A Transition to Advanced Mathematics

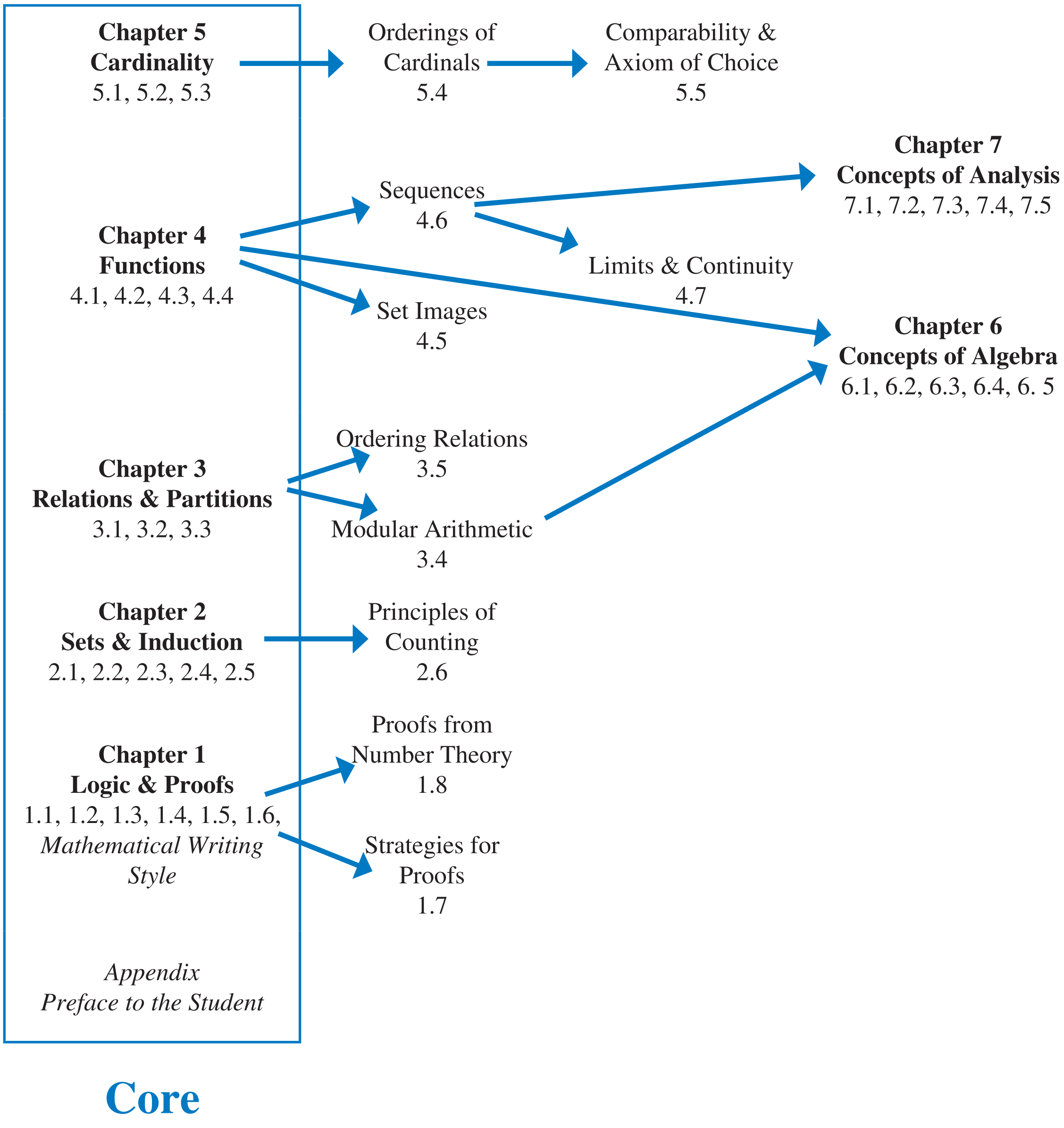
EIGHTH EDITION

DOUGLAS SMITH

MAURICE EGGEN

RICHARD ST. ANDRE

Core Topics and Section Prerequisites



A Transition to Advanced Mathematics

EIGHTH EDITION

A Transition to Advanced Mathematics

Douglas D. Smith

University of North Carolina Wilmington

Maurice Eggen

Trinity University

Richard St. Andre

Central Michigan University

CENGAGE Learning®

Australia • Brazil • Japan • Korea • Mexico • Singapore • Spain • United Kingdom • United States

A Transition to Advanced Mathematics, 8th Edition
Douglas Smith, Maurice Eggen, and Richard St. Andre

Product Director: Liz Covello

Product Manager: Richard Stratton

Content Developer: Erin Brown

Associate Content Developer: Danielle Hallock

Product Assistant: Spencer Arritt

Media Developer: Andrew Coppola

Marketing Brand Manager: Julie Schuster

IP Analyst: Ann Hoffman

IP Project Manager: John N Sarantakis

Manufacturing Planner: Doug Bertke

Art and Design Direction, Production Management, and Composition: Cenveo® Publisher Services

WCN: 02-200-208

Library of Congress Control Number: 2014938401

Student Edition:
ISBN: 978-1-285-46326-1

Cengage Learning
20 Channel Center Street
Boston, MA
USA

Cengage Learning is a leading provider of customized learning solutions with office locations around the globe, including Singapore, the United Kingdom, Australia, Mexico, Brazil and Japan. Locate your local office at **international.cengage.com/region.**

Cengage Learning products are represented in Canada by Nelson Education, Ltd.

For your course and learning solutions, visit **www.cengage.com.**
Purchase any of our products at your local college store or at our preferred online store **www.cengagebrain.com.**
Instructors: Please visit **login.cengage.com** and log in to access instructor-specific resources.

Printed in the United States of America
Print Number: 01 Print Year: 2014

To our wives
Karen, Karen, and Karen

CONTENTS

Preface *ix*
Preface to the Student *xii*

CHAPTER 1 Logic and Proofs 1

1.1 Propositions and Connectives 1
1.2 Conditionals and Biconditionals 10
1.3 Quantified Statements 18
1.4 Basic Proof Methods I 28
1.5 Basic Proof Methods II 42
1.6 Proofs Involving Quantifiers 51
Mathematical Writing Style 64
1.7 Strategies for Constructing Proofs 67
1.8 Proofs from Number Theory 76

CHAPTER 2 Sets and Induction 85

2.1 Basic Concepts of Set Theory 85
2.2 Set Operations 95
2.3 Indexed Families of Sets 104
2.4 Mathematical Induction 114
2.5 Equivalent Forms of Induction 128
2.6 Principles of Counting 137

CHAPTER 3 Relations and Partitions 153

3.1 Relations 153
3.2 Equivalence Relations 163
3.3 Partitions 175
3.4 Modular Arithmetic 182
3.5 Ordering Relations 189

CHAPTER 4 Functions 203

4.1 Functions as Relations 203
4.2 Constructions of Functions 212
4.3 Functions That Are Onto; One-to-One Functions 222
4.4 Inverse Functions 231
4.5 Set Images 237
4.6 Sequences 243
4.7 Limits and Continuity of Real Functions 251

CHAPTER 5 Cardinality 261

5.1 Equivalent Sets; Finite Sets 261
5.2 Infinite Sets 271
5.3 Countable Sets 279
5.4 The Ordering of Cardinal Numbers 287
5.5 Comparability and the Axiom of Choice 295

CHAPTER 6 Concepts of Algebra 303

6.1 Algebraic Structures 303
6.2 Groups 312
6.3 Subgroups 320
6.4 Operation Preserving Maps 326
6.5 Rings and Fields 334

CHAPTER 7 Concepts of Analysis 343

7.1 The Completeness Property 344
7.2 The Heine–Borel Theorem 351
7.3 The Bolzano–Weierstrass Theorem 363
7.4 The Bounded Monotone Sequence Theorem 368
7.5 Equivalents of Completeness 374

APPENDIX Sets, Number Systems, and Functions 379

Answers to Selected Exercises 387
Index 427

PREFACE

This text is intended to bridge the gap between calculus and advanced courses in at least three ways. First, it guides students to think and to express themselves mathematically—to analyze a situation, extract pertinent facts, and draw appropriate conclusions. Second, it provides a firm foundation in the major ideas needed for continued work. Finally, we present introductions to modern algebra and analysis in sufficient depth to capture some of their spirit and characteristics. In summary, our main goals in this text are to improve the student's ability to think and write in a mature mathematical fashion and to provide a solid understanding of the material most useful for advanced courses.

Exercises marked with a solid star (★) have complete answers at the back of the text. Open stars (☆) indicate that a hint or a partial answer is provided. "Proofs to Grade" are a special feature of most of the exercise sets. We present a list of claims with alleged proofs, and the student is asked to assign a letter grade to each "proof" and to justify the grade assigned. Spurious proofs are usually built around a single type of error, which may involve a mistake in logic, a common misunderstanding of the concepts being studied, or an incorrect symbolic argument. Correct proofs may be straightforward, or they may present novel or alternate approaches. We have found these exercises valuable because they reemphasize the theorems and counterexamples in the text and also provide the student with an experience similar to grading papers. Thus, the student becomes aware of the variety of possible errors and develops the ability to read proofs critically.

The eighth edition is based on the same goals as previous editions, with several new or substantially revised sections and many new and revised expositions, examples, and exercises. One of the new features is a mini-section in Chapter 1 on mathematical writing style that describes good practices and some of the special characteristics that distinguish the way mathematics is communicated. In addition to advice on what to include in a proof and what to leave out, this short section offer tips on the use of symbols and other details that help in writing clear, readable proofs.

A listing of useful preliminary concepts of sets, the number systems, and the terminology of functions that students have presumably encountered in prior study is now found in an *Appendix*. This makes the prerequisite material easy to locate and keeps the focus of the text on mathematical reasoning and the core content.

An expanded section on strategies for constructing proofs follows the introductory sections on methods of proof and the discussion on writing style. This section summarizes basic proof methods and includes more than 60 exercises involving proofs. Proofs from elementary number theory appear in a separate section where the Division Algorithm is accepted without proof in order to practice basic proof methods on a coherent set of results about divisibility and the greatest common divisor. We have deliberately placed this early in the text before any discussion of inductive proofs or the Well-Ordering Principle. Later, in Chapter 2, students observe the power of inductive methods to prove the Division Algorithm and other results.

There is a new Section 3.4 on modular arithmetic and a new Section 4.7 on limits of functions and continuity of real functions. Other sections with the most substantial revisions are Section 2.6 on combinatorial counting and Section 4.6 on sequences.

We consider the core material (see the diagram on the inside front cover) to be the first several sections of Chapters 1 through 5. Chapter 1 introduces the propositional and predicate logic required by mathematical arguments, not as formal logic, but as tools of reasoning for more complete understanding of concepts (including some ideas of arithmetic, analytic geometry, and calculus with which the student is already familiar). We present methods of proof and carefully analyze examples of each method, giving special attention to the use of definitions and denials. The techniques in this chapter are used and referred to throughout the text. In Chapters 2, 3, and 4 on sets, relations, and functions, we emphasize writing and understanding proofs that require the student to deal precisely with the concepts of set operations, equivalence relations and partitions, and properties of injective and surjective functions.

Chapter 5 emphasizes a working knowledge of cardinality: finite and infinite sets, denumerable sets and the uncountability of the real numbers, and properties of countable sets. As shown in the diagram on the inside front cover, each of the first five chapters offers opportunities for further study, including basics of number theory, modular arithmetic, limits and continuity of real functions, and the ordering of cardinal numbers.

Chapters 6 and 7 make use of the skills and concepts the student has acquired from the core—and thus are above the earlier work in terms of level and rigor. In Chapter 6, we consider properties of algebras with a binary operation, groups, substructures, and homomorphisms, and relate these concepts to rings and fields. Chapter 7 considers the completeness property of the real numbers by tracing its consequences: the Heine–Borel Theorem, the Bolzano–Weierstrass Theorem, and the Bounded Monotone Sequence Theorem, and back to completeness.

We sincerely thank our reviewers for the eighth edition: Dave Feil, Carroll University; Rita Hibschweiler, University of New Hampshire; William Howland, University of St. Thomas; Katherine Kelm, California State University, Fresno; Irene Loomis, University of Tennessee at Chattanooga; Dale Rohm, University of Wisconsin-Stevens Point; Daniel Seabold, Hofstra University; Boris Shekhtman, University of South Florida; Beimnet Teclezghi, New Jersey City University; and David Walnut, George Mason University.

We also wish to thank Elizabeth Jurisich for her suggestions after accuracy checking the text and the Answers to Selected Exercises, and the staff at Cengage for their exceptional professional assistance in the development of this edition and previous editions.

Finally, we note that instructors who adopt this text can sign up for online access to complete solutions for all exercises via Cengage's Solution Builder, service at www.cengage.com/solutionbuilder.

Douglas D. Smith
Richard St. Andre

"I understand mathematics but I just can't do proofs."

Many students approach the study of mathematical reasoning with some apprehension and uncertainty, perhaps expecting that the study of proofs is something they won't really have to do or won't use later. These feelings, expressed in the remark above, are natural as you move from courses where the goals emphasize performing computations or solving certain equations to more advanced courses where the goal may be to establish whether a mathematical system has certain properties. This textbook is written to help ease the transition between these courses. Let's consider several questions students commonly have at the beginning of a "transition" course.

Why write proofs?

Mathematicians often collect information and make observations about particular cases or phenomena in an attempt to form a theory (a model) that describes patterns or relationships among quantities and structures. This approach to the development of a theory uses **inductive reasoning**. However, the characteristic thinking of the mathematician is **deductive reasoning**, in which one uses logic to develop and extend a theory by drawing conclusions based on statements accepted as true. Proofs are essential in mathematical reasoning because they demonstrate that the conclusions are true. Generally speaking, a mathematical explanation for a conclusion has no value if the explanation cannot be backed up by an acceptable proof.

The first goal of this text is to examine standard proof techniques, especially concentrating on how to get started on a proof, and how to construct correct proofs using those techniques. You will discover how the logical form of a statement can serve as a guide to the structure of a proof of the statement. As you study more advanced courses, it will become apparent that the material in this book is indeed fundamental and the knowledge gained will help you succeed in those courses. Moreover, many of the techniques of reasoning and proof that may seem so difficult at first will become completely natural with practice. In fact, the reasoning that you will study is the essence of advanced mathematics, and the ability to reason abstractly is a primary reason why applicants trained in mathematics are valuable to employers.

Why not just test and repeat enough examples to confirm a theory?

After all, as is typically done in natural and social sciences, the test for truth of a theory is that the results of an experiment conform to predictions and that when the experiment is repeated under the same circumstances, the result is always the same. One major difference is that in mathematics we often need to know whether a given statement is *always* true, so while the statement may be true for many (even infinitely many) examples, we would never know whether another example might show the statement to be false. By studying examples, we might conclude that the statement "$x^2 - 3x + 43$ is a prime number" is true for all positive integers x. We could reach this conclusion testing the first 10 or 20 or even the first 42 integers 1, 2, 3, . . . , 42. In each of these cases and others, such as 44, 45, 47, 48, 49, 50, and more, $x^2 - 3x + 43$ *is* a prime number. But the statement is not always true because $43^2 - 3(43) + 43 = 1763$, which is $41 \cdot 43$. Checking examples is helpful in gaining insight for understanding concepts and relationships in mathematics, but is not a valid proof technique unless we can somehow check *all* examples.

Why not just rely on proofs that someone else has done?

One answer follows from the statement above that deductive reasoning characterizes the way mathematicians think. In the sciences, a new observation may force a complete rethinking of what was thought to be true; in mathematics what we know to be true (by proof) is true forever unless there was a flaw in the reasoning. By learning the techniques of reasoning and proof, you are learning the tools of the trade. A proof is the ultimate test of your understanding of the subject matter and of mathematical reasoning.

What should I know before beginning Chapter 1?

The usual prerequisite for a transition course is at least one semester of calculus. We will sometimes refer to topics that come from calculus and earlier courses (for example, differentiable functions or the graph of a parabola), but we won't be solving equations or finding derivatives.

We assume that you have encountered the basic concepts of sets and subsets; that you are familiar with the natural number system and the integers and rational, real, and complex numbers; and that you have worked with functions—especially with functions defined on sets of real numbers with real number images. See the *Appendix* for a quick review of these essential ideas and notations, which will be used throughout the text.

What am I allowed to assume for a proof?

You may be given specific instructions for some proof-writing exercises, but generally the idea is that you may use what someone studying the topic of your proof would know. That is, when we prove something about intersecting lines, we might use facts about the slope of a line, but we probably would not use properties of derivatives. This really is not much of a problem, except for several of our earliest examples in which we prove well-known facts about even and odd integers. In those few examples we construct proofs using other properties of number systems, but *not* what we already know about evenness and oddness. This is done so that we can study the structure of proofs in a familiar setting.

Remember

We don't expect you to become an expert at proving theorems overnight. With practice—studying lots of examples and exercises—the skills will come. Our goal is to help you write and think as mathematicians do, and to present a solid foundation in material that is useful in advanced courses. We hope you enjoy it.

Douglas D. Smith
Richard St. Andre

C H A P T E R 1

Logic and Proofs

We strongly recommend that you read the entire *Preface to the Student* before beginning this first chapter. As described there, mathematics is concerned with the formation of a **theory** (a collection of true statements called **theorems**) that describes patterns or relationships among quantities and structures. It is characterized by **deductive reasoning**, in which one uses logic to develop and extend a theory. A **proof** of a theorem is a justification (a deduction) of the truth of the theorem. A proof is obtained by drawing conclusions based on statements initially accepted as true (the **axioms**) and statements previously proved. How one puts together a sequence of statements to build an acceptable justification is at the heart of this text. Thus, this chapter begins with the basic logic underlying proofs. It introduces the essential methods used to construct correct proofs.

Writing a proof of a theorem requires thorough understanding of the theorem's mathematical concepts. So that you will be familiar with terminology and notations that will be used throughout this book, we recommend that you review the material in the *Appendix* before beginning this chapter.

1.1 Propositions and Connectives

Our goal in this section and the next is to examine the structure of sentences used in making logical conclusions. Most sentences, such as "$\pi > 3$" and "Earth is the closest planet to the sun," have a truth value. That is, they are either true or false. We call these sentences propositions. Other sentences, such as "What time is it?" (an interrogatory sentence) and "Look out!" (an exclamatory sentence) express complete thoughts but have no truth value.

> **DEFINITION** A **proposition** is a sentence that has exactly one truth value. It is either true, which we denote by T, or false, which we denote by F.

Some propositions, such as "$7^2 = 60$," have easily determined truth values. By contrast, it will take years to determine the truth value of the proposition "The North Pacific right whale will be an extinct species before the year 2525." Other statements, such as "Euclid was left-handed," are propositions whose truth values may never be known.

Sentences like "She lives in New York City" and "$x^2 = 36$" are not propositions because each could be true or false depending on the person to whom "she" refers and what numerical value is assigned to x. We will deal with sentences like these in Section 1.3; until then, when we say that a sentence like "$x > 6$" is a proposition, we are assuming that the variable x has been assigned some specific value.

The statement "This sentence is false" is not a proposition because it is neither true nor false. It is an example of a **paradox**—a situation in which, from premises that look reasonable, one uses apparently acceptable reasoning to derive a conclusion that seems to be contradictory. If the statement "This sentence is false" is true, then by its meaning it must be false. On the other hand, if the given statement is false, then what it claims is false, so it must be true. The study of paradoxes such as this has played a key role in the development of modern mathematical logic. A famous example of a paradox formulated in 1901 by Bertand Russell* is discussed in Section 2.1.

By applying logical connectives to propositions, we can form new propositions.

DEFINITION The **negation** of a proposition P, denoted $\sim P$, is the proposition "not P." The proposition $\sim P$ is true exactly when P is false.

The truth value of the negation of a proposition is the opposite of the truth value of the proposition. For example, the negation of the false proposition "7 is divisible by 2" is the true statement "It is not the case that 7 is divisible by 2," or "7 is not divisible by 2."

DEFINITION Given propositions P and Q, the **conjunction** of P and Q, denoted $P \wedge Q$, is the proposition "P and Q." $P \wedge Q$ is true exactly when *both* P and Q are true.

The English words *but*, *while*, and *although* are usually translated symbolically with the conjunction connective, because they have the same effect on truth value as *and*.

Examples. Let C be the proposition "19 is composite" and M be "45 is a multiple of 3." Then C is false and M is true. Thus the proposition $C \wedge M$ is false.

* Bertrand Russell (1872–1970) was a British philosopher, mathematician, and advocate for social reform. He was a strong voice for precision and clarity of arguments in mathematics and logic. He coauthored *Principia Mathematica* (1910–1913), a monumental effort to derive all of mathematics from a specific set of axioms and well-defined rules of inference.

We read $C \wedge M$ as "19 is composite and 45 is a multiple of 3." On the other hand, if we let C be "Copenhagen is the capital of Denmark" and M be "Madrid is the capital of Spain," then the statement "Copenhagen is the capital of Denmark while Madrid is the capital of Spain" is a true proposition with the same symbolic form, $C \wedge M$. □

The examples above illustrate an important distinction between a statement and the *form* of a statement. *The form* $P \wedge Q$ *itself has no truth value.* Only when the components P and Q are assigned to be specific propositions does $P \wedge Q$ have the value T or F. Those combinations of truth values for P and Q that yield true and those that yield false can be displayed in a *truth table* for $P \wedge Q$.

P	Q	$P \wedge Q$
T	T	T
F	T	F
T	F	F
F	F	F

DEFINITION Given propositions P and Q, the **disjunction** of P and Q, denoted $P \vee Q$, is the proposition "P or Q." $P \vee Q$ is true exactly when *at least one* of P or Q is true.

The truth table for $P \vee Q$ is

P	Q	$P \vee Q$
T	T	T
F	T	T
T	F	T
F	F	F

Example. If R is the proposition "12 is a prime number" and S is "16 is an integer power of 2," we know R is false and S is true. Thus, "12 is a prime number or 16 is an integer power of 2," which has the form $R \vee S$, is true. The false proposition "Either 12 is a prime number or 16 is not an integer power of 2" has the form $R \vee \sim S$. □

The statement "Either 7 is prime and 9 is even, or else 11 is not less than 3" may be symbolized by $(P \wedge Q) \vee \sim R$, where P is "7 is prime," Q is "9 is even," and R is "11 is less than 3." Because the propositional form $(P \wedge Q) \vee \sim R$ has three components (P, Q, and R), it follows that there are $2^3 = 8$ possible combinations of truth values in its truth table. The two main components are $P \wedge Q$ and $\sim R$. We make truth tables for these and combine them by using the truth table for $\vee$.

P	Q	R	$P \wedge Q$	$\sim R$	$(P \wedge Q) \vee \sim R$
T	T	T	T	F	T
F	T	T	F	F	F
T	F	T	F	F	F
F	F	T	F	F	F
T	T	F	T	T	T
F	T	F	F	T	T
T	F	F	F	T	T
F	F	F	F	T	T

In practice, we don't make a complete truth table to determine the truth value of a specific statement such as "Either 7 is prime and 9 is even, or else 11 is not less than 3." We can conclude that this statement is true because

We know P is true, Q is false, and R is false.
Therefore, $P \wedge Q$ is false and $\sim R$ is true.
Thus $(P \wedge Q) \vee \sim R$ is true.

The reasoning here follows the steps necessary to create line 7 of the table.

Some compound forms always yield the value true (or false) just because of the way they are formed.

DEFINITIONS A **tautology** is a propositional form that is true for every assignment of truth values to its components.

A **contradiction** is a propositional form that is false for every assignment of truth values to its components.

The *Law of Excluded Middle*, $P \vee \sim P$, is an example of a tautology because $P \vee \sim P$ is true when P is true and true when P is false. We know that statements like

"20341 is a prime number or 20341 is not a prime number" and
"The absolute value function is continuous or it is not continuous"

must be true because both have the form of this tautology.

Example. Prove that $(P \vee Q) \vee (\sim P \wedge \sim Q)$ is a tautology.

Proof. The truth table for this propositional form is

P	Q	$P \vee Q$	$\sim P$	$\sim Q$	$\sim P \wedge \sim Q$	$(P \vee Q) \vee (\sim P \wedge \sim Q)$
T	T	T	F	F	F	T
F	T	T	T	F	F	T
T	F	T	F	T	F	T
F	F	F	T	T	T	T

Because the last column is all true, $(P \vee Q) \vee (\sim P \wedge \sim Q)$ is a tautology. ■

Both $\sim(P \vee \sim P)$ and $Q \wedge \sim Q$ are examples of contradictions. The negation of a contradiction is, of course, a tautology.

Particularly important in writing proofs will be the ability to recognize or write a statement equivalent to another. Sometimes, knowledge of the mathematical content enables us to write an equivalent statement. For instance, if one step in a proof is the statement "The ones digit of the integer x is zero," a later step could be the equivalent statement "The integer x is divisible by 10."

In other cases, the meaning of a statement does not come into play; it is the *form* of the statement that may be used to find a useful equivalent. We say two propositional forms are **equivalent** if they have the same truth tables.

Some of the most commonly used equivalences are presented in the following theorem. You may wish to make truth tables for each pair of forms to verify that they are equivalent, but in each case you should understand the equivalences by examining their meanings. For example, in part (h), negation is applied to a conjunction. The form $\sim(P \wedge Q)$ is true precisely when $P \wedge Q$ is false. This happens when one of P or Q is false, or, in other words, when one of $\sim P$ or $\sim Q$ is true. Thus, $\sim(P \wedge Q)$ is equivalent to $\sim P \vee \sim Q$. That is, to say "We don't have both P and Q" is the same as saying "We don't have P or we don't have Q."

All parts of Theorem 1.1.1 may be verified by constructing truth tables for each pair of propositional forms. (See Exercise 5.)

Theorem 1.1.1 For propositions P, Q, and R, the following are equivalent:

(a)	P	and	$\sim(\sim P)$	Double Negation Law
(b)	$P \vee Q$	and	$Q \vee P$	Commutative Laws
(c)	$P \wedge Q$	and	$Q \wedge P$	
(d)	$P \vee (Q \vee R)$	and	$(P \vee Q) \vee R$	Associative Laws
(e)	$P \wedge (Q \wedge R)$	and	$(P \wedge Q) \wedge R$	
(f)	$P \wedge (Q \vee R)$	and	$(P \wedge Q) \vee (P \wedge R)$	Distributive Laws
(g)	$P \vee (Q \wedge R)$	and	$(P \vee Q) \wedge (P \vee R)$	
(h)	$\sim(P \wedge Q)$	and	$\sim P \vee \sim Q$	DeMorgan's* Laws
(i)	$\sim(P \vee Q)$	and	$\sim P \wedge \sim Q$	

As an example of how this theorem might be useful, suppose that for some integer x we have determined that the statement "x is even and $x > 10$" is not true. Then its negation,

"It is not the case that the integer x is even and $x > 10$,"

is true and has the form $\sim(P \wedge Q)$, where P is "x is even" and Q is "$x > 10$." By part (h) of Theorem 1.1.1, this is equivalent to $\sim P \vee \sim Q$, which is

"It is not the case that x is even or it is not the case that $x > 10$."

* Augustus DeMorgan (1806–1871) was an English logician and mathematician whose contributions include his notational system for symbolic logic. He also introduced the term *mathematical induction* (see Section 2.4) and developed a rigorous foundation for that proof technique.

An easier way to say this is

"x is not even or x is not greater than 10,"

which may be restated as

"x is odd or $x \leq 10$."

A **denial** of a proposition P is any proposition equivalent to $\sim P$. A proposition has only one negation, $\sim P$, but always has many denials, including $\sim P$, $\sim\sim\sim P$, $\sim\sim\sim\sim\sim P$, etc. Some denials of "x is odd" are "x is not odd," "x is even," and "x is divisible by 2." DeMorgan's Laws provide other ways to construct useful denials.

Example. A denial of "Either the defendant paid a fine or the judge declared a mistrial" is

"The judge did not declare a mistrial and the defendant did not pay a fine."

This can be verified by first writing the two sentences symbolically as $P \vee J$ and $(\sim J) \wedge (\sim P)$, respectively. Then we observe that $P \vee J$ is equivalent to $J \vee P$, so a denial of $P \vee J$ is equivalent to $\sim (J \vee B)$, which we know by DeMorgan's Laws is equivalent to $(\sim J) \wedge (\sim P)$. We could also verify that the sentence is a denial by checking that the truth tables for $P \vee J$ and $(\sim J) \wedge (\sim P)$ have exactly opposite values. □

Example. Suppose L_1 and L_2 are two lines in a coordinate system. Find a denial of the statement

"L_1 and L_2 have the same slope or L_1 and L_2 are vertical lines."

The mathematical concepts expressed determine the form of the statement. The component "L_1 and L_2 have the same slope" cannot mean "L_1 has the same slope" and "L_2 has the same slope." However, "L_1 and L_2 are vertical lines" must mean "L_1 is a vertical line" and "L_2 is a vertical line." The correct symbolization is $S \vee (P \wedge Q)$, where S is "L_1 and L_2 have the same slope,", P is "L_1 is a vertical line," and Q is "L_2 is a vertical line."

The negation of the statement is $\sim[S \vee (P \wedge Q)]$, which is equivalent to $\sim S \wedge \sim(P \wedge Q)$. This form, in turn, is equivalent to $\sim S \wedge (\sim P \vee \sim Q)$. The denial we seek is

"L_1 and L_2 do not have the same slope, and
either L_1 is not a vertical line or L_2 is not a vertical line." □

Does someone who says, "Not P or Q" mean "Neither P nor Q" or "Either not P or else Q"? That is, should the symbolic translation be $\sim(P \vee Q)$ or $(\sim P) \vee Q$? The two translations are not equivalent, so the English sentence needs further explanation. Ambiguities like this can be tolerated in casual conversation but not in situations where precision matters—for example, in mathematics and in legal documents. To avoid ambiguities in symbolic statements, we use parentheses ( ), square brackets [ ], or braces { }.

Propositional forms are often written without all the parentheses you might expect. To correctly understand such a form, use these rules:

First, $\sim$ always is applied to the smallest proposition following it.
Then $\wedge$ connects the smallest propositions surrounding it.
Next, $\vee$ connects the smallest propositions surrounding it.

Also, when the same connective is used two or more times in succession, parentheses are restored from the left. Thus, $\sim P \vee Q$ is an abbreviation for $(\sim P) \vee Q$, but $\sim(P \vee Q)$ is the only way to write the negation of $P \vee Q$. Here are some other examples:

$\sim P \vee \sim Q$ abbreviates $(\sim P) \vee (\sim Q)$
$P \vee Q \wedge R$ abbreviates $P \vee (Q \wedge R)$
$P \wedge \sim Q \vee \sim R$ abbreviates $[P \wedge (\sim Q)] \vee (\sim R)$
$R \wedge P \wedge S \wedge Q$ abbreviates $[(R \wedge P) \wedge S] \wedge Q$

There is no requirement to leave out as many parentheses as possible. For example, $\sim P \wedge \sim R \vee \sim P \wedge R$ is an abbreviation for $[(\sim P) \wedge (\sim R)] \vee [(\sim P) \wedge R]$, but for most readers the form $(\sim P \wedge \sim R) \vee (\sim P \wedge R)$ is easier to read.

Exercises 1.1

1. Which of the following are propositions? Give the truth value of each proposition.
- **(a)** What time is dinner?
- **(b)** It is not the case that π is not a rational number.
- ★ **(c)** $x/2$ is a rational number.
- **(d)** $2x + 3y$ is a real number.
- **(e)** Either π is rational and 17 is a prime, or $7 < 13$ and 81 is a perfect square.
- ★ **(f)** Either 2 is rational and π is irrational, or 2π is rational.
- **(g)** Either 5π is rational and 4.9 is rational, or there are exactly four primes less than 10.
- **(h)** -3.7 is rational, and either $3\pi < 10$ or $3\pi > 15$.
- **(i)** It is not the case that 39 is prime, or that 64 is a power of 2.
- **(j)** There are more than three false statements in this book, and this statement is one of them.

2. For each pair of statements, determine whether the conjunction $P \wedge Q$ and the disjunction $P \vee Q$ are true.
- **(a)** P is "$\sqrt{2} < \pi$" and Q is "97 is a prime number."
- ★ **(b)** P is "The moon is larger than Earth" and Q is "The prime divisors of 12 are 2 and 3."
- **(c)** P is "$5^2 + 12^2 = 13^2$" and Q is "$\sqrt{2} + \sqrt{3} = \sqrt{2 + 3}$."
- ★ **(d)** P is "France is south of Italy" and Q is "New Zealand is in Europe."

(e) P is "0, 5, and 10 are all natural numbers" and Q is "98 has two prime divisors."

(f) P is "Hexagons have 5 sides" and Q is "$\sqrt{2 \cdot 3} = 2\sqrt{3}$."

3. Make a truth table for each of the following propositional forms.

★ **(a)** $P \wedge \sim P$ **(b)** $P \vee \sim P$

★ **(c)** $P \wedge \sim Q$ **(d)** $P \wedge (Q \vee \sim Q)$

★ **(e)** $(P \wedge Q) \vee \sim Q$ **(f)** $\sim(P \wedge Q)$

(g) $(P \vee \sim Q) \wedge R$ **(h)** $\sim P \wedge \sim Q$

★ **(i)** $P \wedge (Q \vee R)$ **(j)** $(P \wedge Q) \vee (P \wedge R)$

(k) $P \wedge P$ **(l)** $(P \wedge Q) \vee (R \wedge \sim S)$

4. If P, Q, and R are true while S and K are false, which of the following are true?

★ **(a)** $Q \wedge (R \wedge S)$ **(b)** $Q \vee (R \wedge S)$

★ **(c)** $(P \vee Q) \wedge (R \vee S)$ **(d)** $(\sim P \vee \sim Q) \vee (\sim R \vee \sim S)$

(e) $\sim P \vee \sim Q$ ★ **(f)** $(\sim Q \vee S) \wedge (Q \vee S)$

★ **(g)** $(P \vee S) \wedge (P \vee K)$ **(h)** $K \wedge \sim(S \vee Q)$

☆ **5.** Use truth tables to verify each part of Theorem 1.1.1.

6. Which of the following pairs of propositional forms are equivalent?

★ **(a)** $\sim P \wedge \sim Q$, $\sim(P \wedge \sim Q)$ **(b)** $(\sim P) \vee (\sim Q)$, $\sim(P \vee \sim Q)$

★ **(c)** $(P \wedge Q) \vee R$, $P \wedge (Q \vee R)$ **(d)** $\sim(P \wedge Q)$, $\sim P \wedge \sim Q$

(e) $(P \wedge Q) \vee R$, $P \vee (Q \wedge R)$ **(f)** $(P \wedge Q) \vee P$, P

7. Determine the propositional form and truth value for each of the following:

(a) It is not the case that gold is not a metal.

(b) 19 and 79 are prime, but 119 is not.

★ **(c)** Julius Caesar was born in 1492 or 1493 and died in 1776.

(d) Perth or Panama City or Pisa is located in Europe.

(e) Although 51 divides 153, it is neither prime nor a divisor of 409.

(f) While the number π is greater than 3, the sum $1 + 2\pi$ is less than 8.

(g) It is not the case that both -5 and 13 are elements of $\mathbb{N}$, but 4 is in the set of rational numbers.

8. Suppose P, Q, and R are propositional forms. Explain why each is true.

★ **(a)** If P is equivalent to Q, then Q is equivalent to P.

(b) If P is equivalent to Q, and Q is equivalent to R, then P is equivalent to R.

(c) If P is equivalent to Q, then $\sim P$ is equivalent to $\sim Q$.

(d) If Q is equivalent to R, then $P \wedge Q$ is equivalent to $P \wedge R$.

(e) If Q is equivalent to R, then $P \vee Q$ is equivalent to $P \vee R$.

9. Suppose P, Q, S, and R are propositional forms, P is equivalent to Q, and S is equivalent to R. For each pair of forms, determine whether they are necessarily equivalent. If they are equivalent, explain why.

★ **(a)** P and R **(b)** P and $\sim \sim Q$

★ **(c)** $P \wedge S$ and $Q \wedge R$ **(d)** $P \vee S$ and $Q \vee R$

(e) $\sim(P \wedge S)$ and $\sim Q \vee \sim R$ **(f)** $P \wedge Q$ and $S \wedge R$

10. Use a truth table to determine whether each of the following is a tautology, a contradiction, or neither.

(a) $(P \wedge Q) \vee (\sim P \wedge \sim Q)$

(b) $\sim(P \wedge \sim P)$

★ **(c)** $(P \wedge Q) \vee (\sim P \vee \sim Q)$

(d) $(P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge Q) \vee (\sim P \wedge \sim Q)$

(e) $(Q \wedge \sim P) \wedge \sim(P \wedge R)$

(f) $P \vee [(\sim Q \wedge P) \wedge (R \vee Q)]$

11. Give a useful denial of each statement. Assume that each variable is some fixed object so that each statement is a proposition.

★ **(a)** x is a positive integer.

(b) Cleveland will win the first game or the second game.

★ **(c)** $5 \geq 3$.

(d) 641,371 is a composite integer.

★ **(e)** Roses are red and violets are blue.

(f) K is not bounded or K is compact.

(g) M is odd and one-to-one.

(h) The matrix M is diagonal and invertible.

(i) The function g has a relative maximum at $x = 2$ or $x = 4$ and a relative minimum at $x = 3$.

(j) Neither $z < s$ nor $z \leq t$ is true.

(k) R is transitive but not symmetric.

12. Restore parentheses to these abbreviated propositional forms.

(a) $\sim\sim P \vee \ \sim Q \wedge \ \sim S$

(b) $Q \wedge \ \sim S \vee \ \sim(\sim P \wedge \ Q)$

(c) $P \wedge \ \sim Q \ \ \vee \ \sim P \wedge \ \sim R \vee \ \sim P \wedge \ S$

(d) $\sim P \vee \ Q \wedge \ \sim\sim P \wedge \ Q \vee \ R$

13–14. Other logical connectives between two propositions P and Q are possible.

13. The word *or* is used in two different ways in English. We have presented the truth table for $\vee$, the **inclusive or,** whose meaning is "one or the other or both." The **exclusive or,** meaning "one or the other but not both" and denoted $\otimes$, has its uses in English, as in "She will marry Heckle or she will marry Jeckle." The "inclusive or" is much more useful in mathematics and is the accepted meaning unless there is a statement to the contrary.

★ **(a)** Make a truth table for the "exclusive or" connective $\otimes$.

(b) Show that $A \otimes B$ is equivalent to $(A \vee B) \wedge \sim(A \wedge B)$.

14. "NAND" and "NOR" circuits are commonly used as a basis for flash memory chips. A NAND B is defined to be the negation of "A and B." A NOR B is defined to be the negation of "A or B."

(a) Write truth tables for the NAND and NOR connectives.

(b) Show that (A NAND B) $\vee$ (A NOR B) is equivalent to (A NAND B).

(c) Show that (A NAND B) $\wedge$ (A NOR B) is equivalent to (A NOR B).

1.2 Conditionals and Biconditionals

Sentences of the form "If P, then Q" are the most important kinds of propositions in mathematics. You have seen many examples of such statements in previous study: from precalculus, "If two lines in a plane have the same slope, then the lines are parallel"; from trigonometry, "If $\sec\theta = \frac{5}{3}$, then $\sin\theta = \frac{4}{5}$."; from calculus, "If f is differentiable at x_0 and $f(x_0)$ is a relative minimum for f, then $f'(x_0) = 0$."

DEFINITIONS For propositions P and Q, the **conditional sentence** $P \Rightarrow Q$ is the proposition "If P, then Q." Proposition P is called the **antecedent** and Q is the **consequent.** The sentence $P \Rightarrow Q$ is true if and only if P is false or Q is true.

The truth table for $P \Rightarrow Q$ is

P	Q	$P \Rightarrow Q$
T	T	T
F	T	T
T	F	F
F	F	T

The only case where $P \Rightarrow Q$ is a false statement occurs on line 3 of its truth table, when P is true and Q is false. This agrees with the way we understand promises.

Example. Suppose someone makes this promise to a friend:

"If the weather is warm, we will go hiking."

The antecedent is "The weather is warm" and the consequent is "We will go hiking." This promise would be broken if the weather turned out to be warm and the friends did *not* go hiking (line 3 of the table.) In every other situation, the statement is true. When the weather was warm and the friends went hiking (line 1 of the table), the promise was kept. Whether the friends go hiking or not, in the event that the weather is *not* warm, we wouldn't say the promise was broken. (These are lines 2 and 4 of the table.) □

Our truth table definition for $P \Rightarrow Q$ captures the same meaning for "If . . ., then . . ." that you have always used in mathematics. For example, if we think of x as some fixed real number, we all know that

"If $x > 8$, then $x > 5$"

is a true statement, no matter what number x we have in mind. Let's examine why we say this sentence is true for some specific values of x, where the antecedent P is "$x > 5$" and the consequent Q is "$x > 5$."

When x is a number greater than both 8 and 5 (for example, 11), both P and Q are true, as in line 1 of the truth table. When x is between 5 and 8 (for example, 7), P is false and Q is true, as in the second line of the table. When x is less than both 8 and 5 (for example, 2), we have the situation in line 4. In all three cases, $P \Rightarrow Q$ is true. In fact, "If $x > 8$, then $x > 5$" is always true because there can be no case corresponding to line 3 of the truth table. We are not claiming that either P or Q is true. What we do say is that no matter what number we think of, *if* it is larger than 8, *then* it is also larger than 5.

One curious consequence of the truth table for $P \Rightarrow Q$ is that a conditional sentence may be true even when there is no connection between the antecedent and the consequent. The reason for this is that the truth value of $P \Rightarrow Q$ depends *only* on the truth value of components P and Q, not on their interpretation. For this reason, all of the following are true:

"If $\sin \pi = 1$, then 6 is prime." (line 4 of the truth table)
"$13 > 7 \Rightarrow 2 + 3 = 5$." (line 1 of the truth table)
"$\pi = 3 \Rightarrow$ Paris is the capital of France." (line 2 of the truth table)

and both of these are false by line 3 of the truth table:

"If Saturn has rings, then $(2 + 3)^2 = 2^2 + 3^2$."
"If $4\pi > 10$, then 1 is a prime number."

Other consequences of the truth table for $P \Rightarrow Q$ are worth noting.

- When P is false (lines 2 and 4), it doesn't matter what truth value Q has: $P \Rightarrow Q$ will be true.
- When Q is true (lines 1 and 2), it doesn't matter what truth value P has: $P \Rightarrow Q$ will be true.
- When P and $P \Rightarrow Q$ are both true (on line 1), Q must also be true.

Two propositions associated with $P \Rightarrow Q$ are its converse and contrapositive.

DEFINITION Let P and Q be propositions.
The **converse** of $P \Rightarrow Q$ is $Q \Rightarrow P$.
The **contrapositive** of $P \Rightarrow Q$ is $(\sim Q) \Rightarrow (\sim P)$.

For the conditional sentence "If π is an integer, then 14 is even," the converse of the sentence is "If 14 is even, then π is an integer" and the contrapositive is "If 14 is not even, then π is not an integer." The sentence and its contrapositive are true, but the converse is false.

For the sentence "If $1 + 1 = 2$, then $\sqrt{10} > 3$," the converse and contrapositive are, respectively, "If $\sqrt{10} > 3$, then $1 + 1 = 2$" and "If $\sqrt{10}$ is not greater than 3, then $1 + 1$ is not equal to 2." In this example, all three sentences are true.

These two examples show that *a conditional sentence and its converse are not always equivalent.* Thus, the truth value of $P \Rightarrow Q$ *cannot* be inferred from its converse $Q \Rightarrow P$. However, a statement and its contrapositive are equivalent, as the following theorem shows.

Theorem 1.2.1 For propositions P and Q, $P \Rightarrow Q$ is equivalent to its contrapositive $(\sim Q) \Rightarrow (\sim P)$.

Proof. The proof is carried out by examination of the truth table.

P	Q	$P \Rightarrow Q$	$\sim P$	$\sim Q$	$(\sim Q) \Rightarrow (\sim P)$
T	T	T	F	F	T
F	T	T	T	F	T
T	F	F	F	T	F
F	F	T	T	T	T

$P \Rightarrow Q$ is equivalent to $(\sim Q) \Rightarrow (\sim P)$ because the third column in the truth table is identical to the sixth column. ■

The biconditional connective, defined next, is symbolized with a double arrow $\Leftrightarrow$, which reminds one of both $\Leftarrow$ and $\Rightarrow$. This is no accident because $P \Leftrightarrow Q$ is equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.

DEFINITION For propositions P and Q, the **biconditional sentence** $P \Leftrightarrow Q$ is the proposition "P if and only if Q." The sentence $P \Leftrightarrow Q$ is true exactly when P and Q have the same truth values.

Mathematicians often abbreviate "P if and only if Q" as "P iff Q." The truth table for $P \Leftrightarrow Q$ is

P	Q	$P \Leftrightarrow Q$
T	T	T
F	T	F
T	F	F
F	F	T

Examples. The proposition "$2^3 = 8$ iff 49 is a perfect square" is true because both components are true. The proposition "$\pi = 22/7$ if and only if $\sqrt{2}$ is a rational number" is also true. The proposition "$6 + 1 = 7$ iff Argentina is north of the equator" is false because the truth values of the components differ. □

Definitions are important examples of biconditional sentences because they describe exactly the condition(s) needed to satisfy the definition. Be aware that definitions in mathematics, however, are not like definitions in ordinary English, which are based on how words are typically used. For example, for a period of at most a few dozen years, the standard meaning of the word *wireless* was a broadcast radio receiver. Definitions in mathematics have precise meanings that stay fixed over time. The definition of an "odd" integer in the *Appendix* tells you exactly what that word always means. You may form a helpful mental image or concept, but the idea that an odd integer ends in 1, 3, 5, 7, or 9 is a consequence of the definition, not the definition.

Definitions may be stated with the "if and only if" wording, but it's also common practice to state a formal definition using the word "if." For example, we could say that "A function f is continuous at a number a if . . . ," leaving the "only if" part understood. Either way it's worded, biconditionality provides the test of whether a statement could serve as a definition or is just a description.

Example. The statement "Horizontal lines have slope 0" could be used as a definition, because a line is horizontal if and only if its slope is 0. However, "A quadratic function is a polynomial" is not a definition, because the sentence "A function is quadratic if and only if it is a polynomial" is false. □

Because the biconditional sentence $P \Leftrightarrow Q$ is true exactly when the truth values of P and Q agree, *the propositional forms P and Q are equivalent precisely when $P \Leftrightarrow Q$ is a tautology.* This means all of the statements in Theorem 1.1.1 may be restated using the $\Leftrightarrow$ connective. For example, the first of DeMorgan's Laws (Theorem 1.1.1(h)) may be written $\sim(P \wedge Q) \Leftrightarrow (\sim P \vee \sim Q)$.

The next theorem contains several additional important pairs of equivalent propositional forms involving implication. They will be used often to construct proofs.

Theorem 1.2.2 For propositions P, Q, and R, the following are equivalent:

(a)	$P \Rightarrow Q$	and	$\sim P \vee Q$
(b)	$P \Leftrightarrow Q$	and	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
(c)	$\sim(P \Rightarrow Q)$	and	$P \wedge \sim Q$
(d)	$\sim(P \wedge Q)$	and	$P \Rightarrow \sim Q$
(e)	$\sim(P \wedge Q)$	and	$Q \Rightarrow \sim P$
(f)	$P \Rightarrow (Q \Rightarrow R)$	and	$(P \wedge Q) \Rightarrow R$
(g)	$P \Rightarrow (Q \wedge R)$	and	$(P \Rightarrow Q) \wedge (P \Rightarrow R)$
(h)	$(P \vee Q) \Rightarrow R$	and	$(P \Rightarrow R) \wedge (Q \Rightarrow R)$

Exercise 8 asks you to prove each part of Theorem 1.2.2. The natural way to proceed is by constructing and then comparing truth tables, but you should also think about the meaning of both sides of each statement of equivalence. With part (a), for example, we reason as follows: $P \Rightarrow Q$ is false exactly when P is true and Q is false, which happens exactly when both $\sim P$ and Q are false. Since this happens exactly when $\sim P \vee Q$ is false, the truth tables for $P \Rightarrow Q$ and $\sim P \vee Q$ are identical.

Note that many of the statements in Theorems 1.1.1 and 1.2.2 are related. For example, once we have established Theorems 1.1.1 and 1.2.2(a), we reason that part (c) is correct as follows:

$\sim(P \Rightarrow Q)$ is equivalent, by part (a), to
$\sim(\sim P \vee Q)$, which is equivalent, by Theorem 1.1.1(i), to
$\sim(\sim P) \wedge \sim Q$, which is equivalent, by Theorem 1.1.1(a), to
$P \wedge \sim Q$.

Recognizing the structure of a sentence and translating the sentence into symbolic form using logical connectives are aids in determining its truth value. The translation of sentences into propositional symbols is sometimes very complicated because some natural languages (such as English) are rich and powerful with many nuances. The ambiguities that we tolerate in English would destroy structure and usefulness if we allowed them in mathematics.

Even the translations of simple sentences can present special problems. Suppose a teacher says to a student,

"If you score 74% or higher on the next test, you will pass this course."

This sentence clearly has the form of a conditional sentence, although almost everyone will interpret the meaning as a biconditional.

Contrast this with the situation in mathematics where "If $x = 2$, then x is a solution to $x^2 = 2x$" must have only the meaning of the connective $\Rightarrow$, because $x^2 = 2x$ does not imply that x is 2.

Here are some phrases in English that are ordinarily translated using the connectives $\Rightarrow$ or $\Leftrightarrow$.

Use $P \Rightarrow Q$ to interpret:	Use $P \Leftrightarrow Q$ to interpret:
If P, then Q.	P if and only if Q.
P is sufficient for Q.	P if, but only if, Q.
P only if Q.	P is equivalent to Q.
Q, if P.	P is necessary and sufficient for Q.
Q whenever P.	P implies Q, and conversely,
Q is necessary for P.	
Q, when P.	

The word *unless* is one of those connective words in English that poses special problems because it has so many different interpretations. See Exercise 11.

Examples. Translate each of these statements into symbols. Think of a as a fixed real number.

Statement:	In symbols:
$a > 5$ is sufficient for $a > 3$.	$a > 5 \Rightarrow a > 3$
$a > 3$ is necessary for $a > 5$.	$a > 5 \Rightarrow a > 3$
$a > 5$ only if $a > 3$.	$a > 5 \Rightarrow a > 3$
$\|a\| = -a$ whenever $a < 0$.	$a < 0 \Rightarrow \|a\| = -a$
$\|a\| = 2$ is necessary and sufficient for $a^2 = 4$.	$\|a\| = 2 \Leftrightarrow a^2 = 4$ □

It is not always necessary to know the meaning of all the words in a statement to determine a correct translation. When you see "S is compact is sufficient for S to be bounded," you understand that the interpretation is "S is compact $\Rightarrow$ S is bounded," even if you don't know what *compact* and *bounded* mean.

There will be more than one way to translate a sentence symbolically. For example, if we let C denote the proposition "S is compact" and B denote the proposition "S is bounded," the statement "If S is compact, then S is bounded" may be translated as $C \Rightarrow B$ or as $\sim B \Rightarrow \sim C$ or as $\sim C \vee B$, because all these forms are equivalent.

On the other hand, the sentence "17 and 35 have no common divisors" shows that the meaning of words must be considered. The translation "17 has no common divisors $\wedge$ 35 has no common divisors" makes no mathematical sense. Compare this to the proposition "17 and 35 have digits totaling 8," which *can* be written as a conjunction.

Example. Suppose b is a fixed real number. The form of the sentence "If b is an integer, then b is either even or odd" is $P \Rightarrow (Q \vee R)$, where P is "b is an integer," Q is "b is even," and R is "b is odd." □

Example. Suppose a, b, and p are fixed integers. "If p is a prime number that divides ab, then p divides a or b" has the form $(P \wedge Q) \Rightarrow (R \vee S)$, where P is "p is a prime," Q is "p divides ab," R is "p divides a," and S is "p divides b." □

The rules presented at the end of the previous section that allow us to sometimes reduce the number of, or restore omitted parentheses to, a propositional form can be extended to the connectives $\Rightarrow$ and $\Leftrightarrow$:

The connectives $\sim$, $\wedge$, $\vee$, $\Rightarrow$, and $\Leftrightarrow$ are always applied in the order listed.

Thus, $\sim$ applies to the smallest possible proposition, then $\wedge$ is applied with the next smallest scope, and so forth. For example,

$$P \Rightarrow \sim Q \vee R \Leftrightarrow S \text{ is an abbreviation for } (P \Rightarrow [(\sim Q) \vee R]) \Leftrightarrow S,$$
$$P \vee \sim Q \Leftrightarrow R \Rightarrow S \text{ is an abbreviation for } [P \vee (\sim Q)] \Leftrightarrow (R \Rightarrow S),$$

and

$$P \Rightarrow Q \Rightarrow R \text{ is an abbreviation for } (P \Rightarrow Q) \Rightarrow R.$$

Whenever you abbreviate a form by eliminating some parentheses, be sure to leave enough to make the form easy to read.

Exercises 1.2

1. Identify the antecedent and the consequent for each of the following conditional sentences. Assume that a, b, and f represent some fixed sequence, integer, or function, respectively.

★ **(a)** If squares have three sides, then triangles have four sides.
(b) If the moon is made of cheese, then 8 is an irrational number.
(c) b divides 3 only if b divides 9.
★ **(d)** The differentiability of f is sufficient for f to be continuous.

(e) A sequence a is bounded whenever a is convergent.
★ (f) A function f is bounded if f is integrable.
(g) $1 + 2 = 3$ is necessary for $1 + 1 = 2$.
(h) The fish bite only when the moon is full.
★ (i) A time of 3 minutes, 48 seconds or less is necessary to qualify for the Olympic team.

☆ **2.** Write the converse and contrapositive of each conditional sentence in Exercise 1.

3. What can be said about the truth value of Q when
(a) P is false and $P \Rightarrow Q$ is true? **(b)** P is true and $P \Rightarrow Q$ is true?
(c) P is true and $P \Rightarrow Q$ is false? **(d)** P is false and $P \Leftrightarrow Q$ is true?
(e) P is true and $P \Leftrightarrow Q$ is false?

4. Identify the antecedent and the consequent for each conditional sentence in the following statements from this book.
(a) Exercise 3 of Section 1.6 **(b)** Theorem 2.1.1(c)
(c) The PMI, Section 2.4 **(d)** Theorem 3.3.1
(e) Theorem 4.7.2 **(f)** Corollary 5.3.6

5. Which of the following conditional sentences are true?
★ **(a)** If triangles have three sides, then squares have four sides.
(b) If hexagons have six sides, then the moon is made of cheese.
★ **(c)** If $7 + 6 = 14$, then $5 + 5 = 10$.
(d) The Nile River flows east only if 64 is a perfect square.
(e) Earth has one moon only if the Amazon River flows into the North Sea.
(f) If Euclid's birthday was April 2, then rectangles have four sides.
(g) 5 is prime if $\sqrt{2}$ is not irrational.
(h) $1 + 1 = 2$ is sufficient for $3 > 6$.

6. Which of the following are true? Assume that x and y are fixed real numbers.
★ **(a)** Triangles have three sides iff squares have four sides.
(b) $7 + 5 = 12$ if and only if $1 + 1 = 2$.
(c) $5 + 6 = 6 + 5$ iff $7 + 1 = 10$.
(d) A parallelogram has three sides iff 27 is prime.
(e) The Eiffel Tower is in Paris if and only if the chemical symbol for helium is H.
(f) $\sqrt{10} + \sqrt{13} < \sqrt{11} + \sqrt{12}$ iff $\sqrt{13} - \sqrt{12} < \sqrt{11} - \sqrt{10}$.
(g) $x^2 \geq 0$ if and only if $x \geq 0$.
(h) $x^2 - y^2 = 0$ iff $(x - y)(x + y) = 0$.
(i) $x^2 + y^2 = 50$ if and only if $(x + y)^2 = 50$.

7. Make truth tables for these propositional forms.
(a) $P \Rightarrow (Q \wedge P)$. ★ **(b)** $(\sim P \Rightarrow Q) \vee (Q \Leftrightarrow P)$.
★ **(c)** $\sim Q \Rightarrow (Q \Leftrightarrow P)$. **(d)** $(P \vee Q) \Rightarrow (P \wedge Q)$.
(e) $(P \wedge Q) \vee (Q \wedge R) \Rightarrow P \vee R$.
(f) $[(Q \Rightarrow S) \wedge (Q \Rightarrow R)] \Rightarrow [(P \vee Q) \Rightarrow (S \vee R)]$.

8. Prove Theorem 1.2.2 by constructing truth tables for each equivalence.

9. Determine whether each statement qualifies as a definition.
(a) $y = f(x)$ is a linear function if its graph is a straight line.
(b) $y = f(x)$ is a quadratic function when it contains an x^2 term.
(c) A quadrilateral is a square when all its sides have equal length.
(d) A triangle is a right triangle if the sum of two of its interior angles is 90°.
(e) Two lines are parallel when their slopes are the same number.
(f) A quadrilateral is a rectangle if all its interior angles are equal.

10. Rewrite each of the following sentences using logical connectives. Assume that each symbol f, x_0, n, x, **B** represents some fixed object.
★ **(a)** If f has a relative minimum at x_0 and if f is differentiable at x_0, then $f'(x_0) = 0$.
(b) If n is prime, then $n = 2$ or n is odd.
★ **(c)** R is symmetric and transitive whenever R is irreflexive.
(d) **B** is square and not invertible whenever $\det B = 0$.
★ **(e)** f has a critical point at x_0 iff $f'(x_0) = 0$ or $f'(x_0)$ does not exist.
(f) $2 < n - 6$ is a necessary condition for $2n < 4$ or $n > 4$.
(g) $6 \geq n - 3$ only if $n > 4$ or $n > 10$.
(h) x is Cauchy implies x is convergent.
(i) f is continuous at x_0 whenever $\lim_{x \to x_0} f(x) = f(x_0)$.
(j) If f is differentiable at x_0 and f is increasing at x_0, then $f'(x_0) > 0$.

11. Dictionaries indicate that the conditional meaning of *unless* is preferred, but there are other interpretations as a converse or a biconditional. Discuss the translation of each sentence.
(a) I will go to the store unless it is raining.
★ **(b)** The Dolphins will not make the playoffs unless the Bears lose all the rest of their games.
(c) You cannot go to the game unless you do your homework first.
(d) You won't win the lottery unless you buy a ticket.

12. Show that the following pairs of statements are equivalent.
(a) $(P \vee Q) \Rightarrow R$ and $\sim R \Rightarrow (\sim P \wedge \sim Q)$.
★ **(b)** $(P \wedge Q) \Rightarrow R$ and $(P \wedge \sim R) \Rightarrow \sim Q$.
(c) $P \Rightarrow (Q \wedge R)$ and $(\sim Q \vee \sim R) \Rightarrow \sim P$.
(d) $P \Rightarrow (Q \vee R)$ and $(P \wedge \sim R) \Rightarrow Q$.
(e) $(P \Rightarrow Q) \Rightarrow R$ and $(P \wedge \sim Q) \vee R$.
(f) $P \Leftrightarrow Q$ and $(\sim P \vee Q) \wedge (\sim Q \vee P)$.

13. Give, if possible, an example of a true conditional sentence for which
★ **(a)** the converse is true. **(b)** the converse is false.
★ **(c)** the contrapositive is false. **(d)** the contrapositive is true.

14. Give, if possible, an example of a false conditional sentence for which
(a) the converse is true. **(b)** the converse is false.
(c) the contrapositive is false. **(d)** the contrapositive is true.

15. Give the converse and contrapositive of each sentence of Exercises 10(a), (b), (f) and (g). Decide whether each converse and contrapositive is true or false.

16. Determine whether each of the following is a tautology, a contradiction, or neither.

★ **(a)** $[(P \Rightarrow Q) \Rightarrow P] \Rightarrow P$.
(b) $P \Leftrightarrow P \wedge (P \vee Q)$.
(c) $P \Rightarrow Q \Leftrightarrow P \wedge \sim Q$.
★ **(d)** $P \Rightarrow [P \Rightarrow (P \Rightarrow Q)]$.
(e) $P \wedge (Q \vee \sim Q) \Leftrightarrow P$.
(f) $[Q \wedge (P \Rightarrow Q)] \Rightarrow P$.
(g) $(P \Leftrightarrow Q) \Leftrightarrow \sim(\sim P \vee Q) \vee (\sim P \wedge Q)$.
(h) $[P \Rightarrow (Q \vee R)] \Rightarrow [(Q \Rightarrow R) \vee (R \Rightarrow P)]$.
(i) $P \wedge (P \Leftrightarrow Q) \wedge \sim Q$.
(j) $(P \vee Q) \Rightarrow Q \Rightarrow P$.
(k) $[P \Rightarrow (Q \wedge R)] \Rightarrow [R \Rightarrow (P \Rightarrow Q)]$.
(l) $[P \Rightarrow (Q \wedge R)] \Rightarrow R \Rightarrow (P \Rightarrow Q)$.

17. The **inverse,** or **opposite,** of the conditional sentence $P \Rightarrow Q$ is $\sim P \Rightarrow \sim Q$.

(a) Show that $P \Rightarrow Q$ and its inverse are not equivalent forms.
(b) For what values of the propositions P and Q are $P \Rightarrow Q$ and its inverse both true?
(c) Which is equivalent to the converse of a conditional sentence, the contrapositive of its inverse, or the inverse of its contrapositive?

1.3 Quantified Statements

Unless there has been prior agreement about the value of x, the statement "$x \geq 3$" is not a proposition because it is neither true nor false. A sentence that contains variables is called an **open sentence** or **predicate** and becomes a proposition only when its variables are assigned specific values. For example, "$x \geq 3$" is true when x is given the value 7 and false when x is 2.

When P is an open sentence with a variable x, the sentence is symbolized by $P(x)$. If P has n variables $x_1, x_2, \ldots, x_n$, we write $P(x_1, x_2, \ldots, x_n)$. For example, if $P(x, y, z)$ represents the open sentence "$x + y = z^2$," then $P(4, 5, 3)$ is the true proposition $4 + 5 = 3^2$, while $P(1, 2, 4)$ is the false proposition $1 + 2 = 4^2$.

The collection of objects that may be substituted to make an open sentence a true proposition is called the **truth set** of the sentence. Before a truth set can be determined, we must be given or must decide what objects are available for consideration; that is, we must have specified a **universe of discourse.** In many cases the universe will be understood from the context. For the sentence "x likes chocolate," the universe is presumably the set of all people. We will often use the number systems $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ as our universes. (See the *Appendix*.)

Example. The truth set of the open sentence "$x^2 < 5$" depends on the collection of objects we choose for the universe of discourse. With the universe specified as the set $\mathbb{N}$, the truth set is $\{1, 2\}$. For the universe $\mathbb{Z}$, the truth set is $\{-2, -1, 0, 1, 2\}$. When the universe is $\mathbb{R}$, the truth set is the open interval $(-\sqrt{5}, \sqrt{5})$. □

DEFINITION With a universe specified, two open sentences $P(x)$ and $Q(x)$ are **equivalent** if they have the same truth set.

Examples. The sentences "$3x + 2 = 20$" and "$x = 6$" are equivalent open sentences in any of the number systems named above. On the other hand, "$x^2 = 4$" and "$x = 2$" are *not* equivalent when the universe is $\mathbb{R}$. They *are* equivalent when the universe is $\mathbb{N}$. □

Although words such as *truth set*, *universe*, and *equivalent open sentence* may be unfamiliar to you, the concepts are not new. The equation $(x^2 + 1)(x - 3) = 0$ is an open sentence. Solving the equation is a matter of finding its truth set. For the universe $\mathbb{R}$, the only solution is $x = 3$ and thus the truth set is $\{3\}$. But if we choose the universe to be $\mathbb{C}$, the equation may be replaced by the equivalent open sentence $(x + i)(x - i)(x - 3) = 0$, which has truth set (solutions) $\{3, i, -i\}$.

To determine whether the sentence

"There is a prime number between 5060 and 5090"

is true in the universe $\mathbb{N}$, we might try to individually examine every natural number, checking whether it is a prime and between 5060 and 5090, until we eventually find any *one* of the primes 5077, 5081, and 5087 and conclude that the sentence is true. (A quicker way is to search through a complete list of the first thousand primes.) The key idea here is that although the open sentence "x is a prime number between 5060 and 5090" is not a proposition, the sentence

"There is a number x such that x is a prime number between 5060 and 5090"

does have a truth value. This sentence is formed from the original open sentence by applying a quantifier.

DEFINITION The symbol $\exists$ is called the **existential quantifier**. For an open sentence $P(x)$, the sentence $(\exists x)P(x)$ is read "**There exists *x* such that *P*(*x*)**" or "**For some *x*, *P*(*x*)**." The sentence $(\exists x)P(x)$ is true if the truth set of $P(x)$ is *nonempty*.

An open sentence $P(x)$ does not have a truth value, but the quantified sentence $(\exists x)P(x)$ does. One way to show that $(\exists x)P(x)$ is true for a particular universe is to identify an object a in the universe such that the proposition $P(a)$ is true. To show that $(\exists x)P(x)$ is false, we must show that the truth set of $P(x)$ is empty.

Examples. Let's examine the truth values of these statements for the universe $\mathbb{R}$:

(a) $(\exists x)(x \geq 3)$ **(b)** $(\exists x)(x^2 = 0)$
(c) $(\exists x)(x \geq 3 \wedge x^2 = -1)$ **(d)** $(\exists x)(x \geq 3 \vee x^2 = -1)$

Statements (a) and (d) are true because 3, 7.02, and many other real numbers are in the truth set of $x \geq 3$, and therefore in the truth set of $x \geq 3 \vee x^2 = -1$. Statement (b) is true because the truth set of $x^2 = 0$ is precisely $\{0\}$ and therefore is nonempty. Because the open sentence $x^2 = -1$ is never true for real numbers, the truth set of $x \geq 3 \wedge x^2 = -1$ is empty. Statement (c) is false in the universe $\mathbb{R}$.

All four statements are false in the universe $\{1, 2\}$, and in the universe $\{-3, 0\}$ only statement (b) is true. □

Sometimes we can say $(\exists x)P(x)$ is true even when we do not know a specific object in the universe in the truth set of $P(x)$, only that there (at least) is one.

Example. Show that $(\exists x)(x^7 - 12x^3 + 16x - 3 = 0)$ is true in the universe of real numbers.

For the polynomial $f(x) = x^7 - 12x^3 + 16x - 3$, we see that $f(0) = -3$ and $f(1) = 2$. From calculus, we know that f is continuous on [0, 1]. The Intermediate Value Theorem tells us there is a zero for f between 0 and 1. Even if we don't know the exact value of the zero, we know it exists. Therefore, the truth set of $x^7 - 12x^3 + 16x - 3 = 0$ is nonempty. Hence $(\exists x)(x^7 - 12x^3 + 16x - 3 = 0)$ is true. □

The sentence "Every number x is greater than 0" needs a different quantifier because it is not enough to find at least one value for x for which "$x > 0$" is true. The open sentence "$x > 0$" must *always* be true—that is, true for every object in the universe. The sentence "Every x is greater than 0" is true when the universe is $\mathbb{N}$ but is false when the universe is the integers.

DEFINITION The symbol $\forall$ is called the **universal quantifier**. For an open sentence $P(x)$, the sentence $(\forall x)\ P(x)$ is read **"For all x, $P(x)$"** or **"For every x, $P(x)$."** The sentence $(\forall x)\ P(x)$ is true if the truth set of $P(x)$ is the *entire universe*.

Examples. In the universe of natural numbers, the sentences $(\forall x)(x + 2 > 1)$ and $(\forall x)(2x$ is an integer) are true. However, the sentence $(\forall x)(x + 2 > 1)$ is false in the universe of real numbers because $-5 + 2 > 1$ is false, and $(\forall x)(2x$ is an integer) is false in $\mathbb{R}$ because 0.6 is not in the truth set.

The sentence $(\forall x)(2x + 1 < 6)$ is false in $\mathbb{N}$ because 3 is not in the truth set, and $(\forall x)(2x - 1 > x)$ is false in $\mathbb{N}$ because 1 is not in the truth set. The two sentences are false in the universe of real numbers for the same reasons.

Some universally quantified sentences that are true in $\mathbb{R}$ are

$$(\forall x)(x < 0 \text{ or } x = 0 \text{ or } x > 0),\ (\forall x)(2^x > 0), \quad \text{and} \quad (\forall x)(x + 2 > x). \qquad \square$$

There are many ways to express a quantified sentence in English. Look for key words such as *for all*, *for every*, *for each*, and similar words that require universal quantifiers. Look for phrases such as *some*, *at least one*, *there exist(s)*, *there is (are)*, and others that indicate existential quantifiers.

You should also be alert for hidden quantifiers because natural languages allow for imprecise quantified statements where the words *for all* and *there exists* are not present. Someone who says "Polynomial functions are continuous" means that "All polynomial functions are continuous," but someone who says "Rational functions have vertical asymptotes" must mean "Some rational functions have vertical asymptotes."

How should the sentence "All apples have spots" be written in symbolic form? If we limit the universe to just apples, a correct symbolization would be $(\forall x)(x \text{ has spots})$. But if the universe is all fruits, we need to be more careful. Let $A(x)$ be "x is an apple" and $S(x)$ be "x has spots." Should we write the sentence as $(\forall x)[A(x) \wedge S(x)]$ or $(\forall x)[A(x) \Rightarrow S(x)]$?

The first quantified form, $(\forall x)[A(x) \wedge S(x)]$, says "For all objects x in the universe, x is an apple and x has spots." Since we don't really intend to say that all fruits are spotted apples, this is not the meaning we want. Our other choice, $(\forall x)[A(x) \Rightarrow S(x)]$, is the correct one because it says "For all objects x in the universe, if x is an apple, then x has spots." In other words, "If a fruit is an apple, then it has spots."

Should the symbolic translation of "Some apples have spots" be $(\exists x)[A(x) \wedge S(x)]$ or $(\exists x)[A(x) \Rightarrow S(x)]$? The first form says "There is an object x such that it is an apple and it has spots," which is correct. On the other hand, $(\exists x)[A(x) \Rightarrow S(x)]$ reads "There is an object x such that, if it is an apple, then it has spots," which does *not* ensure the existence of apples with spots. The sentence $(\exists x)[A(x) \Rightarrow S(x)]$ is true in every universe for which there is an object x such that either x is not an apple or x has spots, which is not the meaning we want.

In general,

$$\text{"All } P(x) \text{ are } Q(x)\text{" should be symbolized } (\forall x)(P(x) \Rightarrow Q(x)),$$

and

$$\text{"Some } P(x) \text{ are } Q(x)\text{" should be symbolized } (\exists x)(P(x) \wedge Q(x)).$$

Here are several examples.

Examples. Translate each of these sentences using quantifiers.

(a) "For every odd prime x less than 10, $x^2 + 4$ is prime."

The sentence means that if x is prime, and odd, and less than 10, then $x^2 + 4$ is prime. It is written symbolically as

$$(\forall x)(x \text{ is prime } \wedge\ x \text{ is odd } \wedge\ x < 10 \Rightarrow x^2 + 4 \text{ is prime}).$$

(b) "Some functions defined at 0 are not continuous at 0."

This is translated as

$$(\exists f)(f \text{ is defined at } 0 \wedge f \text{ is not continuous at } 0).$$

(c) "Some real numbers have a multiplicative inverse."

This statement could be symbolized

$$(\exists x)(x \text{ is a real number} \wedge x \text{ has a real multiplicative inverse}).$$

However, "x has an inverse" means there is some number that is an inverse for x (hidden quantifier), so a more complete symbolic translation is

$$(\exists x)[x \text{ is a real number} \wedge (\exists y)(y \text{ is a real number} \wedge xy = 1)]. \quad \square$$

One correct translation of "Some integers are even and some integers are odd" is

$$(\exists x)(x \text{ is even}) \wedge (\exists x)(x \text{ is odd})$$

because the first quantifier $(\exists x)$ extends only as far as "even." After that, any variable (even x again) may be used to express "Some integers are odd." It would be equally correct and sometimes preferable to write

$$(\exists x)(x \text{ is even}) \wedge (\exists y)(y \text{ is odd}),$$

but it would be incorrect to write

$$(\exists x)(x \text{ is even} \wedge x \text{ is odd}),$$

because there is no integer that is both even and odd.

Definitions in their symbolic forms often use multiple quantifiers. For example, the definition of a rational number may be symbolized as follows:

$$r \text{ is a rational number if } (\exists p)(\exists q)\left(p \in \mathbb{Z} \wedge q \in \mathbb{Z} \wedge q \neq 0 \wedge r = \tfrac{p}{q}\right).$$

Statements of the form "Every element of the set A has the property P" and "Some element of the set A has property P" occur so frequently that abbreviated symbolic forms are used:

"Every element of the set A has the property P" may be restated as "If $x \in A$, then..." and symbolized by

$$(\forall x \in A)P(x).$$

"Some element of the set A has property P" is abbreviated by

$$(\exists x \in A)P(x).$$

Thus the definition of a rational number given above may be written as

$$r \text{ is a rational number if } (\exists p \in \mathbb{Z})(\exists q \in \mathbb{Z})\left(q \neq 0 \wedge r = \tfrac{p}{q}\right).$$

Example. The statement "For every rational number there is a larger integer" may be symbolized by

$$(\forall x)[x \in \mathbb{Q} \Rightarrow (\exists z)(z \in \mathbb{Z} \text{ and } z > x)]$$

or

$$(\forall x \in \mathbb{Q})(\exists z \in \mathbb{Z})(z > x).$$ □

As was noted with propositional forms, it is necessary to make a distinction between a quantified sentence and its logical form. With the universe all integers, the sentence "All integers are odd" is an instance of the logical form $(\forall x)P(x)$, where $P(x)$ is "x is odd." The form itself, $(\forall x)P(x)$, is neither true nor false, but becomes false when "x is odd" is substituted for $P(x)$ and the universe is all integers.

> **DEFINITION** Two quantified sentences are **equivalent in a given universe** if they have the same truth value in that universe. Two quantified sentences are **equivalent** if they are equivalent in every universe.

Example. $(\forall x)(x > 3)$ and $(\forall x)(x \geq 4)$ are equivalent in the universe of integers (because both are false), in the universe of natural numbers greater than 10 (because both are true), and in many other universes. However, if we choose the universe U to be the interval $[3.7, \infty)$, then $(\forall x)(x > 3)$ is true and $(\forall x)(x \geq 4)$ is false in U. The sentences are not equivalent in this universe, so they are not equivalent sentences. □

We can construct equivalent quantified statements using the theorems in Sections 1.1 and 1.2. For example, the statement $(\forall x)(P(x) \Rightarrow Q(x))$ is equivalent to $(\forall x)(\sim Q(x) \Rightarrow \sim P(x))$ by Theorem 1.2.1(a). The two equivalences in the next theorem are essential for building proofs that involve quantifiers.

Theorem 1.3.1 If $A(x)$ is an open sentence with variable x, then

(a) $\sim(\forall x)A(x)$ is equivalent to $(\exists x) \sim A(x)$.
(b) $\sim(\exists x)A(x)$ is equivalent to $(\forall x) \sim A(x)$.

Proof.

(a) Let U be any universe.
The sentence $\sim(\forall x)A(x)$ is true in U
iff $(\forall x)A(x)$ is false in U
iff the truth set of $A(x)$ is not the universe
iff the truth set of $\sim A(x)$ is nonempty
iff $(\exists x) \sim A(x)$ is true in U.

(b) The proof of this part is Exercise 7. ■

Theorem 1.3.1 is helpful for finding a useful **denial** (that is, a simplified form of the negation) of a quantified sentence. In the universe of natural numbers, the sentence "All primes are odd" is symbolized $(\forall x)(x \text{ is prime} \Rightarrow x \text{ is odd})$. The negation is $\sim(\forall x)(x \text{ is prime} \Rightarrow x \text{ is odd})$. When we apply Theorem 1.3.1(a), this becomes $(\exists x)[\sim(x \text{ is prime} \Rightarrow x \text{ is odd})]$. By Theorem 1.2.2(c) this is equivalent to $(\exists x)[x \text{ is prime} \wedge \sim(x \text{ is odd})]$. We read this last statement as "There exists a number that is prime and is not odd" or "Some prime number is even."

Example. For the universe of all real numbers, find a denial of "Every positive real number has a multiplicative inverse."

The sentence is symbolized $(\forall x)[x > 0 \Rightarrow (\exists y)(xy = 1)]$. The negation and successively rewritten equivalents are

$$\sim(\forall x)[x > 0 \Rightarrow (\exists y)(xy = 1)]$$
$$(\exists x) \sim[x > 0 \Rightarrow (\exists y)(xy = 1)]$$
$$(\exists x)[x > 0 \wedge \sim(\exists y)(xy = 1)]$$
$$(\exists x)[x > 0 \wedge (\forall y) \sim(xy = 1)]$$
$$(\exists x)[x > 0 \wedge (\forall y)(xy \neq 1)]$$

This last sentence may be translated as "There is a positive real number that has no multiplicative inverse." □

Example. For the universe of living things, find a denial of "Some children do not like clowns."

The sentence is $(\exists x)\,[x \text{ is a child} \wedge (\forall y)(y \text{ is a clown} \Rightarrow x \text{ does not like } y)]$. Its negation and several equivalents are

$$\sim(\exists x)\,[x \text{ is a child} \wedge (\forall y)(y \text{ is a clown} \Rightarrow x \text{ does not like } y)]$$
$$(\forall x) \sim[x \text{ is a child} \wedge (\forall y)(y \text{ is a clown} \Rightarrow x \text{ does not like } y)]$$
$$(\forall x)\,[x \text{ is a child } \Rightarrow \sim(\forall y)(y \text{ is a clown} \Rightarrow x \text{ does not like } y)]$$
$$(\forall x)\,[x \text{ is a child } \Rightarrow (\exists y) \sim(y \text{ is a clown} \Rightarrow x \text{ does not like } y)]$$
$$(\forall x)\,[x \text{ is a child } \Rightarrow (\exists y)(y \text{ is a clown } \wedge \sim x \text{ does not like } y)]$$
$$(\forall x)\,[x \text{ is a child } \Rightarrow (\exists y)(y \text{ is a clown } \wedge\ x \text{ likes } y)]$$

The denial we seek is "Every child has some clown that he/she likes." □

Example. To find a simplified denial of $(\forall x)(\exists y)(\exists z)(\forall u)(\exists z)(x + y + z > 2u + v)$, we begin with its negation and apply Theorem 1.3.1 five times in succession, working inward from the outermost quantifier $(\forall x)$. Each use of the theorem moves the negation symbol across a quantifier and changes that quantifier to another, and the last use also negates the open sentence with five variables. The result is the simplified form

$$(\exists x)(\forall y)(\forall z)(\exists u)(\forall v)(x + y + z \leq 2u + v). \qquad \square$$

Example. The symbolic form of "All Australians play soccer" is $(\forall x \in A)P(x)$, where A is the set of all Australians and $P(x)$ is "x plays soccer." Determine whether

$$(\exists x)(x \notin A \wedge \sim P(x))$$

is a symbolic form of a denial of the sentence.

Begin by listing several denials of $(\forall x \in A)P(x)$. Start with its negation and its equivalent obtained by using Theorem 1.3.1(a). Then add to the list the unabbreviated form of the denial and some of its equivalents derived from Theorem 1.1.2. Thus we find these denials:

$$\sim(\forall x \in A)P(x) \qquad (\exists x \in A) \sim P(x)$$
$$\sim(\forall x)(x \in A \Rightarrow P(x)) \qquad (\exists x) \sim(x \in A \Rightarrow P(x))$$
$$\sim(\forall x)(x \notin A \vee P(x)) \qquad (\exists x)(x \in A \wedge \sim P(x))$$

To determine whether $(\exists x)(x \notin A \wedge \sim P(x))$ is a denial, make a second list consisting of this sentence and some of its equivalents:

$$(\exists x) \sim(x \in A \vee P(x))$$
$$\sim(\forall x)(x \in A \vee P(x))$$

We suspect that $(\exists x)\ (x \notin A \wedge \sim P(x))$ is not a denial because we haven't found any logically equivalent form from one list in the other list. We can be sure it's not a denial by examining the two forms in blue. The form $(\exists x)\ (x \in A \wedge \sim P(x))$ says "Some Australian does not play soccer." However, the form $(\exists x)\ (x \notin A \wedge \sim P(x))$ says "Some non-Australian does not play soccer." These certainly have different meanings. $\square$

We sometimes hear statements like the complaint one fan had after a great Little League baseball game. "The game was fine," he said, "but everybody didn't get to play." We easily understand that the fan did not mean this literally, because otherwise there would have been no game. The meaning we understand is "Not everyone got to play" or "Some team members did not play." Such misuse of quantifiers, while tolerated in casual conversations, is always to be avoided in mathematics.

A special case of the existential quantifier is defined next.

DEFINITION The symbol $\exists!$ is called the **unique existential quantifier**. For an open sentence $P(x)$, the sentence $(\exists! x)\ P(x)$ is read **"There is a unique *x* such that *P*(*x*)."** The sentence $(\exists! x)\ P(x)$ is true if the truth set of $P(x)$ has *exactly one element*.

Recall that for $(\exists x)P(x)$ to be true it is unimportant how many elements are in the truth set of $P(x)$, as long as there is at least one. For $(\exists! x)P(x)$ to be true, the number of elements in the truth set of $P(x)$ is crucial—there must be exactly one.

In the universe $\mathbb{N}$, $(\exists! x)$ (x is even and x is prime) is true because the truth set of "x is even and x is prime" contains only the number 2. The sentence $(\exists! x)(x^2 = 4)$ is true in the universe $\mathbb{N}$, but false in $\mathbb{Z}$.

Theorem 1.3.2 If $A(x)$ is an open sentence with variable x, then

(a) $(\exists! x)A(x) \Rightarrow (\exists x)A(x)$.
(b) $(\exists! x)A(x)$ is equivalent to $(\exists x)A(x) \wedge (\forall y)(\forall z)[A(y) \wedge A(z) \Rightarrow y = z]$.

Part (a) of Theorem 1.3.2 says that $\exists!$ is indeed a special case of the quantifier $\exists$. Part (b) says that "There exists a unique x such that $A(x)$" is equivalent to "There is an x such that $A(x)$ and if both $A(y)$ and $A(z)$, then $y = z$." The proofs are left to Exercise 11.

Exercises 1.3

1. Translate the following English sentences into symbolic sentences with quantifiers. The universe for each is given in parentheses.

★ **(a)** Not all precious stones are beautiful. (All stones)
☆ **(b)** All precious stones are not beautiful. (All stones)
(c) Some isosceles triangle is a right triangle. (All triangles)
(d) No right triangle is isosceles. (All triangles)
(e) Every triangle that is not isosceles is a right triangle.
(f) All people are honest or no one is honest. (All people)
(g) Some people are honest and some people are not honest. (All people)
(h) Every nonzero real number is positive or negative. (Real numbers)
★ **(i)** Every integer is greater than -4 or less than 6. (Real numbers)
(j) Every integer is greater than some integer. (Integers)
★ **(k)** No integer is greater than every other integer. (Integers)
(l) Between any integer and any larger integer, there is a real number. (Real numbers)
★ **(m)** There is a smallest positive integer. (Real numbers)
★ **(n)** No one loves everybody. (All people)
(o) Everybody loves someone. (All people)
(p) For every positive real number x, there is a unique real number y such that $2^y = x$. (Real numbers)

☆ **2.** For each of the propositions in Exercise 1, write a useful denial, and give a translation into ordinary English.

3. Translate these definitions from the *Appendix* into quantified sentences.

(a) The natural number a **divides** the natural number b.
(b) The natural number n is **prime**.
(c) The natural number n is **composite**.

★ **(d)** The sets A and B are equal.
(e) The set A is a subset of B.
(f) The set A is not a subset of B.

4. Write symbolic translations using quantifiers for each of the five important properties of $\mathbb{Z}$ listed in the *Appendix* under the heading *The Integers*.

☆ **5.** The sentence "People dislike taxes" might be interpreted to mean "All people dislike all taxes," "All people dislike some taxes," "Some people dislike all taxes," or "Some people dislike some taxes." Give a symbolic translation for each of these interpretations.

6. Let $T = \{17\}$, $U = \{6\}$, $V = \{24\}$, and $W = \{2, 3, 7, 26\}$. In which of these four different universes is the statement true?
★ **(a)** $(\exists x)(x \text{ is odd} \Rightarrow x > 8)$.
(b) $(\exists x)(x \text{ is odd} \wedge x > 8)$.
(c) $(\forall x)(x \text{ is odd} \Rightarrow x > 8)$.
(d) $(\forall x)(x \text{ is odd} \wedge x > 8)$.

7. **(a)** Complete the following proof of Theorem 1.3.1(b).
Proof: Let U be any universe.
The sentence $\sim(\exists x)A(x)$ is true in U
iff . . .
iff $(\forall x) \sim A(x)$ is true in U.

☆ **(b)** Give a proof of part (b) of Theorem 1.3.1 that uses part (a) of that theorem.

8. Which of the following are true? The universe for each statement is given in parentheses.
(a) $(\forall x)(x + x \geq x)$. $(\mathbb{R})$
★ **(b)** $(\forall x)(x + x \geq x)$. $(\mathbb{N})$
(c) $(\exists x)(2x + 3 = 6x + 7)$. $(\mathbb{N})$
(d) $(\exists x)(3^x = x^2)$. $(\mathbb{R})$
★ **(e)** $(\exists x)(3^x = x)$. $(\mathbb{R})$
(f) $(\exists x)(3(2 - x) = 5 + 8(1 - x))$. $(\mathbb{R})$
(g) $(\forall x)(x^2 + 6x + 5 \geq 0)$. $(\mathbb{R})$
★ **(h)** $(\forall x)(x^2 + 4x + 5 \geq 0)$. $(\mathbb{R})$
(i) $(\exists x)(x^2 - x + 41 \text{ is prime})$. $(\mathbb{N})$
(j) $(\forall x)(x^2 - x + 41 \text{ is prime})$. $(\mathbb{N})$
(k) $(\forall x)(x^3 + 17x^2 + 6x + 100 \geq 0)$. $(\mathbb{R})$
(l) $(\forall x)(\forall y)[x < y \Rightarrow (\exists w)(x < w < y)]$. $(\mathbb{R})$

9. Give an English translation for each. The universe is given in parentheses.
(a) $(\forall x)(x \geq 1)$. $(\mathbb{N})$
★ **(b)** $(\exists! x)(x \geq 0 \wedge x \leq 0)$. $(\mathbb{R})$
(c) $(\forall x)(x \text{ is prime} \wedge x \neq 2 \Rightarrow x \text{ is odd})$. $(\mathbb{N})$
★ **(d)** $(\exists! x)(\log_e x = 1)$. $(\mathbb{R})$
(e) $\sim(\exists x)(x^2 < 0)$. $(\mathbb{R})$
(f) $(\exists! x)(x^2 = 0)$. $(\mathbb{R})$
(g) $(\forall x)(x \text{ is odd} \Rightarrow x^2 \text{ is odd})$. $(\mathbb{N})$

10. Which of the following are true in the universe of all real numbers?
★ **(a)** $(\forall x)(\exists y)(x + y = 0)$.
(b) $(\exists x)(\forall y)(x + y = 0)$.
(c) $(\exists x)(\exists y)(x^2 + y^2 = -1)$.
★ **(d)** $(\forall x)[x > 0 \Rightarrow (\exists y)(y < 0 \wedge xy > 0)]$.
(e) $(\forall y)(\exists x)(\forall z)(xy = xz)$.
★ **(f)** $(\exists x)(\forall y)(x \leq y)$.
(g) $(\forall y)(\exists x)(x \leq y)$.
(h) $(\exists! y)(y < 0 \wedge y + 3 > 0)$.
★ **(i)** $(\exists! x)(\forall y)(x = y^2)$.
(j) $(\forall y)(\exists! x)(x = y^2)$.
(k) $(\exists! x)(\exists! y)(\forall w)(w^2 > x - y)$.

11. Let $A(x)$ be an open sentence with variable x.
☆ **(a)** Prove Theorem 1.3.2 (a).
☆ **(b)** Show that the converse of Theorem 1.3.2 (a) is false.
(c) Prove Theorem 1.3.2 (b).
(d) Prove that $(\exists! x)A(x)$ is equivalent to $(\exists x)[A(x) \wedge (\forall y)(A(y) \Rightarrow x = y)]$.
★ **(e)** Find a useful denial for $(\exists! x)A(x)$.

12. Suppose the polynomials $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0$ are not equal. Which of the following must be true?
★ **(a)** $a_n \neq b_n$.
(b) $a_i \neq b_i$ whenever $0 \leq i \leq n$.
(c) $a_i \neq b_i$ for every i such that $0 \leq i \leq n$.
(d) $a_i \neq b_i$ for some i such that $0 \leq i \leq n$.
(e) It is not the case that $a_i = b_i$ for all i such that $0 \leq i \leq n$.
(f) It is not the case that $a_i = b_i$ for some i such that $0 \leq i \leq n$.
(g) There is an i such that $0 \leq i \leq n$ and $a_i \neq b_i$.
(h) If $a_i = b_i$ for all i such that $0 \leq i \leq n - 1$, then $a_n \neq b_n$.

13. Which of the following are denials of $(\exists! x)P(x)$?
(a) $(\forall x)P(x) \vee (\forall x) \sim P(x)$.
(b) $(\forall x) \sim P(x) \vee (\exists y)(\exists z)(y \neq z \wedge P(y) \wedge P(z))$.
(c) $(\forall x)[P(x) \Rightarrow (\exists y)(P(y) \wedge x \neq y)]$.
★ **(d)** $\sim(\forall x)(\forall y)[(P(x) \wedge P(y)) \Rightarrow x = y]$.

★ **14.** *Riddle:* What is the English translation of the symbolic statement $\forall\exists\exists\forall$?

1.4 Basic Proof Methods I

A **theorem** in mathematics is a statement that describes a pattern or relationship among quantities or structures. A **proof** of a theorem is a justification of the truth of the theorem that follows the principles of logic.

We cannot define all terms or prove all statements from previous ones. We begin with an initial set of statements, called **axioms** (or **postulates**), that are *assumed to be true*. We then derive theorems that are true in any situation where the axioms are true. The Pythagorean* Theorem, for example, is a theorem whose proof is ultimately based on the five axioms of Euclidean† geometry. In a situation where the Euclidean axioms are not all true (which can happen), the Pythagorean Theorem may not be true.

There must also be an initial set of **undefined terms**—concepts fundamental to the context of study. In geometry, the concept of a point is an undefined term. In this text the real numbers are not formally defined. Instead, they are described in *Appendix* as the decimal numbers along the number line. While a precise definition of a real number could be given,‡ doing so would take us far from our intended goals.

From the axioms and undefined terms, new concepts (new **definitions**) can be introduced. And finally, new theorems can be proved. The structure of a proof for a particular theorem depends greatly on the logical form of the theorem. Proofs may require some ingenuity or insightfulness to put together the right statements to build the justification. Nevertheless, much can be gained in the beginning by studying the fundamental components found in proofs and examples that exhibit them.

The rules that follow provide guidance about what statements are allowed in a proof, and when. The first four rules enable us to replace a statement with an equivalent or to state something that is always true or is assumed to be true.

In any proof at any time you may:
State an axiom, an assumption, or a previously proved result.

The statement of an axiom is usually easily identified as such by the reader because it is a statement about a very fundamental fact assumed about the theory. Sometimes the axiom is so well known that its statement is omitted from proofs, but there are cases (such as the Axiom of Choice in Chapter 5) for which it is prudent to mention the axiom in every proof employing it.

The statement of an assumption generally takes the form "Assume P" to alert the reader that the statement is not derived from a previous step or steps. We must be careful about making assumptions, because it is only *when all the assumptions are true* that we can be certain that what we proved will be true. The most common assumptions are hypotheses given as components in the statement of the theorem to be proved. We will discuss assumptions in more detail later in this section.

Proof steps that use previously proven results help build a rich theory from the basic assumptions. In calculus, for example, before one proves that the derivative of $\sin x$ is $\cos x$, one usually proves that $\lim_{\Delta x \to 0} \dfrac{\sin \Delta x}{\Delta x} = 1$. It is easier to prove this

* Pythagoras, latter half of the 6th century B.C.E., was a Greek mathematician and philosopher who founded a secretive religious society based on mathematical and metaphysical thought. Although Pythagoras is regularly given credit for the theorem named for him, the result was known to Babylonian and Indian mathematicians centuries earlier.

† Euclid of Alexandria, circa 300 B.C.E., made his immortal contribution to mathematics with his famous text on geometry and number theory. His *Elements* sets forth a small number of axioms from which additional definitions and many familiar geometric results were developed in a rigorous way. Other geometries, based on different sets of axioms, did not begin to appear until the 1800s.

‡ See the references cited in Section 7.5.

result first, and then cite the result in the proof of the fact that the derivative of $\sin x$ is $\cos x$. A result that serves as a preliminary step is often called a **lemma**.

In any proof at any time you may use the *tautology rule*:
State a sentence whose symbolic translation is a tautology.

Example. Suppose a proof involves a real number x. You may at any time state "Either $x \geq 0$ or $x < 0$" because this statement is an instance of the tautology $\sim P \vee P$. □

An important skill for writing proofs is the ability to rewrite a statement in an equivalent form that is more useful or helps clarify its meaning.

In any proof at any time you may use the *replacement rule*:
State a sentence equivalent to any statement earlier in the proof.

For example, if a proof contains the statement "It is not the case that x is even and prime," we may deduce that "x is not even or x is not prime." This step is valid because the first statement has the form $\sim(P \wedge Q)$ and the second has the form $\sim P \vee \sim Q$. We have applied the replacement rule, using one of DeMorgan's Laws.

A thorough knowledge of the logical equivalences of Theorems 1.1.1 and 1.2.2 is essential when one uses the replacement rule because these replacements are done routinely, without mentioning the relevant rules of logic.

It is *impossible* to read or write proofs in advanced mathematics without using definitions. Because understanding and using definitions is so crucial, we restate the replacement rule specifically for definitions.

In any proof at any time you may:
Use a definition to state an equivalent to a statement earlier in the proof.

The precise definition of "divides" given in the *Appendix* makes it possible to build proofs involving divisibility properties of $\mathbb{N}$ that are among the first examples we do. An explanation of why a divides b that uses an inexact definition of divisibility, such as "a divides b because it goes in evenly," is practically useless in writing a proof. The key idea is that divisibility is defined as it is, so that in a proof we can replace the statement "a divides b" with the equivalent statement "$b = a \cdot k$ for some integer k." Conversely, if we want to deduce that a divides b, the definition tells us that we need a step that says there is an integer k such that $b = a \cdot k$.

The most fundamental rule of reasoning is **modus ponens**, which is based on the tautology $P \wedge [P \Rightarrow Q)] \Rightarrow Q$. In Section 1.2 we showed that this tautology means that whenever P and $P \Rightarrow Q$ are both true, we may deduce that Q is also true. This rule allows us to make a connection so that we can get from statement P to a *different* statement Q.

In any proof at any time you may use the *modus ponens rule*:
After statements P and $P \Rightarrow Q$ appear in a proof, state Q.

When we use modus ponens to deduce Q from statements P and $P \Rightarrow Q$, the statement P could be an assumption, a compound proposition whose components are hypotheses or previously proved results, or any other statement known to be true at this point. The conditional sentence $P \Rightarrow Q$ could also be a previous theorem or tautology or any other statement that appears earlier in the proof.

You have used these proof rules, at least informally, possibly to answer questions like the one in the next example, which comes from a calculus exam. Notice how the solution (1) states the assumption, (2) replaces the assumption using the definition of differentiability, (3) uses a known result, (4) applies modus ponens to two previous statements, and (5) uses the definition of continuity to deduce the last statement.

Example. Suppose f is a function defined on an interval containing 2, and we know that the derivative of f at 2 exists. Find $\lim_{x \to 0} f(x) = f(2)$. Explain your answer.

We are given that $f'(2)$ exists. Thus f is differentiable at $x = 2$. A theorem from calculus says that if f is differentiable at x, then f is continuous at x. Therefore, f is continuous at $x = 2$. We conclude that $\lim_{x \to 0} f(x) = f(2)$. □

The next example comes from outside mathematics and shows that it may be the *form* of propositions, and not the meaning, that enables us to make a deduction.

Example. You are at a crime scene and have established the following facts:

(1) If the crime did not take place in the billiard room, then Colonel Mustard is guilty.

(2) The lead pipe is not the weapon.

(3) Either Colonel Mustard is not guilty or the weapon used was a lead pipe.

From these facts and modus ponens, you may construct a proof that shows the crime took place in the billiard room:

Proof.

Statement (1)	$\sim B \Rightarrow M$
Statement (2)	$\sim L$
Statement (3)	$\sim M \vee L$
Statements (1) and (2) and (3)	$(\sim B \Rightarrow M) \wedge \sim L \wedge (\sim M \vee L)$
Statements (1), (2), and (3) imply the crime took place in the billiard room.	$[(\sim B \Rightarrow M) \wedge \sim L \wedge (\sim M \vee L)] \Rightarrow B$ is a tautology (see Exercise 2).
Therefore, the crime took place in the billiard room.	B

■

The last three statements above are an application of the modus ponens rule: We deduced Q from the statements P and $P \Rightarrow Q$, where Q is B and P is $(\sim B \Rightarrow M) \wedge \sim L \wedge (\sim M \vee L)$.

Because our proofs are always about mathematical phenomena, we also need to understand the subject matter of the proof—the concepts involved and how they are related. Therefore, when you develop a strategy to construct a proof, keep in mind both the logical form of the theorem's statement and the mathematical concepts involved.

You won't find truth tables displayed or referred to in proofs that you encounter in mathematics: It is expected that readers are familiar with the rules of logic and correct forms of proof. As a general rule, when you write a step in a proof, ask yourself whether deducing that step is valid in the sense that it uses one of the five rules above. If the step follows as a result of the use of a tautology, it is not necessary to cite the tautology in your proof. In fact, with practice you should eventually come to write proofs without purposefully thinking about tautologies. What *is* necessary is that every step be justifiable.

The first—and most important—proof method is the **direct proof** of statement of the form $P \Rightarrow Q$, which proceeds in a step-by-step fashion from the antecedent P to the consequent Q. Since $P \Rightarrow Q$ is false only when P is true and Q is false, it suffices to show that this situation cannot happen. The direct way to proceed is to assume that P is true and show (deduce) that Q is also true. A direct proof of $P \Rightarrow Q$ will have the following form:

DIRECT PROOF OF $P \Rightarrow Q$

Proof.
Assume P.
⋮
Therefore, Q.
Thus, $P \Rightarrow Q$. ■

Some of the examples and exercises in this and the next section involve open statements with variables. The proof techniques to handle open sentences and quantified statements are discussed in detail in Section 1.6. For now, whenever we encounter a sentence with a variable, imagine that the variable represents some fixed object.

You don't need to see a proof to be convinced that the next number after an odd number is an even number, but we'll examine proofs of this and several other obvious results in our first examples. These examples are chosen so that you don't have to deal with new concepts at the same time as you are learning how to write proofs. The important thing to learn from these examples is that *a direct proof proceeds step by step from the antecedent to the consequent*.

Example. Let x be an integer. Prove that if x is odd, then $x + 1$ is even.

The theorem has the form $P \Rightarrow Q$, where P is "x is odd" and Q is "$x + 1$ is even."

Proof.

Let x be an integer.

Given. We may assume this hypothesis because it is given in the statement of the theorem.

Suppose x is odd.	*Assume the antecedent P is true. The goal is to derive the consequent Q.*
Then $x = 2k + 1$ for some integer k.	*We have replaced P with an equivalent statement—the definition of "odd." We now have a statement we can work with.*
Then $x + 1 = (2k + 1) + 1$.	*We add 1 to each side of the equation to get an equivalent statement.*
Then $x + 1 = 2k + 2 = 2(k + 1)$, so $x + 1$ is the product of 2 and the integer $k + 1$.	*We use algebra and the fundamental fact that if k is an integer, then k + 1 is an integer.*
Thus $x + 1$ is even.	*We have deduced Q.*
Therefore, if x is an odd integer, then $x + 1$ is even. ■	*We conclude that $P \Rightarrow Q$.*

The right-hand column was included to describe how the steps are connected. Proofs are not usually written this way because, in practice, such a column is unnecessary. Writing a proof in two-column form can be a good way to begin to understand its structure, but the sequence of statements on the left *is* the complete proof and should be written in shorter form, as follows:

Proof. Let x be an integer. Suppose x is odd. Then $x = 2k + 1$ for some integer k. Then $x + 1 = (2k + 1) + 1$. Because $(2k + 1) + 1 = 2k + 2 = 2(k + 1)$, we see that $x + 1$ is the product of 2 and the integer $k + 1$. Thus $x + 1$ is even. Therefore, if x is an odd integer, then $x + 1$ is even. ■

This form of the proof assumes that the person reading it knows the relevant definitions and the basics of logic. Good proofs include enough detail so that readers with the appropriate background can follow the logical steps and fill in computations as necessary. This can be challenging for someone first learning to read and write proofs, but these skills come with practice. After you've written more proofs using a variety of methods, you'll find useful advice about mathematical writing following Section 1.6.

Writing proofs will come much easier to you when you understand the vital importance of definitions, beginning with this very first example, where the fact that an odd number x is such that "$x = 2k + 1$ for some integer k" gave us something specific to work with: an equation that we could manipulate. That's the first way a definition was used in the example. The second use of a definition came when we concluded that $x + 1$ is even: we determined that $x + 1$ was even because it satisfied the condition given in the definition of *even*.

You'll find that using the definitions often provides a hint about how to begin a proof. Notice how the next theorem uses the definition of *divides*. The proof is again given in two-column style. Try covering the right-hand column the first time you read the proof.

Theorem 1.4.1 Let a, b, and c be integers. If a divides b and b divides c, then a divides c.

The theorem has the form $P \wedge Q \Rightarrow R$, where P is "a divides b," Q is "b divides c," and R is "a divides c."

Proof.

Let a, b, and c be integers.	*Given. That is, we assume this hypothesis is true.*
Suppose a divides b and b divides c.	*Assume the antecedent $P \wedge Q$ is true. The antecedent is a conjunction, so both components are true.*
Then $b = ak$ for some integer k, and $c = bm$ for some integer m.	*Replace each assumption by an equivalent using the definition of "divides." Note that we do not assume that k and m are the same integer.*
Then $c = bm = (ak)m = a(km)$.	*To show "a divides c," we must show c is a multiple of a. We use algebra to obtain an expression for c in terms of a.*
Since k and m are integers, km is an integer.	*We use a fundamental property of the integers.*
Thus a divides c.	*We have deduced R.*
Therefore, if a divides b and b divides c, then a divides c. ■	*We state the conclusion that $P \wedge Q \Rightarrow R$.*

No other proofs in this text will be presented in this two-column format. However, we will sometimes include stylized parenthetical comments offset by $\langle \quad \rangle$ to help explain how and why a proof is proceeding as it is. These comments would not be included in a proof written for experienced readers.

Example. Suppose a, b, and c are integers. Prove that if a divides b and a divides c, then a divides $b - c$.

Proof. Suppose a, b, and c are integers and a divides b and a divides c. ⟨*Now use the definition of divides.*⟩ Then $b = an$ for some integer n and $c = am$ for some integer m. Thus, $b - c = an - am = a(n - m)$. ⟨*We next use the fact that the difference of two integers is an integer.*⟩ Since $n - m$ is an integer, we have that a divides $b - c$. ■

Our next example of a direct proof comes from an exercise in precalculus mathematics about distances between points in the Cartesian plane and uses algebraic properties available to students in such a class.

Example. Prove that if x and y are real numbers such that $x < -4$ and $y > 2$, then the distance from (x, y) to $(1, -2)$ is greater than 6.

Proof. Assume that x and y are real numbers such that $x < -4$ and $y > 2$. ⟨*Then* (x, y) *is a point in the shaded area in Figure 1.4.1.*⟩ Then $x - 1 < -5$, so $(x - 1)^2 > 25$. Also $y + 2 > 4$, so $(y + 2)^2 > 16$. Therefore,

$$\sqrt{(x-1)^2 + (y+2)^2} > \sqrt{25 + 16} > \sqrt{36},$$

so the distance from (x, y) to $(1, -2)$ is greater than 6. ■

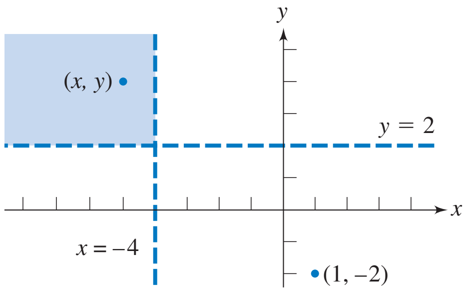


Figure 1.4.1

The proofs above exhibit the strategy for developing a direct proof of a conditional sentence:

1. Determine precisely the hypotheses (if any) and the antecedent and consequent.
2. Replace (if necessary) the antecedent with a more usable equivalent.
3. Replace (if necessary) the consequent by something equivalent and more readily shown.
4. Beginning with the assumption of the antecedent, develop a chain of statements that leads to the consequent. Each statement in the chain must be deducible from its predecessors or other known results.

To discover a chain of statements from the antecedent to the consequent, it is sometimes useful to work backward from what is to be proved: To show that a consequent is true, decide what statement could be used to prove it, another statement that could be used to prove that one, and so forth. Continue until you reach a hypothesis, the antecedent, or a fact known to be true. After doing such preliminary work, construct a proof that progresses forward until it ends with the consequent.

Example. Let a and b be positive real numbers. Prove that if $a < b$, then $b^2 - a^2 > 0$.

Before we write the proof, we work backward from the consequent. First rewrite $b^2 - a^2 > 0$ as $(b - a)(b + a) > 0$. This inequality will be true when both factors are positive. The term $b - a$ is positive because we assumed that $b > a$. Also, $b + a > 0$ because of our hypothesis that both a and b are positive. We now know how to write the proof.

Proof. Assume that a and b are positive real numbers and that $a < b$. Since both a and b are positive, $b + a > 0$. Because $a < b$, we see that $b - a > 0$. Because the product of two positive real numbers is positive, $(b - a)(b + a) > 0$. Therefore, $b^2 - a^2 > 0$. ■

Sometimes, working both ways—backward from what is to be proved and forward from the hypothesis—until you reach a common statement from each direction will help reveal the structure of a proof.

Example. Prove that if $x^2 \leq 1$, then $x^2 - 7x > -10$.

Working backward from $x^2 - 7x > -10$, we note that this can be deduced from $x^2 - 7x + 10 > 0$. This can be deduced from $(x - 5)(x - 2) > 0$, which could be concluded if we knew that $x - 5$ and $x - 2$ were both positive or both negative.

Working forward from $x^2 \leq 1$, we have $-1 \leq x \leq 1$, so $x \leq 1$. Therefore, $x < 5$ and $x < 2$, from which we can conclude that $x - 5 < 0$ and $x - 2 < 0$, which is exactly what we need.

Proof. Assume that $x^2 \leq 1$. Then $-1 \leq x \leq 1$. Therefore, $x \leq 1$. Thus $x < 5$ and $x < 2$, and so we have $x - 5 < 0$ and $x - 2 < 0$. Therefore, $(x - 5)(x - 2) > 0$. Thus $x^2 - 7x + 10 > 0$. Hence $x^2 - 7x > -10$. ■

When either P or Q is a compound proposition, the steps in proving statements of the form $P \Rightarrow Q$ depend on the forms of P and Q. We have already constructed proofs of statements of the form $(P \wedge Q) \Rightarrow R$. When we give a direct proof of a statement of this form, we have the advantage of assuming both P and Q at the beginning of the proof.

A proof of a statement symbolized by $P \Rightarrow (Q \wedge R)$ would probably have two parts. In one part we prove $P \Rightarrow Q$ and in the other part we prove $P \Rightarrow R$. We would use this method to prove the statement "If a circle of radius r is inscribed in a square that is inscribed in a circle, then the center of the outer circle is the center of the inner circle, and the radius of the outer circle is $r\sqrt{2}$."

To prove a conditional sentence whose consequent is a disjunction—that is, a sentence of the form $P \Rightarrow (Q \vee R)$—one often proves either the equivalent $P \wedge \sim Q \Rightarrow R$ or the equivalent $P \wedge \sim R \Rightarrow Q$. For instance, to prove "If the polynomial f has degree 4, then f has a real zero or f can be written as the product of two irreducible quadratics," we would prove "If f has degree 4 and no real zeros, then f can be written as the product of two irreducible quadratics."

A statement of the form $(P \vee Q) \Rightarrow R$ has the meaning "If either P is true or Q is true, then R is true." A natural way to prove such a statement is by cases, so the proof outline would have the form

Case 1. Assume P. . . . Therefore R.
Case 2. Assume Q. . . . Therefore R.

This method is valid because of the tautology

$$[(P \vee Q) \Rightarrow R] \Leftrightarrow [(P \Rightarrow R) \wedge (Q \Rightarrow R)].$$

The statement "If a quadrilateral has opposite sides equal or opposite angles equal, then it is a parallelogram" is proved by showing both "A quadrilateral with opposite sides equal is a parallelogram" and "A quadrilateral with opposite angles equal is a parallelogram."

The two similar statement forms $(P \Rightarrow Q) \Rightarrow R$ and $P \Rightarrow (Q \Rightarrow R)$ have remarkably dissimilar direct proof outlines. For $(P \Rightarrow Q) \Rightarrow R$, we assume $P \Rightarrow Q$ and deduce R. We cannot assume P; we must assume $P \Rightarrow Q$. On the other hand, in a direct proof of $P \Rightarrow (Q \Rightarrow R)$, we do assume P and show $Q \Rightarrow R$. Furthermore, after the assumption of P, a direct proof of $Q \Rightarrow R$ begins by assuming Q is true as well. This is not surprising, because $P \Rightarrow (Q \Rightarrow R)$ is equivalent to $(P \wedge Q) \Rightarrow R$.

The main lesson to be learned from this discussion is that the method of proof you choose will depend on the form of the statement to be proved. The outlines we have given are the most natural, but not the only, ways to construct correct proofs.

Before we begin the proof of the next example, consider integers of the form $n = 2m + 1$ for some integer m. A little experimentation shows that when m is even—for example, when n is $2(-2) + 1$, or $2(0) + 1$, or $2(2) + 1$, $2(4) + 1$, etc.—then n has the form $4j + 1$, and otherwise n has the form $4i - 1$. The statement below has the form $(P \vee Q) \Rightarrow (R_1 \vee R_2)$, where P is "m is even," Q is "m is odd," R_1 is "$n = 4j + 1$ for some integer j," and R_2 is "$n = 4i - 1$ for some integer i." The proof method we choose is to show that $P \Rightarrow R_1$ and $Q \Rightarrow R_2$.

Example. Suppose n is an odd integer. Then $n = 4j + 1$ for some integer j, or $n = 4i - 1$ for some integer i.

Proof. Suppose n is odd. Then $n = 2m + 1$ for some integer m.

Case 1. If m is even, then $m = 2j$ for some integer j, and so $n = 2(2j) + 1 = 4j + 1$.

Case 2. If m is odd, then $m = 2k + 1$ for some integer k. In this case, $n = 2(2k + 1) + 1 = 4k + 3 = 4(k + 1) - 1$. Choosing i to be the integer $k + 1$, we have $n = 4i - 1$. ■

A **proof by exhaustion** consists of an examination of every possible case. The statement to be proved may have any logical form. For example, to prove that every number x in the closed interval $[0, 5]$ has a certain property, we might consider the cases $x = 0$, $0 < x < 5$, and $x = 5$. The exhaustive method was our method in the previous example, and in the suggested proof of Theorem 1.1.1, which requires examination of all combinations of truth values for each pair of propositions. Naturally, the idea of proof by exhaustion is appealing only when the number of cases is small or when large numbers of cases can be systematically handled. Care must be taken to ensure that all possible cases have been considered.

Example. Let x be a real number. Prove that $-|x| \le x \le |x|$.

Proof. ⟨*Because the absolute value of* x *is defined by cases* ($|x| = x$ *if* $x \geq 0$; $|x| = -x$ *if* $x < 0$), *this proof will proceed by cases.*⟩

Case 1. Suppose $x \geq 0$. Then $|x| = x$. Since $x \geq 0$, we have $-x \leq x$. Hence, $-x \leq x \leq x$, which is $-|x| \leq x \leq |x|$ in this case.

Case 2. Suppose $x < 0$. Then $|x| = -x$. Since $x < 0$, $x \leq -x$. Hence, we have $x \leq x \leq -x$, or $-(-x) \leq x \leq -x$, which is $-|x| \leq x \leq |x|$.

Thus, in all cases we have $-|x| \leq x \leq |x|$. ■

There have been instances of truly exhausting proofs involving great numbers of cases. In 1976, Kenneth Appel and Wolfgang Haken of the University of Illinois announced a proof of the famous Four-Color Conjecture. The original version of their proof contains 1,879 cases and took $3\frac{1}{2}$ years to develop.*

Finally, there are proofs by exhaustion with cases so similar in reasoning that we may simply present a single case and alert the reader with the phrase "without loss of generality" that this case represents the essence of arguments for the other cases. Here is an example:

Example. Prove that for the integers m and n, one of which is even and the other odd, $m^2 + n^2$ has the form $4k + 1$ for some integer k.

Proof. Let m and n be integers. Without loss of generality, we may assume that m is even and n is odd. ⟨*The case where m is odd and n is even is similar*.⟩ Then there exist integers s and t such that $m = 2s$ and $n = 2t + 1$. Therefore, $m^2 + n^2 = (2s)^2 + (2t + 1)^2 = 4s^2 + 4t^2 + 4t + 1 = 4(s^2 + t^2 + t) + 1$. Since $s^2 + t^2 + t$ is an integer, $m^2 + n^2$ has the form $4k + 1$ for some integer k. ■

Exercises 1.4

1. Analyze the logical form of each of the following statements and construct just the outline of a proof. Since the statements may contain terms with which you are not familiar, you should not (and perhaps could not) provide any details of the proof.

★ **(a)** Outline a direct proof that if $(G, *)$ is a cyclic group, then $(G, *)$ is abelian.

* The Four-Color Theorem involves coloring regions or countries on a map in such a way that no two adjacent countries have the same color. It states that four colors are sufficient, no matter how intertwined the countries may be. The fact that the proof depended so heavily on the computer for checking cases raised questions about the nature of proof. Verifying the 1,879 cases required more than 10 billion calculations. Many people wondered whether there might have been at least one error in a process so lengthy that it could not be carried out by one human being in a lifetime. Haken and Appel's proof has since been improved, and the Four-Color Theorem is accepted; but the debate about the role of computers in proof continues.