CENGAGE

# MATHEMATICS

## *for* Elementary School Teachers 7e

BASSAREAR • MOSS

# Four Steps for Solving Problems

## *Understanding the Problem*

### Questions that can be useful to ask:

1. Do you understand what the problem is asking for?
2. Can you state the problem in your own words, that is, paraphrase the problem?
3. Have you used all the given information?
4. Can you solve a part of the problem?

### Actions that can be helpful:

1. Reread the problem carefully. (Often it helps to reread a problem a few times.)
2. Try to use the given information to deduce more information.
3. Plug in some numbers to make the problem more concrete, more real.

## *Devising a Plan*

### Several common strategies:

1. Represent the problem with a diagram (carefully drawn and labeled). Check to see if you used (the relevant) given information. Does the diagram "fit" the problem?
2. Guess–check–revise (vs. "grope and hope"). Keep track of "guesses" with a table.
3. Make an estimate. The estimate often serves as a useful "check." A solution plan often comes from the estimation process.
4. Make a table (sometimes the key comes from adding a new column).
5. Look for patterns—in the problem or in your guesses.
6. Be systematic.
7. Look to see if the problem is similar to one already solved.
8. If the problem has "ugly" numbers, you may "see" the problem better by substituting "nice" numbers and then thinking about the problem.
9. Break the problem down into a sequence of simpler "bite-size" problems.
10. Act it out.

## *Carrying Out the Plan*

1. Are you keeping the problem meaningful or are you just "groping and hoping?" On each step ask what the numbers mean. Label your work.
2. Are you bogged down? Do you need to try another strategy?

## *Looking Back*

1. Does your answer make sense? Is the answer reasonable? Is the answer close to your estimate, if you made one?
2. Does your answer work when you check it with the given information? (Note that checking the procedure checks the computation but not the solution.)
3. Can you use a different method to solve the problem?

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This sheet is meant to serve as a starting point. The number of strategies that help the problem-solving process are almost endless and vary according to each person's strengths and preferences.

After you solve a problem that was challenging for you or after you find that your answer was wrong, stop and reflect. Can you describe what you did that got you unstuck or things you did that helped you to solve the problem? If your answer was wrong, can you see what you might have done? *It is the depth of these reflections that connects to your increased ability to solve problems.*

# The Eight Mathematical Practices of the Common Core State Standards

## MP 1 Make sense of problems and persevere in solving them.

- Mathematically proficient students look for a place to get started. That is often the hardest part—where do I start?
- They think, try something, assess if it is helpful, and then continue if it was useful—or try another plan.
- If they recognize this problem as similar to one they have solved, they adapt what they used in that similar problem.
- They simplify the problem—making the numbers smaller or simpler.
- If they are heading down a path that is not solving the problem, they are aware of it and try something different.
- Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves "Does this make sense?"

## MP 2 Reason abstractly and quantitatively.

- Mathematically proficient students make sense of numbers and their context within a problem.
- They are able to "decontextualize" a problem by representing it with numbers and symbols, that abstracts away from the context.
- They are also able to "contextualize" the symbolic manipulations by pausing to go back to the context when needed.
- They are able to make sense of mathematical ideas.

## MP 3 Construct viable arguments and critique the reasoning of others.

- Mathematically proficient students are able to use definitions and previous knowledge to communicate their understanding.
- They are able to build a logical progression of their ideas.
- They are able to use counterexamples to make an argument.
- Elementary students can make sense of math and communicate by using objects, drawings, diagrams, or actions.
- They can listen to the reasoning of others and ask useful questions to clarify.

## MP 4 Model with mathematics.

- Mathematically proficient students can *apply* the mathematics they know to solve problems arising in everyday life, society, other subjects, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation.
- They are able to identify important information in a real-life situation.
- They are able to use objects, pictures, tables, graphs, and equations to explore problems.
- They routinely interpret their model and improve the model when necessary.

## MP 5 Use appropriate tools strategically.

- Mathematically proficient students consider a variety of available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, or a spreadsheet.
- They are able to make good decisions about when each of these tools might be helpful.
- They are able to use a variety of tools to explore and deepen their understanding of ideas.

## MP 6 Attend to precision.

- Mathematically proficient students communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning.
- They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately.
- They are careful about specifying units of measure.
- They calculate accurately and efficiently; they express numerical answers with a degree of precision appropriate for the problem context.

## MP 7 Look for and make use of structure.

- Mathematically proficient students look closely to discern a pattern or structure.
- They use these familiar and known structures to see things in different ways and extend their understanding.
- Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have.
- Later, students will see that $7 \times 8$ equals the well-remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property.

## MP 8 Look for and express regularity in repeated reasoning.

- Mathematically proficient students notice if calculations are repeated, and look for both general methods and shortcuts.
- They continue to look at their process and evaluate their results.
- They develop new methods by generalizing patterns.

The entire Common Core State Standards for mathematics can be viewed at www.corestandards.org/math/

Seventh Edition

# Mathematics for Elementary School Teachers

TOM BASSAREAR
*Keene State College*

MEG MOSS
*Western Governors University*

CENGAGE

Australia • Brazil • Mexico • Singapore • United Kingdom • United States

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**Tom Bassarear, Meg Moss**

Product Director: Mark Santee

Product Manager: Spencer Arritt

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Marketing Manager: Shannon Hawkins

Content Manager: Adrian Daniel

Learning Designer: Mona Zeftel

IP Analyst: Reba Frederics

IP Project Manager: Carly Belcher

Production Service/Compositor: SPi Global

Senior Designer: Diana Graham

Cover Designer: Vernon Boes

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# BRIEF CONTENTS

# CONTENTS

# ABOUT THE AUTHORS

Tom Bassarear

I taught the Mathematics for Elementary School Teachers course for more than 20 years, and in that time, I learned as much from my students as they learned from me. This text was inspired by my students and reflects one of the most important things we have taught one another: that building an understanding of mathematics is an active, exploratory process, and ultimately a rewarding, pleasurable one. My own experience with elementary schoolchildren and my two children, Emily and Josh, has convinced me that young children naturally seek to make sense of the world they live in and for a variety of reasons many people slowly lose that curiosity over time. My hope is that this book will engage your curiosity about mathematics once again. I am fully retired now and enjoying my first grandchild, and I am so excited about the changes that Meg has made in this edition.

Meg Moss

I am excited and honored to be working with Tom Bassarear on this book. I began teaching Mathematics for Elementary Teachers over 20 years ago. I immediately began seeking advice from others who had taught the course and volunteered in elementary classrooms to learn more. Teaching these courses has deepened my mathematical understanding as well as my understanding of how people learn math. Helping future elementary school teachers to truly understand mathematics and see the beauty in mathematics is very rewarding, and I know that all of this will have a major positive impact on their future students. I appreciate you sharing this journey with me!

# A MESSAGE FROM TOM BASSAREAR

We want to introduce ourselves. I began my teaching career in 1973 (I know, "long time ago"), and I just loved teaching. However, I was stunned that so many otherwise intelligent high school students could be so poor at math. Nine years later, after a stint teaching math in Nepal in the Peace Corps, I went to graduate school to get a better understanding of why so many adults had difficulty with math. Four years later, I had a much better understanding, and I became an assistant professor at Keene State College in New Hampshire where I worked until my retirement in 2016.

The idea to write the book came to me in 1993 when a textbook representative asked me if I was happy with the book I was using. I said, "No, but I'm not happy with any of the books." When the rep asked me why and what I was looking for, I explained in detail. Her response was to ask me if I'd consider writing a book. I said, "I don't think what I want would sell because it's out of the mainstream." The rep replied that she felt the time was ripe for a book that was "outside the box." Three months later, I sent out a book proposal. Houghton Mifflin agreed that the time was right for a new kind of book for future teachers of elementary school mathematics, because there had been such a strong positive reaction to the groundbreaking standards by the National Council of Teachers of Mathematics in 1989. Houghton let me write the books I wanted—the textbook and the *Explorations Manual*, which had problems designed to develop stronger problem-solving and reasoning skills. I wrote the textbook in a conversational style (and have received emails from students telling me this was the first math textbook they could actually read). I also use the term "Investigations" for the examples in the book because I had learned that people retain (own) the knowledge they learn in class much better if they are more actively involved as opposed to simply listening to the teacher lecture. After the fourth edition, Houghton Mifflin sold their college textbook division to Cengage, and I have been delighted with all the support from Cengage and the new features that have been added since then.

After I finished the fifth edition, I told my editors that I wanted to bring a new author onto the book. We asked interested teachers to submit their ideas for a new edition, and Meg's work stood out heads and shoulders above the others. Meg and I worked closely together on the sixth edition. I was so impressed by her dedication and integrity in wanting to make mathematics meaningful and accessible to all students. Other college teachers were also impressed because the sixth edition was well received. I am now fully retired and enjoying my first grandchild, and I am so excited at the changes that Meg has made in this edition.

# NEW TO THE SEVENTH EDITION!

We are excited about this edition and the changes made. We maintained the same philosophy in this edition and expanded the ideas of "owning" math knowledge and actively engaged learners. The main changes to the text are:

- More colors have been added to the book for visual appeal as well as to make the concepts clearer for students.
- More Investigations have been added, especially ones that help develop concepts through a concrete, pictorial, and then abstract pathway. This framework for learning math concepts was developed by Jerome Bruner and has shown success in Singapore.
- The information in the sidebars have been incorporated into the text as students tend to find sidebars distracting.
- The WebAssign course now includes explorations, conceptual questions, and classroom videos. We plan yearly updates and welcome your feedback.
- The Explorations Manual is now available to use for free on both the instructor's and student's websites.
- Because students tended to get bogged down in Chapter 1, it has been shortened, with the investigations integrated into the related content chapter. The concept of developing a "mathematical mindset" has been added to this chapter. The section on Sets, which was previously in Chapter 2, has been moved to Chapter 1 as a way to get started.
- Chapters 2–4 have been reorganized with the operations as the framework instead of number sets.
    - Chapter 2 goes in depth into each of the number sets from whole numbers to real numbers.
    - Chapter 3 develops the concepts of addition and subtraction, showing both the relationships between these two operations and how these operations are similar to each of the different number sets. This enables students to better see these connections.
    - Chapter 4 develops the concepts of multiplication and division, again showing the connections between the operations and the number sets.
- Chapter 7 has been reorganized using the National Council of Teachers of Mathematics Principles and Standards of School Mathematics as a framework. The first three sections develop concepts around statistics. The fourth section focuses on how probability and counting techniques are connected. The examples and data sets in Chapter 7 have been updated.
- Number Theory concepts are integrated where they are applicable, such as connecting the greatest common factor with simplifying fractions.
- The pages from elementary school math books have all been updated.
- All of the chapters have been revised and reviewed with attention to detail, brevity, and clarity, as well as ensuring that all examples are current.
- The Geometry chapters (8–10) have been enhanced with added colors and Investigations dedicated to developing concepts.

## Chapter 1 Foundations for Learning Mathematics

This chapter introduces the NCTM standards as well as the Mathematical Practices of the Common Core State Standards. The idea of a mathematical growth mindset has been added. The first section of this chapter lays the groundwork for seeing the standards "in action" and to understanding mathematics as a connected and interesting subject. We have integrated some of the problem-solving investigations into the other chapters where the content was more relevant. The concepts of sets has been moved into Chapter 1 to lay the framework of thinking about the structure and connectedness of concepts.

## Chapter 2 The Number System

This chapter develops the number system to help students to see the connections. Intention was especially put into helping readers to see that fractions are just numbers and not some completely different concept. Section 2.1 includes the development of children's understanding of numeration and its historical development, both of which students find fascinating. Exploration 2.2 (Alphabitia) is one of the most powerful explorations we have used. Most students report this to be the most significant learning and turning point in the semester. The exploration unlocks important understandings related to numeration, which the text supports by discussing the evolution of numeration systems and exploring different bases, giving students a strong understanding of the experience of young students trying to learn math. Section 2.2 is dedicated to developing the concept of fractions; Section 2.3 then develops the concepts of decimals, integers, and real numbers.

## Chapter 3 Understanding Addition and Subtraction

This chapter connects the operations of addition and subtraction, as well as showing how adding fractions is related to adding whole numbers, which is related to adding decimals, and so forth. Section 3.1 develops models and pictorial representations for the addition of whole numbers, including doing so with other bases to further develop place value concepts. Section 3.2 develops the concept of subtraction of whole numbers in a similar manner. Section 3.3 is dedicated to developing addition and subtraction of fractions; Section 3.4 repeats the experience with decimals and integers.

Students see how the concepts of the operations, coupled with an understanding of base ten, enable them to understand how and why procedures that they have performed by memorization for years actually work. In addition to making sense of standard algorithms, we present alternative algorithms in both the text and explorations. Students have found these algorithms to be both enlightening and fascinating.

## Chapter 4 Understanding Multiplication and Division

This chapter connects the operations of multiplication and division with different number sets. It is organized similarly to Chapter 3 in that the first section focuses on multiplication models and understanding with whole numbers and then follows it with division in the second section. Section 4.3 is focused on multiplication and division of fractions, while Section 4.4 does the same with decimals and integers.

## Chapter 5 Proportional Reasoning

The investigations and explorations in Chapter 5 are conceptually rich and provide many real-life examples so that students can enjoy developing an understanding of multiplicative relationships. We look at percentages as ratios and also make connections with fractional thinking.

## Chapter 6 Algebraic Thinking

Chapter 6 explores patterns, the concept of a variable, and solving equations and inequalities using different models, including Singapore bar models. The four sections are arranged under the National Council of Teachers of Mathematics (NCTM) algebra structure of understanding patterns, relations, and functions; representing and analyzing math situations and structures using algebraic symbols; using mathematical models to represent and understand quantitative relationships; and analyzing change in various contexts.

## Chapter 7 Understanding Statistics and Probability

In this chapter, students carefully walk through the stages of defining a question, collecting data, interpreting data, and then presenting data. We are particularly excited that the investigations with the concepts of mean and standard deviation remain successful with students. As a result, students can express these ideas conceptually instead of simply reporting the procedure. The sections in this chapter have been rearranged using the NCTM Standards as a framework. Probability has been put in the same section as counting to help students see how these concepts are related.

## Chapter 8 Geometry as Shape

In this chapter, you have the option of introducing geometry through explorations with tangrams, Geoboards, or pentominoes. This more concrete introduction allows students with unpleasant or failing memories of geometry to build confidence and understanding while engaging in rich mathematical explorations.

## Chapter 9 Geometry as Measurement

This chapter addresses measurement from a conceptual framework (such as identify the attribute, determine a unit, and determine the amount in terms of a unit) and a historical perspective. Both the explorations and investigations get students to make sense of measurement procedures and to grapple with fundamental measurement ideas. The text looks at the larger notion of measurement, presents the major formulas in a helpful way, and illustrates different problem-solving paths.

## Chapter 10 Geometry as Transforming Shapes

The geometric transformations that we explore in Chapter 10 can be some of the most interesting and exciting topics of the course. Quilts and tessellations both spark lots of interest and provoke good mathematical thinking. The text develops concepts and introduces terms that help students to refine understanding that emerges from explorations.

# PREFACE

## Owning versus Renting

This course is about developing and *retaining* the mathematical knowledge that students will need as beginning mathematics teachers. We prefer to say that we are going to *uncover* the material rather than *cover* the material. The analogy to archaeology is useful. When archaeologists explore a site, they carefully *uncover* the site. As time goes on, they see more and more of the underlying structure. This is exactly what can and should happen in a mathematics course. When this happens, students are more likely to *own* rather than to *rent* the knowledge.

There are three ways in which this textbook supports owning versus renting:

1. Knowledge is constructed.
2. Connections are reinforced.
3. Problems appear in authentic contexts.

### 1. Constructing Knowledge

When students are given problems, such as appear in Investigations throughout the textbook, that involve them in grappling with important mathematical ideas, they learn those ideas more deeply than if they are simply presented with the concepts via lecture and then are given problems for practice. There is a need to shift the focus from students studying mathematics to students doing mathematics. That is, students are looking for patterns, making and testing predictions, making their own representations of a problem, inventing their own language and notation, and so on. Developing concepts starting with a concrete investigation, moving to a pictorial mode before moving to the abstract helps students to construct their own understanding of the concepts.

*Investigation 1.1f (Pigs and Chickens)* confronts a common misconception—that there is one right way to solve math problems—by exploring five valid solution paths to the problem. This notion of multiple solution paths is an important part of the book.

**INVESTIGATION 1.1f** **Pigs and Chickens**

A farmer has a daughter who needs more practice in mathematics. One morning, the farmer looks out in the barnyard and sees a number of pigs and chickens. The farmer says to her daughter, "I count 24 heads and 80 feet. How many pigs and how many chickens are out there?"

Before reading ahead, work on the problem yourself or, better yet, with someone else. Close the book or cover the solution paths while you work on the problem.

Compare your answer to the solution paths below.

**DISCUSSION**

**STRATEGY 1** **Use random trial and error**

One way to solve the problem might look like what you see in Figure 1.4.

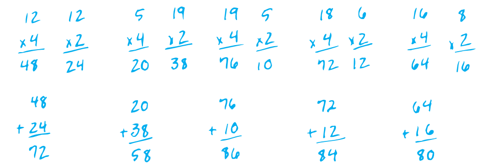



Figure 1.4

## 2. Reinforcing Connections

Understanding can be defined in terms of connections; that is, the extent to which you *understand* a new idea can be seen by the *quality* and *quantity* of connections between that idea and what you already know. There are two ways in which connections are built into the structure of the text.

### 1. Mathematical connections

Owning mathematical knowledge involves connecting new ideas to ideas previously learned. It also involves truly understanding mathematics, not just memorizing formulas and definitions.

- **CONNECTIONS AMONG CONCEPTS ARE EMPHASIZED**

Throughout the book, connections between concepts are developed and presented as crucial for deep understanding. Chapter 2 focuses on developing numeracy by showing the connections between the sets of numbers. Chapters 3 and 4 both develop the connections between the operations and continue the connections between the different number sets by showing that adding one set of numbers is a similar concept to adding another set of numbers. Connections are shown between Statistics and Probability in Chapter 7, while the Geometry chapters also continue the numeracy, as well as connect to the graphs covered in Statistics.

- **THE *HOW* IS CONNECTED TO *WHY***

In this way, students know not only how the procedure works but also why it works. For example, students understand why we move over when we multiply the second row in whole number multiplication; they realize that "carrying" and "borrowing" essentially equate to trading tens for ones or ones for tens; they understand why we first find a common denominator when adding fractions; and they see that $\pi$ is how many times you can wrap any diameter around the circle.

## 2. Connections to children's thinking

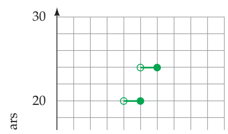
In this book you will see a strong focus on children's thinking, for two reasons. First, much work with teachers focuses on the importance of listening to the students' thinking as an essential part of good teaching. If students experience this in a math course, then by the time they start teaching, it is part of how they view teaching. Second, when students see examples of children's thinking and see connections between problems in this course and problems children solve, both the quality and quantity of the students' cognitive effort increase.

If it is true that we tend to teach the way we were taught, we need to consider the experience we are creating in these courses.

## 3. Authentic Problems

Although most texts have many "real-life" problems, this text differs in how those problems are made and presented.

In Section 6.3, the question of paying a baby-sitter is explored. This situation is often portrayed as a linear function: for example, if the rate is \$10 per hour, $y = 10x$. However, in actuality, it is not a linear function but rather a stepwise function.




**A closer look at paying the baby-sitter** At first glance, the question of how much to pay the baby-sitter is simple: Multiply the hours sat by 8. However, let us use the problem-solving strategy "act it out" to examine this problem more closely. For example, what if Ellen baby-sat from 7 to 11:15? How much would you pay her? Think before reading on. . . .

Some people say \$32. Some people say \$36—they round up to the nearest half-hour. In actuality, different people have different ways of determining how much to pay a baby-sitter. Let us examine the case of a couple, who rounds up the time to the nearest half-hour. We could now represent their process for paying the baby-sitter in each of the ways we have just examined. Is the relationship between time sat and dollars earned a functional relationship in the case of the couple? Think and then read on. . . .

If you were to graph this relationship, what would the graph look like? Try to make your own graph before reading on. . . .

Take a look at the graph on the left and see whether you can make sense of it before reading on.

If you are having trouble making sense of the graph, consider a few examples.

Similarly, when determining the cost of carpeting a room, the solution path is often presented as dividing the area of the room by the cost per square yard; again, this is not how the cost is actually determined.

In this book, you will find many problems—problems we have needed to solve, problems friends have had, problems children have had, problems we have read about—where the content fits with the content of this course.

You will find problems in the text where students are asked to state the assumptions they make in order to solve the problem (such as Section 5.2, Exercises 33 to 38). You also will find problems that have the messiness of "real-life" problems, where the problem statement is ambiguous, too little or too much information is given, or the information provided is contradictory.

*Problems 33–38 require you to make some assumptions in order to determine an answer. Describe and justify the assumptions you make in determining your answer.*

**33.** Let's say that you read in the newspaper that last year's rate of inflation was 7.2%.

**a.** If your grocery bill averaged \$325 per month last year, about how much would you expect your grocery bill to be this year?

**b.** Let's say you received a \$1200 raise, from \$23,400 per year to \$24,600 per year. Did your raise keep you ahead of the game, or are you falling behind?

**34.** A government agency is recommending that the legal definition of drunk driving be reduced from an alcohol blood content of 0.08 to 0.05. Explain why some might consider this a little drop and others might consider it a big drop. What do you think?

**35.** Which would you prefer to see on a sale sign at a store: \$10 off or 10% off? Explain your choice.

**36.** Refer to Investigation 5.2b. Jane still doesn't understand the problem. Roberto tries to help her make sense of the problem by saying that the 8% means that if we were to select 100 students at the college, 8 of them would be working full-time. What do you think?

# Features

## What do you think?

What-do-you-think questions appear at the start of each section to help students focus on key ideas or concepts that appear within the sections.

**6.1 Understanding Patterns, Relations, and Functions**

***What do you think?***

- How are patterns related to algebraic thinking?
- What are some examples of functions in everyday life?
- What is a reason for developing algebraic thinking in elementary school?

## Investigations

Investigations are the primary means of instruction, uniquely designed to promote active thinking, reasoning, and construction of knowledge. Each investigation presents a problem statement or scenario that students work through, often to uncover a mathematical principle relevant to the content of the section. The "Discussion" that follows the problem statement provides a framework for insightful solution logic.

**INVESTIGATION 3.1h Children's Mistakes**

The problem below illustrates a common mistake made by many children as they learn to add. Understanding how a child might make that mistake and then going back to look at what lack of knowledge of place value, of the operation, or of properties of that operation contributed to this mistake is useful. What error on the part of the child might have resulted in this wrong answer?

The problem: $38 + 4 = 78$

**DISCUSSION**

In this case, it is likely that the child lined up the numbers incorrectly:

$$\begin{array}{r} 4 \\ +\ 38 \\ \hline 78 \end{array}$$

Giving other problems where the addends do not all have the same number of places will almost surely result in the wrong answer. For example, given $45 + 3$, this child would likely get the answer 75. Given $234 + 42$, the child would likely get 654. In this case, the child has not "owned" the notion of place value. Probably, part of the difficulty is not knowing expanded form (for example, that 38 means $30 + 8$—that is, 3 tens and 8 ones). An important concept here is that we need to add ones to ones, tens to tens, and so on. Base ten blocks provide an excellent visual for this concept as students can literally see why they cannot add 4 ones to 3 tens.

## Questions in the Text

To encourage active learning outside of the Investigations, questions appear embedded within the text, often accompanied by the icon . These "thinking" questions require students to pause in their reading to reflect or to complete a short exercise before continuing. Answers to these questions can be found in Appendix B (available in MindTap).

Translate the following Babylonian numerals into our system. Check your answers in Appendix B.

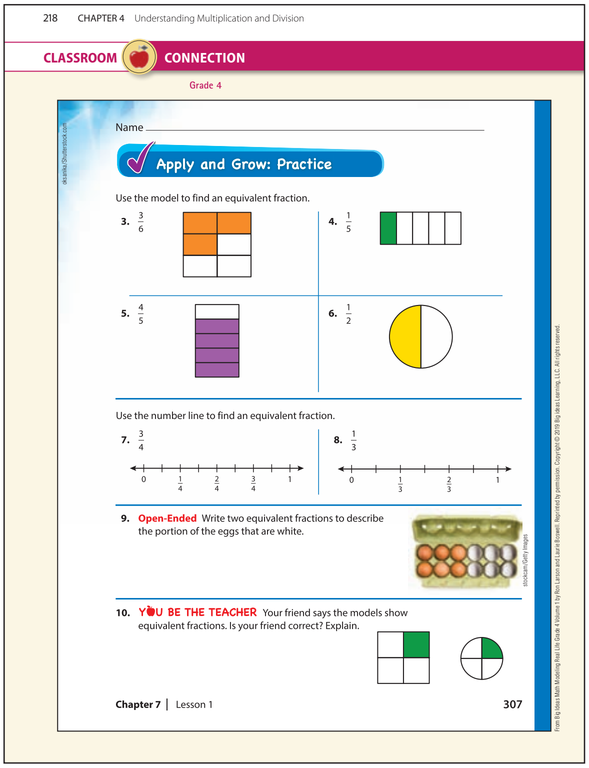
**1.**   **2.**   **3.** 

Translate the following amounts into Babylonian numerals.

**4.** 1202  **5.** 304

## Classroom Connections

Assignments from actual elementary/middle school books appear throughout so that students can see how the material they are learning will directly apply. Connections are also found in the exercises that highlight children's work.


218 CHAPTER 4 Understanding Multiplication and Division

CLASSROOM CONNECTION

Grade 4

Name

Apply and Grow: Practice

Use the model to find an equivalent fraction.

3. $\frac{3}{6}$  4. $\frac{1}{5}$

5. $\frac{4}{5}$  6. $\frac{1}{2}$

Use the number line to find an equivalent fraction.

7. $\frac{3}{4}$  8. $\frac{1}{3}$

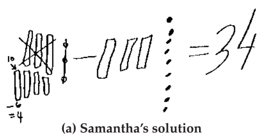
9. Open-Ended Write two equivalent fractions to describe the portion of the eggs that are white.

10. YOU BE THE TEACHER Your friend says the models show equivalent fractions. Is your friend correct? Explain.

Chapter 7 | Lesson 1  307

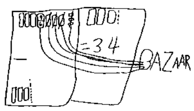
**11.** Following are two children's solutions to 73 − 39. Study each child's work, and then describe how that child would find the answer to 65 − 28. Then explain why that method works.

a.



(a) Samantha's solution

b.



(b) Alice's solution

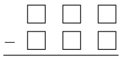
Source: *Teaching Children Mathematics*, by the National Council of Teachers of Mathematics December, 2001, p. 231.

## Section Exercises

The exercises are designed to give students a deeper sense and awareness of the kinds of problems their future students are expected to solve at various grade levels, as well as to increase their own proficiency with the content. *From Standardized Assessments* exercises derive from exams such as the NECAP and NAEP to give students a sense of the types of questions found on diverse national exams at various grade levels. Questions are also included from the Smarter Balanced Assessment Consortium which is developing assessments for Common Core State Standards.

Source: National Council of Teachers of Mathematics.

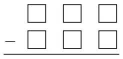
**28.** Place the digits 1, 2, 3, 6, 7, and 8 in the boxes to obtain



**a.** The greatest difference

**b.** The least difference

**29.** Choose among the digits 1, 2, 3, 4, 5, 6, 7, 8, and 9 to make the difference 234. You can use each digit only once. How many different ways can you make 234?



**30.** With three boys on a large scale, it read 170 pounds. When Adam stepped off, the scale read 115 pounds. When both Adam and Ben stepped off, the scale read 65 pounds. What is the weight of each boy?

**31.** A mule and a horse were carrying some bales of cloth. The mule said to the horse, "If you give me one of your bales, I shall carry as many as you." "If you give me one of yours," replied the horse, "I will be carrying twice as many as you." How many bales was each animal carrying?

**FROM STANDARDIZED ASSESSMENTS**

NECAP 2006, Grade 5

**36.** Mrs. Lombardi had 2 hours to prepare for a party. The chart below shows the amount of time she spent completing different tasks.

**TIME MRS. LOMBARDI SPENT ON DIFFERENT TASKS**

| Task | Time |
|---|---|
| Decorated cake | 20 minutes |
| Made punch | 15 minutes |
| Made sandwiches | 50 minutes |
| Put up balloons | ? |

How much time did Mrs. Lombardi have to put up the balloons? (1 hour = 60 minutes)

**a.** 15 minutes **b.** 25 minutes

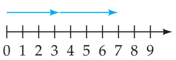
**c.** 35 minutes **d.** 45 minutes

## Exercises 3.1

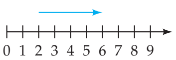
**1.** Reread the 8 Mathematical Practices of the CCSS in Section 1.1, or on the corestandards.org website and write about relationships between them and what we did in this section.

**2.** For each number line problem below, identify the computation it models and briefly justify your answer.

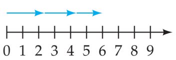
**a.**



**b.**



**c.**



**3. a.** Draw a picture that shows that addition is commutative.

**b.** Draw a picture that shows that addition is associative.

**9.** Through illustrations, demonstrate how to solve these problems with manipulatives.

**a.** 76 + 47 **b.** 524 + 268

**10.** Below is an addition algorithm from an old text. Explain why it works.

```
 36
+48
 14
 7
 84
```

**11.** How does this addition algorithm work? Imagine someone asking, "How do you know where to put the 1 and the 3 in the 13?"

## Section Summary

Each section ends with a summary that reviews the main ideas and important concepts discussed.

**SUMMARY 3.2**

We have now examined addition and subtraction rather carefully. In what ways do you see similarities between the two operations? In what ways do you see differences? Think and then read on. . . .

One way in which the two processes are alike is illustrated with the part–whole diagram used to describe each operation. These representations help us to see connections between addition and subtraction. In one sense, addition consists of adding two parts to make a whole. In one sense, subtraction consists of having a whole and a part and needing to find the value of the other part.

We see another similarity between the two operations when we watch children develop methods for subtraction; it involves the "missing addend" concept. That is, the problem $c - a$ can be seen as $a + ? = c$.

We saw a related similarity in children's strategies. Just as some children add large numbers by "adding up," some children subtract larger numbers by "subtracting down."

Earlier in this section, subtraction was formally defined as $c - b = a$ if $a + b = c$. The negative numbers strategy that some children invent brings us to another way of defining subtraction, which we will examine further in Section 3.2 when we examine negative numbers. That is, we can define subtraction as adding the inverse: $a - b = a + -b$.

A very important way in which the two operations are different is that the commutative and associative properties hold for addition but not for subtraction.

## Looking Back

Each chapter concludes with *Looking Back*—a study tool that brings together all the important points from the chapter. *Looking Back* includes *Questions to Summarize Big Ideas*, which ask students to reflect on the main ideas from the chapter; *Chapter Summary*, which lists major take-aways and terminology from the chapter; and *Review Exercises*, which provide an opportunity for students to put concepts from the chapter into practice.

**LOOKING BACK on Chapter 3**

**QUESTIONS TO SUMMARIZE BIG IDEAS**

1. What are some of the different models for addition and subtraction?
2. How can you use base ten blocks to model the algorithms for addition and subtraction of whole numbers and decimals?
3. How are these models similar and different in a base other than ten?
4. Which algorithms for the operations are different from what you learned in elementary school?
5. Look back at the Mathematical Practices of the Common Core State Standards. In what ways did you engage in those practices during this c
6. What parts of this chapter are less clea

**CHAPTER 3 SUMMARY**

1. There are many concrete and pictorial models for addition and subtraction.
2. Each operation has multiple meanings.
3. Many algorithms have been developed to enable us to compute more efficiently.
4. The standard algorithm for each operation does not connect equally well to each meaning of the operation.
5. Being able to make sense of algorithms requires:
   - The ability to apply base ten and place value concepts
   - The ability to compose and decompose the numbers (for example, to use expanded form)
6. Patterns enable us to understand the operations more deeply.
7. In many real-life problems, the answer depends on knowing how to interpret one's computation.
8. Being able to perform mental math and to estimate requires:
   - The ability to apply base ten and place value concepts
   - The ability to compose and decompose the numbers (for example, to use expanded form)
   - The ability to apply properties of the operations, especially the commutative, associative, and distributive properties
9. In real-life problem solving, one needs to know when to find an exact answer and when to find an estimate.

**Addition properties:**
identity 106 commutative 107
associative 107 closure 107

**Algorithms for addition:**
common 116 lattice 117

**Other terminology:**
matrix 107 composed 117
algorithm 114 decompose 117

**Section 3.2: Understanding Subtraction of Whole Numbers**

**Subtraction contexts:**
take-away 128 comparison 129
missing addend 129

**Subtraction terminology:**
subtraction 130 minuend 130
subtrahend 130 difference 130

**Subtraction models:**
number line 132

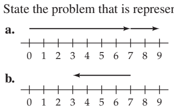
**Section 3.3: Understanding Addition and Subtraction with Fractions**

**REVIEW EXERCISES Chapter 3**

1. State the problem that is represented in each case below:

a.

b.




Explain why it works, as though to a parent who only the traditional right-to-left algorithm.

832
+ 549
1300
70

## Explorations

The icon that appears throughout the text references additional activities that may be found in the *Explorations Manual*. Explorations present new ideas and concepts for students to engage with "hands on." The *Explorations Manual* is now available for free use on both the Instructors and Students website for the book. It can be accessed at login.cengage.com.

Explorations Manual 1.5

### MP 4: Model with Mathematics.

- Mathematically proficient students can solve real-life problems, which in elementary school includes being able to write a multiplication equation to solve a problem.
- They are able to identify important information in a real-life problem and analyze relationships using tools.
- They can use models to draw conclusions, make predictions, and reflect on and adjust the effectiveness of the model.

### MP 5: Use Appropriate Tools Strategically.

- Mathematically proficient students can use a variety of tools such as concrete models and technology to and solutions.
- They are able to make good decisions about when to use each of these tools and how to effectively use them.
- They are able to use a variety of tools to investigate and develop their understanding of ideas.

| SUPPLEMENTS | |
|---|---|
| **FOR THE STUDENT** | **FOR THE INSTRUCTOR** |
| | **Instructor's Edition**<br>**(ISBN: 978-0-357-02551-2)**<br>The Instructor's Edition includes answers to all exercises in the text, including those not found in the student edition. (Print) |
| **Student Solutions Manual**<br>**(ISBN: 978-0-357-02553-6)**<br>Go beyond the answers—see what it takes to get there and improve your grade! This manual provides worked-out, step-by-step solutions to the odd-numbered problems in the text. This gives you the information you need to truly understand how these problems are solved. | **Complete Solutions Manual**<br>The Complete Solutions Manual provides worked-out solutions to all of the problems in the text. In addition, instructors will find helpful aids such as "Teaching the Course," which shows how to teach in a constructivist manner. "Chapter by Chapter Notes" provide commentary for the *Explorations* manual as well as solutions to exercises that appear in the supplement. This manual can be found on the Instructor Companion Site. |
| ***Explorations, Mathematics for Elementary School Teachers,* 7e**<br>**(ISBN: 978-0-357-02552-9)**<br>This manual contains open-ended activities for you to practice and apply the knowledge you learn from the main text. When you begin teaching, you can use the activities as models in your own classrooms. Posted on the Student Resource Center. | ***Explorations, Mathematics for Elementary School Teachers,* 7e**<br>**(ISBN: 978-0-357-02552-9)**<br>This manual contains open-ended activities for students to practice and apply the knowledge they learn from the main text. When students begin teaching, they can use the activities as models in their own classrooms. Posted on the Instructor Companion Site at login.cengage.com |
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| | **Instructor Companion Site**<br>Everything you need for your course is in one place! This collection of book-specific lecture and classroom tools is available online via www.cengage.com/login. Access and download Lecture Notes PowerPoints, images, Complete Solutions Manual, and more. |
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# 1 Foundations for Learning Mathematics



*Knowing mathematics means being able to use it in purposeful ways. To learn mathematics, students must be engaged in exploring, conjecturing, and thinking rather than only in rote learning of rules and procedures. Mathematics learning is not a spectator sport. When students construct personal knowledge derived from meaningful experiences, they are much more likely to retain and use what they have learned. This fact underlies [the] teacher's new role in providing experiences that help students make sense of mathematics, to view and use it as a tool for reasoning and problem solving.*[1]

—National Council of Teachers of Mathematics

## SECTION 1.1 What Is Mathematics?

### *What do you think?*

- Respond to the prompt: Mathematics is _______________.
- Describe a few of your experiences learning mathematics.
- What does it mean to learn mathematics?

You are at the beginning of a course where you will re-examine elementary school mathematics to understand these concepts on a much deeper level, and to learn why the mathematical procedures and formulas actually work. On this journey, you will learn several ways to see and think about concepts and procedures that you may have previously simply memorized. This deeper understanding will lead to increased confidence and comfort level with mathematics. Mathematics is both beauty and truth. Two plus two always equals four. The distance around a circle is always a little more than three (actually pi) times the distance across the circle. Throughout this book, we hope you will appreciate more and more of the beautiful truths of mathematics.

Your approach to learning and teaching mathematics depends on the attitudes and beliefs you bring; in subtle and not so subtle ways, they may affect your learning of math and you may pass these beliefs along when you enter the classroom as a teacher. Reflect on how you answered the questions above. Whatever your feelings about mathematics, consider where these feelings come from. Research suggests that people who have math anxiety can relate it back to a teacher and/or experience in their elementary or middle school years. Think about the best math teacher you have had as well as the worst math teacher you have had. Consider the skills and qualities that each of them had that led to your experience of them. What skills and qualities do you have and need to further develop to become an excellent math learner and teacher?

## Mathematical Growth Mindset

Do you have a fixed mindset or a growth mindset when it comes to learning math? Take this survey to find out.

Seven pairs of statements are given in Table 1.1. Score your beliefs in the following manner:

- If you strongly agree with the statement in column A, record a 1.
- If you agree with the statement in column A more than column B, record a 2.
- If you agree with the statement in column B more than column A, record a 3.
- If you strongly agree with the statement in column B, record a 4.

**Table 1.1**

| Column A | | Column B |
|---|---|---|
| **1.** There will be many problems in this book that I won't be able to solve, even if I try really hard. | 1 2 3 4 | **1.** I believe that if I try really hard, I can solve virtually every problem in this book. |
| **2.** There is only one way to solve most problems. | 1 2 3 4 | **2.** There is usually more than one way to solve most problems. |
| **3.** The best way to learn is to memorize the different kinds of problems and the steps to solve them. | 1 2 3 4 | **3.** The best way to learn is to make sure that I understand the concepts and each step I take to solve the question. |
| **4.** Some people have mathematical minds and some don't. Nothing they do can *really* make a difference. | 1 2 3 4 | **4.** Everyone can learn math with the right opportunities to learn and hard work. |
| **5.** I get frustrated when I make a mistake and want to give up. | 1 2 3 4 | **5.** I am comfortable making mistakes because mistakes help me to learn. |
| **6.** Mathematics is about getting the right answer by quickly recalling math facts. | 1 2 3 4 | **6.** Mathematics is about problem solving and critical thinking. |
| **7.** I only plan to teach very young children, so all I need to know is basic numbers and operations. | 1 2 3 4 | **7.** I need to develop a deeper understanding of all elementary school math topics. |
| | Total_______ | |

In Table 1.1, the statements in column A indicate a fixed mindset, and the statements in column B indicate the corresponding growth mindset. If you take the arithmetic average, or *mean*, of your scores (by adding up your scores and dividing by seven), you will get a number that we could call your belief index. If your belief index is low, you are currently in more of a fixed mindset.

A main goal of this book is to help you understand math in a conceptual and flexible way to help you foster a "growth mindset" toward math and increase your self-confidence about math, which in turn you can pass on to your students. Many people see math as a set of rules and formulas to memorize instead of connected concepts that make sense. Deepening your understanding of mathematics through your studies will help you to have a positive attitude about math. The essence of a math growth mindset is to approach math conceptually, seek understanding, and believe that everyone can learn math.

If you have been told that you do not have a math brain, or learned to believe this—it is not true. Recent brain research shows us that, with the right learning opportunities and beliefs, everyone can learn math and reach high levels of math. Henry Ford is quoted as saying, "Whether you believe you can do a thing or not, you are right." As you engage with this book, believe you can make sense of the math concepts. If you catch yourself saying anything like "I am not good at math," change that to "I can learn math." By changing your words, you can change your mindset. That is the first step in learning and creating a growth mindset. Approach the concepts in this book with an open mind toward learning and a belief that you can learn the concepts. Your future students are depending on you to deepen your understanding of math and develop a growth mindset so that you can teach them.

In our current performance model of schools, where getting the right answer is valued, mistakes are seen as bad. However, recent brain research provides evidence that mistakes actually help our brain to grow. Our brains grow even more by making a mistake and learning from it (Moser, et al., 2011). When you follow a path on a math problem that does not get you to the answer, go back, see where you went down a nonproductive path, and correct the way you were thinking about it. Studies of successful people show that they make many more mistakes than unsuccessful people. Therefore, as you read through this book, be comfortable exploring ideas and making mistakes, learn from your mistakes, persevere through the solution, and success will be yours.

## Owning Versus Renting

You will notice that I often ask you to stop, think, and write some notes. I really mean it! Many students rent what they have learned just long enough to pass the test. However, within days or weeks of the final exam, it's gone. If you make sense of what you are learning and make connections, then you will remember the concepts. One of the important differences between students who own what they learn and those who simply rent is that those who own the knowledge tend to be active readers.

## Mathematical Knowledge for Teaching

Having a mathematical growth mindset is especially important for teachers, both in their own learning and their ability to help their students.

### INVESTIGATION 1.1a More than One Way to Multiply?

First, multiply $49 \times 25$ using any method you choose. Then consider how the following students solved the problem.

What method do you think each student is using in each solution below? Do you think each method will work for any two whole numbers?

**Note:** Whenever you see the pencil icon in this book, stop and think and briefly write your thoughts before reading on. Students who take the time to think and write after these points (or at least to pause and think) say that it makes a big difference in how much they learn.

Student A:

$$\begin{array}{r} 49 \\ \times\ 25 \\ \hline 245 \\ 98\phantom{0} \\ \hline 1225 \end{array}$$

Student B:

$$\begin{array}{r} 49 \\ \times\ 25 \\ \hline 45 \\ 200 \\ 180 \\ 800 \\ \hline 1225 \end{array}$$

#### DISCUSSION

We will look much more deeply into multiplication later in this book. We use this example to illustrate how part of the teacher's role is to first have a math mindset that there are different methods for solving problems; to be able to develop, understand, and communicate different strategies; and to see the connections between them. This book is not about methods of teaching; it is about developing a deep and flexible understanding of math and developing a mathematical growth mindset.

Student A used the method of $5 \times 49$ and $20 \times 49$ that many of us learned in elementary school, especially if we went to school in the United States. Different and equally effective methods are taught in other countries, and we will look at some of these in this text to deepen our understanding. Some of us learned to write the second row in the multiplication as 980, and some were taught to move over a space as is shown here. Why do we do this? Because we are multiplying $49 \times 20$, not $49 \times 2$, so we write it as either 980 or as 98 with a space in the ones place. Either way, the 8 is in the tens place and the 9 is in the hundreds place when we add to get our final answer. Place value is a key concept to deeply understand in mathematics.

We can use the properties and split the numbers up into their parts to figure this out.

$$49 \times 25 = 49 \times (20 + 5) = 49 \times 20 + 49 \times 5 = 980 + 245 = 245 + 980 = 1225$$

Student B used a method that is sometimes called partial products of $9 \times 5 = 45$, $40 \times 5 = 200$, $9 \times 20 = 180$, and $40 \times 20 = 800$. Another way to write this, which uses the distributive property more clearly, is:

$$49 \times 25 = (40 + 9) \times (20 + 5) = 40 \times 20 + 40 \times 5 + 9 \times 20 + 9 \times 5 =$$
$$600 + 200 + 180 + 45 = 1225$$

The methods of both students are valid. One of the major ideas in this text is that there are multiple ways (often called "solution paths") to get to an answer. There is no ONE "right" way to solve any math problem. This may be different from what you have always thought about mathematics. You may have even had teachers who marked you "wrong" if you did not solve it their way.

As a teacher of elementary school mathematics, a deep understanding of mathematics will enable you to respond to the above type of situation in an elementary school classroom. Simply being able to get the right answer is not sufficient. Teachers need a specialized understanding of mathematics that is flexible, connected, and conceptual.

Teachers use mathematics every day in the classroom, but in different ways than other people. In 1986, Lee Shulman used the term "pedagogical content knowledge" to refer to this specialized understanding of mathematics, which includes an understanding of multiple representations and examples, plus an understanding of what ideas may be more difficult for students and why these ideas are more difficult. In 2008, Deborah Loewenberg Ball, Mark Thames, and Geoffrey Phelps developed the framework depicted in Figure 1.1 showing different types of knowledge. This book is focused on the specialized content knowledge, which includes understanding multiple representations and multiple student procedures, and analyzing student errors. It is the type of math content knowledge that teachers draw on every day.

Explorations Manual 1.2

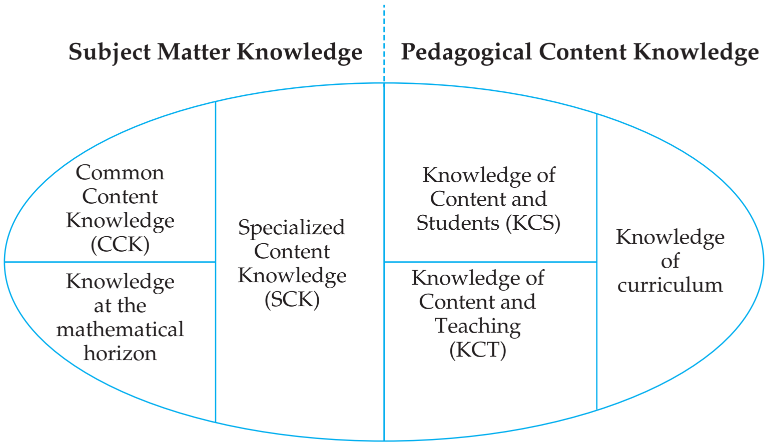



**Figure 1.1**
Source: Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, 39(4), 372–400.

## INVESTIGATION 1.1b Understanding Students' Errors

How do you think the answer was produced? What do you think the student was thinking that led to this error?

$$\begin{array}{r} 49 \\ \times\ 25 \\ \hline 245 \\ 98 \\ \hline 343 \end{array}$$

How does this type of scenario draw upon specialized content knowledge?

**DISCUSSION**

Analyzing student errors draws upon a specialized understanding of mathematics, in that the teacher needs to understand the mathematics deeply in order to identify the error and then to help the student to correct the misunderstanding.

This student does not understand that in the second step, we are actually multiplying 20 times 49, so it should be 980, not 98.

While most people could get a correct answer for $49 \times 25$, teachers need to understand the concept much more deeply. Many of us experienced elementary school mathematics as a series of procedures to memorize (like multiplying $49 \times 25$). In order to teach true understanding of mathematics, teachers must develop this specialized content knowledge and a mathematical growth mindset.

## Mathematical Problem Solving

A useful metaphor for problem solving is a **toolbox**. Some useful tools to explore in solving math problems include drawing a picture, making a table, using algebra, guess-check-revise, using concrete materials, looking for patterns, solving a simpler problem, and visualization tools. As you go through this course, you will learn new tools and how to more skillfully use tools you already have.

**INVESTIGATION 1.1c**

## Real-Life Problem Solving

Consider a few problems you have had in your life, and not necessarily math problems.

What steps did you take to solve these problems?

Use this recollection to make a list of general steps that you take to solve a problem. Then read on to learn about a mathematician that made a similar list. Your list will likely be pretty similar to his.

**DISCUSSION**

There are many kinds of real-life problems that you may have considered here. One that many of us have dealt with is what kind of car to buy. The first step would be to understand what you need. How many seats do you need? What is your budget? Is gas mileage a priority? What models do you like? This part might be called something like "understand the problem." Did you have a step like this in your list? The second step would be to develop a plan. Where will you look? What research will you do? The third step might be to carry out the plan. Research the best models, look around for the best deals, test drive some models, and find the one you like. The next step, hopefully before actually buying the one you have your heart set on, might be to look back and reflect on whether it really meets your needs, fits your budget, and so on. Read on to see how this process is the same for solving a math problem.

## Polya's Four Steps

George Polya developed a framework that breaks down problem solving into four distinguishable steps. In 1945, he outlined these steps in a now-classic book called *How to Solve It*.

### Four Steps for Solving Problems

#### Understanding the Problem

Questions that can be useful to ask:

1. Do you understand what the problem is asking?
2. Can you state the problem in your own words—that is, paraphrase the problem?
3. Have you used all the given information?
4. Can you solve a part of the problem?

Actions that can be helpful:

1. Reread the problem carefully. (Often it helps to reread a problem a few times.)
2. Try to use the given information to deduce more information.
3. Plug in some numbers to make the problem more concrete.

#### Devising a Plan

Several common strategies:

1. Represent the problem with a diagram (carefully drawn and labeled). Check to see if you used the (relevant) given information. Does the diagram "fit" the problem?
2. Guess–check–revise. Keep track of "guesses" with a table.
3. Make an estimate. The estimate often serves as a useful "check." A solution plan often comes from the estimation process.
4. Make a table (sometimes the clue comes from adding a new column).
5. Look for patterns—in the problem or in your guesses.
6. Be systematic.
7. Look to see if the problem is similar to one you have already solved.
8. If the problem has "ugly" numbers, you may "see" the problem better by substituting "nice" numbers and then thinking about the problem.
9. Break the problem down into a sequence of simpler "bite-size" problems.
10. Act it out.

#### Carrying Out the Plan

1. Are you keeping the problem meaningful? On each step, ask what the numbers mean. Label your work.
2. Are you bogged down? Do you need to try another strategy?

#### Looking Back

1. Does your answer make sense? Is the answer reasonable? Is the answer close to your estimate, if you made one?
2. Does your answer work when you check it with the given information? (Note that checking the procedure checks the computation but not the solution.)
3. Can you use a different method to solve the problem?

### Using Polya's Four Steps

I encourage you to use Polya's four steps in all of the following ways:

1. Use them as a guide when you get stuck.
2. Don't rent them, buy them. Buying them involves paraphrasing my language and adding new strategies that you and your classmates discover. For example, many of my students have added a step to help reduce anxiety: First take a deep breath and remind yourself to slow down!
3. After you have solved a problem, stop and reflect on the tools you used. Over time, you should find that you are using the tools more skillfully.

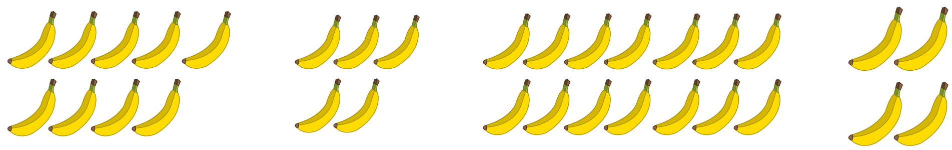
## Why Emphasize Problem Solving?

Although Polya described his problem-solving strategies back in 1945, it was quite some time before they had a significant impact on the way mathematics was taught.

## CLASSROOM CONNECTION

Grade 2

3. **Structure** There are 9 bananas in a bunch. A monkey eats some. There are 5 bananas left. Which picture shows how many bananas the monkey ate?



4. **Modeling Real Life** You make 4 snow angels and Newton makes 5. Descartes makes 3 more snow angels. How many snow angels are there in all?

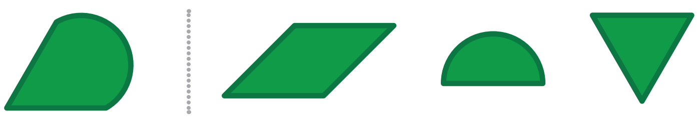
______ snow angels

5. **Modeling Real Life** Newton has 9 glitter pens. Descartes has 2 fewer than Newton. How many glitter pens do they have in all?

______ glitter pens

### Review & Refresh

6. Which two shapes combine to make the shape on the left?



One of the reasons is that until recently, "problems" were generally defined too narrowly. Many of you learned how to do different kinds of problems—mixture problems, distance problems, percent problems, age problems, coin problems—but never realized that they have many similarities and connections. There has been too great a focus on single-step problems and routine problems instead of on developing a growth mindset in math. Consider the examples from the National Assessment of Educational Progress shown in Table 1.2.

**Table 1.2**

| Problem | Percent correct Grade 11 |
|---|---|
| **1.** Here are the ages of six children: 13, 10, 8, 5, 3, 3. What is the average age of these children? | 72 |
| **2.** Edith has an average (mean) score of 80 on five tests. What score does she need on the next test to raise her average to 81? | 24 |

Source: Mary M. Lindquist, ed., Results from the *Fourth Mathematics Assessment of the National Assessment of Educational Progress* (Reston, VA: NCTM, 1989), pp. 30, 32.

To solve the first problem, one only has to remember the procedure for finding an average and then use it:

$$\frac{13+10+8+5+3+3}{6}$$

However, there is no simple formula for solving the second problem. Try to solve it on your own and then read on. . . .

To solve this one, you have to have a better understanding of what an average means. One approach is to see that if her average for 5 tests is 80, then her total score for the 5 tests is 400. If her average for the 6 tests is to be 81, then her total score for the 6 tests must be 486 (that is, $81 \times 6$). Because she had a total of 400 points after 5 tests and she needs a total of 486 points after 6 tests, she needs to get an 86 on the sixth test to raise her overall average to 81.

Many students consider the second question to be a "trick" question unless the teacher has explicitly taught them how to solve that kind of problem. However, many employers note that problems that occur in work situations are rarely just like the ones in the book. What employers need is more people who can solve the "trick" problems, because, as some may say, "Life is a trick problem!" Let's work through the next investigation with a focus on putting Polya's steps into practice.

## INVESTIGATION 1.1d Coin Problem

Variations of this problem are often found in elementary school textbooks because it provides an opportunity to move beyond random guess and test.

If 8 coins total 50 cents, what are the coins?

Solve this problem intentionally using and writing out Polya's four steps of problem solving.

Explorations Manual 1.1

### DISCUSSION

#### STEP 1: UNDERSTAND THE PROBLEM

So often students will jump into a problem without stopping to really understand it. Read a problem more than once before attempting to solve it. Take a few deep breaths and know that you can figure it out with some thinking. Write down the important information and pay attention to what the question is before starting. Here, you have 8 coins, which might be pennies, nickels, dimes, quarters, or half dollars. All together they equal 50 cents. We need to determine what kind of coins we have.

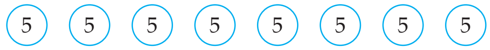
### STEP 2: DEVISE A PLAN

There is more than one strategy to solve any problem. Here we could use a diagram, make a table, use reasoning, or use a bag of coins to help us solve it. Let's consider two strategies of making a diagram and using reasoning.

### STEP 3: MONITOR THE PLAN

**STRATEGY 1** **Use a diagram**

We could make 8 circles and begin with all nickels: 8 coins = 40¢. What might be the next step?




With a bit of thinking, we can conclude that each time we substitute a dime for a nickel, the total increases by 5 cents. Thus, we need to trade 2 nickels for 2 dimes, and the answer is 6 nickels and 2 dimes.

**STRATEGY 2** **Use reasoning**

Eight nickels would make 40 cents, and 8 dimes would make 80 cents. Because the 8 coins make 50 cents, your first guess will have more nickels than dimes. Even if the guess is wrong—for example, 5 nickels and 3 dimes make 55 cents—you are almost there.

### STEP 4: LOOK BACK AT YOUR WORK

We are almost done, but we need to ask questions like: Does my answer make sense? Did I answer the question? Is this the only answer?

We can go back and reread the problem, make sure our solution answers the questions, make sure our answer makes sense, see if we missed any information, and think about alternate ways to get to the solution.

The only other possible solutions are that there might be 1 quarter or 5 pennies. Do you see why? If we make 1 of the coins a quarter, then the other 7 coins must be worth 25 cents. If 5 of those coins are pennies, then we need 2 coins that are worth 20 cents. Aha—2 dimes. Do you see that we could have arrived at the same answer if we had begun with 5 pennies?

## Mathematical Bioreactions

Some students have the attitude, "Why do I have to learn this?" Others may run away from learning math. I have heard students say that they read a math question and freeze, not knowing what to do. These are biological reactions called fight, flight, or freeze that get triggered in us when we are faced with something we are afraid of. We have these bioreactions to keep us safe. Whether you have experienced this with math, or something else, the good news is that you have control over this once you understand what is happening. Realizing that math is not going to hurt you, that mistakes are good because they help your brain to grow, and taking deep breaths will all help you to overcome these bioreactions, which is a first step in problem solving and in developing a mathematical growth mindset.

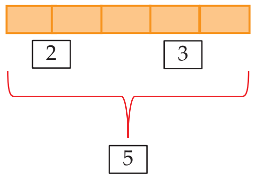
## Mathematical Visualization

Many educational psychologists such as Piaget and Bruner have talked about the progression of learning from the concrete to the abstract. This concept of developing ideas through concrete, pictorial, and then abstract (CPA) has helped Singapore's math education system to become one of the best in the world. We will develop many concepts in this book through these CPA steps. Let's look at basic addition and how this progression works.

Concrete: I have 2 pencils and then get 3 more. How many do I have now?

We can lay out 2 pencils and then 3 pencils and easily count 5 pencils together.

Pictorial: There are many pictures we could draw to show this. Here is one:




Abstract: This is where the symbolic language comes in of $2 + 3 = 5$. Approaching math this way helps us to make sense of the concepts before the abstract notation and concept. Throughout this book we will develop concepts through these steps to help us understand them better.

### Mathematical Connections

How are addition and subtraction related? What is the connection between finding the area and finding the volume? Seeing these types of connections in math is critical to developing a big picture sense of mathematics instead of seeing it as a bunch of unrelated procedures to memorize. As an example, we will look at addition and subtraction in one chapter to see not only how these are connected, but also to see connections between how we add and subtract whole numbers, decimals, fractions, and integers. It is also important to see connections between the concrete, pictorial, and abstract representations of a concept to connect the symbolic language with the concrete concepts. I urge you to look for these connections as you go through this book.

## Mathematical Process, Practice, and Content Standards

What does math in elementary schools look like?

Let's take a brief historical look at the development of math standards, which define what is taught in school. From the "new math" of the 1960s and 1970s to the "back to basics" movement of the 1980s, the pendulum has swung between many ideas of how mathematics should be taught and learned. In 1989, the National Council of Teachers of Mathematics (NCTM) published a landmark book called *The Curriculum and Evaluation Standards for School Mathematics*. This was the first document that detailed curriculum standards for elementary, middle, and high school mathematics, and led to individual states creating their own curriculum standards. In 2000, NCTM published a revised version, *Principles and Standards for School Mathematics*, which outlined both content standards and process standards. The content standards are about what math topics should be taught at different grade levels, while the process standards are about how students and teachers will engage in the math.

| **Content standards** | **Process standards** |
|---|---|
| Standard 1: Number and Operation | Standard 6: Problem Solving |
| Standard 2: Patterns, Functions, and Algebra | Standard 7: Reasoning and Proof |
| Standard 3: Geometry and Spatial Sense | Standard 8: Communication |
| Standard 4: Measurement | Standard 9: Connections |
| Standard 5: Data Analysis, Statistics, and Probability | Standard 10: Representation |

This document also outlines principles of school mathematics that address equity, curriculum, teaching, learning, assessment, and technology.

Based on these NCTM standards, other recommendation documents, and international comparisons, the Common Core State Standards (CCSS) were developed in 2010 by educators nationwide to establish clear, consistent standards that states could then voluntarily adopt. Currently 42 states plus the District of Columbia and four U.S. territories are following the CCSS. Even states like Texas, which developed their own standards instead of adopting

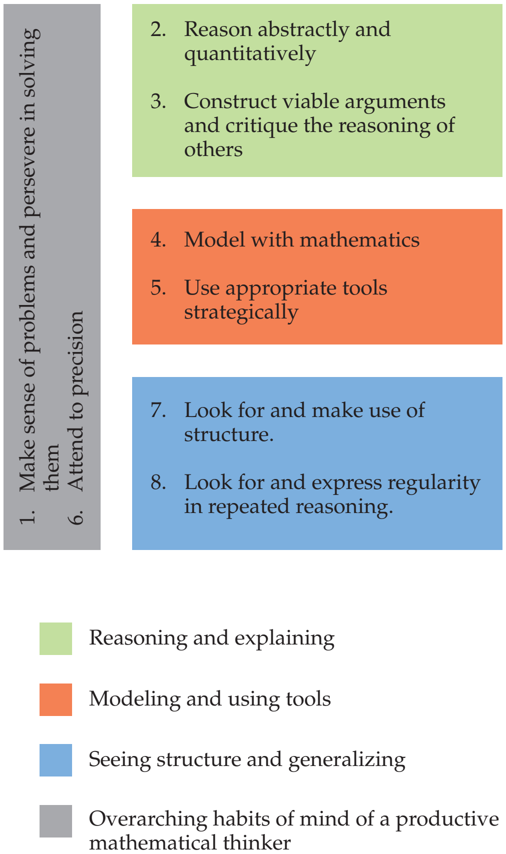



**Figure 1.2**
Source: © Copyright 2010. National Governors Association Center for Best Practices and Council of Chief State School Officers. All rights reserved.

the CCSS, or South Carolina, which backed out of the CCSS, still have language and intent very similar to the CCSS. Similar to the NCTM content standards, the CCSS defines content standards for which topics should be taught at which grade level. In this book, we will explore topics found in the CCSS K-8 content standards, but in a much deeper way.

Similar to the NCTM Process Standards, the CCSS describe eight Mathematical Practices (MPs). These practices are interconnected; any worthwhile mathematical activity will involve several of them. Figure 1.2 helps to organize the higher-order thinking skills of the Mathematical Practices and provides an interesting grouping that may help you to think about what these practices actually look like in a mathematics classroom. How do these Mathematical Practices relate to a mathematical growth mindset?

You will learn more about teaching the curriculum standards as you proceed through your education program. For now, it is important for you to experience thinking about and understanding mathematics through these practices. If it is true that we teach the way we were taught, then it is really important for you to learn math in these flexible, connected ways so you can eventually teach it this way.

## Overarching Habits of Mind of a Productive Mathematical Thinker: MP 1 and MP 6

As discussed in the previous section, a growth mindset rather than a fixed mindset is critical for both the learner and teacher. Perseverance, being able to think outside of the box, and clear communication are important in becoming proficient problem solvers. MP 1 and MP 6 involve these important habits of mind, which are necessary to be a productive mathematical thinker.

### MP 1: Make Sense of Problems and Persevere in Solving Them.

- Mathematically proficient students look for a place to get started. Often that is the hardest part—where do I start?
- They think, try something, assess whether it is helpful, and then continue if it was useful or try another plan.
- If they recognize the current problem as similar to one they have solved, they adapt what they used in the similar problem.
- They simplify the problem—making the numbers smaller or simpler.
- If they are heading down a path that is not solving the problem, they become aware of it and try something different.
- They check their solution and strategy and often ask themselves, "Does this make sense?"

How are Polya's four steps of problem solving related to this Mathematical Practice? With so many of us exposed to the Internet and the speed and ease of getting quick answers to questions, the art of persevering and "staying the course" is growing increasingly difficult. For students to be successful problem solvers, patience on the part of both the teacher and the student is required to allow the time to grapple with a problem and to be okay with not getting an immediate answer. Having students engaged in sharing solution methods, and creating a safe space where mistakes are part of the learning process, can help develop these skills.

### MP 6: Attend to Precision.

1. Mathematically proficient students are able to communicate their mathematical thinking to others.
2. They are able to articulate clear definitions and the meaning of symbols.
3. They are careful to make sure they use correct units of measure.
4. They calculate accurately and efficiently, and communicate precise answers.

## Reasoning and Explaining: MP 2 and MP 3

Mathematical thinking and communication are important aspects of being a mathematically proficient student. As you continue through this course, there will be opportunities for you to analyze your reasoning and communicate your thinking through writing and discussions. Young children naturally make generalizations: they conclude that when the doorbell rings, someone is outside the door; and when they get into the car, they must be strapped in. Elementary school children also regularly make mathematical generalizations: when you add two numbers, you get the same amount when you "add forward as well as backwards"; if you add two odd numbers together, you get an even number. Reasoning like this and communication of the reasoning are at the heart of these Mathematical Practices.

Explorations Manual 1.3

### MP 2: Reason Abstractly and Quantitatively.

- Mathematically proficient students make sense of numbers and their context within a problem.
- They are able to "decontextualize" a problem by representing it with numbers and symbols that abstracts away from the context.
- They are also able to "contextualize" the symbolic manipulations by pausing to go back to the context when needed.

Explorations Manual 1.4

### MP 3: Construct Viable Arguments and Critique the Reasoning of Others.

- Mathematically proficient students are able to use definitions and previous knowledge to communicate their understanding.
- They are able to build a logical progression of their ideas.
- They are able to use counterexamples to make an argument.
- Elementary students can make sense of math and communicate by using objects, drawings, diagrams, or actions.
- They can listen to the reasoning of others and ask useful questions to clarify.

## Modeling and Using Tools: MP 4 and MP 5

These two Mathematical Practices are particularly related to using math in the workplace and in practical real-life ways. Modeling mathematics problems may involve using tools such as graphs, pictures, concrete materials, and verbal descriptions.

Explorations Manual 1.5

### MP 4: Model with Mathematics.

- Mathematically proficient students can solve real-life problems, which in elementary school includes being able to write a multiplication equation to solve a problem.
- They are able to identify important information in a real-life problem and analyze relationships using tools.
- They can use models to draw conclusions, make predictions, and reflect on and adjust the effectiveness of the model.

### MP 5: Use Appropriate Tools Strategically.

- Mathematically proficient students can use a variety of tools such as concrete models and technology to and solutions.
- They are able to make good decisions about when to use each of these tools and how to effectively use them.
- They are able to use a variety of tools to investigate and develop their understanding of ideas.

## Seeing Structure and Generalizing: MP 7 and MP 8

These two practices involve seeing patterns and then using those patterns to generalize ideas. These two practices are very closely intertwined, and even math educators struggle with how they are different. MP 7 is seeing the structure in math to be able to build understanding. MP 8 is more about using those patterns to generalize for solving any problem.

Explorations Manual 1.6

### MP 7: Look for and Make Use of Structure.

- Mathematically proficient students are able to find and use patterns or structures.
- They might notice that 5 times 8 has the same answer as 8 times 5 and extend this to discover that the order in which we multiply numbers does not affect the answer (commutative property).
- Young children naturally use structure. They may use the word "eated" as past tense for "eat" or "swimmed" instead of "swam" since they have noticed the structure of adding "ed" to a word to make it past tense, such as in words like "played," "talked," and "worked."

The structure in mathematics is far more consistent than in language and can help with developing understanding. For example, if children learn that 3 apples plus 4 apples equals 7 apples, and 3 hundreds plus 4 hundreds equals 7 hundreds, they can use this structure to see that 3 sevenths plus 4 sevenths would naturally equal 7 sevenths $\frac{3}{7}+\frac{4}{7}=\frac{7}{7}$, or that 3 $x$'s plus 4 $x$'s equals 7 $x$'s ($3x + 4x = 7x$). Noticing structure can be useful in being able to do arithmetic quickly without memorizing. For example, 9 can be added easily to any number by realizing that we can add 10 and subtract 1. This can be generalized to adding $39 + 52$ (add 40 and subtract 1). Using structure makes memorizing facts less important.

Explorations Manual 1.7

### MP 8: Look for and Express Regularity in Repeated Reasoning.

- Mathematically proficient students notice when computations are repeated and use this to create methods and shortcuts.
- They continue to look at their process and evaluate their results.
- They develop new methods by generalizing patterns.

Lower elementary students might notice that when counting by 5's, the ones digit is always a 0 or 5. They may notice that adding 0 and multiplying by 1 does not change the number, therefore developing the identity property. Upper elementary students might notice when dividing a number by 10 that the numbers move one place value to the right, or in other words that the decimal point moves one place to the left, and then be able to use this whenever dividing by 10.

In each of the following four Investigations, practice using Polya's four steps of problem solving and consider how you are engaging in the Mathematical Practices in each of these.

## CLASSROOM CONNECTION

Grade 3

### Think and Grow: Modeling Real Life

Newton earns the same amount of money each week. The multiplication table shows the amount (in dollars) he earns after 2 weeks, 4 weeks, and 6 weeks. If the pattern continues, how much money will he earn after 10 weeks?

Describe the pattern.

| × | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 2 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
| 3 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 |
| 4 | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 | 40 |
| 5 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 |
| 6 | 6 | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 | 60 |
| 7 | 7 | 14 | 21 | 28 | 35 | 42 | 49 | 56 | 63 | 70 |
| 8 | 8 | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 | 80 |
| 9 | 9 | 18 | 27 | 36 | 45 | 54 | 63 | 72 | 81 | 90 |
| 10 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |

Newton will earn $_____ after 10 weeks.

### Show and Grow

**8.** You plant a 10-inch-tall bamboo cane in the ground. It grows the same number of inches each day. Find and shade the pattern in the multiplication table above. How many inches does the bamboo grow each day?

| Day | Inches Grown |
|---|---|
| 2 | 6 |
| 4 | 12 |
| 6 | 18 |

How tall is the bamboo after 6 days? If the pattern continues, how tall will the bamboo be after 8 days?

**DIG DEEPER!** A different type of bamboo grows two times as fast. Explain how you can use the multiplication table to find how many inches this bamboo will grow in 8 days.

## INVESTIGATION 1.1e The Nine Dots Problem

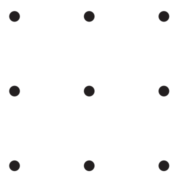


Figure 1.3

Not all problem solving involves computation and formulas, as this investigation shows.

Without lifting your pencil, can you go through all nine dots in Figure 1.3 with only four lines?

### DISCUSSION

This is a very famous problem, which some of you may have already encountered because of its moral: This problem is impossible to solve as long as you "stay inside the box." In order to solve the problem, you need to go "outside the box." If you haven't solved the problem yet, try to work with this hint. . . .

The solution to the problem can be seen in Appendix B.

## INVESTIGATION 1.1f Pigs and Chickens

A farmer has a daughter who needs more practice in mathematics. One morning, the farmer looks out in the barnyard and sees a number of pigs and chickens. The farmer says to her daughter, "I count 24 heads and 80 feet. How many pigs and how many chickens are out there?"

Before reading ahead, work on the problem yourself or, better yet, with someone else. Close the book or cover the solution paths while you work on the problem.

Compare your answer to the solution paths below.

### DISCUSSION

**STRATEGY 1** Use random trial and error

One way to solve the problem might look like what you see in Figure 1.4.

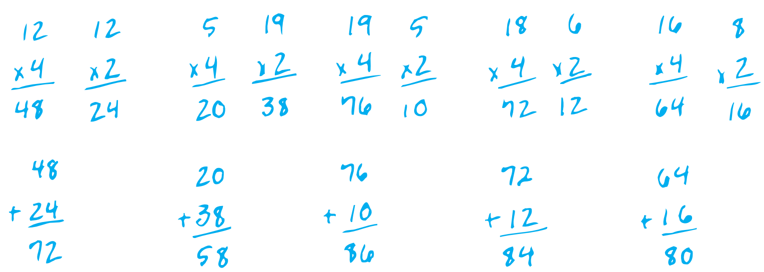



Figure 1.4

The words *trial* and *error* do not sound very friendly. However, this strategy is often very appropriate and can help students to make sense of the problem as MP 2 suggests. In fact, many advances in technology have been made by engineers and scientists who were guessing with the help of powerful computers using **what-if programs**. A what-if program is a logically structured guessing program. Informed trial and error, which I call **guess–check–revise**, is like a systematic what-if program. Random trial and error is what the student who wrote the solution in Figure 1.4 was doing. In this case, the student finally got the right answer. In many cases though, trial and error does not produce an answer—or if it does produce an answer, it is after many trials.

**STRATEGY 2** Use guess–check–revise (with a table)

One major difference between this strategy and trial and error is that we record our guesses (or hypotheses) in a table and look for patterns in that table. Such a strategy is a powerful new tool for many students because a table often reveals patterns. Look at Table 1.3. A key to "seeing" the patterns is to make a fourth column called "Difference." Do you see how this column helps?

**Table 1.3**

| | Numbers of pigs | Number of chickens | Total number of feet | Difference | Thinking Process |
|---|---|---|---|---|---|
| **First guess** | 10 | 14 | 68 | | We need more feet, so the next guess needs to have more pigs. |
| **Second guess** | 11 | 13 | 70 | +2 | Increasing the number of pigs by 1 adds 2 feet to the total. What if we add 2 more pigs? |
| **Third guess** | 13 | 11 | 74 | +4 | Increasing the number of pigs by 2 adds 4 feet to the total. Because we need 6 more feet, let's increase the number of pigs by 3 in the next guess. |
| **Fourth guess** | 16 | 8 | 80 | +6 | Yes! |

The left side of Table 1.3 "decontextualizes" the problem by representing the information numerically. The right side "contextualizes" the problem by returning to the context of the problem to make sense of the numbers.

From the table, we observe that if you add 1 pig (and subtract 1 chicken), you get 2 more feet. Similarly, if you add 2 pigs (and subtract 2 chickens), you get 4 more feet. Do you see why? Think before reading on. . . .

Because pigs have 2 more feet than chickens, each trade (substitute 1 pig for 1 chicken) will produce 2 more legs in the total number of feet. This observation would enable us to solve the problem in the second guess. Do you see how . . . ? After the first guess, we need 12 more feet to get to the desired 80 feet. Because each trade gives us 2 more feet, we need to increase the number of pigs by 6.

It is important to note that the guesses shown in Table 1.3 represent one of many variations of a guess–check–revise strategy.

### STRATEGY 3 Make a diagram

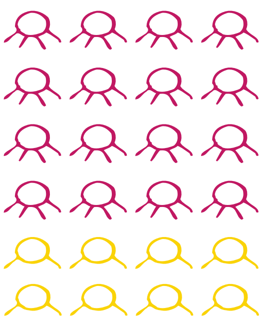


Figure 1.5

Sometimes making a diagram can lead to a solution to a problem. Figure 1.5 shows how one student solved this problem. How do you think she solved the problem? Write your thoughts before reading on. . . .

She made 24 chickens, which gave her 48 feet. Then she kept turning chickens into pigs (adding 2 feet each time) until she had 80 feet! I was thrilled because she represented the problem visually and used reasoning instead of trial and error. She was embarrassed because she felt she had not done it "mathematically." However, she engaged in the reasoning that MP 2 suggests.

Upon reflection, we realize the enormous potential of this solution path. For example, what if the problem were 82 heads and 192 feet? No way you say? True, it would be tedious to draw 82 heads and then 2 feet underneath each head. This is exactly the power of mathematical thinking—you don't have to do all the drawing. Think about what the diagram tells us, and then see whether you can solve the problem. . . .

*If* we drew 82 heads and then drew 2 feet below each head, that would tell us how many feet would be used by 82 chickens: 164 feet. Because $192 - 164 = 28$, we need 28 more feet—that is, 14 more pigs. Therefore, the answer is 14 pigs and 68 chickens. Check it out!

### STRATEGY 4 Use algebra

Because the range of abilities present among students taking this course is generally wide, it is likely that some of you fully understand the following algebraic strategy and some of you do not. Let's look at an algebraic solution and then see how it connects to other strategies and to the goals of this course and the MPs.

Go back and review strategy 2. Each guess involved a total of 24 pigs and chickens. Can you explain in words why this is so? Think about this before reading on. . . .

Most students say something like “Because the total number of animal is 24.” Therefore, if we say that

$$p = \text{the number of pigs and } c = \text{the number of chickens}$$

then the *number* of pigs plus the *number* of chickens will be 24. Hence, the first equation is

$$p + c = 24$$

Looking back at how we checked our guesses in the previous methods helps us to write the second equation. We multiplied the number of pigs by 4 and the number of chickens by 2 and then added those two numbers to see how close that sum was to 80. In other words, we were doing the following:

$$\begin{array}{ccc} 4\times(\text{the guess for number of pigs}) & + & 2\times(\text{the guess for number of chickens}) \\ (4\times p) & + & (2\times c) \end{array}$$

More conventionally, this would be written as $4p + 2c$.

Using guess–check–revise, we had the right answer when this sum was 80. Thus the second equation is $4p + 2c = 80$.

Let’s solve these equations step by step:

$$p + c = 24 \qquad 4p + 2c = 80.$$

First, take the equation $p + c = 24$ and solve for $p$ by subtracting $c$ from both sides.

$$\begin{array}{l} p + c = 24 \\ \;\;-c \quad -c \end{array}$$

The $c$ subtracts out on the right side, and you are left with

$$p = 24 - c$$

Now, substitute $24 - c$ where $p$ is in the second equation since you know that $p = 24 - c$.

$4p + 2c = 80$, replacing $p$ with $(24 - c)$ you get

| | |
|---|---|
| $4(24 - c) + 2c = 80$ | now distribute, or multiply, the 4 through the parentheses to get |
| $96 - 4c + 2c = 80$ | now combine the $-4c + 2c = -2c$ and the equation becomes |
| $96 - 2c = 80$ | now subtract 96 from both sides of the equation, so it subtracts away from the left side |
| $96 - 2c - 96 = 80 - 96$ | |
| $-2c = -16$ | now divide both sides by $-2$ to get the $c$ by itself and get |
| $c = 8$ | |

This tells us that we have 8 chickens. To find the number of pigs, since we know that we have a total of 24 animals, subtract 8 chickens from 24 total animals to get 16 pigs.

### STRATEGY 5 Visualization by Pictorial Representation

At different places in this book, we will use models like the ones used in Singapore, which has some of the most successful math students in the world. A Singaporean colleague, Alice Ho, has shared these methods and also has developed a five-color–coded model that helps communicate mathematical reasoning. You will see the colored arrows showing that green is the first step, then blue, and so on. You will see this five-color communication tool used in other places in this book. Examine the model in Figure 1.6, and then we will discuss it.

The green in the first step shows that 4 times the number of pigs plus 2 times the number of chickens together equals 80. Then the blue shows how we can use the fact that the number of pigs plus the number of chickens equals 24, so both of the blue boxes represent 48. This leaves 32 that the red box represents, so the number of pigs must equal 32 divided by 2.

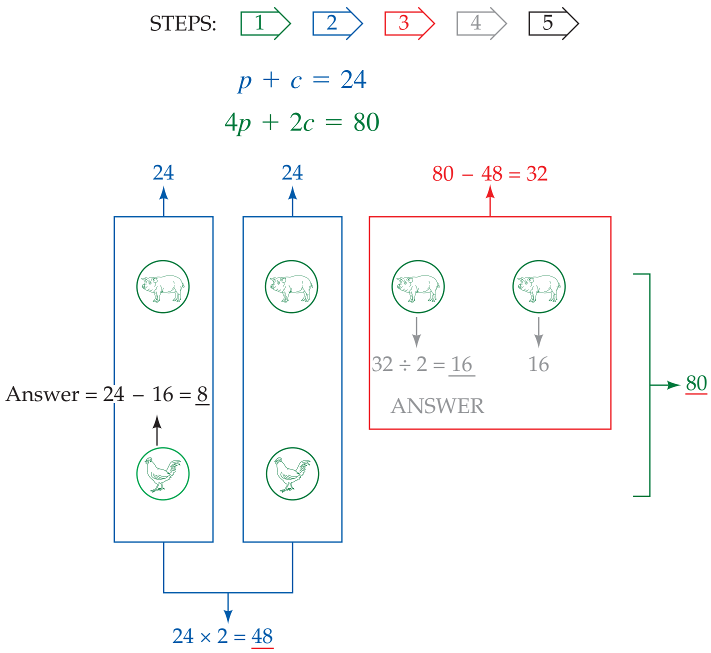



**Figure 1.6**
Source: Alice Ho of Math Teach Singapore.

## INVESTIGATION 1.1g Darts, Proof, and Communication

Suppose you have a dart board like the one in Figure 1.7. You throw four darts, all of which land on the dart board. One of the questions I asked fifth-graders was, "What kinds of scores would be possible, and what kinds of scores would be impossible?" What do you think?

Explorations Manual 1.4

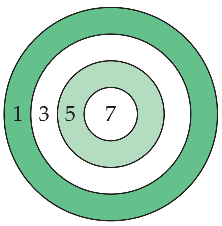



**Figure 1.7**

### DISCUSSION

After a few minutes, one of the students, Erika, suddenly said, "Only even numbers are possible." I asked her how she came to that conclusion, and she said, "Well, I know that an odd plus an odd is even and an odd plus an even is odd. [At this point, she held up four fingers to represent the four darts.] The first two darts are odd, and so when you add them, you have an even number. [She joined two of her fingers together to indicate the combined score from two darts.] Now, this number (even) plus the next dart (odd) will make an odd number. [She now joined three of her fingers together to indicate the combined score from the first three darts.] Now, this number (odd) plus the last dart (odd) will make an even number. Therefore, the only possible scores you can get are even numbers."

We can represent Erika's proof as shown in Figure 1.8.

(odd + odd) + odd + odd
(even + odd) + odd
odd + odd
even

**Figure 1.8**

Erika was able to build a logical progression, make sense of the problem, and use actions and objects (her fingers) to communicate her ideas. Because not everyone learns and understands the same way, we have shown another way to represent and communicate the solution above.

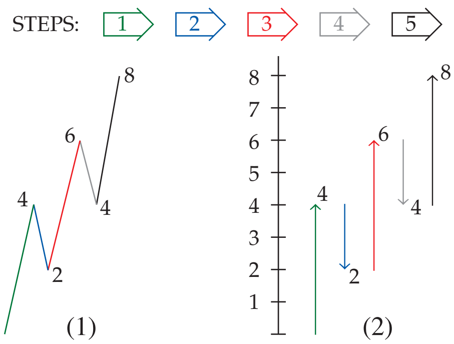
## INVESTIGATION 1.1h How Long Will It Take the Frog to Get Out of the Well?

**A.** A frog is climbing out of a well that is 8 feet deep. The frog can climb 4 feet per hour, but then it rests for an hour, during which it slips back 2 feet. How long will it take for the frog to get out of the well?

**B.** What if the well is 40 feet deep, the frog climbs 6 feet per hour, and it slips back 1 foot while resting? Work on the problem before reading on. . . .

### DISCUSSION

**A.** One of the amazing things about this problem is that, in a class of 25 students, I will often see 10 or more different valid models of the problem. Below are two graph models that both lead to the same answer. Each step is color coded to help you follow them. First, examine them to see if you understand them. . . .




Both models show the frog's progress for each hour and that the frog reaches 8 feet after 5 hours.

**B.** As we found in the pigs-and-chickens problem, some models can be "scaled up" and others cannot. Each of the two models shown *could* be used to solve B, but they would be somewhat tedious. In this case, we look for a more efficient model. A table to explore the problem is shown below:

| **Hour** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Height** | 6 | 5 | 11 | 10 | 16 | 15 | 21 | 20 | 26 | 25 | 31 | 30 | 36 | 35 | 41 |

Even this model is a bit tedious. However, if we are always looking for patterns, we can actually get the answer by making only part of the table.

| **Hour** | 2 | 4 | 6 | 8 | 10 | 12 | 14 |
|---|---|---|---|---|---|---|---|
| **Height** | 5 | 10 | 15 | 20 | 25 | 30 | 35 |

We can see that the numbers when the hours are even are simply multiples of 5. We can then count by 2s to get close to the 40-foot height, or we can see that the height (in even hours) is always $2\frac{1}{2}$ times the number representing the hour. Therefore, we can jump to 14 hours when the frog has climbed 35 feet and know that on the 15th hour the frog will get out.

## SUMMARY 1.1

In this first section, we have examined the importance of developing a mathematical growth mindset. We have also broadened the question of what is mathematics and what specialized math knowledge is needed for teaching. We examined Polya's four steps as the foundation for a toolbox for problem-solving strategies. We discussed developing ideas through concrete, pictorial, and abstract development and communicating ideas with the five-color communicator. Exploring the Mathematical Practices helps us to see how the vision of elementary school mathematics is now a deep, connected, reasoning endeavor. I encourage you to continue to experience mathematics in these ways as you proceed though this course and beyond into your teaching career. As you continue to deepen your mathematical understanding, and experience mathematics differently, you will continue to gain skills and confidence that will help you to become a successful math student and teacher.

## Exercises 1.1

**1.** Write down some personal goals in this course. Keep these in a prominent place in your notebook so that you can refer back to them periodically.

**2.** Interview a friend, a child, and a parent or grandparent (if possible), asking them the first four questions in the "*What do you think?*" list at the beginning of this section.

**3.** Make a list of uses of mathematics in your own life. Ask others to discuss where mathematics is useful in their lives and add these to your list.

**4.** Read the elementary mathematics curriculum standards for your state/district. For many of you, these can be found at corestandards.org. Write about your impression of them and how they are similar and different from your current vision of elementary school mathematics.

**5.** What habits and attitudes does a mathematically proficient student have?

**6.** Explain what it means to have a mathematical growth mindset.

**7.** Multiply $86 \times 47$ in each of the two valid ways in Investigation 1.1a.

**8.** Write about your past experiences in math classes. How have these experiences influenced your current beliefs and attitudes about mathematics?

**9.** Read the Common Core State Standards (www.corestandards.org) for math for the grade levels you teach, or your state curriculum math standards if you live in a state that is not following the CCSS. Write a short paragraph describing what you found interesting.

**10.** Go to the website of the National Council of Teachers of Mathematics (nctm.org) to read more about this organization and the resources there. Write about what you learn.

*Solve the following using Polya's steps for problem solving. See if you can find more than one way to get to the answer.*

**11.** Sally works 40 hours a week and makes \$10 an hour, but her kids are in child care each day, and the day care center charges her \$20 per day. If you deduct her child care expenses, how many dollars per hour does she actually make?

**12.** Since its beginning, the U.S. Mint has produced over 288 billion pennies.

**a.** What if we lined these pennies up? How long would the line be?

**b.** The mint currently makes about 30 million pennies a day. How many is this per second? How many is this per year?

**13.** The record for the longest migration is held by the arctic tern, which flies a round trip that can be as long as 20,000 miles per year, from the Arctic to the Antarctic and back. If the bird flies an average of 25 miles per hour and an average of 12 hours per day, how many days would it take for a one-way flight?

**14.** A family is planning a three-week vacation for which they will drive across the country. They have a van that gets 18 miles per gallon, and they have a sedan that gets 32 miles per gallon. How much more will they pay for gasoline if they take the van?

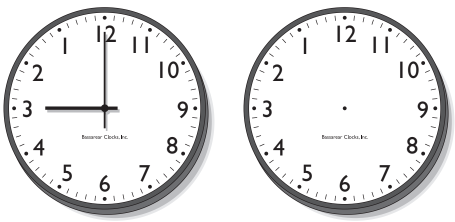
**a.** First describe the assumptions you need to make in order to solve the problem.

**b.** Solve the problem and show your work.

**c.** What if the price of gas rose by 40¢ between the planning of the trip and the actual trip? How much more would the gas cost for the trip?

*As you work through the following exercises, consider how you are using the Mathematical Practices and the problem-solving process.*

**15.** Look at the clock below.




**a.** How do this clock and a normal clock differ?

**b.** What time is it?

**c.** Draw the hour and minute hands in the positions they will occupy when it is 9:45.

**d.** Why do you think the direction we call clockwise was selected?

**16.** Aunt Katrina just moved into a new house and wants to brighten up the dining room wall by hanging 6 of her favorite decorative plates. Each plate is 8 inches in diameter. Aunt Katrina is having trouble deciding how to hang the plates in a straight line and evenly space them along her 104-inch wall.

**a.** Draw a diagram to show the placement and spacing and explain how Aunt Katrina should hang her 6 decorative plates.

**b.** What if she only had 5 plates?

**c.** What if she had 6 plates and she wanted to have the two end plates 12 inches from the end of the wall?

Source: Kim Hartweg, *Teaching Children Mathematics*, December 2004/January 2005, pp. 280–284. National Council of Teachers of Mathematics.

**17.** A farmer looks out into the barnyard and sees the pigs and the chickens. He says to his daughter, "I count 40 heads and 100 feet. How many pigs and how many chickens are out there?"

**18.** Jacky was riding her new bike at the playground with other children. Everyone was riding a bicycle or a tricycle. If Jacky counted 12 wheels, not including her own, how many children could be riding and what would they be riding?

Source: *Teaching Children Mathematics*, April 2003, pp. 457–460.

**19.** What if the farmer in Exercise 17 sees 169 heads and 398 feet? How many pigs and how many chickens are out there?

**20.** Look back at Investigation 1.1g with the dart board. What if the scores were 1, 5, 10, 25?

**a.** List all possible scores from 1 to 100.

**b.** Select one score that is more than 4 and less than 100 that is impossible, and prove that it is impossible.

**21. a.** A frog is climbing out of a well that is 11 feet deep. The frog can climb 3 feet per hour but then it rests for an hour, during which it slips back 1 foot. How long will it take for the frog to get out of the well?

**b.** What if the well is 50 feet deep, the frog climbs 7 feet per hour, and it slips back 3 feet while resting?

**c.** What if a caterpillar is climbing out of a glass that is 12 inches high and can climb $1\frac{1}{2}$ inches in an hour but slides and falls back $\frac{1}{4}$ inch during the hour it rests. How long will it take the caterpillar to climb to the top of the jar?

**22.** A boatman is to transport a fox, a goose, and a sack of corn across the river. There is room in his boat for only one of the three at a time. Furthermore, if the fox and the goose are left together, the fox will eat the goose. If the goose and the corn are left together, the goose will eat the corn. How can the boatman do the job?

**23. a.** In how many different ways can you make change for 50 cents?

**b.** In how many different ways can you make change for one dollar without using pennies or half dollars?

**24.** The following pattern, called **Pascal's triangle**, was "discovered" by the mathematician Blaise Pascal (1623–1662). Examine the triangle, play with it, and describe the patterns you see. You may choose to describe patterns with words, in a table or sequence, or by using mathematical notation.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

**25.** For each of the triangles below:
(1) Describe as many patterns as you can find in the triangle.
(2) Write the numbers that would be in the next row of the triangle, and explain how you arrived at your answer.

**a.**
1
2 4
3 6 9
4 8 12 16
5 10 15 20 25

**b.**
1
1 1
1 3 1
1 5 5 1
1 7 13 7 1
1 9 25 25 9 1
1 11 41 63 41 11 1

**c.**
1
2 3
4 5 6
7 9 11 12
13 16 20 23 24
25 29 36 43 47 48

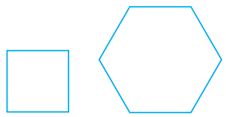
**d.**
3
3 3
3 6 3
3 9 9 3
3 12 18 12 3
3 15 30 30 15 3
3 18 45 60 45 18 3

**26.** Sheila has a lot of stamps in her drawer and will be mailing a number of 2-oz letters costing 61¢ of postage. How many different ways can she make 61¢ postage using a 44¢ stamp and combinations of 5¢, 3¢, and 2¢ stamps?

## FROM STANDARDIZED ASSESSMENTS

### 2006 NECAP, Grade 5

**27.** Karen used toothpicks to make the two shapes shown below.


Square Hexagon

She used a total of 24 toothpicks to make the square. She made the hexagon so that its sides are the same length as the sides of the square. How many toothpicks did Karen use to make the hexagon?

**28.** Suppose you have 8 coins and you have at least one each of a quarter, a dime, and a penny. What is the least amount of money you could have? (23% of seventh-graders got this correct)

Source: Results of the *Fourth NAEP Mathematics Assessment*, p. 16. U.S. Department of Education, National Center for Education Statistics.

### 2008 NECAP, Grade 4

**29.** Harold is planning a party for 32 people. He has some round tables and some square tables.

- There are 6 chairs at each round table.
- There are 4 chairs at each square table.

What is the *fewest* number of tables Harold needs for 32 people with no chairs left over? Show your work or explain how you know.

### 2007 NECAP, Grade 6

**30.** Trisha has a job walking 16 dogs for her neighbors. She can walk one, two, or three dogs on each trip to the park. What is the fewest number of trips Trisha takes to walk all 16 dogs?

**a.** 16 **b.** 15 **c.** 6 **d.** 5

# SECTION 1.2 Sets

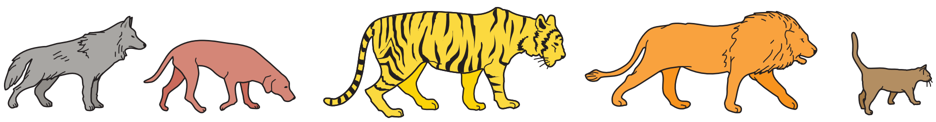
### *What do you think?*

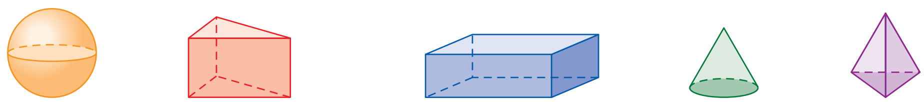
- How do we use and apply set concepts in everyday life?
- Do we have to use circles to make Venn diagrams?

## Sets as a Classification Tool

Children use set ideas in everyday life as they look for similarities and differences between sets; for example, they want to know why lions are in the cat family and wolves are in the dog family. Children also look for similarities and differences within sets; for example, they look within a set of blocks for the blocks that they can stack and the blocks that they can't.

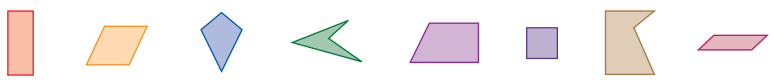
Whether we realize it or not, we are classifying many times each day, and our lives are shaped by classifications that we and others have made.





## INVESTIGATION 1.2a Classifying Quadrilaterals

Look at the eight shapes below. How can you classify these shapes into two groups so that each group has a common characteristic? Give a name to each group if you can. Work and then read on. . . .



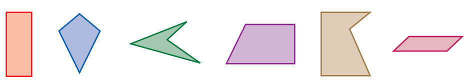
**DISCUSSION**

There are many possible answers to this question. Let us examine some common ones.

*One answer:*

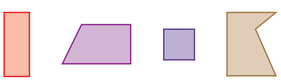
Those shapes in which all sides are equal

Those shapes in which not all sides are equal

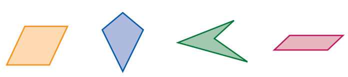


*Another answer:*

Those shapes with at least one right angle

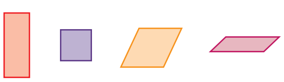


Those shapes with no right angles

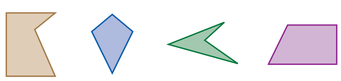


*Another answer:*

Those shapes in which opposite sides are equal (parallelograms)



Those shapes in which opposite sides are not equal



## Describing Sets

We speak of individual objects in a given set as **members** or **elements** of the set. The symbol $\in$ means "is a member of." The symbol $\notin$ means "is not a member of." For example, if $E$ is the set of even numbers, then $4 \in E$ but $3 \notin E$.

There are three different ways to describe sets:

1. We can use words.
2. We can make a list.
3. We can use set-builder notation.

In many cases, one of these representations is simpler or easier than the others. Let us describe different sets of numbers.

We can use words or a list to describe natural numbers, whole numbers, and integers.

$N$ is the set of natural numbers or counting numbers.
$N = \{1, 2, 3, \ldots\}$

We use braces to indicate a set. The three dots are referred to as an ellipsis and are used to indicate that the established pattern continues forever.

$W$ is the set of natural numbers and zero.
$W = \{0, 1, 2, 3, \ldots\}$

I is the set of positive and negative whole numbers:

$I = \{\ldots\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ \ldots\}$

**Set-Builder Notation** Another important set is the set of **rational numbers** ($Q$), which we can describe in words:

> $Q$ is the set of all numbers that can be represented as the ratio of two integers as long as the denominator is not zero. Another way to say this is the set of fractions where the numerator and denominator are integers and the denominator is not zero. This includes whole numbers, since any whole number can be written in the form of a fraction—for example, $3 = \frac{3}{1} = \frac{6}{2}$, and so on.

We cannot represent this set by making a list. Do you understand why? Try listing all the rational numbers between 0 and 1 and it may help you think about why we cannot list them all (there are infinitely many rational numbers between 0 and 1).

In this case, and in many other cases, we describe the set using set-builder notation:

$$Q = \left\{ \frac{a}{b} \mid a \in I \text{ and } b \in I, b \neq 0 \right\}$$

This statement is read in English as "$Q$ is the set of all numbers of the form $\frac{a}{b}$ such that $a$ and $b$ are both integers, but $b$ is not equal to zero."

**Set-builder notation** always takes the form $\{x \mid x \text{ has a certain property}\}$.

Let us now apply these different ways of describing sets.

## INVESTIGATION 1.2b Describing Sets

Consider the following set:

$T = \{10,\ 20,\ 30,\ 40,\ 50,\ 60,\ 70,\ 80,\ 90,\ 100,\ 110,\ 120\}$

Describe this set with words and with set-builder notation. What do you see as advantages and disadvantages of each of the three ways to describe this set?

### DISCUSSION

Verbal description:

$T$ is the set of all multiples of 10 that are less than or equal to 120.

Set-builder notation:

$$T = \{x \mid x = 10n,\ 1 \leq n \leq 12,\ n \in N\}.$$

This notation tells us that we are looking at all numbers that have the form "$10n$," that is, multiples of 10. Because we are not talking about *all* multiples of 10, we have to let the reader know which multiples of 10 are in $T$. The mathematical phrase $1 \leq n \leq 12$ tells us that we are looking for multiples of 10 beginning with $10 \times 1$ and ending with $10 \times 12$. The last part of the description, "$n \in N$," simply lets us know that $n$ must be a natural number

**Finite and Infinite Sets** If the number of elements in a set is a whole number, that set is said to be **finite**. Some finite sets are small—for example, the set of Nobel Prize winners. Some finite sets are very large—for example, the grains of sand on all the beaches in the world. An **infinite** set has an unlimited number of members.

Consider the following infinite set. Does each way of describing the set make sense?

| | |
|---|---|
| Verbal description | $E$ is the set of positive even numbers. |
| List | $E = \{2, 4, 6, 8 \ldots\}$ |
| Set-builder notation | $E = \{x \mid x = 2n,\ n \in N\}$ |

## Defining Subsets

Many practical applications of sets involve subsets. What do you think a subset is? Write your current definition of a subset and then read on. . . .

These examples may help you develop a definition:

The set of even numbers is a subset of the set of whole numbers.
The set $\{1, 2, 3\}$ is a subset of $\{1, 2, 3, 4, 5\}$.

One informal definition of a subset is that a set is a subset of another set if all of the elements in the subset are also in the set. We can say this more formally as:

A set $X$ is a **subset** of a set $Y$ if every member of $X$ is also a member of $Y$.

The symbol $\subseteq$ means "is a subset of." Thus, we say $X \subseteq Y$. On the other hand, if a set $X$ is not a subset of a set $Y$, we say $X \not\subseteq Y$.

There is another symbol that we can use when talking about subsets. This symbol ($\subset$) is used when we want to emphasize that the subset is a **proper subset**. A subset $X$ is a proper subset of set $Y$ if the two sets are not equal *and* every member of $X$ is also a member of $Y$. In the case of finite sets, this means that the proper subset has fewer elements than the given set.

The difference in the two subset symbols is very similar to the difference between the $<$ "less than" and the $\leq$ "less than or equal to" symbol. We can say $3 < 4$ or $3 \leq 4$. We can say $4 \leq 4$, as we have to use the "or equal to" since they are equal. We can say $\{1,2\} \subset \{1,2,3\}$ or $\{1,2\} \subseteq \{1,2,3\}$. We can say $\{1,2,3\} \subseteq \{1,2,3\}$, and again we have to use the "or equal to" since they are equal.

## INVESTIGATION 1.2c How Many Subsets?

Let's say that you and your friends decide to go out and get a large pizza. Let $T$ represent the set of toppings that this restaurant offers:

$$T = \{\text{onions, sausage, mushrooms, peppers}\}$$

List all the possible different combinations of pizza that you could order, such as a mushroom and onion pizza. Then read on. . . .

**DISCUSSION**

The text will contain occasional reminders about the process of problem solving. In this investigation, we focus on understanding the problem and looking back. Before you made your list, did you make sure you understood the question? After you make your list and before reading on, take a few moments to look back: How can you check your solution to make sure you didn't miss any combinations?. . .

**STRATEGY 1 List all the combinations systematically**

Begin with all combinations that involve onion, from the simplest to the most complex, and then do the same for each of the other toppings (see Table 1.4).

There are many patterns within Table 1.4. How many can you see?

**Table 1.4**

| Onion | Sausage | Mushroom | Pepper |
|---|---|---|---|
| o | s | m | p |
| o, s | s, m | m, p | |
| o, m | s, p | | |
| o, p | s, m, p | | |
| o, s, m | | | |
| o, s, p | | | |
| o, m, p | | | |
| o, s, m, p | | | |

**STRATEGY 2 List all the combinations using another system**

Begin with all the ways to have one topping, then all the ways to have two toppings, and so on (Table 1.5).

There are many patterns within Table 1.5. How many can you see?

**Table 1.5**

| One of the four | Two of the four | Three of the four | All of the choices |
|---|---|---|---|
| o | o, s | o, s, m | o, s, m, p |
| s | o, m | o, s, p | |
| m | o, p | o, m, p | |
| p | s, m | s, m, p | |
| | s, p | | |
| | m, p | | |

One of the mistakes that students commonly make in such problems is to overlook a combination. There are patterns in both of the tables above that could help you to find all the combinations. Write down all the patterns you see and then read on. . . .

One pattern (in the "Three of the four" column) is that all of the four toppings are equally represented. Let's say you somehow omitted the last combination: s, m, p. Looking over the three combinations you found, you would notice that onion occurred *three* times but that sausage, mushroom, and pepper each occurred only *two* times. This observation itself names the combination you missed: sausage, mushroom, and pepper.

Mathematically, there are 16 possible subsets of a set containing four elements. However, if you look at either of the tables above, there are only 15 subsets. What is the missing subset?

## The Empty Set

The missing subset is plain pizza. How would we represent this subset using set notation?

One way is to put nothing inside the brackets { }. We also use the following symbol to represent a set that is empty: $\varnothing$. We use the terms **empty set** and **null set** interchangeably to mean the set with no elements.

This pizza problem also illustrates two mathematical statements that will be left to explore as exercises:

- Every set is a subset of itself.
- The empty set is a subset of every set.

## Equal and Equivalent Sets

In many situations, the relationship between two sets is important.

Two sets are said to be **equal** if they contain the same elements. For example, $\{1, 2, 3, 4, 5\} = \{5, 4, 3, 2, 1\}$.

Two sets are **equivalent** if they have the same number of elements. More precisely, two sets are equivalent if their elements can be placed in a **one-to-one correspondence**. In such a correspondence, an element of either set is paired with exactly one element in the other set. We use the symbol $\sim$ to designate set equivalence. For example, {United States, Canada, Mexico} $\sim \{1, 2, 3\}$.

### Equivalence and Counting

One reason for defining sets in terms of a one-to-one correspondence has to do with the difficulty young children have counting objects accurately. Initially, they do not realize that each number has to match one and only one object, and so they can count a set several times and arrive at a different number each time! At some point, they realize (and you can see it in their pointing) that there must be a one-to-one correspondence between their words and the physical objects. Thus, although our definition of equivalence appears unnecessarily formal to many students, it reflects what children actually go through in learning to count objects accurately.

## Venn Diagrams

Explorations Manual 1.8

One way to represent sets is to use **Venn diagrams**, which are named after John Venn, the Englishman who invented these diagrams to illustrate ideas in logic. You have probably seen Venn diagrams used in other contexts, and we used them in the previous section.

An elementary teacher explained how she used the Venn diagram in Figure 1.9 to help her students understand the similarities and differences between butterflies and moths. In this Venn diagram, one region represents the set of moths' characteristics, another region represents the set of butterflies' characteristics, and the overlapping region represents the set of characteristics common to both.

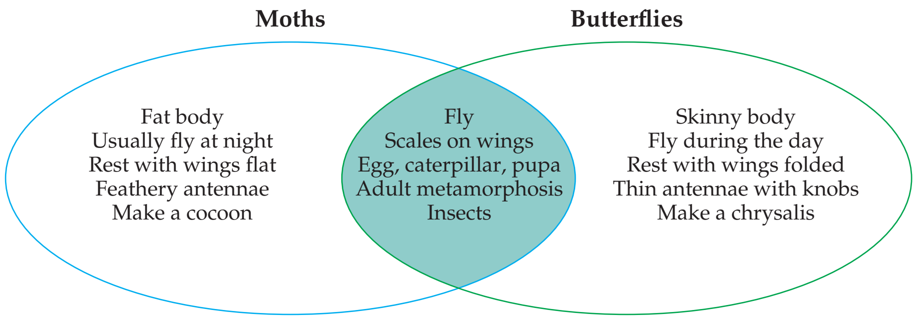



**Figure 1.9**

## Operations on Sets

We will use Venn diagrams as we examine three operations on sets: intersection, union, and complement.

When we perform operations on sets of objects, it is often useful to refer to the set that consists of all the elements being considered as the **universal set**, or the **universe**, and to symbolize it as $U$. We represent $U$ in the Venn diagram with a rectangle.

In the following discussions, we will let $U$ be the set of students in a small class:

$U = \{$Amy, Uri, Tia, Eli, Pam, Sue, Tom, Riki$\}$

We begin with two subsets of $U$:

$B = \{$students who have at least one brother$\}$
$S = \{$students who have at least one sister$\}$