MATHEMATICS

for Elementary School Teachers 6e

BASSAREAR • MOSS

Four Steps for Solving Problems

Understanding the Problem

Questions that can be useful to ask:

1. Do you understand what the problem is asking for?
2. Can you state the problem in your own words, that is, paraphrase the problem?
3. Have you used all the given information?
4. Can you solve a part of the problem?

Actions that can be helpful:

1. Reread the problem carefully. (Often it helps to reread a problem a few times.)
2. Try to use the given information to deduce more information.
3. Plug in some numbers to make the problem more concrete, more real.

Devising a Plan

Several common strategies:

1. Represent the problem with a diagram (carefully drawn and labeled). Check to see if you used (the relevant) given information. Does the diagram "fit" the problem?
2. Guess–check–revise (vs. "grope and hope"). Keep track of "guesses" with a table.
3. Make an estimate. The estimate often serves as a useful "check." A solution plan often comes from the estimation process.
4. Make a table (sometimes the key comes from adding a new column).
5. Look for patterns—in the problem or in your guesses.
6. Be systematic.
7. Look to see if the problem is similar to one already solved.
8. If the problem has "ugly" numbers, you may "see" the problem better by substituting "nice" numbers and then thinking about the problem.
9. Break the problem down into a sequence of simpler "bite-size" problems.
10. Act it out.

Carrying Out the Plan

1. Are you keeping the problem meaningful or are you just "groping and hoping?" On each step ask what the numbers mean. Label your work.
2. Are you bogged down? Do you need to try another strategy?

Looking Back

1. Does your answer make sense? Is the answer reasonable? Is the answer close to your estimate, if you made one?
2. Does your answer work when you check it with the given information? (Note that checking the procedure checks the computation but not the solution.)
3. Can you use a different method to solve the problem?

This sheet is meant to serve as a starting point. The number of strategies that help the problem-solving process are almost endless and vary according to each person's strengths and preferences.

After you solve a problem that was challenging for you or after you find that your answer was wrong, stop and reflect. Can you describe what you did that got you unstuck or things you did that helped you to solve the problem? If your answer was wrong, can you see what you might have done? *It is the depth of these reflections that connects to your increased ability to solve problems.*

Mathematics for Elementary School Teachers

6e

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***Mathematics for Elementary School Teachers,* Sixth Edition**

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ABOUT THE AUTHORS

Tom Bassarear

I have been teaching the Mathematics for Elementary School Teachers course for more than 20 years, and in that time, I have learned as much from my students as they have learned from me. This text was inspired by my students and reflects one of the most important things we have taught one another: that building an understanding of mathematics is an active, exploratory process, and ultimately a rewarding, pleasurable one. My own experience with elementary schoolchildren and my two children, Emily and Josh, has convinced me that young children naturally seek to make sense of the world they live in and for a variety of reasons many people slowly lose that curiosity over time. My hope is that this book will engage your curiosity about mathematics once again.

Meg Moss

I am excited and honored to be working with Tom Bassarear on this book. I began teaching Mathematics for Elementary Teachers over 20 years ago. I immediately began seeking advice from others who had taught the course, and volunteered in elementary classrooms to learn more. Teaching these courses has deepened my mathematical understanding as well as my understanding of how people learn math. Helping future elementary school teachers to truly understand mathematics, and see the beauty in mathematics is very rewarding, and I know that all of this will have a major positive impact on their future students. I appreciate you sharing this journey with me!

NEW TO THE SIXTH EDITION!

I am pleased to welcome Meg Moss, Ph.D., to this textbook and to introduce her to you. I have known Meg for about 10 years through conversations and presentations at conferences. I have admired the quality of her work, depth of thought, and commitment to students, so I was delighted when she agreed to join me in continuing this book as I transition toward retirement. While I have been actively involved in the current revision, you will find Meg's footprints throughout the book.

She has done a magnificent job of framing the Common Core State Standards in mathematics (CCSM), which have been adopted by 45 states, in a way that helps readers to see where their future students will learn these concepts and to help them see the importance of such concepts. The CCSM articulates eight mathematical practices (MPs) that replace standards by the National Council of Teachers of Mathematics (NCTM). While I believe that the NCTM standards are more clearly stated and more user-friendly, the eight MPs correlate strongly with NCTM's framework. Meg's genius was not to try to incorporate all the details of the MPs, which would be overwhelming, but to focus on and articulate the big ideas embedded in those practices. She articulates them in the first chapter, connecting them directly to Investigations, and then refers to them in appropriate ways throughout the book.

After many discussions, she constructed a revised and streamlined Chapter 1, which I love. She integrated the number theory concepts—which previously comprised a separate chapter—into the textbook, as those concepts are needed. This connects to research on learning that indicates students are more willing and able to retain ideas if they see how they are connected and if they use them immediately.

With CCSM's emphasis on algebraic thinking, we decided to have a separate Chapter 6 on algebra. Meg did a heroic job of researching best practices in algebra in schools and then organized the material into a coherent framework that addresses the important algebraic ideas articulated by CCSM. She took many Investigations from the fifth edition's algebra section and some from Chapter 1 and has added many of her own.

Meg also wrote Questions to Summarize Big Ideas for the end-of-chapter summaries. These questions help students reflect on what they have learned and articulate major "take-away" ideas from the chapter, ultimately supporting one of the most important ideas of the textbook—this is to OWN knowledge.

In addition, Meg went through every page of the textbook and you will see her work in many places, such as in

- revising text to make points more clear and concise;
- adding extra steps and more concreteness when she felt it would be helpful, especially for students who tend to struggle with those ideas;
- more visual representations including Singaporean bar models; and
- more technology, including references to virtual manipulatives, Geogebra investigations, and several other websites.

I hope you will welcome and appreciate Meg's contributions to the sixth edition as much as I do.

—Tom Bassarear

ANNOTATED CONTENTS

Chapter 1 Foundations for Learning Mathematics

This chapter continues the theme from the fifth edition, but with a new emphasis on the Mathematical Practices of the Common Core State Standards (CCSS). While references to the NCTM standards remain, one of the goals of the revised Chapter 1 is to lay the groundwork for the CCSS so that students can see some of those standards “in action” while they are learning the mathematics throughout the textbook. The explorations in the *Explorations* manual offer diverse types of problems to grapple with that support the strategies used in the rest of the course.

Chapter 2 Fundamental Concepts

Chapter 2 has been shortened, with former Section 2.2 now included in a newly developed Chapter 6 that is devoted to algebraic thinking. Sections 2.1 and 2.3 from the fifth edition remain, with revisions in these sections focused on enhancing discussions of sets and numeration.

Section 2.1 gives students tools that enable them to talk about sets and subsets and to use Venn diagrams when the need arises in other chapters, such as to understand the relationship between different sets of numbers.

Section 2.2 includes the development of children’s understanding of numeration and its historical development, both of which students find fascinating. Exploration 2.3 (Alphabitia) is one of the most powerful we have used. Most of our students report this to be the most significant learning and/or turning point in the semester. The exploration unlocks powerful understandings related to numeration, which the text supports by discussing the evolution of numeration systems over time and exploring different bases.

Chapter 3 The Four Fundamental Operations of Arithmetic

Portions of the fifth edition’s Chapter 4, Number Theory, are now integrated into Chapter 3, as appropriate. For example, divisibility may now be found in Section 3.4, Understanding Division. Several discussions are now rewritten with more emphasis on place value and visual representations of numbers. The goals of Chapter 3 otherwise are the same. Students see how the concepts of the operations, coupled with an understanding of base ten, enable them to understand how and why procedures that they have performed by memorization for years actually work. In addition to making sense of standard algorithms, we present alternative algorithms in both the text and explorations. Our students have found these algorithms to be both enlightening and fascinating.

Chapter 4 Extending the Number System

The sets of integers, fractions, and decimals represent three historically significant extensions to the set of whole numbers. To enhance the discussion of fractions, Singaporean bar models are used. The concepts of least common multiple and greatest common divisor are integrated into the fraction section when needed for simplifying and for common denominators.

In Exploration 4.5 (Making Manipulatives), students construct fraction manipulatives and then look for rules when ordering fractions, a critically important first step in seeing fractions as more than numerator and denominator. In Exploration 4.19 (Meanings of Operations with Fractions), having students represent problem situations with diagrams requires them to adapt their understanding of the four operations to fraction situations. Having first constructed this concept through exploration, students can approach Investigation 4.2k (Ordering Rational Numbers) with a richer understanding of what it really means to say that one fraction is greater than another.

While Chapters 3 and 4 have been arranged according to the manner in which many instructors prefer this content to appear, it is not fixed. For those instructors who prefer a more “operations-centric” approach to the course, we offer an alternative organization of topics as follows:

Chapter 1 Foundations for Learning Mathematics

Chapter 2 Fundamental Concepts

4.1 Integers

4.2 Fractions and Rational Numbers

3.1 Understanding Addition

3.2 Understanding Subtraction

4.3 Understanding Operations with Fractions (first half addition and subtraction of fractions)

3.3 Understanding Multiplication

3.4 Understanding Division

4.3 Understanding Operations with Fractions (second half, multiplication and division of fractions)

4.4 Beyond Integers and Fractions

Chapter 5 Proportional Reasoning

Chapter 6 Algebraic Thinking, and so on

Chapter 5 Proportional Reasoning

The investigations and explorations in Chapter 5 are conceptually rich and provide many real-life examples so that students can enjoy developing an understanding of multiplicative relationships.

Chapter 6 Algebraic Thinking

In response to requests from reviewers, we have included a new chapter devoted to algebraic thinking. Chapter 6 explores patterns, the concept of a variable, and solving equations and inequalities using different models, including Singaporean bar models. The four sections are arranged under the National Council of Teachers of Mathematics (NCTM) algebra structure of understanding patterns, relations, and functions; representing and analyzing math situations and structures using algebraic symbols; using mathematical models to represent and understand quantitative relationships; and analyzing change in various contexts.

Chapter 7 Uncertainty: Data and Chance

In this chapter, students carefully walk through the stages of defining a question, collecting data, interpreting data, and then presenting data. We are particularly excited that the investigations with the concepts of mean and standard deviation remain successful with students. As a result, students can express these ideas conceptually instead of simply reporting the procedure.

Chapter 8 Geometry as Shape

In Chapter 8, you have the option of introducing geometry through explorations with tangrams, Geoboards, or pentominoes. This more concrete introduction allows students with unpleasant or failing memories of geometry to build confidence and understanding while engaging in rich mathematical explorations.

Chapter 9 Geometry as Measurement

This chapter addresses measurement from a conceptual framework (i.e., identify the attribute, determine a unit, and determine the amount in terms of a unit) and a historical perspective. Both the explorations and investigations get students to make sense of measurement procedures and to grapple with fundamental measurement ideas. Exploration 9.2 (How Tall?) generates many different solution paths and ideas and many discussions about indirect measurement and precision. Exploration 9.5 (What Does π Mean?) has demystified π in the minds of many students and is a wonderful exercise in communication. Exploration 9.11 (Irregular Areas) requires students to apply notions of measuring area to a novel situation. Students will hypothesize many different strategies, some of which are valid and some of which are not. The text looks at the larger notion of measurement, presents the major formulas in a helpful way, and illustrates different problem-solving paths.

Some of the most significant revisions to this chapter have been made to increase conceptual understanding of the concepts of measurement such as perimeter, area, and volume.

Chapter 10 Geometry as Transforming Shapes

The geometric transformations that we explore in Chapter 10 can be some of the most interesting and exciting topics of the course. Quilts and tessellations both spark lots of interest and provoke good mathematical thinking. The text develops concepts and introduces terms that help students to refine understanding that emerges from explorations.

PREFACE

Owning versus Renting

This course is about developing and *retaining* the mathematical knowledge that students will need as beginning mathematics teachers. We prefer to say that we are going to *uncover* the material rather than *cover* the material. The analogy to archaeology is useful. When archaeologists explore a site, they carefully *uncover* the site. As time goes on, they see more and more of the underlying structure. This is exactly what can and should happen in a mathematics course. When this happens, students are more likely to *own* rather than to *rent* the knowledge.

There are three ways in which this textbook supports owning versus renting:

1. Knowledge is constructed.
2. Connections are reinforced.
3. Problems appear in authentic contexts.

1. Constructing Knowledge

When students are given problems, such as appear in Investigations throughout the textbook, that involve them in grappling with important mathematical ideas, they learn those ideas more deeply than if they are simply presented with the concepts via lecture and then are given problems for practice. Additionally, there is a need to shift the focus from students studying mathematics to students doing mathematics. That is, students are looking for patterns, making and testing predictions, making their own representations of a problem, inventing their own language and notation, etc.

Investigation 1.2d (Pigs and Chickens) confronts a common misconception—that there is one right way to solve math problems—by exploring five valid solution paths to the problem. This notion of multiple solution paths is an important part of the book.

INVESTIGATION **1.2d** Pigs and Chickens

A farmer has a daughter who needs more practice in mathematics. One morning, the farmer looks out in the barnyard and sees a number of pigs and chickens. The farmer says to her daughter, “I count 24 heads and 80 feet. How many pigs and how many chickens are out there?”

Before reading ahead, work on the problem yourself or, better yet, with someone else. Close the book or cover the solution paths while you work on the problem.

Compare your answer to the solution paths below.

DISCUSSION

STRATEGY 1 Use random trial and error

One way to solve the problem might look like what you see in Figure 1.3.

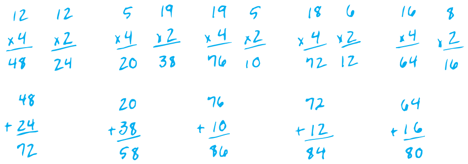


Figure 1.3

2. Reinforcing Connections

Understanding can be defined in terms of connections; that is, the extent to which you *understand* a new idea can be seen by the *quality* and *quantity* of connections between that idea and what you already know. There are two ways in which connections are built into the structure of the text.

1. Mathematical connections

Owning mathematical knowledge involves connecting new ideas to ideas previously learned. It also involves truly understanding mathematics, not just memorizing formulas and definitions.

- **CONNECTIONS AMONG CONCEPTS ARE EMPHASIZED**

Investigation 1.1d helps students see how the algebraic formula is closely connected to guess–check–revise. Investigation 1.2i is later connected both to fractions and to remainder. In Chapter 3, the four operations are constantly connected to each other in their development. Then in Chapter 4, the connections between operations with fractions and operations with whole numbers are discussed, as are how decimals are connected to whole numbers and to fractions. In Chapter 5, we look back at some problems done in Chapter 4 and see how they can now be solved more efficiently with the concepts of ratio. In Chapter 10, students see how our work with numbers and shapes is similar.

- **THE *HOW* IS CONNECTED TO *WHY***

In this way, students know not only how the procedure works but also why it works. For example, students understand why we move over when we multiply the second row in whole number multiplication; they realize that “carrying” and “borrowing” essentially equate to trading tens for ones or ones for tens; they understand why we first find a common denominator when adding fractions; and they see that π is how many times you can wrap any diameter around the circle.

2. Connections to children's thinking

In this book you will see a strong focus on children's thinking, for two reasons. First, much work with teachers focuses on the importance of listening to the students' thinking as an essential part of good teaching. If students experience this in a math course, then by the time they start teaching, it is part of how they view teaching. Second, when students see examples of children's thinking and see connections between problems in this course and problems children solve, both the quality and quantity of the students' cognitive effort increase.

3. Authentic Problems

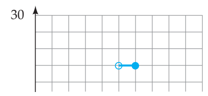
Although most texts have many "real-life" problems, this text differs in how those problems are made and presented.

In Section 6.3, the question of paying a baby-sitter is explored. This situation is often portrayed as a linear function: for example, if the rate is $10 per hour, $y = 10x$. However, in actuality, it is not a linear function but rather a stepwise function.

Figure 6.3

MATHEMATICS

The graph that represents this baby-sitting function is often confusing to students who see it for the first time. However, it is relatively common in real-world mathematics and is a member of the subset of functions called step functions.



A closer look at paying the baby-sitter At first glance, the question of how much to pay the baby-sitter is simple: Multiply the hours sat by 8. However, let us use the problem-solving strategy "act it out" to examine this problem more closely. For example, what if Ellen baby-sat from 7 to 11:15? How much would you pay her? Think before reading on. . . .

Some people say $32. Some people say $36—they round up to the nearest half-hour. In actuality, different people have different ways of determining how much to pay a baby-sitter. Let us examine the case of a couple, who rounds up the time to the nearest half-hour. We could now represent their process for paying the baby-sitter in each of the ways we have just examined. Is the relationship between time sat and dollars earned a functional relationship in the case of the couple? Think and then read on. . . .

If you were to graph this relationship, what would the graph look like? Try to make your own graph before reading on. . . .

Similarly, when determining the cost of carpeting a room, the solution path is often presented as dividing the area of the room by the cost per square yard; again, this is not how the cost is actually determined.

In this book, you will find many problems—problems we have needed to solve, problems friends have had, problems children have had, problems we have read about—where the content fits with the content of this course.

You will find problems in the text where students are asked to state the assumptions they make in order to solve the problem (e.g., Section 5.2, Exercises 33 to 38). You also will find problems that have the messiness of "real-life" problems, where the problem statement is ambiguous, too little or too much information is given, or the information provided is contradictory.

Problems 33–38 require you to make some assumptions in order to determine an answer. Describe and justify the assumptions you make in determining your answer.

33. Let's say that you read in the newspaper that last year's rate of inflation was 7.2%.

- **a.** If your grocery bill averaged $325 per month last year, about how much would you expect your grocery bill to be this year?
- **b.** Let's say you received a $1200 raise, from $23,400 per year to $24,600 per year. Did your raise keep you ahead of the game, or are you falling behind?

34. There was a proposal in New Hampshire in 1991 to reduce the definition of "drunk driving" from an alcohol blood content of 0.1 to 0.08. Explain why some might consider this a little drop and others might consider it a big drop. What do you think?

35. Which would you prefer to see on a sale sign at a store: $10 off or 10% off? Explain your choice.

36. ***Classroom Connection*** Refer to Investigation 5.2b. Jane still doesn't understand the problem. Roberto tries to help her make sense of the problem by saying that the 8% means that if we were to select 100 students at the college, 8 of them would be working full-time. What do you think?

37. Annie has just received a 5% raise from her current wage of $9.80 per hour.

- **a.** What is her new wage?
- **b.** What would this amount to over a year?
- **c.** What assumptions did you make in order to answer part (b)?
- **d.** What if the raise had been 5.4%?

Features

What do you think?

What-do-you-think questions appear at the start of each section to help students focus on key ideas or concepts that appear within the sections.

6.1 Understanding Patterns, Relations, and Functions

What do you think?

- How are patterns related to algebraic thinking?
- What are some examples of functions in everyday life?
- What is a reason for developing algebraic thinking in elementary school?

Investigations

Investigations are the primary means of instruction, uniquely designed to promote active thinking, reasoning, and construction of knowledge. Each investigation presents a problem statement or scenario that students work through, often to uncover a mathematical principle relevant to the content of the section. The "Discussion" that follows the problem statement provides a framework for insightful solution logic.

INVESTIGATION 3.1f Children's Mistakes

The problem below illustrates a common mistake made by many children as they learn to add. Understanding how a child might make that mistake and then going back to look at what lack of knowledge of place value, of the operation, or of properties of that operation contributed to this mistake is useful. What error on the part of the child might have resulted in this wrong answer?

The problem: 38 + 4 = 78

DISCUSSION

In this case, it is likely that the child lined up the numbers incorrectly:

$$\begin{array}{r} 4 \\ +\ 38 \\ \hline 78 \end{array}$$

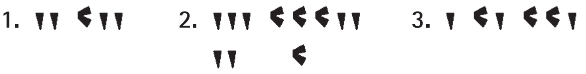
Giving other problems where the addends do not all have the same number of places will almost surely result in the wrong answer. For example, given 45 + 3, this child would likely get the answer 75. Given 234 + 42, the child would likely get 654. In this case, the child has not "owned" the notion of place value. Probably, part of the difficulty is not knowing expanded form (for example, that 38 means 30 + 8—that is, 3 tens and 8 ones). An important concept here is that we need to add ones to ones, tens to tens, etc. Base ten blocks provide an excellent visual for this concept as students can literally see why they cannot add 4 ones to 3 tens.

CLASSROOM CONNECTION

A friend of mine, David Sobel, was talking about mathematics with his six-year-old daughter, Tara. David had just shown Tara that 20 + 20 = 40. Tara thought for a moment and then proudly announced that 50 + 50 must be 70. When David asked how she had got that answer, she said, "When you add the same numbers that have a zero at the end, you just skip ten!"

Questions in the Text

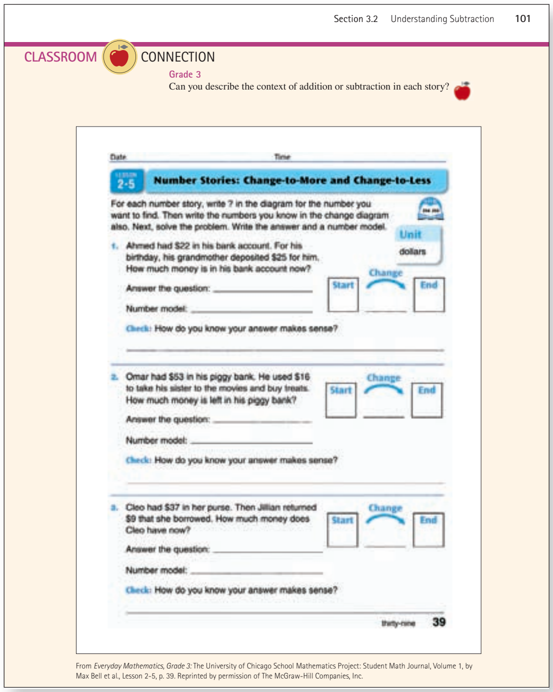
To encourage active learning outside of the Investigations, questions appear embedded within the text, often accompanied by the icon. These "thinking" questions require students to pause in their reading to reflect or to complete a short exercise before continuing. Answers to these questions can be found in Appendix B in the back of the textbook.

> Translate the following Babylonian numerals into our system. Check your answers in Appendix B.
>
> 
>
> Translate the following amounts into Babylonian numerals.
>
> **4.** 1202 **5.** 304

Classroom Connections

Connections to the Classroom, denoted with the icon, are found throughout the textbook. The boxed Connections that appear in the margins provide observations, tips, and notes about the elementary/middle-school classroom. Assignments from actual elementary/middle-school books appear throughout as well so that students can see how the material they are learning will directly apply. Connections are also found in the exercises that highlight children's work.

> **CLASSROOM CONNECTION**
>
> This question asking how two things are alike and how they are different is an important teaching structure and one that we will revisit over the course of the book. You may remember it in a common *Sesame Street* feature: Three of These Things Belong Together. For example, they might show a triangle, a square, a hexagon, and a circle. The answer is that the circle doesn't belong because it doesn't have line segments. We will examine this idea of asking how things are alike and how they are different throughout the textbook.



Section 3.2 Understanding Subtraction 101

CLASSROOM CONNECTION

Grade 3

Can you describe the context of addition or subtraction in each story?

Date Time

LESSON 2-5 **Number Stories: Change-to-More and Change-to-Less**

For each number story, write ? in the diagram for the number you want to find. Then write the numbers you know in the change diagram also. Next, solve the problem. Write the answer and a number model.

Unit: dollars

1. Ahmed had $22 in his bank account. For his birthday, his grandmother deposited $25 for him. How much money is in his bank account now?
Answer the question: ____
Number model: ____
Check: How do you know your answer makes sense?

2. Omar had $53 in his piggy bank. He used $16 to take his sister to the movies and buy treats. How much money is left in his piggy bank?
Answer the question: ____
Number model: ____
Check: How do you know your answer makes sense?

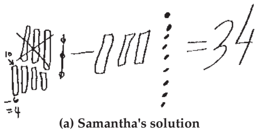
3. Cleo had $37 in her purse. Then Jillian returned $9 that she borrowed. How much money does Cleo have now?
Answer the question: ____
Number model: ____
Check: How do you know your answer makes sense?

thirty-nine 39

From *Everyday Mathematics, Grade 3:* The University of Chicago School Mathematics Project: Student Math Journal, Volume 1, by Max Bell et al., Lesson 2-5, p. 39. Reprinted by permission of The McGraw-Hill Companies, Inc.

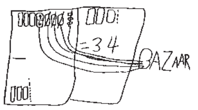
11. ***Classroom Connection*** Following are two children's solutions to 73 − 39, taken from the December 2001 issue of *Teaching Children Mathematics* (p. 231). Study each child's work, and then describe how that child would find the answer to 65 − 28. Then explain why that method works.

a.



(a) Samantha's solution

b.



(b) Alice's solution

Source: *Teaching Children Mathematics,* by the National Council of Teachers of Mathematics.

Margin Notes

To help round out the mathematics education of pre-service teachers, other special margin notes are provided.

▼ ***Outside the Classroom*** boxes highlight applications and uses of mathematical concepts and procedures in the business world, science, and everyday life.

OUTSIDE THE CLASSROOM

Skateboarders and snowboarders use reflex angles to describe some of their moves. What do you think they mean when they talk about doing a frontside 540 or a backside 720?

LANGUAGE

The term *ratio* comes from the Latin verb *ratus,* which means "to think or estimate." Many mathematicians in the sixteenth and seventeenth centuries used the word *proportion* for ratio. Even today you hear the two terms used interchangeably; for example, instructions for making a certain color might say, "Mix the two colors in the following proportion—3 : 2."

◀ ***Language*** boxes include the etymology of selected terms and/or describe nuances of terms.

HISTORY

Much of the mathematical notation we use is actually quite recent. The symbols for addition and subtraction, $+$ and $-$, first appeared in Germany in the late 1400s. These symbols were first used to indicate sacks that were surplus or minus in weight. In 1631, William Oughtred first used the letter x to represent multiplication. Italian merchants introduced the symbol for division ($\div$) in the 1400s to indicate a half. For example, they wrote $4 \div$ to indicate $4\frac{1}{2}$. The equals sign first appeared in the late 1500s in a book by Robert Recorde.

◀ ***History*** boxes present interesting side notes, relevant to concepts developed in the text.

▼ ***Mathematics*** boxes associate a previously discussed concept or other related math idea with the topic under discussion.

MATHEMATICS

If you did Exploration 2.3, recall how strange the Alphabitian system was to you. What observations made you more comfortable with adding? Can you think of analogous observations that might make the learning of base ten addition facts easier for young children?

Section Exercises

The exercises are designed to give students a deeper sense and awareness of the kinds of problems their future students are expected to solve at various grade levels, as well as to increase their own proficiency with the content. Special subcategories appear toward the end of each set of exercises. *Deepening Your Understanding* exercises go a step beyond, encouraging students to extend their thinking beyond the basics. *From Standardized Assessments* exercises derive from exams such as the NECAP and NAEP to give students a sense of the types of questions found on diverse national exams at various grade levels. Questions are also included from the Smarter Balanced Assessment Consortium which is developing assessments for Common Core State Standards.

DEEPENING YOUR UNDERSTANDING

26. Place the digits 1, 2, 3, 6, 7, and 8 in the boxes to obtain

$\square\square\square - \square\square\square$

a. The greatest difference

b. The least difference

27. Choose among the digits 1, 2, 3, 4, 5, 6, 7, 8, and 9 to make the difference 234. You can use each digit only once. How many different ways can you make 234?

$\square\square\square - \square\square\square$

28. With three boys on a large scale, it read 170 pounds. When Adam stepped off, the scale read 115 pounds. When both Adam and Ben stepped off, the scale read 65 pounds. What is the weight of each boy?

29. A mule and a horse were carrying some bales of cloth. The mule said to the horse, "If you give me one of your bales, I shall carry as many as you." "If you give me one of yours," replied the horse, "I will be carrying twice as many as you." How many bales was each animal carrying?

30. From each of the following lists, select two numbers whose difference will be closest to the target difference.

	Numbers			Target
a.	315	475	764	300
b.	185	372	953	650
c.	382	723	793	350

31. One of my students asked me this question after s the text: "Why do we have addition and multiplica but not subtraction and division tables?" Write yo to her question.

3.2 Exercises

1. Make up a subtraction story problem for each of the following contexts. Briefly *explain* why the story problem is an example of the particular model.

a. Take-away

b. Missing addend

c. Comparison

2. Which model of subtraction best illustrates each of the problems below:

a. Reena had 25 quilts and sold 10 of them at the show. How many does she have left?

b. The hall holds 100 people. Currently 65 tickets have been sold. How many more tickets can be sold?

c. The sixth grade has sold 58 raffle tickets and the fifth grade has sold only 45. How many more has the sixth grade sold?

4. Represent the following problems on a number line. Explain each problem as though you were talking to someone who is not taking this class.

a. $5 + 4$ **b.** $8 - 3$

5. a. Explain why the operation of subtraction is not commutative.

b. Explain why the operation of subtraction is not associative.

6. Determine the following differences mentally. Briefly describe how you obtained your estimate.

a. $87 - 29$ **b.** $70 - 23$ **c.** $82 - 34$ **d.** $500 - 134$ **e.** 502

f. $625 - 475$ **g.** $4000 - 555$

Determine these differences mentally by a means other standard algorithm.

FROM STANDARDIZED ASSESSMENTS

NECAP 2006, Grade 5

34. Mrs. Lombardi had 2 hours to prepare for a party. The chart below shows the amount of time she spent completing different tasks.

TIME MRS. LOMBARDI SPENT ON DIFFERENT TASKS

Task	Time
Decorated cake	20 minutes
Made punch	15 minutes
Made sandwiches	50 minutes
Put up balloons	?

How much time did Mrs. Lombardi have to put up the balloons? (1 hour = 60 minutes)

a. 15 minutes **b.** 25 minutes

c. 35 minutes **d.** 45 minutes

NECAP 2005, Grade 5

35. The students at Maple Grove School are selling flowers. Their goal is to sell 1500 flowers.

- On the first day, the students sold 547 flowers.
- On the second day, the students sold 655 flowers.

How many flowers must the students sell on the third day to meet their goal?

a. 298 **b.** 308 **c.** 1202 **d.** 2702

Section Summary

Each section ends with a summary that reviews the main ideas and important concepts discussed.

SUMMARY 3.2

We have now examined addition and subtraction rather carefully. In what ways do you see similarities between the two operations? In what ways do you see differences? Think and then read on. . . .

One way in which the two processes are alike is illustrated with the part–whole diagram used to describe each operation. These representations help us to see connections between addition and subtraction. In one sense, addition consists of adding two parts to make a whole. In one sense, subtraction consists of having a whole and a part and needing to find the value of the other part.

We see another similarity between the two operations when we watch children develop methods for subtraction; it involves the "missing addend" concept. That is, the problem $c - a$ can be seen as $a + ? = c$.

We saw a related similarity in children's strategies. Just as some children add large numbers by "adding up," some children subtract larger numbers by "subtracting down."

Earlier in this section, subtraction was formally defined as $c - b = a$ if $a + b = c$. The negative numbers strategy that some children invent brings us to another way of defining subtraction, which we will examine further in Chapter 4 when we examine negative numbers. That is, we can define subtraction as adding the inverse: $a - b = a + -b$.

A very important way in which the two operations are different is that the commutative and associative properties hold for addition but not for subtraction.

Looking Back

Each chapter concludes with *Looking Back*—a study tool that brings together all the important points from the chapter. *Looking Back* includes *Questions to Summarize Big Ideas* (NEW!), which ask students to reflect on the main ideas from the chapter; *Chapter Summary*, which lists major take-aways and terminology from the chapter; and *Review Exercises*, which provide an opportunity for students to put concepts from the chapter into practice.

LOOKING BACK on chapter three

QUESTIONS TO SUMMARIZE BIG IDEAS

1. What are some of the different models for addition, subtraction, multiplication, and division?
2. How can you use base ten blocks to model the algorithms for each of the operations?
3. How are these models similar and different in a base other than ten?
4. Which algorithms for the operations are different from what you learned in elementary school?
5. What are the tools for determining divisibility and why do they work?
6. Look back at the Mathematical Practices of the Common Core State Standards. In what ways did you engage in those practices during this chapter?
7. What parts of this chapter are less clear to you at

CHAPTER 3 SUMMARY

1. Many students have said that really understanding base ten and the four operations, was, for them, the beginning of a new attitude toward mathematics. We will continue to examine new and important mathematical ideas throughout this book, but the foundation for much of elementary mathematics has now been laid.
2. Each operation has multiple meanings.
3. Many algorithms have been developed to enable us to compute more efficiently.
4. The standard algorithm for each operation does not connect equally well to each meaning of the operation.
5. Being able to make sense of algorithms requires:
   - The ability to apply base ten and place value concepts
   - The ability to compose and decompose the numbers (for example, to use expanded form)
6. Patterns enable us to understand the operations more deeply.
7. In many real-life problems, the answer depends on knowing how to interpret one's computation.
8. Being able to perform mental math and to estimate requires
   - The ability to apply base ten and place value concepts
   - The ability to compose and decompose the numbers (for example, to use expanded form)
   - The ability to apply properties of the operations, especially the commutative, associative, and distributive properties
9. Numbers in real-life settings are sometimes exact, sometimes rounded, and sometimes estimates.
10. In real-life problem-solving, one needs to know when to find an exact answer and when to find an estimate.
11. Real-life problem solvers need to know whether their estimates are reasonable.
12. People may use rounded numbers rather than exact numbers for a variety of reasons.

BASIC CONCEPTS

Section 3.1: Understanding Addition

Addition terminology:

addition **78** sum **78**
addends **78**

Addition contexts:

discrete **76** continuous, measured **76**

Addition models:

pictorial **76** number line **78**
tables **79**

REVIEW EXERCISES chapter three

1. State the problem that is represented in each case below:

Explorations

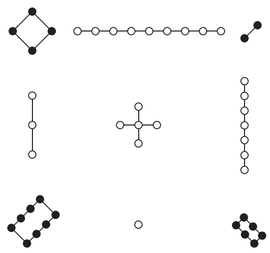
The icon that appears throughout the text references additional activities that may be found in the *Explorations Manual*. Explorations present new ideas and concepts for students to engage with "hands on."

Mathematics for Elementary School Teachers p. 27

EXPLORATION 1.6 Magic Squares

Magic squares have fascinated human beings for thousands of years. The oldest recorded magic square, the Lo Shu magic square, dates to 2200 B.C. and is supposed to have been marked on the back of a divine tortoise that appeared before Emperor Yu when he was standing on the bank of the Yellow River. In the Middle Ages, many people believed magic squares would protect them against illness! Even in the twenty-first century, people in some countries still use magic squares as amulets.

As a teacher, you will find that many of your students love working with magic squares and other magic figures.



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Foundations for Learning Mathematics

1

Knowing mathematics means being able to use it in purposeful ways. To learn mathematics, students must be engaged in exploring, conjecturing, and thinking rather than only in rote learning of rules and procedures. Mathematics learning is not a spectator sport. When students construct personal knowledge derived from meaningful experiences, they are much more likely to retain and use what they have learned. This fact underlies [the] teacher's new role in providing experiences that help students make sense of mathematics, to view and use it as a tool for reasoning and problem solving.[1]

—National Council of Teachers of Mathematics

SECTION 1.1 Getting Started and Problem Solving

What do you think?

- Respond to the prompt: Mathematics is ___________.
- Describe a few of your experiences learning mathematics as an elementary school student.
- Describe your attitudes toward mathematics and where you think these attitudes come from.
- What attitudes do you have about taking this course?

You are at the beginning of a course where you will re-examine elementary school mathematics to understand these concepts on a much deeper level, and to learn why the mathematical procedures and formulas actually work. On this journey, you will learn several ways to see and think about concepts and procedures that you may have previously simply memorized. This deeper understanding will lead to increased confidence and comfort level with mathematics. Your approach to this course, and to teaching mathematics, depends on the attitudes and beliefs you bring to the classroom; in subtle and not so subtle ways, you may pass these beliefs along when you enter the classroom as a teacher. Reflect on how you answered the questions above. Whatever your feelings about mathematics, consider where these feelings come from. Research suggests that people who have mathematics anxiety can relate it back to a teacher and/or experience in their elementary or middle school years. Think about the best math teacher you have had as well as the worst math teacher you have had. Consider the skills and qualities that each of them had that led to your experience of them. What skills and qualities do you have and need to further develop to become an excellent math teacher?

BELIEFS AND ATTITUDES ABOUT MATHEMATICS

This preliminary exercise is designed with two purposes in mind. First, it will help you examine and reflect on your beliefs and attitudes at the beginning of the course. Second, it will help you see a practical use of mathematics.

Rate your attitudes

Seven pairs of statements concerning attitudes and beliefs about mathematics are given in Table 1.1. Score your beliefs in the following manner:

- If you strongly agree with the statement in column A, record a 1.
- If you agree with the statement in column A more than with the statement in column B, record a 2.
- If you agree with the statement in column B more than with the statement in column A, record a 3.
- If you strongly agree with the statement in column B, record a 4.

Adaptive and maladaptive beliefs

Before we discuss your responses to Table 1.1 let us examine attitudes. I have worked with thousands of students during my time as a teacher, and I know from experience and from reading research that one's beliefs about mathematics can influence how one learns and teaches.

TABLE 1.1

Column A		Column B
1. There will be many problems in this book that I won't be able to solve, even if I try really hard.	1 2 3 4	**1.** I believe that if I try really hard, I can solve virtually every problem in this book.
2. There is only one way to solve most "word" problems.	1 2 3 4	**2.** There is usually more than one way to solve most "word" problems.
3. The best way to learn is to memorize the different kinds of problems—rate problems, mixture problems, coin problems, etc.—and how to solve them.	1 2 3 4	**3.** The best way to learn is to make sure that I understand each step.
4. Some people have mathematical minds and some don't. Nothing they do can *really* make a difference.	1 2 3 4	**4.** Some students may have more aptitude for mathematics than others, but everyone can become competent in mathematics.
5. The teacher's job is to show us how to do problems and then give us similar problems to practice.	1 2 3 4	**5.** The teacher's job is more like that of a coach or guide—to help us develop the problem-solving tools we need.
6. A good test consists of problems that are just like the ones we have done in class.	1 2 3 4	**6.** A good test has problems at a variety of levels of difficulty, including some that are not just like the ones in the book.
7. I don't need to know all the ideas covered in this book because I'm going to teach younger children.	1 2 3 4	**7.** Even teachers of young children need to have a good understanding of the ideas in this book.
	Total ______	

OUTSIDE THE CLASSROOM[2]

The pervasiveness of negative attitudes toward mathematics was powerfully illustrated in 1992 when Mattel introduced a new talking Barbie doll that said, "Math is tough." Now this may be true for some people, but having Barbie say it only reinforced that stereotypical perception of mathematics in the United States, especially among females. Mattel was persuaded to change Barbie's statement.

Some students have **adaptive beliefs** that help them approach math with a positive and confident attitude. Some have **maladaptive beliefs** that keep them from thinking of learning as an evolving and enjoyable process.

In Table 1.1, the statements in column A indicate maladaptive beliefs, and the statements in column B indicate the corresponding adaptive beliefs. If you take the arithmetic average, or *mean,* of your scores (by adding up your scores and dividing by 7), you will get a number that we could call your belief index. If your belief index is low and you encounter difficulties in this course, it may be because some of your beliefs are hindering your ability to learn the material. If you find this course frustrating, try to discuss your beliefs with your professor, with someone at a math center (if your college has one), or with a friend who is doing well in the course. Deepening your understanding of mathematics through this course and beyond will help you to have a positive attitude about mathematics. Approach this course with an open mind toward learning and a belief that everyone (including yourself) can understand mathematics. Your future students are depending on you to deepen your understanding of math and to have positive attitudes about it.

LANGUAGE

Whenever you see the pencil icon, stop and think and briefly write your thoughts before reading on. Students who take the time to think and write after these points (or at least to pause and think) say that it makes a big difference in how much they learn.

WHAT IS MATHEMATICS?

What is mathematics? Think about this question for a minute and then read on. . . .

You may be surprised to learn that not all mathematicians give the same response to this question. *On the Shoulders of Giants: New Approaches to Numeracy*[3] was written partly to help expand people's views of mathematics beyond the common stereotype of "mathematics is a bunch of formulas and rules for numbers." A group of mathematicians and mathematics educators brainstormed a number of possible themes for that book. In the end, it was agreed that the idea of *pattern* permeates all fields of mathematics. Five mathematicians were asked to write chapters on the following themes:

Dimension. In school, you have studied two- and three-dimensional shapes. Mathematicians have gone far beyond three dimensions for years. Recently, a field of mathematics has opened up the exploration of fractional dimensions. For example, the coastline of Britain (which can be modeled by a long, squiggly line) has been calculated to have a dimension of 1.26.

Quantity. This begins (with children) with the question "how many," for which the counting numbers (1, 2, 3, . . .) are appropriate; it moves in complexity to the question "how much," for which fractions and decimals were invented, and then to questions far more complex, for which other numbers and systems were invented.

Uncertainty. Questions of uncertainty permeate everyday life: How long will I live? What are my chances of getting a job after I graduate? What are the chances that my baby will be "normal"?

Shape. Humans' relationship with shape has a fascinating history—the shape of one's environment (desert, forest, mountain), what shape is best for packaging, the shapes that artists make, and the shapes we manufacture for quilts, clothing, and so on.

Change. We live in a world that is constantly changing. The development of computers enables us better to understand and manage change, whether it be the changing climate, the change in epidemics (such as AIDS), the change in populations (human and animal), or changes in the economy.

Mathematics is far more than titles of courses and chapters in textbooks—whole numbers, fractions, decimals, percents, algebra, geometry, etc. The topics in textbooks represent tools that are needed in order to answer important questions about dimension, quantity, uncertainty, shape, and change.

> The numbers, lines, angles, shapes, dimensions, averages, probabilities, ratios, operations, cycles, and correlations that make up the world of mathematics enable people to make sense of a universe that otherwise might seem to be hopelessly complicated.[5]

MATHEMATICS

Keith Devlin has written several fascinating and readable books on this subject,[4] one of which is a companion to a PBS series entitled *Life by the Numbers* (your college or local library probably has this book). The chapter titles for *Mathematics: The Science of Patterns* are "Counting," "Reasoning and Communicating," "Motion and Change," "Shape, Symmetry and Regularity," and "Position." Devlin discusses (among many other things) how mathematicians helped us to understand why leopards have spots and tigers have stripes, how mathematicians helped American ice skaters learn how to perform triple axel jumps, and how we use mathematics to measure the heights of mountains.

CLASSROOM CONNECTION

Deborah Meier, an award-winning principal, urged her teachers to think critically and often[6] asked them, "So what?" I encourage this of you, too. There are many wonderful teachers in this country, but there are also many teachers for whom mathematics is just another subject to be "covered." Do you want your students to form this view of mathematics and carry it from your classroom? If not, then you need to ask yourself the "So what?" question frequently, and when you cannot give a satisfactory answer, talk with your instructor or go elsewhere—to the Web, to articles in *Teaching Children Mathematics,* to classrooms—and see for yourself the kinds of mathematical experiences that we want young children to have in school.

Mathematics is both beauty and truth. Two plus two always equals four. The distance around a circle is always a little more than three (actually pi) times the distance across the circle. Throughout this text, we hope you will appreciate more and more of the beautiful truths of mathematics.

USING MATHEMATICS

Let us now turn our attention to how people use mathematics in everyday situations and in work situations. Take a few minutes to jot down some instances in which you have used mathematics in your life and some instances in which you know that mathematics is used in different careers and work situations. Then read on. . . .

People use mathematics for various purposes, for example:

- To persuade a boss that our idea will make money
- To persuade a potential customer that our product will save money
- To predict—tomorrow's weather or who will win the election
- To make a personal decision—whether we can afford to buy a house
- To make a business decision—how much to charge for a new product or whether a new medicine (for example, a cure for AIDS) really works
- To help us understand how the world works (for example, why leopards have spots and tigers have stripes)
- To relax—many people enjoy Sudoku and other math puzzles

Solving problems

In each of these examples, people are using mathematics as a tool for solving problems. To decide whether you can afford a new car, you have to collect data (on insurance, for example), add decimals, and work with percents (such as sales tax and interest on the loan). The mathematics you use will help you solve the problem of whether to buy a car.

Think about this situation involving weather forecasters. In 1994 a hurricane brought severe rains to Georgia. Forecasters predicted that the Flint River would crest at 20 feet above flood level; the river actually crested at 13 feet above flood level, much to the relief of many residents. The forecasters used decimals, volume formulas, and conversions to determine the maximum volume of water that would be flowing. They also used computer models of flooding rivers, and the computer models were based on data collected on previous flooding.

Now that we have discussed mathematics in general, let's focus on the specialized mathematical understanding that teachers need.

MATHEMATICAL KNOWLEDGE FOR TEACHING

INVESTIGATION 1.1a More Than One Way to Multiply?

First, multiply 49 × 25 using any method you choose. Then consider how the following students solved the problem.

What method do you think each student is using in each solution below? Do you think each method will work for any two whole numbers?

Student A:

$$\begin{array}{r} 49 \\ \times\, 25 \\ \hline 245 \\ 98 \\ \hline 1225 \end{array}$$

Student B:

$$\begin{array}{r} 49 \\ \times\, 25 \\ \hline 45 \\ 200 \\ 180 \\ 800 \\ \hline 1225 \end{array}$$

DISCUSSION

We will look much more deeply into multiplication later in this course. For now, we use this example to illustrate how part of the role of a teacher is to be able to first understand that there are different methods for solving problems, and then be able to understand different strategies, and to know what to do with these different methods to develop deeper understanding. This is the mathematical understanding that is important for elementary teachers, and this deeper understanding is the focus of this course.

Student A used the method that many of us learned in elementary school of 5×49 and 20×49. Some of us learned to write the second row in the multiplication as 980, some were taught to move over a space as is shown here. Why do we do this? Because we are multiplying 25×40, not 25×4, we write it as 980, or as 98 with a space in the ones place.

We can use the properties and split the numbers up into their parts to figure this out.

$$49 \times 25 = 49 \times (20 + 5) = 49 \times 20 + 49 \times 5 = 980 + 245 = 245 + 980 = 1225$$

Student B used a method that is sometimes called partial products of $9 \times 5 = 45$, $40 \times 5 = 200$, $9 \times 20 = 180$, and $40 \times 20 = 800$. Another way to write this, which uses the distributive property more clearly is:

$$49 \times 25 = (40 + 9) \times (20 + 5) = 40 \times 20 + 40 \times 5 + 9 \times 20 + 9 \times 5 = 600 + 200 + 180 + 45 = 1225$$

The methods of both students are valid. One of the major ideas in this text is that there are multiple ways (often called "solution paths") to get to an answer. There is no ONE "right" way to solve any math problem. This may be different from what you have always thought about mathematics. You may have even had teachers who marked you "wrong" if you did not solve it their way.

As a teacher of elementary school mathematics, a deep understanding of mathematics will enable you to respond to the above type of situation that arises in an elementary school classroom. Simply being able to get the right answer is not sufficient. Teachers need a specialized understanding of mathematics that is flexible, connected, and conceptual. This course will help you develop that.

Teachers use mathematics every day in the classroom, but in different ways than others. In 1986, Lee Shulman used the term "pedagogical content knowledge" to refer to this specialized understanding of mathematics, which includes an understanding of multiple representations and examples, plus an understanding of what ideas may be more difficult for students and why these ideas are more difficult. In 2008, Deborah Loewenberg Ball, Mark Thames, and Geoffrey Phelps developed the framework depicted in Figure 1.1 showing different types of knowledge. This book is focused on the specialized content knowledge, which includes understanding multiple representations and multiple student procedures, and analyzing student errors. It is the type of math knowledge that teachers draw on every day.

CLASSROOM CONNECTION

Many students have told me that before they took this course, they thought there was just *one right way* to do a problem, and so they never looked for patterns but instead looked for formulas or procedures. Once the students started looking for patterns, they found them everywhere, and over time they learned how to use their awareness of patterns more powerfully. We can refer to different ways to solve a problem as different **solution paths**.

Explorations Manual 1.2

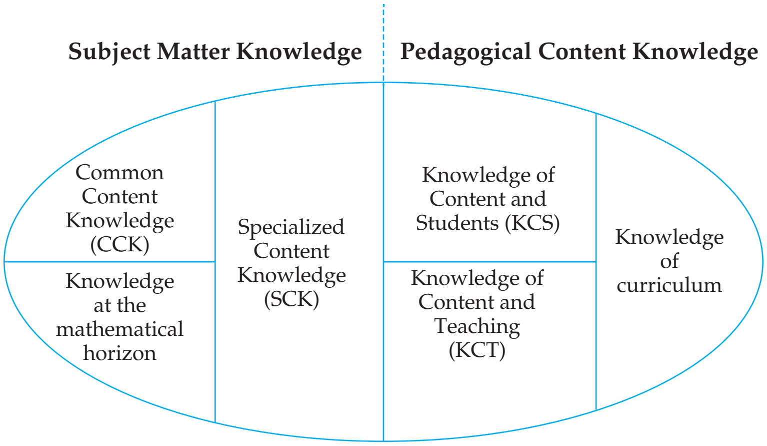


Figure 1.1
Source: Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, 39(4), 372-400.

INVESTIGATION 1.1b Understanding Students' Errors

How do you think the answer was produced? What do you think the student was thinking that led to this error?

$$\begin{array}{r} 49 \\ \times\ 25 \\ \hline 245 \\ 98 \\ \hline 343 \end{array}$$

How does this type of scenario draw upon specialized content knowledge?

DISCUSSION

Analyzing student errors draws upon a specialized understanding of mathematics in that the teacher needs to understand the mathematics deeply in order to identify the error and then to help the student to correct the misunderstanding.

This student does not understand that in the second step, we are actually multiplying 20 times 49 so it should be 980, not 98.

While most people could get a correct answer for 49 × 25, teachers need to understand the concept much more deeply. Many of us experienced elementary school mathematics as a series of procedures to memorize (like multiplying 49 × 25). In order to teach true understanding of mathematics, teachers must develop this specialized content knowledge.

LEARNING STYLES

Consider how you best learn, particularly mathematics. Do you prefer visuals? Do you need to experience it by "getting your hands on it"? Do you need to experience it through sound? Do you prefer to talk with others while solving a problem, or think about it alone? There are several theories about how people learn and think differently, including the Kolb learning styles, VAK (Visual Auditory Kinesthetic) learning styles, and multiple intelligences theory. As you go through this course, it may be helpful to think about how you best learn mathematics. There are several online surveys that can help you determine your own learning styles. You can locate free surveys upon searching the Internet for "learning styles surveys"; once you understand your own learning styles more deeply, you can adjust your learning experiences to accommodate them.

LANGUAGE

Many authors define understanding in terms of connections. That is, you truly understand an idea only if it is well-connected to other ideas, and your depth of understanding is connected to how many connections you are making and the quality of those connections.

PROBLEM SOLVING AND TOOLBOXES

A useful metaphor for problem solving is a **toolbox**. Imagine that your car breaks down and is towed to the garage, where a novice mechanic is the first to look at it. The novice will probably try a few standard procedures: Insert the key to see what happens, check the battery connections, look for a loose wire, and so on. If none of those strategies work, the novice mechanic will be stumped and will have to summon the senior mechanic. The senior mechanic may try the same basic procedures and may solve the problem by *interpreting* the results more skillfully. As you go through this course, you will learn new tools and how to use tools you already have more skillfully.

INVESTIGATION 1.1c Real-Life Problem Solving

Consider a few problems you have had in your life, and not necessarily math problems.

What steps did you take to solve these problems?

Use this recollection to make a list of general steps that you take to solve a problem. Then read on to learn about a mathematician that made a similar list. Your list will likely be pretty similar to his.

DISCUSSION

There are many kinds of real-life problems that you may have considered here. One that many of us have dealt with is what kind of car to buy. The first step would be to understand what you need. How many seats do you need? What is your budget? Is gas mileage a priority? What models do you like? This part might be called something like "understand the problem." Did you have a step like this in your list? The second step would be to develop a plan. Where will you look? What research will you do? The third step might be to carry out the plan. Research the best models, look around for the best deals, test drive some models and find the one you like. The next step, hopefully before actually buying the one you have your heart set on, might be to reflect on whether it really meets your needs, fits your budget, and so on. Read on to see how this process is the same for solving a math problem.

POLYA'S FOUR STEPS

George Polya developed a framework that breaks down problem solving into four distinguishable steps. In 1945, he outlined these steps in a now-classic book called *How to Solve It*.

When you approach a problem, if you think that you have to come up with an answer immediately and that there is only one "right" way to reach that answer, a solution may seem to be beyond your grasp. But if you break the problem down and thoughtfully approach each *step* of the problem, it generally becomes more manageable. Polya suggests that you first need to make sure you **understand the problem**. Once you understand the problem, you **devise a plan** for solving the problem. Then you **monitor your plan**; you check frequently to see whether it is productive or is going down a dead-end street. Finally, you **look back at your work**. This last step involves more than just checking your computation; for example, it includes making sure that your answer makes sense. For each of these four steps, there are specific strategies that we will explore in this chapter and that you will refine throughout this course.

Owning versus renting

Instead of just listing Polya's strategies, we are going to discover them by putting them into action. You will notice that I often ask you to stop, think, and write some notes. I really

CLASSROOM CONNECTION

A colleague was working through a word problem with her class one day. Suddenly one of the students said, "But you don't need to do all this stuff you are teaching us; you just know the answer." She was stunned, and discovered that many students believe that the difference between a student and a teacher is that the teacher just knows the answer or automatically knows how to get the answer and thus doesn't need such strategies as guess–check–revise, make a table, draw a diagram, look for patterns, etc. The truth is that we do! Virtually all engineers, scientists, businesspeople, carpenters, researchers, and entrepreneurs approach complex problems by using the very tools that are being stressed in this text.

mean it! I have come to distinguish between those students who *own* what they learn and those who simply *rent* what they learn. Many students rent what they have learned just long enough to pass the test. However, within days or weeks of the final exam, it's gone, just like a video that has been returned to the store. One of the important differences between owners and renters is that those who own the knowledge tend to be *active readers.*

Using Polya's four steps

I encourage you to use Polya's four steps (on the inside front cover of this book and *Explorations*) in all of the following ways:

1. Use them as a guide when you get stuck.
2. Don't rent them, buy them. Buying them involves paraphrasing my language and adding new strategies that you and your classmates discover. For example, many of my students have added a step to help reduce anxiety: First take a deep breath and remind yourself to slow down!
3. After you have solved a problem, stop and reflect on the tools you used. Over time, you should find that you are using the tools more skillfully.

Think and then read on . . .

Throughout the book, I will often pose a question and ask you to "think and then read on. . . ." Rather than just look to the next paragraph and see the answer, you will learn more if you immediately cover up the next paragraph . . . stop . . . think . . . write down your thoughts . . . and then read on. The phrase "think and read on . . ." is there to remind you to read the book actively rather than passively. An **active reader** stops and thinks about the material just read and asks questions: Does this make sense? Have I had experiences like this? The active reader does the examples with pen or pencil, rather than just reading the author's description.

WHY EMPHASIZE PROBLEM SOLVING?

Although Polya described his problem-solving strategies back in 1945, it was quite some time before they had a significant impact on the way mathematics was taught.

One of the reasons is that until recently, "problems" were generally defined too narrowly. Many of you learned how to do different kinds of problems—mixture problems, distance problems, percent problems, age problems, coin problems—but never realized that they have many principles in common. There has been too great a focus on single-step problems and routine problems. Consider the examples from the National Assessment of Educational Progress shown in Table 1.2.

To solve the first problem, one only has to remember the procedure for finding an average and then use it:

$$\frac{13 + 10 + 8 + 5 + 3 + 3}{6}$$

TABLE 1.2

Problem	Percent correct Grade 11
1. Here are the ages of six children: 13, 10, 8, 5, 3, 3. What is the average age of these children?	72
2. Edith has an average (mean) score of 80 on five tests. What score does she need on the next test to raise her average to 81?	24

Source: Mary M. Lindquist, ed., Results from the *Fourth Mathematics Assessment of the National Assessment of Educational Progress* (Reston, VA: NCTM, 1989), pp. 30, 32.

TABLE 1.3

Textbook word problems	Real-life problems
1. The problem is given.	**1.** Often, you have to figure out what the problem really is.
2. All the information you need to solve the problem is given.	**2.** You have to determine the information needed to solve the problem.
3. There is always enough information to solve the problem.	**3.** Sometimes you will find that there is not enough information to solve the problem.
4. There is no extraneous information.	**4.** Sometimes there is too much information, and you have to decide what information you need and what you don't.
5. The answer is in the back of the book, or the teacher tells you whether your answer is correct.	**5.** You, or your team, decides whether your answer is valid. Your job may depend on how well you can "check" your answer.
6. There is usually a right or best way to solve the problem.	**6.** There are usually many different ways to solve the problem.

However, there is no simple formula for solving the second problem. Try to solve it on your own and then read on. . . .

To solve this one, you have to have a better understanding of what an average means. One approach is to see that if her average for 5 tests is 80, then her total score for the 5 tests is 400. If her average for the 6 tests is to be 81, then her total score for the 6 tests must be 486 (that is, 81×6). Because she had a total of 400 points after 5 tests and she needs a total of 486 points after 6 tests, she needs to get an 86 on the sixth test to raise her overall average to 81.

OUTSIDE THE CLASSROOM[2]

Many students still consider the second question to be a "trick" question unless the teacher has explicitly taught them how to solve that kind of problem. However, many employers note that problems that occur in work situations are rarely *just* like the ones in the book. What employers desperately need is more people who can solve the "trick" problems, because, as some may say, "life is a trick problem!"

The difference between traditional word problems and many real-life problems

Table 1.3 lists differences between the word problems generally found in textbooks and real-life problems.

When students undertake more authentic problems, they realize that mathematics is more than just memorizing and using formulas, and they begin to value their own thinking.

INVESTIGATION 1.1d Coin Problem

Variations of this problem are often found in elementary school textbooks because it provides an opportunity to move beyond random guess and test.

If 8 coins total 50 cents, what are the coins?

Solve this problem intentionally using and writing out Polya's four steps of problem solving.

Explorations Manual 1.1

DISCUSSION

STEP 1: UNDERSTAND THE PROBLEM

So often students will jump into a problem without stopping to really understand it. Read a problem more than once before attempting to solve. Write down the important information and pay attention to what the question is before starting. Here, you have 8 coins, which might be pennies, nickels, dimes, quarters, or half dollars. All together they equal 50 cents. We need to determine what kind of coins we have.

STEP 2: DEVISE A PLAN

There is more than one strategy to solve any problem. Here we could use a diagram, make a table, use reasoning, or use a bag of coins to help us solve it. Let's consider two strategies of making a diagram and using reasoning.

STEP 3: MONITOR THE PLAN

STRATEGY 1 Use a diagram

We could make 8 circles and begin with all nickels: 8 coins = 40¢. What might be the next step?

(5) (5) (5) (5) (5) (5) (5) (5)

With a bit of thinking, we can conclude that each time we substitute a dime for a nickel, the total increases by 5 cents. Thus, we need to trade 2 nickels for 2 dimes, and the answer is 6 nickels and 2 dimes.

STRATEGY 2 Use reasoning

Eight nickels would make 40 cents, and 8 dimes would make 80 cents. Because the 8 coins make 50 cents, your first guess will have more nickels than dimes. Even if the guess is wrong—for example, 5 nickels and 3 dimes make 55 cents—you are almost there.

STEP 4: LOOK BACK AT YOUR WORK

We are almost done but we need to ask questions like: Does my answer make sense? Did I answer the question? Is this the only answer?

We can go back and reread the problem, make sure our solution answers the questions, make sure our answer makes sense, see if we missed any information, and think about alternate ways to get to the solution.

The only other possible solutions are that there might be 1 quarter or 5 pennies. Do you see why? If we make 1 of the coins a quarter, then the other 7 coins must be worth 25 cents. If 5 of those coins are pennies, then we need 2 coins that are worth 20 cents. Aha—2 dimes. Do you see that we could have arrived at the same answer if we had begun with 5 pennies?

SUMMARY 1.1

In this first section, we have examined the importance of adaptive attitudes and beliefs toward mathematics. We have also broadened the question of what is mathematics and specialized math knowledge needed for teaching. We examined Polya's four steps as the foundation for a toolbox for problem-solving strategies.

1.1 Exercises

1. Write down some personal goals in this course. Keep these in a prominent place in your notebook so that you can refer back to them periodically.
2. Interview a friend, a child, and a parent or grandparent (if possible), asking them the first four questions in the What Do You Think? list at the beginning of this section.
3. Make a list of uses of mathematics in your own life. Ask others to discuss where mathematics is useful in their lives and add these to your list.
4. Read the elementary mathematics curriculum standards for your state/district. For many of you, these can be found at corestandards.org. Write about your impression of them, and how they are similar and different from your current vision of elementary school mathematics.
5. Multiply 86 × 47 in each of the two valid ways in Investigation 1.1a.
6. Write about your past experiences in math classes. How have these experiences influenced your current beliefs and attitudes about mathematics?

DEEPENING YOUR UNDERSTANDING

Solve the following using Polya's steps for problem solving. See if you can find more than one way to get to the answer.

7. A video store charges $4 per movie, and the fifth movie is free. How much do you actually pay per movie?

8. At 60 miles per hour, my car's dashboard shows 3000 rpm. This means that the crankshaft, which drives the car, is turning at 3000 revolutions per minute. If a car went 60 miles per hour for 100,000 miles, how many revolutions would the crankshaft have done?

9. Sally works 40 hours a week and makes $6.85 an hour, but her kids are in child care each day and the day care center charges her $15 per day. If you deduct her child care expenses, how many dollars per hour does she actually make?

10. A farmer needs to fence a rectangular piece of land. She wants the length of the field to be 80 feet longer than the width. If she has 1080 feet of fencing material, what should be the length and the width of the field?

11. Since its beginning, the U.S. Mint has produced over 288.7 billion pennies.

a. What if we lined these pennies up? How long would the line be?

b. The mint currently makes about 30 million pennies a day. How many is this per second? How many is this per year?

12. The record for the longest migration is held by the arctic tern, which flies a round trip that can be as long as 20,000 miles per year, from the Arctic to the Antarctic and back. If the bird flies an average of 25 miles per hour and an average of 12 hours per day, how many days would it take for a one-way flight?

13. A family is planning a three-week vacation for which they will drive across the country. They have a van that gets 18 miles per gallon, and they have a sedan that gets 32 miles per gallon. How much more will they pay for gasoline if they take the van?

a. First describe the assumptions you need to make in order to solve the problem.

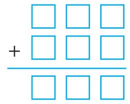
b. Solve the problem and show your work.

c. What if the price of gas rose by 40¢ between the planning of the trip and the actual trip? How much more would the gas cost for the trip?

14. This problem was explored in the September 2007 issue of *Teaching Children Mathematics,* pp. 102–106. In September, ruby-throated hummingbirds fly across the Gulf of Mexico to spend the winter on the Yucatan peninsula. The migration takes them 525 miles across the Gulf of Mexico and another 1000 miles farther into Central America. Ruby-throated hummingbirds typically fly about 25 mph. How many hours would it take a hummingbird to make this migration?

15. A hummingbird's wings beat about 60 times per second. How many would this be in a minute? In an hour?

16. **a.** Using each of the numbers 1–9 exactly once, fill in the blanks below:

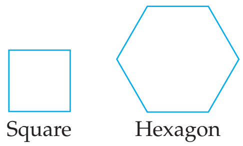


b. How many other solutions can you find?

FROM STANDARDIZED ASSESSMENTS

2006 NECAP, Grade 5

17. Karen used toothpicks to make the two shapes shown below.



She used a total of 24 toothpicks to make the square. She made the hexagon so that its sides are the same length as the sides of the square. How many toothpicks did Karen use to make the hexagon?

18. Suppose you have 8 coins and you have at least one each of a quarter, a dime, and a penny. What is the least amount of money you could have? (23% of seventh-graders got this correct)

Source: Results of the *Fourth NAEP Mathematics Assessment*, p. 16. U.S. Department of Education, National Center for Education Statistics.

SECTION 1.2 Process, Practice, and Content Standards

What do you think?

- If you were to write a vision for the mathematics the students in elementary school should learn, what would it look like?
- What habits and attitudes does a mathematically proficient student have?

First let's take a brief historical look at the development of math standards, which define what is taught in school. From the "new math" of the 1960s and 1970s to the "back to basics" movement of the 1980s, the pendulum has swung between many ideas of how and what mathematics should be taught. In 1989, the National Council of Teachers of Mathematics (NCTM) published a landmark book called *The Curriculum and Evaluation Standards for School Mathematics*. This was the first document that detailed curriculum standards for elementary, middle, and high school mathematics, and led to individual states creating their own curriculum standards. In

2000, NCTM published a revised version, *Principles and Standards for School Mathematics*, which outlined both content standards and process standards. The content standards are about what math topics should be taught at different grade levels while the process standards are about how students and teachers will engage in the math.

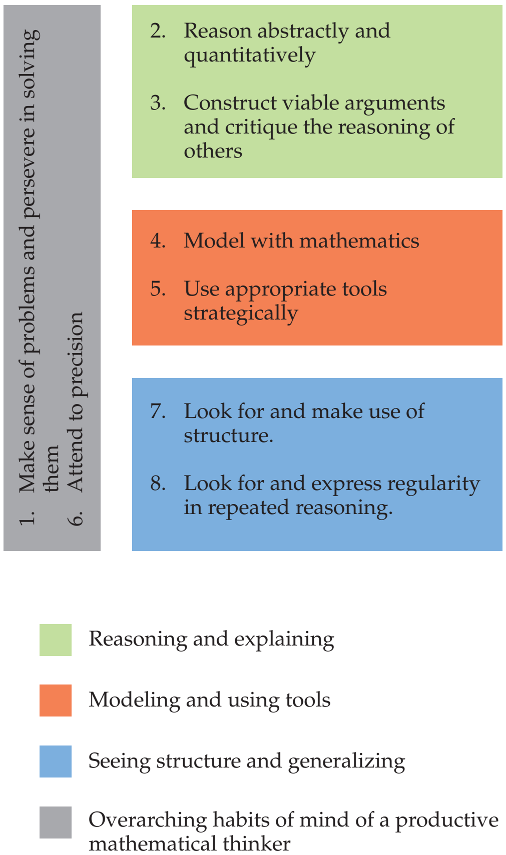
Content standards	**Process standards**
Standard 1: Number and Operation	Standard 6: Problem Solving
Standard 2: Patterns, Functions, and Algebra	Standard 7: Reasoning and Proof
Standard 3: Geometry and Spatial Sense	Standard 8: Communication
Standard 4: Measurement	Standard 9: Connections
Standard 5: Data Analysis, Statistics, and Probability	Standard 10: Representation

This document also outlines principles of school mathematics that address equity, curriculum, teaching, learning, assessment, and technology.

Based on these NCTM standards, other recommendation documents and international comparisons, the Common Core State Standards (CCSS) were developed in 2010 by educators nationwide to establish clear consistent standards that states could then voluntarily adopt. Currently 45 states plus the District of Columbia have adopted these standards.

Similar to the NCTM content standards, the CCSS defines content standards for which topics should be taught at which grade level. The topics included in this book are found in the CCSS K–8 content standards. The introduction to each section discusses which grade levels look at the content according to the CCSS. If you are in a state that offers a K–5 or K–6 teaching license, your instructor may choose to focus on that content. However, knowing what math is on the horizon for your students is also helpful.

Similar to the NCTM Process Standards, the CCSS describe eight mathematical practices (MPs). These practices are interconnected and although we will look at investigations closely associated with each of the practices below, any worthwhile mathematical activity will involve several of them. These mathematical practices are embedded throughout this course. The following figure helps to organize the higher-order thinking skills of the mathematical practices and provides an interesting grouping that may help us to think about what these practices actually look like in a mathematics classroom.



Source: © Copyright 2010. National Governors Association Center for Best Practices and Council of Chief State School Officers. All rights reserved.

You will learn more about teaching the curriculum standards as you proceed through your education program. For now, it is important for you to experience thinking about and understanding mathematics through these practices. In the rest of this section we will experience each of these MPs through investigations. Again, all of the practices are quite interrelated and all the investigations connect to more than one. However, in each investigation, we will focus on one to help you to understand these practices better.

OVERARCHING HABITS OF MIND OF A PRODUCTIVE MATHEMATICAL THINKER: MP 1 AND MP 6

As discussed in the previous section, positive attitudes and adaptive beliefs about mathematics are critical for the student and teacher. Perseverance, being able to think outside of the box, and clear communication are important in becoming proficient problem solvers. MP 1 and MP 6 involve these important habits of mind that are necessary to be a productive mathematical thinker.

MP 1: Make sense of problems and persevere in solving them.

- Mathematically proficient students look for a place to get started. Often that is the hardest part—where do I start?
- They think, try something, assess whether it is helpful, and then continue if it was useful or try another plan.
- If they recognize the current problem as similar to one they have solved, they adapt what they used in the similar problem.
- They simplify the problem—making the numbers smaller or simpler.
- If they are heading down a path that is not solving the problem, they become aware of it and try something different.
- They check their solution and strategy and often ask themselves, "Does this make sense?"

Recall in Section 1.1 that we discussed Polya's four steps of problem solving. Those four steps—understand the problem, devise a plan, monitor your plan, and look back at your work—are embedded in this MP. With so many of us exposed to the Internet and the speed and ease of getting quick answers to questions, the art of persevering and "staying the course" is growing increasingly compromised. For students to be successful problem solvers, patience on the part of both the teacher and the student is required to allow the time to grapple with a problem and to be okay with not getting an immediate answer. Having students engaged in sharing solution methods and creating a safe space where mistakes are part of the learning process can help develop the skills that this MP suggests.

INVESTIGATION 1.2a The Nine Dots Problem

Not all problem solving involves computation and formulas, as this investigation shows.

Without lifting your pencil, can you go through all nine dots in Figure 1.2 with only four lines?

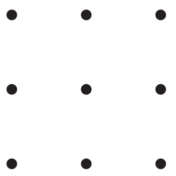


Figure 1.2

DISCUSSION

This is a very famous problem, which some of you may have already encountered because of its moral: This problem is impossible to solve as long as you "stay inside the box." In order to solve the problem, you need to go "outside the box." If you haven't solved the problem yet, try to work with this hint. . . .

The solution to the problem can be seen in Appendix B.

How does this problem relate to MP 1? Being able to "think outside the box" is closely related to good problem solving. In many real-life problems, the solution requires that people think about the problem differently and that they persevere. This relates to MP 1: Make sense of problems and persevere in solving them. In what ways do you see the connection?

INVESTIGATION 1.2b Does Your Answer Make Sense?

This problem has a history that I will share after you solve it. All 261 fifth-graders in a school are going on a field trip, and each bus can carry 36 children and 4 adults. How many buses are needed? Do this problem before reading on. . . .

DISCUSSION

If you divide 261 by 36, you get 7.25. When this problem was given to seventh-graders in one of the National Assessments of Educational Progress, a majority of children gave 7 as the answer. What do you think they did? Do you think they stopped to ask "Does this make sense?" as MP 1 suggests? Many of them had been taught the rules of rounding mechanically, and thus they rounded 7.25 to 7. While they followed the mathematical rule correctly, the real-life application of this computation requires us to round 7.25 up to 8. When solving problems, it is always necessary not only to check your answer but your reasoning. The MP 1 suggests that mathematical thinkers question whether their reasoning as well as their answer makes sense.

MP 6: Attend to precision.

1. Mathematically proficient students are able to communicate their mathematical thinking to others.
2. They are able to articulate clear definitions and the meaning of symbols.
3. They are careful to make sure they use correct units of measure.
4. They calculate accurately and efficiently, and communicate precise answers.

INVESTIGATION 1.2c Precision with Definitions and Symbols

A. Which of the following definitions for even numbers is more precise?

Definition one: An even number has 0, 2, 4, 6, or 8 in the ones place.
Definition two: When an even number is divided by 2, you get a whole number with none left over.

B. Research has shown that when asked to fill in the blank to the following, some students will insert a 5. Why do you think they do this?

$3 + 2 = \_\_\_\_ + 1$

DISCUSSION

A. Although definition one does provide a way to recognize even numbers, it is not very precise for a couple of reasons. For one, a number like 2.4 would be even under the first definition as there is a 2 in the ones place. However, 2.4 is not an even number. This definition also does not tell what an even number really is. Definition two is more precise, because it excludes numbers like 2.4 from meeting the definition and it explains precisely what an even number is.

B. Some students have the misconception that an equals sign is a call to action to perform the operation, instead of realizing that the two sides of the equals sign have to be balanced. These students will see the sign as a call to add 3 + 2 without paying attention to the 1. We will explore this concept further in Chapter 6. For now it is an illustration of how the equals sign may not be as intuitive as you previously believed.

While one aspect of MP 6 is that students are able to perform computations with accuracy (such as finding the correct answer to 35 + 58), MP 6 has greater depth. Students also need to be able to understand precisely what things mean and to be able to consider symbols and units with precision.

REASONING AND EXPLAINING: MP 2 AND MP 3

Mathematical thinking and communication are important aspects of being a mathematically proficient student. As you continue through this course, there will be opportunities for you to analyze your reasoning and communicate your thinking through writing and discussions. Young children naturally make generalizations: They conclude that when the doorbell rings, someone is outside the door; and when they get into the car, they must be strapped in. Elementary school children also regularly make mathematical generalizations: When you add two numbers, you get the same amount when you "add forward as well as backwards" and that if you add two odd numbers together, you get an even number. Reasoning like this and communication of the reasoning are at the heart of these mathematical practices.

Explorations Manual 1.3

MP 2: Reason abstractly and quantitatively.

- Mathematically proficient students make sense of numbers and their context within a problem.
- They are able to "decontextualize" a problem by representing it with numbers and symbols that abstracts away from the context.
- They are also able to "contextualize" the symbolic manipulations by pausing to go back to the context when needed.

INVESTIGATION 1.2d Pigs and Chickens

A farmer has a daughter who needs more practice in mathematics. One morning, the farmer looks out in the barnyard and sees a number of pigs and chickens. The farmer says to her daughter, "I count 24 heads and 80 feet. How many pigs and how many chickens are out there?"

Before reading ahead, work on the problem yourself or, better yet, with someone else. Close the book or cover the solution paths while you work on the problem.

Compare your answer to the solution paths below.

DISCUSSION

STRATEGY 1 Use random trial and error

One way to solve the problem might look like what you see in Figure 1.3.

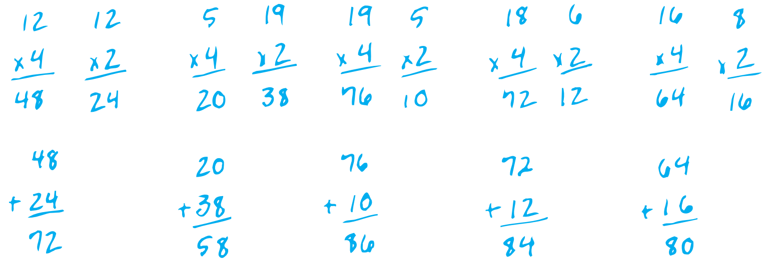


Figure 1.3

LANGUAGE

The full description of this strategy is "Think–guess–check–think–revise (if necessary), and repeat this process until you get an answer that makes sense." I will refer to this strategy throughout the book simply as guess–check–revise, but I urge you not to let the strategy become mechanical.

The words *trial* and *error* do not sound very friendly. However, this strategy is often very appropriate and can help students to make sense of the problem as the MP 2 suggests. In fact, many advances in technology have been made by engineers and scientists who were guessing with the help of powerful computers using **what-if programs**. A what-if program is a logically structured guessing program. Informed trial and error, which I call **guess–check–revise**, is like a systematic what-if program. Random trial and error, which I call **grope-and-hope**, is what the student who wrote the solution in Figure 1.3 was doing. In this case, the student finally got the right answer. In many cases, though, grope-and-hope does not produce an answer, or if it does produce an answer, it is after many trials.

STRATEGY 2 Use guess–check–revise (with a table)

One major difference between this strategy and grope-and-hope is that we record our guesses (or hypotheses) in a table and look for patterns in that table. Such a strategy is a powerful new tool for many students because a table often reveals patterns. Look at Table 1.4. A key to "seeing" the patterns is to make a fourth column called "Difference." Do you see how this column helps?

TABLE 1.4

	Number of pigs	Number of chickens	Total number of feet	Difference	Thinking process
First guess	10	14	68		We need more feet, so the next guess needs to have more pigs.
Second guess	11	13	70	+2	Increasing the number of pigs by 1 adds 2 feet to the total. What if we add 2 more pigs?
Third guess	13	11	74	+4	Increasing the number of pigs by 2 adds 4 feet to the total. Because we need 6 more feet, let's increase the number of pigs by 3 in the next guess.
Fourth guess	16	8	80	+6	Yes!

The left side of Table 1.4 "decontextualizes" the problem by representing the information numerically. The right side "contextualizes" the problem by returning to the context of the problem to make sense of the numbers.

From the table, we observe that if you add 1 pig (and subtract 1 chicken), you get 2 more feet. Similarly, if you add 2 pigs (and subtract 2 chickens), you get 4 more feet. Do you see why? Think before reading on. . . .

Because pigs have 2 more feet than chickens, each trade (substitute 1 pig for 1 chicken) will produce 2 more legs in the total number of feet. This observation would enable us to solve the problem in the second guess. Do you see how . . . ? After the first guess, we need 12 more feet to get to the desired 80 feet. Because each trade gives us 2 more feet, we need to increase the number of pigs by 6.

It is important to note that the guesses shown in Table 1.4 represent one of many variations of a guess–check–revise strategy.

STRATEGY 3 Make a diagram

Sometimes making a diagram can lead to a solution to a problem. Figure 1.4 shows how one student solved this problem. How do you think she had solved the problem? Write your thoughts before reading on. . . .

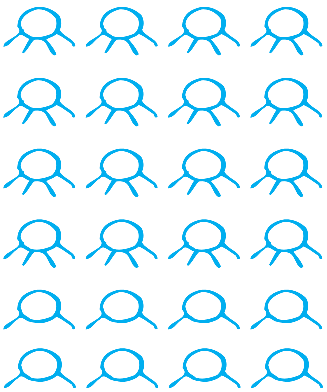


Figure 1.4

She had made 24 chickens, which gave her 48 feet. Then she kept turning chickens into pigs (adding 2 feet each time) until she had 80 feet! I was thrilled because she had represented the problem visually and had used reasoning instead of grope-and-hope. She was embarrassed because she felt she had not done it "mathematically." However, she had engaged in reasoning that MP 2 suggests.

MATHEMATICS

Most complex mathematical problems are solved first by making a model of the problem—the model makes the problem simpler to see and thus to solve. The model enables us to do things that we could not do in the original situation. In this case, the model enables us to do what is biologically impossible but mathematically possible—we turn chickens into pigs until we get the right answer!

Upon reflection, we realize the enormous potential of this solution path. For example, what if the problem were 82 heads and 192 feet? No way you say? True, it would be tedious to draw 82 heads and then 2 feet underneath each head. This is exactly the power of mathematical thinking—you don't have to do all the drawing. Think about what the diagram tells us, and then see whether you can solve the problem. . . .

If we drew 82 heads and then drew 2 feet below each head, that would tell us how many feet would be used by 82 chickens: 164 feet. Because $192 - 164 = 28$, we need 28 more feet—that is, 14 more pigs. So the answer is 14 pigs and 68 chickens. Check it out!

STRATEGY 4 Use algebra

Because the range of abilities present among students taking this course is generally wide, it is likely that some of you fully understand the following algebraic strategy and some of you do not. Let's look at an algebraic solution and then see how it connects to other strategies and to the goals of this course and the MPs.

Go back and review strategy 2. Each guess involved a total of 24 pigs and chickens. Can you explain in words why this is so? Think about this before reading on. . . .

Most students say something like "Because the total number of animals is 24." Therefore, if we say that

$p =$ the number of pigs and $c =$ the number of chickens

then the *number* of pigs plus the *number* of chickens will be 24. Hence, the first equation is

$$p + c = 24$$

CLASSROOM CONNECTION

Many students have told me that before they took this course, they thought there was just *one right way* to do a problem, and so they never looked for patterns but instead looked for formulas or procedures. Once the students started looking for patterns, they found them everywhere, and over time they learned how to use their awareness of patterns more powerfully. We can refer to different ways to do a problem as different **solution paths**.

Many students have difficulty coming up with the second equation. If this applies to you, look back at how we checked our guesses: We multiplied the number of pigs by 4 and the number of chickens by 2 and then added those two numbers to see how close that sum was to 80. In other words, we were doing the following:

$$4 \times (\text{the guess for number of pigs}) + 2 \times (\text{the guess for number of chickens})$$
$$(4 \times p) \qquad + \qquad (2 \times c)$$

More conventionally, this would be written as $4p + 2c$.

Using guess–check–revise, we had the right answer when this sum was 80. Thus the second equation is $4p + 2c = 80$.

If solving these two equations, we see that $p = 16$ and $c = 8$.

STRATEGY 5 Visualization by Pictorial Representation

At different places in this book we will use models like the ones used in Singapore, which has some of the most successful math students in the world. A Singaporean colleague, Alice Ho, has shared these methods and also has developed a five-color–coded model that helps communicate mathematical reasoning. Examine the model on the next page and then we will discuss it below.

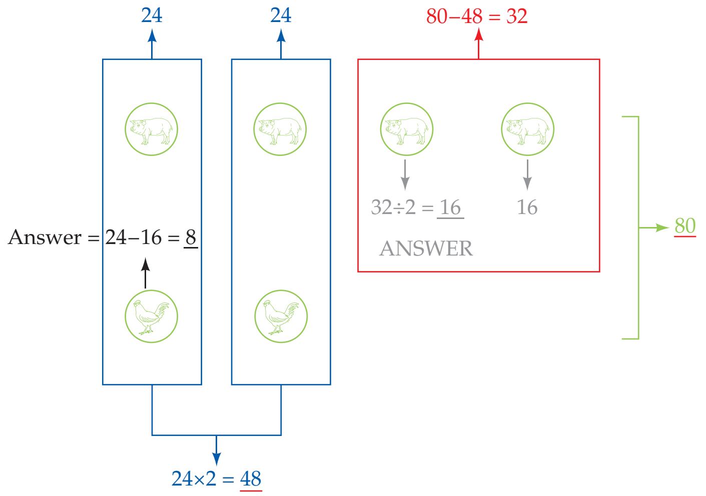
The green in the first step shows that 4 times the number of pigs plus 2 times the number of chickens together equals 80. Then the blue shows how we can use the fact that the number of pigs plus the number of chickens equals 24, so both of the blue boxes represent 48. This leaves 32 that the red box represents, so the number of pigs must equal 32 divided by 2.

How do each of these strategies relate to MP 2? In each of them, students make sense of quantities and their relationships by considering the numbers of legs, along with the numbers of animals. In the algebra strategy, the context is used to create the equations, but then it is decontextualized as the symbol manipulation occurs, and then it is put back into the context to make sense of the algebraic solution. Reasoning strategies are used in the visualization model, which also helps us to make sense of the numbers within the context of the problem.

STEPS: 1 2 3 4 5

Source: Alice Ho of Math Teach Singapore.

$$p + c = 24$$
$$4p + 2c = 80$$



Source: Alice Ho of Math Teach Singapore.

Explorations Manual 1.4

MP 3: Construct viable arguments and critique the reasoning of others.

- Mathematically proficient students are able to use definitions and previous knowledge to communicate their understanding.
- They are able to build a logical progression of their ideas.
- They are able to use counterexamples to make an argument.
- Elementary students can make sense of math and communicate by using objects, drawings, diagrams, or actions.
- They can listen to the reasoning of others and ask useful questions to clarify.

INVESTIGATION 1.2e Why Is the Sum of Two Even Numbers an Even Number?

In the 2009 NCTM Yearbook entitled *Teaching and Learning Proof Across the Grades: A K–16 perspective,* Deborah Schifter describes third-graders working on the question of how to prove that the sum of two even numbers is even. Examine the following responses by the students and think about whether they constitute a proof:

> Paul: I know that the sum is even because my older sister told me it always happens that way.
>
> Zoe: I know it will add to an even number because $4 + 4 = 8$ and $8 + 8 = 16$.
>
> Evan: We really can't know! Because we might not know about an even number and if we add it with 2 it might equal an odd number!
>
> Melody: (Pointing to two sets of cubes she had arranged) This number is in pairs (pointing to the light-colored cubes), and this number is in pairs (pointing to the dark-colored cubes), and when you put them together, it's still in pairs.

DISCUSSION

How are these students engaging in MP 3? Are they communicating their understanding, building a logical progression of ideas, and using drawings to communicate their thinking? Shifter describes four categories of justification common to elementary students (and, I find, with college students too):

appeal to authority (Paul),
inference from instances (Zoe),
assertion that claims about an infinite class cannot be proven (Evan), and
reasoning from representation or context (Melody).

Do you see these categories in the students above?

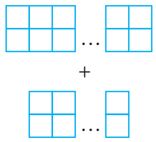
In the diagrams below, you can see how Melody's assertion closely parallels a more formal proof.

An integer is even if it can be represented as 2 times another integer.

If a and b are even numbers, then we can find two integers x and y such that $a = 2x$ and $b = 2y$.

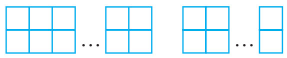
A number is even if it can be broken up into 2 pairs.

These two numbers are even because they can be broken into pairs.



So $a + b = 2x + 2y$; but then $a + b = 2(x + y)$

If you put them together, you still have 2 pairs.



Thus $a + b$ is equal to 2 times an integer, but that is the definition of an even number.

Therefore the sum of the two numbers is also even.

Schifter asserts that young children are capable of making and justifying mathematical generalizations and that making arguments from representations (physical objects, pictures, diagrams, or story contexts) is an effective way to help students develop such reasoning capacity. She proposed three criteria for such representations:

1. The meaning of the operation(s) involved is represented in diagrams, manipulatives, or story contexts.
2. The representation can accommodate a class of examples.
3. The conclusion of the claim follows from the structure of the representation.

Do you see how Melody's argument satisfied these criteria?

1. Her representation modeled two whole numbers.
2. Her language did not say 10 + 16 but rather two whole numbers. That is, her argument did not depend on the actual value of the two numbers (as Zoe's did).
3. When you place the two diagrams together, the resulting amount can also be represented in pairs.

INVESTIGATION 1.2f Darts, Proof, and Communication

Suppose you have a dart board like the one in Figure 1.5. You throw four darts, all of which land on the dart board. One of the questions I asked fifth-graders was what kinds of scores would be possible and what kinds of scores would be impossible. What do you think?

Explorations Manual 1.4

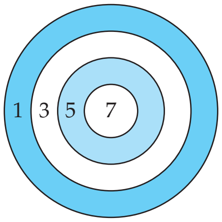


Figure 1.5

DISCUSSION

After a few minutes, one of the students, Erika, suddenly said, "Only even numbers are possible." I asked her how she came to that conclusion, and she said, "Well I know that an odd plus an odd is even and an odd plus an even is odd. [At this point, she held up four fingers to represent the four darts.] The first two darts are odd and so when you add them, you have an even number. [She joined two of her fingers together to indicate the combined score from two darts.] Now this number (even) plus the next dart (odd) will make an odd number. [She now joined three of her fingers together to indicate the combined score from the first three darts.] Now this number (odd) plus the last dart (odd) will make an even number. So the only possible scores you can get are even numbers."

We can represent Erika's proof as shown in Figure 1.6.

(odd + odd) + odd + odd
(even + odd) + odd
odd + odd
even

Figure 1.6

Reflect on this problem for a moment along with MP 3. Erika was able to build a logical progression, make sense of the problem, and use actions and objects (her fingers) to communicate her ideas. Because not everyone learns and understands the same way, we have shown another way to represent and communicate the solution above.

We are using this investigation to highlight MP 3, but this is also a great place to consider how the practices work interactively.

INVESTIGATION 1.2g Using Counterexamples

Is the following statement true?

If a number is divisible by 2 and divisible by 6, then it is also divisible by 12.

DISCUSSION

Here we can look at examples and see if we can find a "counterexample," that is, an example where this statement is not true.

What about 24? Divisible by 2 and by 6 and also 12.

What about 36? Divisible by 2 and by 6 and also 12. Looks good.

What about 30? Divisible by 2 and by 6, but not by 12. So we found a counterexample and therefore the statement is not true.

This illustrates how counterexamples can help us to make the argument that this statement is not true since we found a case where it is not true. There are actually many cases that show it is not true, but finding one is sufficient.

MODELING AND USING TOOLS: MP 4 AND MP 5

These two mathematical practices are particularly related to using math in the workplace and in practical real-life ways. Modeling mathematics problems may involve using tools such as graphs, pictures, concrete materials, and verbal descriptions.

Explorations Manual 1.5

MP 4: Model with mathematics.

- Mathematically proficient students can solve real-life problems, which in elementary school includes being able to write a multiplication equation to solve a problem.
- They are able to identify important information in a real-life problem and analyze relationships using tools.
- They can use models to draw conclusions, make predictions, and reflect on and adjust the effectiveness of the model.

INVESTIGATION 1.2h How Long Will It Take the Frog to Get Out of the Well?

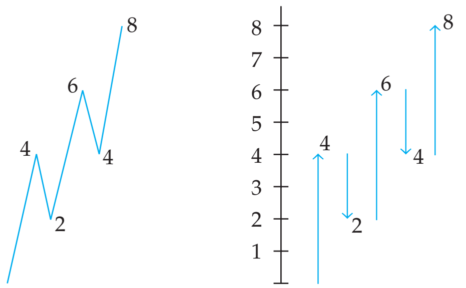
Variations of this problem can be found in the 2001 NCTM Yearbook, p. 78 and *Teaching Children Mathematics,* February 1997, p. 326.

A. A frog is climbing out of a well that is 8 feet deep. The frog can climb 4 feet per hour but then it rests for an hour, during which it slips back 2 feet. How long will it take for the frog to get out of the well?

B. What if the well was 40 feet deep, the frog climbs 6 feet per hour, and it slips back 1 foot while resting? Work on the problem before reading on. . . .

DISCUSSION

A. One of the amazing things about this problem is that, in a class of 25 students, I will often see 10 or more different valid models of the problem. Below are two graph models that both lead to the same answer. First, examine them to see if you understand them. . . .



(1) (2)

Both models show the frog's progress for each hour and that the frog reaches 8 feet after 5 hours. Let us look more closely: How are models (1) and (2) alike and how are they different?

Alike	Different
1. They have line segments.	1. In the first strategy the line segments are not vertical and in the second they are.

CLASSROOM CONNECTION

This question asking how two things are alike and how they are different is an important teaching structure and one that we will revisit over the course of the book. You may remember it in a common *Sesame Street* feature: Three of These Things Belong Together. For example, they might show a triangle, a square, a hexagon, and a circle. The answer is that the circle doesn't belong because it doesn't have line segments. We will examine this idea of asking how things are alike and how they are different throughout the textbook.

2. They have numbers: 4, 2, 6, 4, 8.

2. The second strategy has a number line at the left.
3. The second strategy has arrows.

B. As we found in the pigs-and-chickens problem, some models can be "scaled up" and others cannot. Each of the two models shown *could* be used to solve B, but they would be somewhat tedious. In this case, we look for a more efficient model. Using a table to explore the problem is shown below:

Hour	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Height	6	5	11	10	16	15	21	20	26	25	31	30	36	35	41

Even this model is a bit tedious. However, if we are always looking out for patterns, we can actually get the answer by making only part of the table.

Hour	2	4	6	8	10	12	14
Height	5	10	15	20	25	30	35

We can see that the numbers when the hours are even are simply multiples of 5. We can then count by 2s to get close to the 40-foot height, or we can see that the height (in even hours) is always $2\frac{1}{2}$ times the number representing the hour. So we can jump to 14 hours when the frog has climbed 35 feet and know that on the 15th hour the frog will get out.

While this problem is not exactly a real-life problem, as MP 4 discusses, it is a great example of how different models can help us to go about solving a problem and how students will come up with different models to think about problems. The next investigation uses math to model a real-life problem.

INVESTIGATION 1.2i How Many Pieces of Wire?

A jewelry artisan is making earring hoops. Each hoop requires a piece of wire that is $3\frac{3}{4}$ inches long. If the wire comes in 50-inch coils, how many $3\frac{3}{4}$-inch pieces can be made from one coil, and how much wire is wasted? Solve this problem on your own and then read on. . . .

But before you do, ask yourself, "Do I understand the problem? What is the important information in this situation, and how can I apply what I know to solve it? Does the problem's wording help me devise a plan for solving it? Once I have a solution, can I check it?"

DISCUSSION

STRATEGY 1 Divide

Some people quickly realize that you can divide "to get the answer." If you use a calculator, it shows 13.333333. . . . If you use fractions, you get $13\frac{1}{3}$ Many people interpret these numbers to mean that you can get 13 pieces and you will have $\frac{1}{3}$ inch wasted. Unfortunately, that is not correct. Can you explain why $\frac{1}{3}$ inch is not the correct answer and what the correct answer is. Then read on. . . .

One of the reasons why math teachers stress the importance of labels is that they illustrate the meaning of what we are doing. The *meaning* of $13\frac{1}{3}$ is 13 whole hoops and $\frac{1}{3}$ of a hoop. That is, what we have left would make $\frac{1}{3}$ of a hoop. Because one whole piece is $3\frac{3}{4}$ inches long, $\frac{1}{3}$ of a piece is $\frac{1}{3}$ of $3\frac{3}{4}$; that is, $1\frac{1}{4}$ inches is wasted.

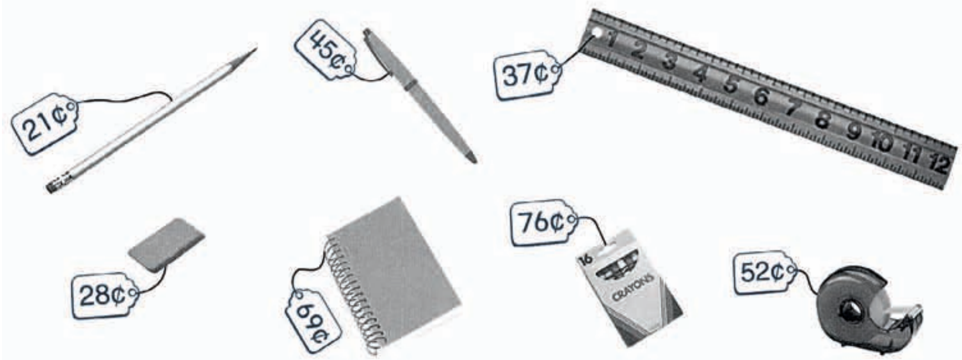
CLASSROOM CONNECTION

Grade 2

These are challenging problems for many second-graders. If you had a class of 20 students and $20 to spend, what might you purchase?

Date Time

LESSON 4•5 **School Supply Store**



You have $1.00 to spend at the School Store.
Use estimation to answer each question.

Can you buy:	**Write *yes* or *no*.**
1. a notebook and a pen?	________
2. a pen and a pencil?	________
3. a box of crayons and a roll of tape?	________
4. a pencil and a box of crayons?	________
5. 2 rolls of tape?	________
6. a pencil and 2 erasers?	________

7. You want to buy two of the same item.
List items you could buy two of with $1.00.

________ ________

________ ________

8. How many pencils could you buy with $1.00? ____

ninety-three **93**

From *Everyday Mathematics, Grade 2:* The University of Chicago School Mathematics Project: Student Math Journal, Volume 1, by Max Bell et al., Lesson 4-5, p. 93. Reprinted by permission of The McGraw-Hill Companies, Inc.

How would we check this answer? Think and then read on. . . .

One way to check would be to multiply $3\frac{3}{4} \times 13$. This would tell us the length of the 13 whole pieces. If this number plus $1\frac{1}{4}$ equals 50, then our answers are correct. In fact, $3\frac{3}{4} \times 13 = 48\frac{3}{4}$ and $48\frac{3}{4} + 1\frac{1}{4} = 50$.

STRATEGY 2 "Act it out"

Some people understand the problem better when they model the problem with a diagram like that in Figure 1.7.

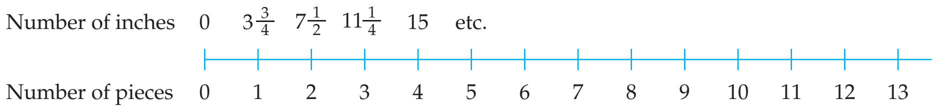


Figure 1.7

STRATEGY 3 Make a table

Yet other people solve the problem by starting with one piece and building up as shown in Table 1.5.

All of these models help us to apply the mathematics to this scenario, as MP 4 suggests. Again, we are focused on MP 4 here, but several MPs come into play here. What others do you see?

TABLE 1.5

Number of pieces	Number of inches	Thinking process
1	$3\frac{3}{4}$	
2	$7\frac{1}{2}$	
3	$11\frac{1}{4}$	
4	15	Using the concept of ratio and proportion, we can reason that if 4 pieces make 15 inches, then 12 pieces would make 45 inches.
12	45	So 1 more piece works.
13	$48\frac{3}{4}$	There are only $1\frac{1}{4}$ inches left, so that is the wasted part.

MP 5: Use appropriate tools strategically.

- Mathematically proficient students can use a variety of tools such as concrete models and technology to find solutions.
- They are able to make good decisions about when to use each of these tools and how to effectively use them.
- They are able to use a variety of tools to investigate and develop their understanding of ideas.

INVESTIGATION 1.2j How Many Handshakes?

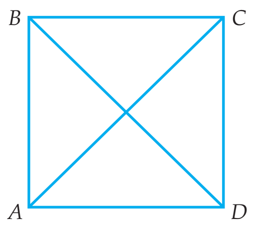
If there are nine people and every person shakes hands once with each person in the room, how many handshakes take place?

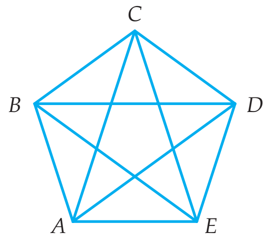
What tools could you use to help?

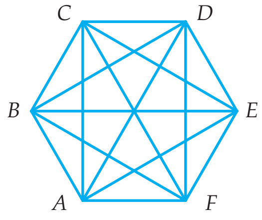
DISCUSSION

Several tools can be used to answer this. One way may be to have nine people in the class stand up and consider what happens, or you could use beads to represent the people, or a picture. These tools are being used strategically to see that the first person will shake hands with 8 people, the next person will shake hands with 7 other people, until the answer is seen to be 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 which equals 36 handshakes in all.

Another tool that can be used is a geometric model and looking at patterns. The following diagram helps us to see that with 2 people there would be 1 handshake, with 3 people there would be 3 handshakes, and so on. Consider these geometric models that illustrate the number of handshakes for 4 people, 5 people, and 6 people.







While it is possible to draw geometric models for 7, 8 and 9 people, it might become cumbersome. Therefore a table is another tool that can help us see the pattern.

Number of people	2	3	4	5	6	7	8	9
Total number of handshakes	1	3	6	10	15			

By extending the pattern of adding 2, then 3, then 4, then 5 handshakes each time a person is added, we would continue this pattern by adding 6, then 7, then 8 to get 36 handshakes with 9 people.

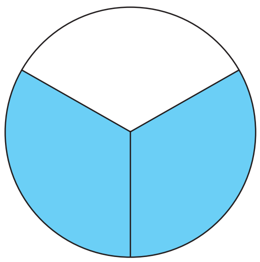
Using these tools leads us to a deeper understanding of the problem. Through this investigation, some may intuit a formula for this. Since everyone is shaking hands with everyone else, each of the 9 people will be shaking hands with 8 other people. If we multiply 8 times 9 there is duplication as we are double counting each handshake and so the final answer is $\frac{1}{2}$ of 8 times 9.

INVESTIGATION 1.2k Tools for Defining a Unit Whole

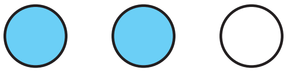
You are teaching a third-grade classroom and are going to draw a picture of $\frac{2}{3}$ on the board. When you ask your students how many circles to draw to illustrate $\frac{2}{3}$, you get two proposals. One student suggests that you start with 1 circle, another student suggests you start with 3 circles. Which student is correct?

DISCUSSION

Each student is using a model that could be used to illustrate $\frac{2}{3}$. If one circle represents the unit whole, then $\frac{2}{3}$ would look like:



If three circles represent the unit whole, then $\frac{2}{3}$ would look like:

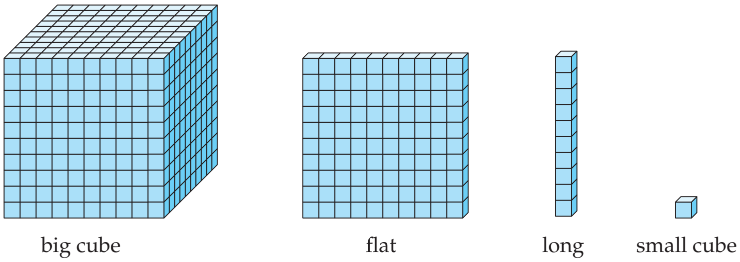


Depending on the context of the problem, which model makes sense to the learner, and the teacher's goals for the discussion in this situation, a strategic decision would be made as to which model to use or to use both models to gain a deeper understanding. Using these pictures strategically illustrates using appropriate tools to solve a problem, as MP 5 suggests.

INVESTIGATION 1.2l Using Base-10 Blocks Strategically

How could you use these base-10 blocks to model 34?

How could you use the base-10 blocks to model 3.4?

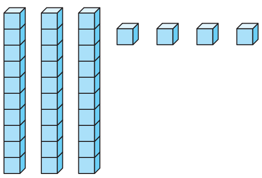


DISCUSSION

These base-10 blocks will be used throughout this course to help us to understand the mathematics more deeply. As we progress through the course, we will see how these tools can be useful. This is a preliminary look at the tools to illustrate this mathematical practice.

As in Investigation 1.2j, strategic choice of unit is important here.

To represent 34, let's define the small cube as "1." Then, 34 would look like:



However, when we try to represent 3.4, we find that using the small cube as "1" will not work since there is no way to represent four-tenths. We need to define one of the larger blocks as "1." What if we defined the long one to be "1"? What would 3.4 look like?

Three longs then would equal 3 and each small cube would each equal .1, so we would need 4 of those. Therefore, it would be the same picture as above!

What would 3.4 look like if we defined the big cube as "1"? Then 3 big cubes would model the 3 and 4 flats would equal 4 tenths.

This example illustrates how you as a learner and as a teacher might choose to use these blocks strategically in diverse ways depending on whether you are working with whole numbers or decimals. Mathematical learners (and teachers) must be flexible with how they use tools in order to use them well.

SEEING STRUCTURE AND GENERALIZING: MP 7 AND MP 8

These two practices involve seeing patterns and then using those patterns to generalize ideas. These two practices are very closely intertwined, and even math educators struggle with how they are different. MP 7 is seeing the structure in math to be able to build understanding. MP 8 is more about using those patterns to generalize for solving any problem. Let's take a closer look.

Explorations Manual 1.6

MP 7: Look for and make use of structure.

- Mathematically proficient students are able to find and use patterns or structures.
- They might notice that 5 times 8 has the same answer as 8 times 5 and extend this to discover that the order we multiple numbers in does not affect the answer (commutative property).
- Young children naturally use structure. They may use the word "eated" as past tense for "eat" or "swimmed" instead of "swam" since they have noticed the structure of adding "ed" to a word to make it past tense, such as in words like "played," "talked," and "worked."

The structure in mathematics is far more consistent than in language and can help with developing understanding. For example, if children learn that 3 apples plus 4 apples equals 7 apples, and 3 hundreds plus 4 hundreds equals 7 hundreds, they can use this structure to see that 3 sevenths plus 4 sevenths would naturally equal 7 sevenths $\frac{3}{7} + \frac{4}{7} = \frac{7}{7}$, or that 3 x's plus 4 x's equals 7 x's ($3x + 4x = 7x$). Noticing structure can be useful in being able to do arithmetic quickly without memorizing. For example, 9 can be added easily to any number by realizing that we can add 10 and subtract 1. This can be generalized to adding 39 + 52 (add 40 and subtract 1). Using structure makes memorizing facts less important.

INVESTIGATION 1.2m Using Structure to Do Mental Math

How could you find the answer to the following without finding a common denominator?

$$1\frac{2}{3} - \frac{1}{4} + 2 + \frac{1}{3} - \frac{3}{4}$$

How could you find the answer to the following without doing the pencil-and-paper calculations you traditionally use to multiply and add?

$$25 \times 6 + 25 \times 4$$

DISCUSSION

In the first problem, rearranging the numbers makes this difficult-looking problem quite easy to do in your head. If you look at the parts of this problem, knowing the flexibility to move around, it can be seen as

$$1\frac{2}{3} + \frac{1}{3} - \frac{1}{4} - \frac{3}{4} + 2 = 2 - 1 + 2 = 3$$

In the second problem, if we notice that we can use the distributive property, it becomes $25 \times (6 + 4) = 25 \times (10) = 250$. We can use words to discern this structure: we are adding 6 groups of 25 to 4 groups of 25, so we have 10 groups of 25, which is 250.

Both strategies use the structure of math (the distributive and commutative properties) to simplify these problems.

Explorations Manual 1.7

MP 8: Look for and express regularity in repeated reasoning.

- Mathematically proficient students notice when computations are repeated and use this to create methods and shortcuts.
- They continue to look at their process and evaluate their results.
- They develop new methods by generalizing patterns.

Lower elementary students might notice that when counting by 5's, the ones digit is always a 0 or 5. They may notice that adding 0 and multiplying by 1 do not change the number, therefore developing the identity property. Upper elementary students might notice when dividing a number by 10 that the numbers move one place value to the right and then be able to use this whenever dividing by 10.

INVESTIGATION 1.2n Patterns in Multiplying by 11

Let us investigate what happens when we multiply a number by 11. From examining the first three problems in Table 1.6, can you predict the answer to 53 × 11? Make your prediction and then multiply 53 × 11 to check your prediction.

DISCUSSION

TABLE 1.6

The problem	The product
26 × 11	286
35 × 11	385
42 × 11	462
53 × 11	?
62 × 11	?
73 × 11	?
75 × 11	?

If you had trouble, the *algorithm* for multiplying a two-digit number by 11 is to add the two digits together, and that number is the middle digit of the product. Test this out for 62 × 11.

When we get to 73 × 11, the problem becomes more challenging because the sum of the two digits is more than 9. Can you modify your thinking to predict the product of 73 × 11?

First, we know that the digit in the ones place will stay the same. Because 3 + 7 = 10, the digit in the tens place will be a zero and the digit in the hundreds place increases by 1, like "carrying." This is how we get the answer of 803. Using this analysis, predict the product of 75 × 11.

We find that the algorithm works: 7 + 5 = 12. We still have 5 in the ones place, 2 in the tens place, and the hundreds place is now 1 more, 8, giving us the predicted product of 825.

In a classroom this is when we see who has really engaged. Has the problem gotten under their skin? Some people now will ask "what-if" questions: What if we are multiplying by a three-digit number? What if we multiply by 111? These will be left as exercises, if you can wait!

You can make up your own problems with a spreadsheet. The directions below work on Excel with a PC.