = (3 + 9)x^2 \quad \text{• Distributive property}$$
$$= 12x^2$$

Note from this example that like terms are combined by adding the coefficients of the like terms.

Question • Are the terms $2x^2$ and $3x$ like terms?

EXAMPLE 12 Simplify Variable Expressions

Simplify.

a. $5 + 3(2x - 6)$

b. $4x - 2[7 - 5(2x - 3)]$

Solution

a. $5 + 3(2x - 6) = 5 + 6x - 18$ • Use the distributive property.

$= 6x - 13$ • Add the constant terms.

b. $4x - 2[7 - 5(2x - 3)]$

$= 4x - 2[7 - 10x + 15]$ • Use the distributive property to remove the inner parentheses.

$= 4x - 2[-10x + 22]$ • Simplify.

$= 4x + 20x - 44$ • Use the distributive property to remove the brackets.

$= 24x - 44$ • Simplify.

Try Exercise 120, page 15

An **equation** is a statement of equality between two numbers or two expressions. There are four basic properties of equality that relate to equations.

Properties of Equality

Let a, b, and c be real numbers.

Reflexive	$a = a$
Symmetric	If $a = b$, then $b = a$.
Transitive	If $a = b$ and $b = c$, then $a = c$.
Substitution	If $a = b$, then a may be replaced by b in any expression that involves a.

EXAMPLE 13 Identify Properties of Equality

Identify the property of equality illustrated in each statement.

a. If $3a + b = c$, then $c = 3a + b$.

b. $5(x + y) = 5(x + y)$

c. If $4a - 1 = 7b$ and $7b = 5c + 2$, then $4a - 1 = 5c + 2$.

d. If $a = 5$ and $b(a + c) = 72$, then $b(5 + c) = 72$.

Solution

a. Symmetric **b.** Reflexive **c.** Transitive **d.** Substitution

Try Exercise 106, page 15

Answer • No. The variable parts are not the same. The variable part of $2x^2$ is $x \cdot x$. The variable part of $3x$ is x.

EXERCISE SET P.1

Concept Check

1. Which of the following numbers are prime numbers?
i. 39 **ii.** 53 **iii.** 102 **iv.** 97

2. Give an example of a rational number that is not an integer.

3. If $A = \{-7, -3, 0, 2, 5, 8\}$ and $B = \{-3, -1, 0, 1, 3, 5, 7\}$, what numbers are common to both A and B?

4. Use the numbers $-12, -5, 0, 3, 6$, and 9.
a. Which number has the greatest absolute value?
b. Which number has the least absolute value?

5. If $a < 0$, is a^2 positive or negative?

6. **a.** Name the endpoints of the interval $[-2, 5)$.
b. Is $0 \in [-2, 5)$?
c. Is $-2 \in [-2, 5)$?
d. Is $5 \in [-2, 5)$?

In Exercises 7 and 8, determine whether each number is an integer, a rational number, an irrational number, a prime number, or a real number.

7. $-\frac{1}{5}, 0, -44, \pi, 3.14, 5.05005000500005\ldots, \sqrt{81}, 53$

8. $\frac{5}{\sqrt{7}}, \frac{5}{7}, 31, -2\frac{1}{2}, 4.235653907493, 51, 0.888\ldots$

In Exercises 9 to 14, list the four smallest elements of each set.

9. $\{2x \mid x \in \text{positive integers}\}$

10. $\{|x| \mid x \in \text{integers}\}$

11. $\{y \mid y = 2x + 1, x \in \text{natural numbers}\}$

12. $\{y \mid y = x^2 - 1, x \in \text{integers}\}$

13. $\{z \mid z = |x|, x \in \text{integers}\}$

14. $\{z \mid z = |x| - x, x \in \text{negative integers}\}$

In Exercises 15 to 24, perform the operations given that $A = \{-3, -2, -1, 0, 1, 2, 3\}$, $B = \{-2, 0, 2, 4, 6\}$, $C = \{0, 1, 2, 3, 4, 5, 6\}$, and $D = \{-3, -1, 1, 3\}$.

15. $A \cup B$

16. $C \cup D$

17. $A \cap C$

18. $C \cap D$

19. $B \cap D$

20. $B \cup (A \cap C)$

21. $D \cap (B \cup C)$

22. $(A \cap B) \cup (A \cap C)$

Indicates Try It Exercises

23. $(B \cup C) \cap (B \cup D)$

24. $(A \cap C) \cup (B \cap D)$

In Exercises 25 to 36, graph each set. Write sets given in interval notation in set-builder notation, and write sets given in set-builder notation in interval notation.

25. $(-2, 3)$

26. $[1, 5]$

27. $[-5, -1]$

28. $(-3, 3)$

29. $[2, \infty)$

30. $(-\infty, 4)$

31. $\{x \mid 3 < x < 5\}$

32. $\{x \mid x < -1\}$

33. $\{x \mid x \geq -2\}$

34. $\{x \mid -1 \leq x < 5\}$

35. $\{x \mid 0 \leq x \leq 1\}$

36. $\{x \mid -4 < x \leq 5\}$

In Exercises 37 to 52, graph each set.

37. $(-\infty, 0) \cup [2, 4]$

38. $(-3, 1) \cup (3, 5)$

39. $(-4, 0) \cap [-2, 5]$

40. $(-\infty, 3] \cap (2, 6)$

41. $(1, \infty) \cup (-2, \infty)$

42. $(-4, \infty) \cup (0, \infty)$

43. $(1, \infty) \cap (-2, \infty)$

44. $(-4, \infty) \cap (0, \infty)$

45. $[-2, 4] \cap [4, 5]$

46. $(-\infty, 1] \cap [1, \infty)$

47. $(-2, 4) \cap (4, 5)$

48. $(-\infty, 1) \cap (1, \infty)$

49. $\{x \mid x < -3\} \cup \{x \mid 1 < x < 2\}$

50. $\{x \mid -3 \leq x < 0\} \cup \{x \mid x \geq 2\}$

51. $\{x \mid x < -3\} \cup \{x \mid x < 2\}$

52. $\{x \mid x < -3\} \cap \{x \mid x < 2\}$

In Exercises 53 to 62, write each expression without absolute value symbols.

53. $-|-5|$

54. $-|-4|^2$

55. $|3| \cdot |-4|$

56. $|3| - |-7|$

57. $|\pi^2 + 10|$

58. $|\pi^2 - 10|$

59. $|x - 4| + |x + 5|$, given $0 < x < 1$

60. $|x + 6| + |x - 2|$, given $0 < x < 2$

61. $|2x| - |x - 1|$, given $0 < x < 1$

62. $|x + 1| + |x - 3|$, given $x > 3$

In Exercises 63 to 74, use absolute value notation to describe the given situation.

63. $d(m, n)$

64. $d(p, 8)$

65. The distance between x and 3

66. The distance between a and -2

67. The distance between x and -2 is 4.

68. The distance between z and 5 is 1.

69. The distance between a and 4 is less than 5.

70. The distance between z and 5 is greater than 7.

71. The distance between x and -2 is greater than 4.

72. The distance between y and -3 is greater than 6.

73. The distance between x and 4 is greater than 0 and less than 1.

74. The distance between y and -3 is greater than 0 and less than 0.5.

In Exercises 75 to 82, evaluate the expression.

75. $-5^3(-4)^2$

76. $-\dfrac{-6^3}{(-3)^4}$

77. $4 + (3 - 8)^2$

78. $-2 \cdot 3^4 - (6 - 7)^6$

79. $28 \div (-7 + 5)^2$

80. $(3 - 5)^2(3^2 - 5^2)$

81. $7 + 2[3(-2)^3 - 4^2 \div 8]$

82. $5 - 4[3 - 6(2 \cdot 3^2 - 12 \div 4)]$

In Exercises 83 to 94, evaluate the variable expression for *x* = 3, *y* = –2, and *z* = –1.

83. $-y^3$

84. $-y^2$

85. $2xyz$

86. $-3xz$

87. $-2x^2y^2$

88. $2y^3z^2$

89. $xy - z(x - y)^2$

90. $(z - 2y)^2 - 3z^3$

91. $\dfrac{x^2 + y^2}{x + y}$

92. $\dfrac{2xy^2z^4}{(y - z)^4}$

93. $\dfrac{3y}{x} - \dfrac{2z}{y}$

94. $(x - z)^2(x + z)^2$

In Exercises 95 to 108, state the property of real numbers or the property of equality that is used.

95. $(ab^2)c = a(b^2c)$

96. $2x - 3y = -3y + 2x$

97. $4(2a - b) = 8a - 4b$

98. $6 + (7 + a) = 6 + (a + 7)$

99. $(3x)y = y(3x)$

100. $4ab + 0 = 4ab$

101. $1 \cdot (4x) = 4x$

102. $7(a + b) = 7(b + a)$

103. $x^2 + 1 = x^2 + 1$

104. If $a + b = 2$, then $2 = a + b$.

105. If $2x + 1 = y$ and $y = 3x - 2$, then $2x + 1 = 3x - 2$.

106. If $4x + 2y = 7$ and $x = 3$, then $4(3) + 2y = 7$.

107. $4 \cdot \dfrac{1}{4} = 1$

108. $ab + (-ab) = 0$

109. Is division of real numbers an associative operation? Give a reason for your answer.

110. Is subtraction of real numbers a commutative operation? Give a reason for your answer.

111. Which of the properties of real numbers are satisfied by the integers?

112. Which of the properties of real numbers are satisfied by the rational numbers?

In Exercises 113 to 122, simplify the variable expression.

113. $2 + 3(2x - 5)$

114. $4 + 2(2a - 3)$

115. $5 - 3(4x - 2y)$

116. $7 - 2(5n - 8m)$

117. $3(2a - 4b) - 4(a - 3b)$

118. $5(4r - 7t) - 2(10r + 3t)$

119. $5a - 2[3 - 2(4a + 3)]$

120. $6 + 3[2x - 4(3x - 2)]$

121. $\dfrac{3}{4}(5a + 2) - \dfrac{1}{2}(3a - 5)$

122. $-\dfrac{2}{5}(2x + 3) + \dfrac{3}{4}(3x - 7)$

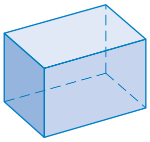
123. Area of a Triangle The area of a triangle is given by

$$\text{Area} = \frac{1}{2}bh$$

where b is the base of the triangle and h is its height. Find the area of a triangle whose base is 3 inches and whose height is 4 inches.

124. Volume of a Box The volume of a rectangular box is given by

$$\text{Volume} = lwh$$

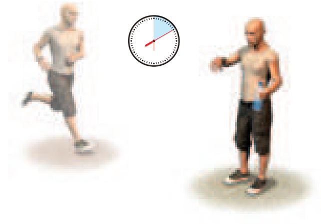


where l is the length, w is the width, and h is the height of the box. Find the volume of a classroom that has a length of 40 feet, a width of 30 feet, and a height of 12 feet.

125. Heart Rate The heart rate, in beats per minute, of a certain runner during a cool-down period can be approximated by

$$\text{Heart rate} = 65 + \frac{53}{4t + 1}$$

where t is the number of minutes after the start of cool-down. Find the runner's heart rate after 10 minutes. Round to the nearest natural number.



126. Body Mass Index According to the National Institutes of Health, body mass index (BMI) is a measure of body fat based on height and weight that applies to both adult men and women, with values between 18.5 and 24.9 considered healthy. BMI is calculated as $\text{BMI} = \frac{705w}{h^2}$, where w is the person's weight in pounds and h is the person's height in inches. Find the BMI for a person who weighs 160 pounds and is 5 feet 10 inches tall. Round to the nearest natural number.

127. Physics The height, in feet, of a ball t seconds after it is thrown upward is given by

$$\text{Height} = -16t^2 + 80t + 4$$

Find the height of the ball 2 seconds after it has been thrown upward.

128. Chemistry Salt is being added to water in such a way that the concentration of salt, in grams per liter, is given by $\text{concentration} = \frac{50t}{t + 1}$, where t is the time in minutes after the introduction of the salt. Find the concentration of salt after 24 minutes.

129. Sabermetrics *Slugging percentage* (SLG) is one of the measurements of a baseball player's performance. It is given by the ratio $\frac{\text{singles} + 2 \cdot 2B + 3 \cdot 3B + 4 \cdot 4B}{AB}$, where singles is the number of singles, $2B$ is the number of doubles, $3B$ is the number of triples, and $4B$ is the number of home runs hit by a player. The abbreviation AB is the number of at bats the player had. In 2011, Miguel Cabrera had 197 singles, 48 doubles, 0 triples, 30 home runs, and 572 at bats. Find his SLG. Round to the nearest thousandth.

130. Sabermetrics *Pythagorean expectation* is a formula that tries to determine how many games a team "should have" won during a season. It is based on the number of runs scored by a team in one season and the number of runs allowed by the team for the season. Pythagorean expectation is given by the ratio $\frac{(\text{runs scored})^2}{(\text{runs scored})^2 + (\text{runs allowed})^2}$. Multiplying this ratio by the number of games played in a season (162) gives the number of games the team "should have" won. In 2011, the Boston Red Sox won 90 games, scored 875 runs, and allowed 757 runs. According to the Pythagorean expectation, how many games should the Red Sox have won? Round to the nearest whole number.

Enrichment Exercises

In Exercises 131 and 132, let *A* and *B* be any two sets.

131. If $A \cap B = B$, what can be said about B?

132. If $A \cup B = B$, what can be said about A?

In Exercises 133 to 136, let *A* be any set. Perform the given operation.

133. $A \cup A$

134. $A \cap A$

135. $A \cup \varnothing$

136. $A \cap \varnothing$

137. If a and b are the coordinates of two points on a number line, give an example of a point whose coordinates are between a and b.

138. Define an operation denoted by $\oplus$ and given by $a \oplus b = a^2 + b^2$. Does $\oplus$ satisfy the commutative property? Does $\oplus$ satisfy the associative property?

139. A *deleted delta neighborhood* of a number a on a number line is the set of all points x that are within δ (the Greek letter delta) units of a but not including a. Write the deleted delta neighborhood of a using absolute value notation.

SECTION P.2

Integer Exponents
Scientific Notation
Rational Exponents and Radicals
Simplifying Radical Expressions

Integer and Rational Number Exponents

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A1.

PS1. Simplify: $2^2 \cdot 2^3$ [P.1]

PS2. Simplify: $\frac{4^3}{4^5}$ [P.1]

PS3. Simplify: $(2^3)^2$ [P.1]

PS4. Simplify: $3.14(10^5)$ [P.1]

PS5. True or false: $3^4 \cdot 3^2 = 9^6$ [P.1]

PS6. True or false: $(3 + 4)^2 = 3^2 + 4^2$ [P.1]

Integer Exponents

Recall that if n is a natural number, then $b^n = \overbrace{b \cdot b \cdot b \cdot \cdots \cdot b}^{b \text{ is a factor } n \text{ times}}$. We can extend the definition of exponent to all integers. We begin with the case of zero as an exponent.

Definition of b^0

For any nonzero real number b, $b^0 = 1$.

EXAMPLE

$3^0 = 1$ $\qquad \left(\frac{3}{4}\right)^0 = 1$ $\qquad -7^0 = -1$ $\qquad (a^2 + 1)^0 = 1$

Note

Note that $-7^0 = -(7^0) = -1$.

Now we extend the definition to include negative integers.

Definition of b^{-n}

If $b \neq 0$ and n is a natural number, then $b^{-n} = \frac{1}{b^n}$ and $\frac{1}{b^{-n}} = b^n$.

EXAMPLE

$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$ $\qquad \frac{1}{4^{-3}} = 4^3 = 64$ $\qquad \frac{5^{-2}}{7^{-1}} = \frac{7}{5^2} = \frac{7}{25}$

EXAMPLE 1 Evaluate an Exponential Expression

Evaluate.

a. $(-2^4)(-3)^2$

b. $\frac{(-4)^{-3}}{(-2)^{-5}}$

c. $-\pi^0$

(continued)

Solution

a. $(-2^4)(-3)^2 = -(2 \cdot 2 \cdot 2 \cdot 2)(-3)(-3) = -(16)(9) = -144$

b. $\dfrac{(-4)^{-3}}{(-2)^{-5}} = \dfrac{(-2)(-2)(-2)(-2)(-2)}{(-4)(-4)(-4)} = \dfrac{-32}{-64} = \dfrac{1}{2}$

c. $-\pi^0 = -(\pi^0) = -1$

▶ **Try Exercise 24, page 28**

When working with exponential expressions containing variables, we must ensure that a value of the variable does not result in an undefined expression. Take, for instance, $x^{-2} = \dfrac{1}{x^2}$. Because division by zero is not allowed, for the expression x^{-2}, we must assume that $x \neq 0$. Therefore, to avoid problems with undefined expressions, we will use the following restriction agreement.

Restriction Agreement

The expressions 0^0, 0^n (where n is a negative integer), and $\dfrac{a}{0}$ are all undefined expressions. Therefore, all values of variables in this text are restricted to avoid any one of these expressions.

EXAMPLE

In the expression $\dfrac{x^0y^{-3}}{z-4}$, $x \neq 0$, $y \neq 0$, and $z \neq 4$.

In the expression $\dfrac{(a-1)^0}{b+2}$, $a \neq 1$ and $b \neq -2$.

Exponential expressions containing variables are simplified using the following properties of exponents.

Properties of Exponents

If m, n, and p are integers and a and b are real numbers, then

Product $\quad b^m \cdot b^n = b^{m+n}$

Quotient $\quad \dfrac{b^m}{b^n} = b^{m-n}, \qquad b \neq 0$

Power $\quad (b^m)^n = b^{mn}$

$(a^mb^n)^p = a^{mp}b^{np}$

$\left(\dfrac{a^m}{b^n}\right)^p = \dfrac{a^{mp}}{b^{np}}, \qquad b \neq 0$

EXAMPLE

$a^4 \cdot a \cdot a^3 = a^{4+1+3} = a^8$ • Add the exponents of the like bases. Recall that $a = a^1$.

$(x^4y^3)(xy^5z^2) = x^{4+1}y^{3+5}z^2 = x^5y^8z^2$ • Add the exponents of the like bases.

$\dfrac{a^7b}{a^2b^5} = a^{7-2}b^{1-5} = a^5b^{-4} = \dfrac{a^5}{b^4}$ • Subtract the exponents of the like bases.

$(uv^3)^5 = u^{1\cdot 5}v^{3\cdot 5} = u^5v^{15}$ • Multiply the exponents.

$\left(\frac{2x^5}{5y^4}\right)^3 = \frac{2^{1\cdot 3}x^{5\cdot 3}}{5^{1\cdot 3}y^{4\cdot 3}} = \frac{2^3x^{15}}{5^3y^{12}} = \frac{8x^{15}}{125y^{12}}$ • Multiply the exponents.

Question • Can the exponential expression x^5y^3 be simplified using the properties of exponents?

Integrating Technology

Exponential expressions such as a^{b^c} can be confusing. The generally accepted meaning of a^{b^c} is $a^{(b^c)}$. However, some graphing calculators do not evaluate exponential expressions in this way. Enter 2^3^4 in a graphing calculator. If the result is approximately 2.42×10^{24}, then the calculator evaluated $2^{(3^4)}$. If the result is 4096, then the calculator evaluated $(2^3)^4$. To ensure that you calculate the value you intend, we strongly urge you to use parentheses. For instance, entering 2^(3^4) will produce 2.42×10^{24} and entering (2^3)^4 will produce 4096.

To simplify an expression involving exponents, write the expression in a form in which *each base occurs at most once* and *no powers of powers or negative exponents occur.*

EXAMPLE 2 Simplify Exponential Expressions

Simplify.

a. $(5x^2y)(-4x^3y^5)$ **b.** $(3x^2yz^{-4})^3$ **c.** $\frac{-12x^5y}{18x^2y^6}$ **d.** $\left(\frac{4p^2q}{6pq^4}\right)^{-2}$

Solution

a. $(5x^2y)(-4x^3y^5) = [5(-4)]x^{2+3}y^{1+5}$ • Multiply the coefficients. Multiply the variables by adding the exponents of the like bases.

$= -20x^5y^6$

b. $(3x^2yz^{-4})^3 = 3^{1\cdot 3}x^{2\cdot 3}y^{1\cdot 3}z^{-4\cdot 3}$ • Use the power property of exponents.

$= 3^3x^6y^3z^{-12} = \frac{27x^6y^3}{z^{12}}$

c. $\frac{-12x^5y}{18x^2y^6} = -\frac{2}{3}x^{5-2}y^{1-6}$ • Simplify $\frac{-12}{18} = -\frac{2}{3}$. Divide the variables by subtracting the exponents of the like bases.

$= -\frac{2}{3}x^3y^{-5}$

$= -\frac{2x^3}{3y^5}$

d. $\left(\frac{4p^2q}{6pq^4}\right)^{-2} = \left(\frac{2p^{2-1}q^{1-4}}{3}\right)^{-2} = \left(\frac{2pq^{-3}}{3}\right)^{-2}$ • Use the quotient property of exponents.

$= \frac{2^{1(-2)}p^{1(-2)}q^{-3(-2)}}{3^{1(-2)}} = \frac{2^{-2}p^{-2}q^6}{3^{-2}}$ • Use the power property of exponents.

$= \frac{9q^6}{4p^2}$ • Write the answer in simplest form.

▶ Try Exercise 50, page 29

Answer • No. The bases are not the same.

Math Matters

- Approximately 3.1×10^6 orchid seeds weigh 1 ounce.
- Computer scientists measure an operation in nanoseconds. 1 nanosecond $= 1 \times 10^{-9}$ second
- If a spaceship traveled at 25,000 mph, it would require approximately 2.7×10^9 years to travel from one end of the universe to the other.

Scientific Notation

The exponent theorems provide a compact method of writing very large or very small numbers. The method is called *scientific notation.* A number written in **scientific notation** has the form $a \cdot 10^n$, where n is an integer and $1 \le a < 10$. The following procedure is used to change a number from its decimal form to scientific notation.

For numbers greater than 10, move the decimal point to the position to the right of the first digit. The exponent n will equal the number of places the decimal point has been moved. For example,

$$\underbrace{7{,}430{,}000}_{6 \text{ places}} = 7.43 \times 10^6$$

For numbers less than 1, move the decimal point to the right of the first nonzero digit. The exponent n will be negative, and its absolute value will equal the number of places the decimal point has been moved. For example,

$$\underbrace{0.00000078}_{7 \text{ places}} = 7.8 \times 10^{-7}$$

To change a number from scientific notation to its decimal form, reverse the procedure. That is, if the exponent is positive, move the decimal point to the right the same number of places as the exponent. For example,

$$3.5 \times 10^5 = \underbrace{350{,}000}_{5 \text{ places}}$$

If the exponent is negative, move the decimal point to the left the same number of places as the absolute value of the exponent. For example,

$$2.51 \times 10^{-8} = \underbrace{0.0000000251}_{8 \text{ places}}$$

Most calculators display very large and very small numbers in scientific notation. The number $450{,}000^2$ is displayed as `2.025 E 11`. This means $450{,}000^2 = 2.025 \times 10^{11}$.

EXAMPLE 3 Simplify an Expression Using Scientific Notation

One of the purposes of the Apollo 15 mission was to place a lunar Laser Ranging RetroReflector (LRRR) on the moon. The purpose of the LRRR is to precisely measure the distance from Earth to the moon. A laser beam is sent from a station on Earth to the LRRR, which then reflects the laser beam back to Earth.

Assuming the laser beam travels at 3.0×10^8 meters per second and the distance to the moon is 3.8×10^8 meters, find the round-trip time for the laser beam to reach the moon and the reflected beam to return to Earth. Round to the nearest hundredth of a second.

Solution

To find the time, divide the distance to the moon by the speed of the laser beam. Then multiply that result by 2 to obtain the round-trip time.

$$t = \frac{3.8 \times 10^8}{3.0 \times 10^8} = \frac{3.8}{3.0} \times 10^{8-8} \approx 1.267 \times 10^0 = 1.267 \times 1 = 1.267$$

We multiply 1.267 by 2 and see that the round-trip time for the laser beam is approximately 2.53 seconds.

Try Exercise 58, page 29

Rational Exponents and Radicals

To this point, the expression b^n has been defined for real number b and integers n. Now we wish to extend the definition of exponents to include rational numbers so that expressions such as $2^{1/2}$ will be meaningful. Not just any definition will do. We want a definition of rational exponents for which the properties of integer exponents are true. The following example shows the direction we can take to accomplish our goal.

If the product property for exponential expressions is to hold for rational exponents, then for rational numbers p and q, $b^p b^q = b^{p+q}$. For example,

$$9^{1/2} \cdot 9^{1/2} \quad \text{must equal} \quad 9^{1/2+1/2} = 9^1 = 9$$

Thus $9^{1/2}$ must be a square root of 9. That is, $9^{1/2} = 3$.

The example suggests that $b^{1/n}$ can be defined in terms of roots according to the following definition.

Definition of $b^{1/n}$

If n is an even positive integer and $b \geq 0$, then $b^{1/n}$ is the nonnegative real number such that $(b^{1/n})^n = b$.

If n is an odd positive integer, then $b^{1/n}$ is the real number such that $(b^{1/n})^n = b$.

EXAMPLE

- $25^{1/2} = 5$ because $5^2 = 25$.
- $(-64)^{1/3} = -4$ because $(-4)^3 = -64$.
- $16^{1/2} = 4$ because $4^2 = 16$.
- $-16^{1/2} = -(16^{1/2}) = -4$.
- $(-16)^{1/2}$ is not a real number.
- $(-32)^{1/5} = -2$ because $(-2)^5 = -32$.

If n is an even positive integer and $b < 0$, then $b^{1/n}$ is a *complex number*. Complex numbers are discussed in Section P.6.

To define expressions such as $8^{2/3}$, we will extend our definition of exponents even further. Because we want the power property $(b^p)^q = b^{pq}$ to be true for rational exponents also, we must have $(b^{1/n})^m = b^{m/n}$. With this in mind, we make the following definition.

Definition of $b^{m/n}$

For all positive integers m and n such that m/n is in simplest form, and for all real numbers b for which $b^{1/n}$ is a real number,

$$b^{m/n} = (b^{1/n})^m = (b^m)^{1/n}$$

Because $b^{m/n}$ is defined as $(b^{1/n})^m$ and as $(b^m)^{1/n}$, we can evaluate expressions such as $8^{4/3}$ in more than one way. For example, because $8^{1/3}$ is a real number, $8^{4/3}$ can be evaluated in either of the following ways.

$$8^{4/3} = (8^{1/3})^4 = 2^4 = 16$$

$$8^{4/3} = (8^4)^{1/3} = 4096^{1/3} = 16$$

Of the two methods, the $b^{m/n} = (b^{1/n})^m$ method is usually easier to apply, provided you can evaluate $b^{1/n}$.

EXAMPLE 4 Evaluate a Number with a Rational Exponent

Simplify.

a. $64^{2/3}$ **b.** $32^{-3/5}$ **c.** $\left(\frac{16}{81}\right)^{-3/4}$

Solution

a. $64^{2/3} = (64^{1/3})^2 = 4^2 = 16$

b. $32^{-3/5} = (32^{1/5})^{-3} = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

c. $\left(\frac{16}{81}\right)^{-3/4} = \left(\frac{81}{16}\right)^{3/4} = \left[\left(\frac{81}{16}\right)^{1/4}\right]^3 = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$

Try Exercise 62, page 29

The following exponent properties were stated earlier, but they are restated here to remind you that they have now been extended to apply to rational exponents.

Properties of Rational Exponents

If p, q, and r represent rational numbers and a and b are positive real numbers, then

Product	$b^p \cdot b^q = b^{p+q}$	
Quotient	$\frac{b^p}{b^q} = b^{p-q}$	
Power	$(b^p)^q = b^{pq}$	$(a^p b^q)^r = a^{pr} b^{qr}$
	$\left(\frac{a^p}{b^q}\right)^r = \frac{a^{pr}}{b^{qr}}$	$b^{-p} = \frac{1}{b^p}$

Recall that an exponential expression is in simplest form when no powers of powers or negative exponents occur and each base occurs at most once.

EXAMPLE 5 Simplify Exponential Expressions

Simplify.

a. $(2x^{1/3}y^{3/5})^2\ (9x^3y^{3/2})^{1/2}$ **b.** $\frac{(a^{3/4}b^{1/2})^2}{(a^{2/3}b^{3/4})^3}$

Solution

a. $(2x^{1/3}y^{3/5})^2(9x^3y^{3/2})^{1/2} = (2^2x^{2/3}y^{6/5})(9^{1/2}x^{3/2}y^{3/4})$ • Use the power property.

$= (4x^{2/3}y^{6/5})(3x^{3/2}y^{3/4})$

$= 12x^{\frac{2}{3}+\frac{3}{2}}y^{\frac{6}{5}+\frac{3}{4}} = 12x^{\frac{4}{6}+\frac{9}{6}}y^{\frac{24}{20}+\frac{15}{20}}$ • Add the exponents on like bases.

$= 12x^{13/6}y^{39/20}$

b. $\dfrac{(a^{3/4}b^{1/2})^2}{(a^{2/3}b^{3/4})^3} = \dfrac{a^{3/2}b}{a^2b^{9/4}}$ • Use the power property.

$= a^{\frac{3}{2}-2}b^{1-\frac{9}{4}}$ • Subtract the exponents on like bases.

$= a^{\frac{3}{2}-\frac{4}{2}}b^{\frac{4}{4}-\frac{9}{4}} = a^{-1/2}b^{-5/4}$

$= \dfrac{1}{a^{1/2}b^{5/4}}$

Try Exercise 68, page 29

Math Matters

The formula for kinetic energy (energy of motion) that is used in Einstein's Theory of Relativity involves a radical,

$$\text{K.E}_r = mc^2\left(\frac{1}{\sqrt{1-\dfrac{v^2}{c^2}}} - 1\right)$$

where m is the mass of the object at rest, v is the speed of the object, and c is the speed of light.

Simplifying Radical Expressions

Radicals, expressed by the notation $\sqrt[n]{b}$, are also used to denote roots. The number b is the **radicand,** and the positive integer n is the **index** of the radical.

Definition of $\sqrt[n]{b}$

If n is a positive integer and b is a real number such that $b^{1/n}$ is a real number, then $\sqrt[n]{b} = b^{1/n}$.

If the index n equals 2, then the radical $\sqrt[2]{b}$ is written as simply $\sqrt{b}$, and it is referred to as the **principal square root of *b***, or simply the **square root of *b***.

The symbol $\sqrt{b}$ is reserved to represent the nonnegative square root of b. To represent the negative square root of b, write $-\sqrt{b}$. For example, $\sqrt{25} = 5$, whereas $-\sqrt{25} = -5$.

Definition of $(\sqrt[n]{b})^m$

For all positive integers n, all integers m, and all real numbers b such that $\sqrt[n]{b}$ is a real number, $(\sqrt[n]{b})^m = \sqrt[n]{b^m} = b^{m/n}$.

When $\sqrt[n]{b}$ is a real number, the equations

$$b^{m/n} = \sqrt[n]{b^m} \qquad \text{and} \qquad b^{m/n} = (\sqrt[n]{b})^m$$

can be used to write exponential expressions such as $b^{m/n}$ in radical form. Use the denominator n as the index of the radical and the numerator m as the power of the radicand or as the power of the radical. For example,

$(5xy)^{2/3} = (\sqrt[3]{5xy})^2 = \sqrt[3]{25x^2y^2}$ • Use the denominator 3 as the index of the radical and the numerator 2 as the power of the radical.

The equations

$$b^{m/n} = \sqrt[n]{b^m} \qquad \text{and} \qquad b^{m/n} = (\sqrt[n]{b})^m$$

also can be used to write radical expressions in exponential form. For example,

$\sqrt{(2ab)^3} = (2ab)^{3/2}$ • Use the index 2 as the denominator of the power and the exponent 3 as the numerator of the power.

The definition of $(\sqrt[n]{b})^m$ often can be used to evaluate radical expressions. For instance,

$$(\sqrt[3]{8})^4 = 8^{4/3} = (8^{1/3})^4 = 2^4 = 16$$

Care must be exercised when simplifying even roots (square roots, fourth roots, sixth roots, and so on) of variable expressions. Consider $\sqrt{x^2}$ when $x = 5$ and when $x = -5$.

Case 1 If $x = 5$, then $\sqrt{x^2} = \sqrt{5^2} = \sqrt{25} = 5 = x$.

Case 2 If $x = -5$, then $\sqrt{x^2} = \sqrt{(-5)^2} = \sqrt{25} = 5 = -x$.

These two cases suggest that

$$\sqrt{x^2} = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

TO REVIEW

Absolute Value
See pages 7–8.

Recalling the definition of absolute value, we can write this more compactly as $\sqrt{x^2} = |x|$.

Simplifying odd roots of a variable expression does not require using the absolute value symbol. Consider $\sqrt[3]{x^3}$ when $x = 5$ and when $x = -5$.

Case 1 If $x = 5$, then $\sqrt[3]{x^3} = \sqrt[3]{5^3} = \sqrt[3]{125} = 5 = x$.

Case 2 If $x = -5$, then $\sqrt[3]{x^3} = \sqrt[3]{(-5)^3} = \sqrt[3]{-125} = -5 = x$.

Thus $\sqrt[3]{x^3} = x$.

Although we have illustrated this principle only for square roots and cube roots, the same reasoning can be applied to other cases. The general result is given below.

Definition of $\sqrt[n]{b^n}$

If n is an even natural number and b is a real number, then

$$\sqrt[n]{b^n} = |b|$$

If n is an odd natural number and b is a real number, then

$$\sqrt[n]{b^n} = b$$

EXAMPLE

$\sqrt[4]{16z^4} = 2|z|$ $\qquad$ $\sqrt[5]{32a^5} = 2a$

Because radicals are defined in terms of rational powers, the properties of radicals are similar to those of exponential expressions.

Properties of Radicals

If m and n are natural numbers and a and b are positive real numbers, then

Product $\quad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

Quotient $\quad \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$

Index $\quad \sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

A radical is in **simplest form** if it meets all of the following criteria:

1. The radicand contains only powers less than the index. ($\sqrt{x^5}$ does not satisfy this requirement because 5, the exponent, is greater than 2, the index.)
2. The index of the radical is as small as possible. ($\sqrt[9]{x^3}$ does not satisfy this requirement because $\sqrt[9]{x^3} = x^{3/9} = x^{1/3} = \sqrt[3]{x}$.)
3. The denominator has been rationalized. That is, no radicals occur in the denominator. ($1/\sqrt{2}$ does not satisfy this requirement.)
4. No fractions occur under the radical sign. ($\sqrt[4]{2/x^3}$ does not satisfy this requirement.)

Radical expressions are simplified by using the properties of radicals. Here are some examples.

EXAMPLE 6 Simplify Radical Expressions

Simplify.

a. $\sqrt{48x^7y^2}$ **b.** $\sqrt[3]{162x^4y^6}$ **c.** $\sqrt[4]{32x^3y^4}$

Solution

a. $\sqrt{48x^7y^2} = \sqrt{(2^4 \cdot 3)x^7y^2} = \sqrt{(2^2x^3y)^2 \cdot 3x}$
• Factor and group factors that can be written as a power of the index, 2.

$= \sqrt{(2^2x^3y)^2} \cdot \sqrt{3x}$
• Use the product property of radicals.

$= 4|x^3y|\sqrt{3x}$
• Recall that for n even, $\sqrt[n]{b^n} = |b|$.

b. $\sqrt[3]{162x^4x^6} = \sqrt[3]{(2 \cdot 3^4)x^4y^6}$
• Factor and group factors that can be written as a power of the index.

$= \sqrt[3]{(3xy^2)^3 \cdot (2 \cdot 3x)}$

$= \sqrt[3]{(3xy^2)^3} \cdot \sqrt[3]{6x}$
• Use the product property of radicals.

$= 3xy^2\sqrt[3]{6x}$
• Recall that for n odd, $\sqrt[n]{b^n} = b$.

c. $\sqrt[4]{32x^3y^4} = \sqrt[4]{2^5x^3y^4} = \sqrt[4]{(2^4y^4) \cdot (2x^3)}$
• Factor and group factors that can be written as a power of the index.

$= \sqrt[4]{2^4y^4} \cdot \sqrt[4]{2x^3}$
• Use the product property of radicals.

$= 2|y|\sqrt[4]{2x^3}$
• Recall that for n even, $\sqrt[n]{b^n} = |b|$.

Try Exercise 84, page 29

Like radicals have the same radicand and the same index. For instance,

$$3\sqrt[3]{5xy^2} \quad \text{and} \quad -4\sqrt[3]{5xy^2}$$

are like radicals. Addition and subtraction of like radicals are accomplished by using the distributive property. For example,

$$4\sqrt{3x} - 9\sqrt{3x} = (4 - 9)\sqrt{3x} = -5\sqrt{3x}$$
$$2\sqrt[3]{y^2} + 4\sqrt[3]{y^2} - \sqrt[3]{y^2} = (2 + 4 - 1)\sqrt[3]{y^2} = 5\sqrt[3]{y^2}$$

The sum $2\sqrt{3} + 6\sqrt{5}$ cannot be simplified further because the radicands are not the same. The sum $3\sqrt[3]{x} + 5\sqrt[4]{x}$ cannot be simplified because the indices are not the same.

Sometimes it is possible to simplify radical expressions that do not appear to be like radicals by simplifying each radical expression.

EXAMPLE 7 Combine Radical Expressions

Simplify: $5x\sqrt[3]{16x^4} - \sqrt[3]{128x^7}$

Solution

$5x\sqrt[3]{16x^4} - \sqrt[3]{128x^7}$

$= 5x\sqrt[3]{2^4x^4} - \sqrt[3]{2^7x^7}$ • Factor.

$= 5x\sqrt[3]{2^3x^3} \cdot \sqrt[3]{2x} - \sqrt[3]{2^6x^6} \cdot \sqrt[3]{2x}$ • Group factors that can be written as a power of the index.

$= 5x(2x\sqrt[3]{2x}) - 2^2x^2 \cdot \sqrt[3]{2x}$ • Use the product property of radicals.

$= 10x^2\sqrt[3]{2x} - 4x^2\sqrt[3]{2x}$ • Simplify.

$= 6x^2\sqrt[3]{2x}$

▶ Try Exercise 92, page 29

Multiplication of radical expressions is accomplished by using the distributive property. For instance,

$\sqrt{5}(\sqrt{20} - 3\sqrt{15}) = \sqrt{5}(\sqrt{20}) - \sqrt{5}(3\sqrt{15})$ • Use the distributive property.

$= \sqrt{100} - 3\sqrt{75}$ • Multiply the radicals.

$= 10 - 3 \cdot 5\sqrt{3}$ • Simplify.

$= 10 - 15\sqrt{3}$

Finding the product of more complicated radical expressions may require repeated use of the distributive property.

EXAMPLE 8 Multiply Radical Expressions

Perform the indicated operation.

a. $(5\sqrt{6} - 7)(3\sqrt{6} + 2)$

b. $(3 - \sqrt{x-7})^2,\ x \geq 7$

Solution

a. $(5\sqrt{6} - 7)(3\sqrt{6} + 2)$

$= 5\sqrt{6}(3\sqrt{6} + 2) - 7(3\sqrt{6} + 2)$ • Use the distributive property.

$= (15 \cdot 6 + 10\sqrt{6}) - (21\sqrt{6} + 14)$ • Use the distributive property.

$= 90 + 10\sqrt{6} - 21\sqrt{6} - 14$ • Simplify.

$= 76 - 11\sqrt{6}$

b. $(3 - \sqrt{x-7})^2$

$= (3 - \sqrt{x-7})(3 - \sqrt{x-7})$

$= 9 - 3\sqrt{x-7} - 3\sqrt{x-7} + (\sqrt{x-7})^2$ • Use the distributive property.

$= 9 - 6\sqrt{x-7} + (x - 7)$ • $(\sqrt{x-7})^2 = x - 7$, since $x \geq 7$.

$= 2 - 6\sqrt{x-7} + x$

▶ Try Exercise 102, page 29

To **rationalize the denominator** of a fraction means to write the fraction in an equivalent form that does not involve any radicals in the denominator. This is accomplished by multiplying the numerator and denominator of the radical expression by an expression that will cause the radicand in the denominator to be a perfect root of the index.

$$\frac{5}{\sqrt{3}} = \frac{5}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{\sqrt{3^2}} = \frac{5\sqrt{3}}{3}$$

• Recall that $\sqrt{3}$ means $\sqrt[2]{3}$. Multiply numerator and denominator by $\sqrt{3}$ so that the radicand is a perfect root of the index of the radical.

$$\frac{2}{\sqrt[3]{7}} = \frac{2}{\sqrt[3]{7}} \cdot \frac{\sqrt[3]{7^2}}{\sqrt[3]{7^2}} = \frac{2\sqrt[3]{7^2}}{\sqrt[3]{7^3}} = \frac{2\sqrt[3]{49}}{7}$$

• Multiply numerator and denominator by $\sqrt[3]{7^2}$ so that the radicand is a perfect root of the index of the radical.

$$\frac{5}{\sqrt[4]{x^5}} = \frac{5}{\sqrt[4]{x^5}} \cdot \frac{\sqrt[4]{x^3}}{\sqrt[4]{x^3}} = \frac{5\sqrt[4]{x^3}}{\sqrt[4]{x^8}} = \frac{5\sqrt[4]{x^3}}{x^2}$$

• Multiply numerator and denominator by $\sqrt[4]{x^3}$ so that the radicand is a perfect root of the index of the radical.

EXAMPLE 9 Rationalize the Denominator

Rationalize the denominator.

a. $\dfrac{5}{\sqrt[3]{a}}$ **b.** $\sqrt{\dfrac{3}{32y}}, y > 0$

Solution

a. $\dfrac{5}{\sqrt[3]{a}} = \dfrac{5}{\sqrt[3]{a}} \cdot \dfrac{\sqrt[3]{a^2}}{\sqrt[3]{a^2}} = \dfrac{5\sqrt[3]{a^2}}{\sqrt[3]{a^3}} = \dfrac{5\sqrt[3]{a^2}}{a}$

• Use $\sqrt[3]{a} \cdot \sqrt[3]{a^2} = \sqrt[3]{a^3} = a$.

b. $\sqrt{\dfrac{3}{32y}} = \dfrac{\sqrt{3}}{\sqrt{32y}} = \dfrac{\sqrt{3}}{4\sqrt{2y}} = \dfrac{\sqrt{3}}{4\sqrt{2y}} \cdot \dfrac{\sqrt{2y}}{\sqrt{2y}} = \dfrac{\sqrt{6y}}{8y}$

▸ **Try Exercise 112, page 29**

To rationalize the denominator of a fractional expression such as

$$\frac{1}{\sqrt{m} + \sqrt{n}}$$

we use the conjugate of $\sqrt{m} + \sqrt{n}$, which is $\sqrt{m} - \sqrt{n}$. The product of these conjugate pairs does not involve a radical.

$$(\sqrt{m} + \sqrt{n})(\sqrt{m} - \sqrt{n}) = m - n$$

EXAMPLE 10 Rationalize the Denominator

Rationalize the denominator.

a. $\dfrac{3 + 2\sqrt{5}}{1 - 4\sqrt{5}}$

b. $\dfrac{2 + 4\sqrt{x}}{3 - 5\sqrt{x}}, x > 0$

Solution

a. $\dfrac{3 + 2\sqrt{5}}{1 - 4\sqrt{5}} = \dfrac{3 + 2\sqrt{5}}{1 - 4\sqrt{5}} \cdot \dfrac{1 + 4\sqrt{5}}{1 + 4\sqrt{5}}$

• Multiply numerator and denominator by the conjugate of the denominator.

$$= \frac{3(1 + 4\sqrt{5}) + 2\sqrt{5}(1 + 4\sqrt{5})}{1^2 - (4\sqrt{5})^2}$$

(continued)

$$= \frac{3 + 12\sqrt{5} + 2\sqrt{5} + 8 \cdot 5}{1 - 16 \cdot 5}$$ • Simplify.

$$= \frac{43 + 14\sqrt{5}}{-79}$$

$$= -\frac{43 + 14\sqrt{5}}{79}$$

b. $$\frac{2 + 4\sqrt{x}}{3 - 5\sqrt{x}} = \frac{2 + 4\sqrt{x}}{3 - 5\sqrt{x}} \cdot \frac{3 + 5\sqrt{x}}{3 + 5\sqrt{x}}$$ • Multiply numerator and denominator by the conjugate of the denominator.

$$= \frac{2(3 + 5\sqrt{x}) + 4\sqrt{x}(3 + 5\sqrt{x})}{3^2 - (5\sqrt{x})^2}$$

$$= \frac{6 + 10\sqrt{x} + 12\sqrt{x} + 20x}{9 - 25x}$$

$$= \frac{6 + 22\sqrt{x} + 20x}{9 - 25x}$$

Try Exercise 116, page 29

EXERCISE SET P.2

Concept Check

In Exercises 1 to 8, evaluate each expression.

1. -5^3

2. $(-5)^3$

3. $\left(\frac{2}{3}\right)^0$

4. -6^0

5. 4^{-2}

6. 3^{-4}

7. $\frac{1}{2^{-5}}$

8. $\frac{1}{3^{-3}}$

In Exercises 9 to 12, write the number in scientific notation.

9. 2,011,000,000,000

10. 49,100,000,000

11. 0.000000000562

12. 0.000000402

In Exercises 13 to 16, change the number from scientific notation to decimal notation.

13. 3.14×10^7

14. 4.03×10^9

15. -2.3×10^{-6}

16. 6.14×10^{-8}

In Exercises 17 to 22, evaluate each exponential expression.

17. $4^{3/2}$

18. $-16^{3/2}$

19. $-64^{2/3}$

20. $125^{4/3}$

21. $9^{-3/2}$

22. $32^{-4/5}$

In Exercises 23 to 52, write the exponential expression in simplest form.

23. $\frac{2^{-3}}{6^{-3}}$

24. $\frac{4^{-2}}{2^{-3}}$

25. $-2x^0$

26. $\frac{x^0}{4}$

27. $2x^{-4}$

28. $3y^{-2}$

29. $\frac{5}{z^{-6}}$

30. $\frac{8}{x^{-5}}$

31. $(x^3y^2)(xy^5)$

32. $(uv^6)(u^2v)$

33. $(-2ab^4)(-3a^2b^5)$

34. $(9xy^2)(-2x^2y^5)$

35. $(-4x^{-3}y)(7x^5y^{-2})$

36. $(-6x^4y)(7x^{-3}y^{-5})$

37. $\frac{6a^4}{8a^8}$

38. $\frac{12x^3}{16x^4}$

39. $\frac{12x^3y^4}{18x^5y^2}$

40. $\frac{5v^4w^{-3}}{10v^8}$

41. $\frac{36a^{-2}b^3}{3ab^4}$

42. $\frac{-48ab^{10}}{-32a^4b^3}$

43. $(-2m^3n^2)(-3mn^2)^2$

44. $(2a^3b^2)^3(-4a^4b^2)$

Indicates Try It Exercises

45. $(x^{-2}y)^2(xy)^{-2}$

46. $(x^{-1}y^2)^{-3}(x^2y^{-4})^{-3}$

47. $\left(\frac{3a^2b^3}{6a^4b^4}\right)^2$

48. $\left(\frac{2ab^2c^3}{5ab^2}\right)^3$

49. $\frac{(-4x^2y^3)^2}{(2xy^2)^3}$

50. $\frac{(-3a^2b^3)^2}{(-2ab^4)^3}$

51. $\left(\frac{a^{-2}b}{a^3b^{-4}}\right)^2$

52. $\left(\frac{x^{-3}y^{-4}}{x^{-2}y}\right)^{-2}$

In Exercises 53 to 60, perform the indicated operation and write the answer in scientific notation.

53. $(3 \times 10^{12})(9 \times 10^{-5})$

54. $(8.9 \times 10^{-5})(3.4 \times 10^{-6})$

55. $\frac{9 \times 10^{-3}}{6 \times 10^{8}}$

56. $\frac{2.5 \times 10^{8}}{5 \times 10^{10}}$

57. $\frac{(3.2 \times 10^{-11})(2.7 \times 10^{18})}{1.2 \times 10^{-5}}$

58. $\frac{(6.9 \times 10^{27})(8.2 \times 10^{-13})}{4.1 \times 10^{15}}$

59. $\frac{(4.0 \times 10^{-9})(8.4 \times 10^{5})}{(3.0 \times 10^{-6})(1.4 \times 10^{18})}$

60. $\frac{(7.2 \times 10^{8})(3.9 \times 10^{-7})}{(2.6 \times 10^{-10})(1.8 \times 10^{-8})}$

In Exercises 61 to 76, evaluate each exponential expression.

61. $\left(\frac{4}{9}\right)^{1/2}$

62. $\left(\frac{16}{25}\right)^{3/2}$

63. $\left(\frac{1}{8}\right)^{-4/3}$

64. $\left(\frac{8}{27}\right)^{-2/3}$

65. $(4a^{2/3}b^{1/2})(2a^{1/3}b^{3/2})$

66. $(6a^{3/5}b^{1/4})(-3a^{1/5}b^{3/4})$

67. $(-3x^{2/3})(4x^{1/4})$

68. $(-5x^{1/3})(-4x^{1/2})$

69. $(81x^8y^{12})^{1/4}$

70. $(27x^3y^6)^{2/3}$

71. $\frac{16z^{3/5}}{12z^{1/5}}$

72. $\frac{6a^{2/3}}{9a^{1/3}}$

73. $(2x^{2/3}y^{1/2})(3x^{1/6}y^{1/3})$

74. $\frac{x^{1/3}y^{5/6}}{x^{2/3}y^{1/6}}$

75. $\frac{9a^{3/4}b}{3a^{2/3}b^2}$

76. $\frac{12x^{1/6}y^{1/4}}{16x^{3/4}y^{1/2}}$

In Exercises 77 to 86, simplify each radical expression.

77. $\sqrt{45}$

78. $\sqrt{75}$

79. $\sqrt[3]{24}$

80. $\sqrt[3]{135}$

81. $\sqrt[3]{-135}$

82. $\sqrt[3]{-250}$

83. $\sqrt{24x^2y^3}$

84. $\sqrt{18x^2y^5}$

85. $\sqrt[3]{16a^3y^7}$

86. $\sqrt[3]{54m^2n^7}$

In Exercises 87 to 94, simplify each radical and then combine like radicals.

87. $2\sqrt{32} - 3\sqrt{98}$

88. $5\sqrt[3]{32} + 2\sqrt[3]{108}$

89. $-8\sqrt[4]{48} + 2\sqrt[4]{243}$

90. $2\sqrt[3]{40} - 3\sqrt[3]{135}$

91. $4\sqrt[3]{32y^4} + 3y\sqrt[3]{108y}$

92. $-3x\sqrt[3]{54x^4} + 2\sqrt[3]{16x^7}$

93. $x\sqrt[3]{8x^3y^4} - 4y\sqrt[3]{64x^6y}$

94. $4\sqrt{a^5b} - a^2\sqrt{ab}$

In Exercises 95 to 104, find the indicated product and express each result in simplest form.

95. $(\sqrt{5} + 3)(\sqrt{5} + 4)$

96. $(\sqrt{7} + 2)(\sqrt{7} - 5)$

97. $(\sqrt{2} - 3)(\sqrt{2} + 3)$

98. $(2\sqrt{7} + 3)(2\sqrt{7} - 3)$

99. $(3\sqrt{z} - 2)(4\sqrt{z} + 3)$

100. $(4\sqrt{a} - \sqrt{b})(3\sqrt{a} + 2\sqrt{b})$

101. $(\sqrt{x} + 2)$

102. $(3\sqrt{5y} - 4)^2$

103. $(\sqrt{x-3} + 2)^2$

104. $(\sqrt{2x+1} - 3)^2$

In Exercises 105 to 126, simplify each expression by rationalizing the denominator. Write the result in simplest form. Assume $x > 0$ and $y > 0$.

105. $\frac{2}{\sqrt{2}}$

106. $\frac{3x}{\sqrt{3}}$

107. $\sqrt{\frac{5}{18}}$

108. $\sqrt{\frac{7}{40}}$

109. $\frac{3}{\sqrt[3]{2}}$

110. $\frac{2}{\sqrt[3]{4}}$

111. $\frac{4}{\sqrt[3]{8x^2}}$

112. $\frac{2}{\sqrt[4]{4y}}$

113. $\frac{3}{\sqrt{3} + 4}$

114. $\frac{2}{\sqrt{5} - 2}$

115. $\frac{6}{2\sqrt{5} + 2}$

116. $\frac{-7}{3\sqrt{2} - 5}$

117. $\dfrac{3 + 2\sqrt{5}}{5 - 3\sqrt{5}}$

118. $\dfrac{6 - 3\sqrt{2}}{5 - \sqrt{2}}$

119. $\dfrac{6\sqrt{3} - 11}{4\sqrt{3} - 7}$

120. $\dfrac{2\sqrt{7} + 8}{12\sqrt{7} - 6}$

121. $\dfrac{2 + \sqrt{x}}{3 - 2\sqrt{x}}$

122. $\dfrac{4 - 2\sqrt{x}}{5 + 3\sqrt{x}}$

123. $\dfrac{x - \sqrt{5}}{x + 2\sqrt{5}}$

124. $\dfrac{x + 3\sqrt{7}}{x + 2\sqrt{7}}$

125. $\dfrac{3}{\sqrt{5} + \sqrt{x}}$

126. $\dfrac{5}{\sqrt{y} - \sqrt{3}}$

127. **Weight of an Orchid Seed** An orchid seed weighs approximately 3.2×10^{-8} ounce. If a package of seeds contains 1 ounce of orchid seeds, how many seeds are in the package?

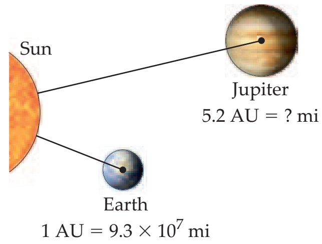
128. **Biology** The weight of one *E. coli* bacterium is approximately 670 femtograms, where 1 femtogram $= 1 \times 10^{-15}$ gram. If one *E. coli* bacterium can divide into two bacteria every 20 minutes, then after 24 hours there would be (assuming all bacteria survived) approximately 4.7×10^{21} bacteria. What is the weight, in grams, of these bacteria?

129. **Doppler Effect** Astronomers can approximate the distance to a galaxy by measuring its *red shift,* which is a shift in the wavelength of light due to the velocity of the galaxy. This is similar to the way the sound of a siren coming toward you seems to have a higher pitch than the sound of the siren moving away from you. A formula for red shift is $\dfrac{\lambda_r - \lambda_s}{\lambda_s}$, where λ_r and λ_s are wavelengths of a certain frequency of light. Calculate the red shift for a galaxy for which $\lambda_r = 5.13 \times 10^{-7}$ meter and $\lambda_s = 5.06 \times 10^{-7}$ meter.

130. **Laser Wavelength** The wavelength of a certain helium-neon laser is 800 nanometers. (1 nanometer is 1×10^{-9} meter.) The frequency, in cycles per second, of this wave is $\dfrac{1}{\text{wavelength}}$. What is the frequency of this laser?

131. **Astronomy** The Sun is approximately 1.44×10^{11} meters from Earth. If light travels 3×10^{8} meters per second, how many minutes does it take light from the sun to reach Earth?

132. **Astronomical Unit** Earth's mean distance from the Sun is 9.3×10^{7} miles. This distance is called the *astronomical unit* (AU). Jupiter is 5.2 AU from the Sun. Find the distance in miles from Jupiter to the Sun.



133. **Medicine** *Body surface area* (BSA) is a measure of the surface area of an adult human. A calculation of this number is important in prescribing medications for patients. One formula given by E. A. Gehan and S. L. George is $\text{BSA} = 0.0235\,h^{0.3964} \cdot w^{0.51456}$, where BSA is measured in meter^2, h is the height of a person in centimeters, and w is the weight of a person in kilograms. Find the BSA of a person who is 178 cm tall and weighs 73 kg. Round to the nearest hundredth.

134. **Drug Potency** The amount A (in milligrams) of digoxin, a drug taken by cardiac patients, remaining in the blood t hours after a patient takes a 2-milligram dose is given by $A = 2(10^{-0.0078t})$.

a. How much digoxin remains in the blood of a patient 4 hours after taking a 2-milligram dose?

b. Suppose that a patient takes a 2-milligram dose of digoxin at 1:00 P.M. and another 2-milligram dose at 5:00 P.M. How much digoxin remains in the patient's blood at 6:00 P.M.?

135. **Oceanography** The percent P of light that will pass to a depth d, in meters, at a certain place in the ocean is given by $P = 10^{2-(d/40)}$. Find, to the nearest percent, the amount of light that will pass to a depth of **a.** 10 meters and **b.** 25 meters below the surface of the ocean.

136. **Learning Theory** In a psychology experiment, students were given a nine-digit number to memorize. The percent P of students who remembered the number t minutes after it was read to them can be given by $P = 90 - 3t^{2/3}$. What percent of the students remembered the number after 1 hour?

Enrichment Exercises

In Exercises 137 to 140, rationalize the numerator, a technique that is occasionally used in calculus. For Exercises 137 and 138, begin by writing the expression with a 1 in the denominator.

137. $\sqrt{n^2 + 4} - n$

138. $\sqrt{n^2 + 3n} - n$

139. $\dfrac{\sqrt{4 + h} - 2}{h}$

140. $\dfrac{\sqrt{9 + h} - 3}{h}$

SECTION P.3

Operations on Polynomials
Applications of Polynomials

Polynomials

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A2.

PS1. Simplify: $-3(2a - 4b)$ [P.1]

PS2. Simplify: $5 - 2(2x - 7)$ [P.1]

PS3. Simplify: $2x^2 + 3x - 5 + x^2 - 6x - 1$ [P.1]

PS4. Simplify: $4x^2 - 6x - 1 - 5x^2 + x$ [P.1]

PS5. True or false: $4 - 3x - 2x^2 = 2x^2 - 3x + 4$ [P.1]

PS6. True or false: $\dfrac{12 + 15}{4} = \dfrac{\overset{3}{\cancel{12}} + 15}{\cancel{4}} = 18$ [P.1]

Operations on Polynomials

A **monomial** is a constant, a variable, or the product of a constant and one or more variables, with the variables having only *nonnegative* integer exponents.

-8	z	$7y$	$-12a^2bc^3$
A number	A variable	The product of a constant and one variable	The product of a constant and several variables

The expression $3x^{-2}$ is *not* a monomial because it is the product of a constant and a variable with a *negative* integer exponent.

The constant multiplying the variables is called the **numerical coefficient** or **coefficient**. For $7y$, the coefficient is 7; for $-12a^2bc^3$, the coefficient is -12. The coefficient of z is 1 because $z = 1 \cdot z$. Similarly, the coefficient of $-x$ is -1 because $-x = -1 \cdot x$.

The **degree of a monomial** is the sum of the exponents of the variables. The degree of a nonzero constant is 0. The constant 0 has no degree.

$7y$	$-12a^2bc^3$	-8
Degree is 1 because $y = y^1$.	Degree is $2 + 1 + 3 = 6$.	Degree is 0.

A **polynomial** is the sum of a finite number of monomials. Each monomial is called a **term** of the polynomial. The **degree of a polynomial** is the greatest of the degrees of the terms. See Table P.1.

TO REVIEW

Terms
See page 10.

Note

The sign of a term is the sign that precedes the term.

Table P.1 **Terms and Degree of a Polynomial**

Polynomial	Terms	Degree
$5x^4 - 6x^3 + 5x^2 - 7x - 8$	$5x^4, -6x^3, 5x^2, -7x, -8$	4
$-3xy^2 - 8xy + 6x$	$-3xy^2, -8xy, 6x$	3

Terms that have exactly the same variables raised to the same powers are called **like terms**. For example, $14x^2$ and $-x^2$ are like terms. $7x^2y$ and $5yx^2$ are like terms; the order of the variables is not important. The terms $6xy^2$ and $6x^2y$ are not like terms; the exponents on the variables are different.

A polynomial is said to be in simplest form if all its like terms have been combined. For example, the simplified form of $4x^2 + 3x + 5x - x^2$ is $3x^2 + 8x$. A **binomial** is a simplified polynomial with two terms; $3x^4 - 7$, $2xy - y^2$, and $x + 1$ are binomials. A **trinomial** is a simplified polynomial with three terms; $3x^2 + 6x - 1$, $2x^2 - 3xy + 7y^2$, and $x + y + 2$ are trinomials. A nonzero constant, such as 5, is a **constant polynomial**.

Definition of the Standard Form of a Polynomial

The **standard form of a polynomial** of degree n in the variable x is

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

where $a_n \neq 0$ and n is a nonnegative integer. The coefficient a_n is the **leading coefficient**, and a_0 is the **constant term**.

EXAMPLE

Polynomial	Standard Form	Leading Coefficient
$6x - 7 + 2x^3$	$2x^3 + 6x - 7$	2
$4z^3 - 2z^4 + 3z - 9$	$-2z^4 + 4z^3 + 3z - 9$	-2
$y^5 - 3y^3 + 1 - 2y - y^2$	$y^5 - 3y^3 - y^2 - 2y + 1$	1

EXAMPLE 1 Identify Terms Related to a Polynomial

Write the polynomial $6x^3 - x + 5 - 2x^4$ in standard form. Identify the degree, terms, constant term, leading coefficient, and coefficients of the polynomial.

Solution

A polynomial is in standard form when the terms are written in decreasing powers of the variable. The standard form of the polynomial is $-2x^4 + 6x^3 - x + 5$. In this form, the degree is 4; the terms are $-2x^4$, $6x^3$, $-x$, and 5; the constant term is 5. The leading coefficient is -2; the coefficients are -2, 6, -1, and 5.

Try Exercise 16, page 36

To add polynomials, add the coefficients of the like terms.

EXAMPLE 2 Add Polynomials

Add: $(3x^3 - 2x^2 - 6) + (4x^2 - 6x - 7)$

Solution

$$\begin{aligned}(3x^3 - 2x^2 - 6) &+ (4x^2 - 6x - 7)\\ &= 3x^3 + (-2x^2 + 4x^2) + (-6x) + [(-6) + (-7)]\\ &= 3x^3 + 2x^2 - 6x - 13\end{aligned}$$

Try Exercise 28, page 36

The **additive inverse of the polynomial** $3x - 7$ is

$$-(3x - 7) = -3x + 7$$

Question • What is the additive inverse of $3x^2 - 8x + 7$?

Answer • The additive inverse is $-3x^2 + 8x - 7$.

To subtract a polynomial, we add its additive inverse. For example,

$$\begin{aligned}(2x - 5) - (3x - 7) &= (2x - 5) + (-3x + 7)\\ &= [2x + (-3x)] + [(-5) + 7]\\ &= -x + 2\end{aligned}$$

The distributive property is used to multiply polynomials. For instance,

$$\begin{aligned}(2x^2 - 5x + 3)(3x + 4) &= (2x^2 - 5x + 3)(3x) + (2x^2 - 5x + 3)4\\ &= (6x^3 - 15x^2 + 9x) + (8x^2 - 20x + 12)\\ &= 6x^3 - 7x^2 - 11x + 12\end{aligned}$$

Although we could always multiply polynomials using the preceding procedure, we frequently use a vertical format. Here is the same product as shown previously using that format.

$$\begin{array}{rrrrl} & 2x^2 & -\ 5x & +\ 3 & \\ & & 3x & +\ 4 & \\ \hline & 8x^2 & -\ 20x & +\ 12 & = (2x^2 - 5x + 3)4\\ 6x^3 & -\ 15x^2 & +\ 9x & & = (2x^2 - 5x + 3)(3x)\\ \hline 6x^3 & -\ 7x^2 & -\ 11x & +\ 12 & \end{array}$$

EXAMPLE 3 Multiply Polynomials

Multiply: $(2x - 5)(x^3 - 4x + 2)$

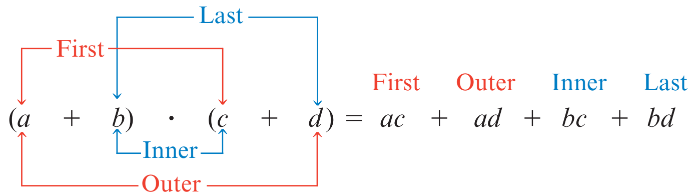
Solution

Note in the following solution how like terms are placed in columns.

$$\begin{array}{rrrrr} & x^3 & & -\ 4x & +\ 2\\ & & & 2x & -\ 5\\ \hline & -\ 5x^3 & & +\ 20x & -\ 10\\ 2x^4 & & -\ 8x^2 & +\ 4x & \\ \hline 2x^4 & -\ 5x^3 & -\ 8x^2 & +\ 24x & -\ 10\end{array}$$

▶ Try Exercise 42, page 37

If the terms of the binomials $(a + b)$ and $(c + d)$ are labeled as shown below, then the product of the two binomials can be computed mentally by the **FOIL method**.



In the following illustration, we find the product of $(7x - 2)$ and $(5x + 4)$ by the FOIL method.

$$\begin{aligned}(7x - 2)(5x + 4) &= \overset{\text{First}}{(7x)(5x)} + \overset{\text{Outer}}{(7x)(4)} + \overset{\text{Inner}}{(-2)(5x)} + \overset{\text{Last}}{(-2)(4)}\\ &= 35x^2 + 28x - 10x - 8\\ &= 35x^2 + 18x - 8\end{aligned}$$

EXAMPLE 4 Multiply Binomials

Multiply.

a. $(4x + 5)(3x - 7)$

b. $(2x - 3y)(4x - 5y)$

Solution

a. $(4x + 5)(3x - 7) = (4x)(3x) - (4x)7 + 5(3x) - 5(7)$
$= 12x^2 - 28x + 15x - 35$
$= 12x^2 - 13x - 35$

b. $(2x - 3y)(4x - 5y) = (2x)(4x) - (2x)(5y) - (3y)(4x) + (3y)(5y)$
$= 8x^2 - 10xy - 12xy + 15y^2$
$= 8x^2 - 22xy + 15y^2$

Try Exercise 54, page 37

Certain products occur so frequently in algebra that they deserve special attention. See Table P.2.

Table P.2 Special Product Formulas

Special Form	Formula(s)
(Sum)(Difference)	$(x + y)(x - y) = x^2 - y^2$
$(\text{Binomial})^2$	$(x + y)^2 = x^2 + 2xy + y^2$ $(x - y)^2 = x^2 - 2xy + y^2$

The variables x and y in these special product formulas can be replaced by other algebraic expressions, as shown in Example 5.

EXAMPLE 5 Use the Special Product Formulas

Find each special product.

a. $(7x + 10)(7x - 10)$ **b.** $(2y^2 + 11z)^2$

Solution

a. $(7x + 10)(7x - 10) = (7x)^2 - (10)^2 = 49x^2 - 100$

b. $(2y^2 + 11z)^2 = (2y^2)^2 + 2[(2y^2)(11z)] + (11z)^2 = 4y^4 + 44y^2z + 121z^2$

Try Exercise 60, page 37

Many application problems require you to *evaluate polynomials*. To **evaluate a polynomial**, substitute the given value or values for the variable or variables and then perform the indicated operations using the Order of Operations Agreement.

EXAMPLE 6 Evaluate a Polynomial

Evaluate the polynomial $2x^3 - 6x^2 + 7$ for $x = -4$.

Solution

$2x^3 - 6x^2 + 7$
$2(-4)^3 - 6(-4)^2 + 7 = 2(-64) - 6(16) + 7$ • Substitute -4 for x. Evaluate the powers.

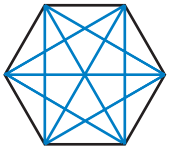
$= -128 - 96 + 7$ • Perform the multiplications.

$= -217$ • Perform the additions and subtractions.

▶ Try Exercise 70, page 37

Applications of Polynomials

EXAMPLE 7 Solve an Application



A diagonal of a polygon is a line segment from one vertex to any other nonadjacent vertex. The diagonals of a regular hexagon (one whose sides are equal) are shown at the left. The number of distinct diagonals of a polygon is given by $\frac{1}{2}n^2 - \frac{3}{2}n$, where n is the number of sides of the polygon. Just as an artist or musician may view a painting or composition as elegant, mathematicians view regular polygons that can be constructed with a straightedge and compass as elegant. In 1796, Carl Friedrich Gauss, one of the greatest mathematicians who ever lived, proved that it was possible to draw a regular 17-sided polygon with just a straightedge and compass. How many distinct diagonals are in a 17-gon?

Solution

$$\frac{1}{2}n^2 - \frac{3}{2}n$$

$$\frac{1}{2}(17)^2 - \frac{3}{2}(17) = \frac{1}{2}(289) - \frac{3}{2}(17) = 119$$ • Substitute 17 for n. Then simplify.

There are 119 diagonals in a 17-gon.

▶ Try Exercise 80, page 37

Math Matters

The procedure used by the computer to determine whether a number is prime or composite is a *polynomial time algorithm*, because the time required can be estimated using a polynomial. The procedure used to factor a number is an *exponential time algorithm*. In the field of *computational complexity*, it is important to distinguish between polynomial time algorithms and exponential time algorithms. Example 8 illustrates that the polynomial time algorithm can be run in about 2 seconds, whereas the exponential time algorithm requires about 44 minutes!

EXAMPLE 8 Solve an Application

A scientist determines that the average time in seconds that it takes a particular computer to determine whether an n-digit natural number is prime or composite is given by

$$0.002n^2 + 0.002n + 0.009, \quad 20 \le n \le 40$$

The average time in seconds that it takes the computer to factor an n-digit number is given by

$$0.00032(1.7)^n, \quad 20 \le n \le 40$$

Estimate the average time it takes the computer to

a. determine whether a 30-digit number is prime or composite

b. factor a 30-digit number

(continued)