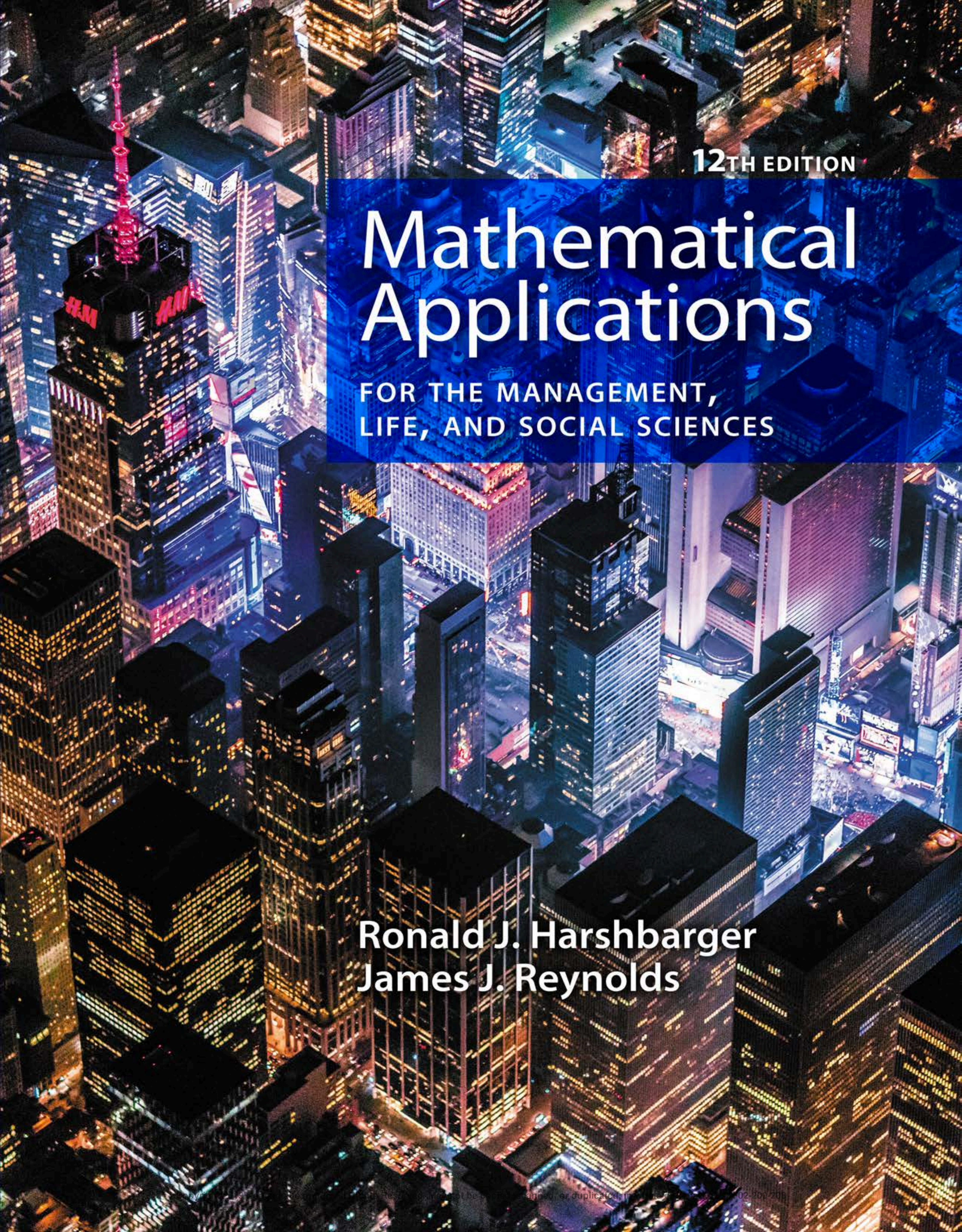


12TH EDITION

Mathematical Applications

FOR THE MANAGEMENT, LIFE, AND SOCIAL SCIENCES

Ronald J. Harshbarger
James J. Reynolds

Mathematical Applications

for the Management, Life, and Social Sciences

12th Edition

Ronald J. Harshbarger
University of South Carolina

James J. Reynolds
Clarion University of Pennsylvania

CENGAGE

Australia • Brazil • Mexico • Singapore • United Kingdom • United States

Mathematic Applications for the Management, Life, and Social Sciences, Twelfth Edition

Ronald J. Harshbarger, James J. Reynolds

Senior Vice President: Erin Joyner
Product Director: Mark Santee
Product Manager: Rita Lombard
Content Developer: Morgan Mendoza
Product Assistant: Abby DeVeuve
Marketing Manager: Ana Albinson and Giana Manzi
Senior Content Project Manager: John Fuller
Art Director: Vernon Boes
Manufacturing Planner: Douglas Bertke
IP Analyst: Ann Hoffman
IP Project Manager: Erika Mugavin
Production Service and Compositor: Lumina Datamatics
Text and Cover Design: Hespenheide Design
Cover Image: Times Square from helicopter at night. Image courtesy of WIN-Initiative/ Getty Images.

Library of Congress Control Number: 2017953697

Student Edition:
978-1-337-62534-0

Loose-leaf Edition:
978-1-337-63053-5

Instructor's Edition:
978-1-337-63045-0

Cengage Learning
20 Channel Center Street
Boston, MA 02210
USA

Cengage is a leading provider of customized learning solutions with employees residing in nearly 40 different countries and sales in more than 125 countries around the world. Find your local representative at **www.cengage.com.**

Cengage products are represented in Canada by Nelson Education, Ltd.

To learn more about Cengage platforms and services, visit **www.cengage.com**.

To register or access your online learning solution or purchase materials for your course, visit **www.cengagebrain.com**.

Printed in the United States of America
Print Number: 01 Print Year: 2017

Contents

Preface

To paraphrase English mathematician, philosopher, and educator Alfred North Whitehead, the purpose of education is not to fill a vessel but to kindle a fire. In particular, Whitehead encouraged students to be creative and imaginative in their learning and to continually form ideas into new and more exciting combinations. This desirable goal is not always an easy one to realize in mathematics with students whose primary interests are in areas other than mathematics. The purpose of this text, then, is to present mathematical skills and concepts and to apply them to ideas that are important to students in the management, life, and social sciences. We hope that this look at the relevance of mathematical ideas to a broad range of fields will help inspire the imaginative thinking and excitement for learning about which Whitehead spoke. The applications included allow students to view mathematics in a practical setting relevant to their intended careers. Almost every chapter of this book includes a section or two devoted to the applications of mathematical topics, and every section contains a number of application examples and problems. An index of these applications on the front and back inside covers demonstrates the wide variety used in examples and exercises. Although intended for students who have completed two years of high school algebra or its equivalent, this text begins with a brief review of algebra that, if covered, will aid in preparing students for the work ahead.

Pedagogical Features

In this new edition, we have incorporated many suggestions that reflect the needs and wishes of our users, including effective pedagogical features from previous editions.

Intuitive Viewpoint. The book is written from an intuitive viewpoint, with emphasis on concepts and problem solving rather than on mathematical theory. Yet each topic is carefully developed and explained, and examples illustrate the techniques involved.

Flexibility. At different colleges and universities, the coverage and sequencing of topics may vary according to the purpose of the course and the nature of the student audience. To accommodate alternative approaches, the text has a great deal of flexibility in the order in which topics may be presented and the degree to which they may be emphasized.

Applications. We have found that integrating applied topics into the discussions and exercises helps provide motivation within the sections and demonstrates the relevance of each topic. Numerous real-life application examples and exercises represent the applicability of the mathematics, and each application problem is identified so that the instructor or student can select applications that are of special interest. In addition, we have found that offering separate lessons on applied topics such as cost, revenue, and profit functions brings the preceding mathematical discussions into clear, concise focus and provides a thread of continuity as mathematical sophistication increases. There are 10 such sections throughout the book and two application-focused chapters: Chapter 4, devoted to linear programming, and Chapter 6, devoted to financial applications. Of the more than 5500 exercises in the book, more than 2000 are applied.

Chapter Warm-ups. With the exception of Chapter 0, a Warm-up appears at the beginning of each chapter and invites students to test themselves on the skills needed for that chapter. The Warm-up sections present many prerequisite problem types that are keyed to the appropriate sections in the upcoming chapter where those skills are needed. Students who have difficulty with any particular skill are directed to specific sections of the text for review. Instructors may also find the Warm-ups useful in creating a course syllabus that includes an appropriate scope and sequence of topics.

CHAPTER WARM-UP

Prerequisite Problem Type	For Section
Evaluate: (a) $2(-1)^3 - 3(-1)^2 + 1$ (b) $3(-3) - 1$ (c) $14(10) - 0.02(10^2)$	**1.1–1.3, 1.5, 1.6**

Application Previews and Associated Examples. Each section begins with an Application Preview that establishes the context and direction for the concepts that will be presented. Each of these Previews presents the mathematics in the section and references a completely worked Application Preview Example appearing later in the section.

SECTION 1.1

Solutions of Linear Equations and Inequalities in One Variable

OBJECTIVES

- To solve linear equations in one variable
- To solve applied problems using linear equations
- To solve linear inequalities in one variable

APPLICATION PREVIEW

Using data from 1980 and projected to 2050, the number of Hispanics ir
civilian non-institutional population can be approximated by

$$y = 0.876x + 6.084$$

million, where x is the number of years after 1980 (*Source*: U.S. Census Bu
find the year when this
we solve 36.74 = 0.876
linear equations and ine

EXAMPLE 2 U.S. Hispanic Population | APPLICATION PREVIEW |

Using data from 1980 and projected to 2050, the number of Hispanics
non-institutional population can be approximated by $y = 0.876x + 6.08$
is the number of years after 1980 (*Source:* U.S. Census Bureau). Accordi
in what year will the Hispanic population equal 36.74 million?

Solution

To answer this question, we solve

$$\begin{aligned} 36.74 &= 0.876x + 6.084 \\ 30.656 &= 0.876x \\ 34.995 &\approx x \end{aligned}$$

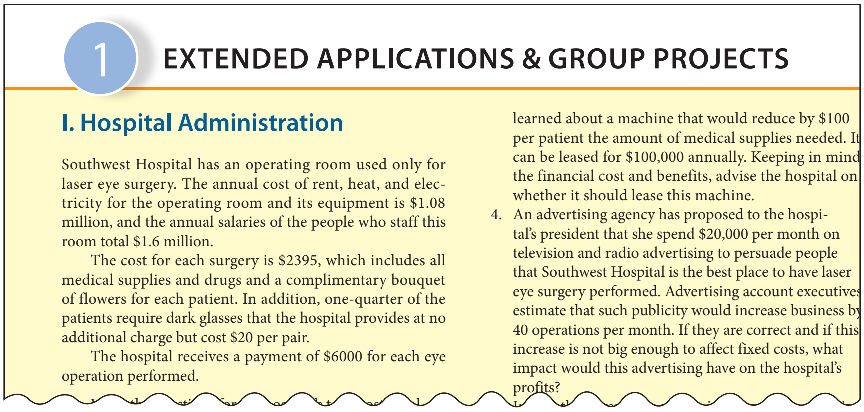
Recall that x is the number of years past 1980, so we use $x = 35$ when c
to which this corresponds: $1980 + 35 = 2015$.

Checking reveals that $36.74 \approx 0.876(34.995) + 6.084$. ✔

Thus the number of Hispanics in the United States is estimated to
36.74 million in 2015.

Comprehensive Exercise Sets. The overall variety and grading of drill and application exercises offer problems for different skill levels, and there are enough challenging problems to stimulate students in thoughtful investigations. Many exercise sets contain critical-thinking and thought-provoking multistep problems that extend students' knowledge and skills.

Extended Applications and Group Projects. Starting with Chapter 1, each chapter ends with at least two case studies, which further illustrate how mathematics can be used in business and personal decision making. In addition, many applications are cumulative in that solutions require students to combine the mathematical concepts and techniques they learned in some of the preceding chapters.



1 EXTENDED APPLICATIONS & GROUP PROJECTS

I. Hospital Administration

Southwest Hospital has an operating room used only for laser eye surgery. The annual cost of rent, heat, and electricity for the operating room and its equipment is $1.08 million, and the annual salaries of the people who staff this room total $1.6 million.

The cost for each surgery is $2395, which includes all medical supplies and drugs and a complimentary bouquet of flowers for each patient. In addition, one-quarter of the patients require dark glasses that the hospital provides at no additional charge but cost $20 per pair.

The hospital receives a payment of $6000 for each eye operation performed.

learned about a machine that would reduce by $100 per patient the amount of medical supplies needed. It can be leased for $100,000 annually. Keeping in mind the financial cost and benefits, advise the hospital on whether it should lease this machine.

4. An advertising agency has proposed to the hospital's president that she spend $20,000 per month on television and radio advertising to persuade people that Southwest Hospital is the best place to have laser eye surgery performed. Advertising account executives estimate that such publicity would increase business by 40 operations per month. If they are correct and if this increase is not big enough to affect fixed costs, what impact would this advertising have on the hospital's profits?

Graphical, Numerical, and Symbolic Methods. A large number of real data modeling applications are included in the examples and exercises throughout the text and are denoted by the header ***Modeling.*** Many sections include problems with functions that are modeled from real data, and some problems ask students to model functions from the data given. These problems are solved by using one or more graphical, numerical, or symbolic methods.

Graphing Calculators and Excel. Instructors differ on how they use technology in their course. The icon on the left denotes the many examples, applications, Technology Notes, Calculator Notes, and Spreadsheet Notes throughout the text where technology use is featured or appropriate. Many of these notes reference detailed step-by-step instructions in Appendix A (Graphing Calculator Guide) and Appendix B (Excel Guide) and in the Online Guide for Excel. Discussions of the use of technology are placed in subsections and examples in many sections so that they can be emphasized or de-emphasized at the option of the instructor.

The discussions of graphing calculator technology highlight its most common features and uses, such as graphing, window setting, Trace, Zoom, Solver, tables, finding points of intersection, numerical derivatives, numerical integration, matrices, solving inequalities, and modeling (curve fitting). While technology never replaces the mathematics, it does supplement and extend the mathematics by providing opportunities for generalization and alternative ways of understanding, doing, and checking. Some exercises that are better worked with the use of technology—including graphing calculators and Excel—are highlighted with the technology icon. Of course, many additional exercises can benefit from the use of technology, at the option of the instructor. Technology can be used to graph functions and to discuss the generalizations, applications, and implications of problems being studied.

Excel is useful in solving problems involving linear equations; systems of equations; quadratic equations; matrices; linear programming; output comparisons of $f(x)$, $f'(x)$, and $f''(x)$; and maxima and minima of functions subject to constraints. Excel is also a useful problem-solving tool when studying the mathematics of finance in Chapter 6.

Checkpoints. The Checkpoints ask questions and pose problems within each section's discussion, allowing students to check their understanding of the skills and concepts under discussion before they proceed. Answers to these Checkpoints appear before the section exercises. Complete solutions are available on the textbook's companion web site (www.cengagebrain.com).

EXAMPLE 4 **Solving an Equation for One of Two Variables**

Solve $4x + 3y = 12$ for y.

Solution

No fractions or parentheses are present, so we subtract $4x$ from both sides to get only term that contains y on one side.

$$3y = -4x + 12$$

Dividing both sides by 3 gives the solution.

$$y = -\frac{4}{3}x + 4$$

Check: $4x + 3\left(\frac{-4}{3}x + 4\right) \stackrel{?}{=} 12$

$4x + (-4x + 12) = 12$ ✔

CHECKPOINT 2. Solve for y: $y - 4 = -4(x + 2)$

CHECKPOINT 1. Solve the following for x.

(a) $4(x - 3) = 10x - 12$ (b) $\frac{5(x - 3)}{6} - x = 1 - \frac{x}{9}$

Objective Lists. Every section begins with a brief list of objectives that outlines the goals of that section for the student.

SECTION 1.1 **Solutions of Linear Equations and Inequalities in One Variable**

OBJECTIVES

- To solve linear equations in one variable
- To solve applied problems using

APPLICATION PREVIEW

Using data from 1980 and projected to 2050, the number of Hispan civilian non-institutional population can be approximated by

Procedure/Example and Property/Example Tables. Appearing throughout the text, these tables aid student understanding by giving step-by-step descriptions of important procedures and properties with illustrative examples worked out beside them.

SOLVING A LINEAR EQUATION

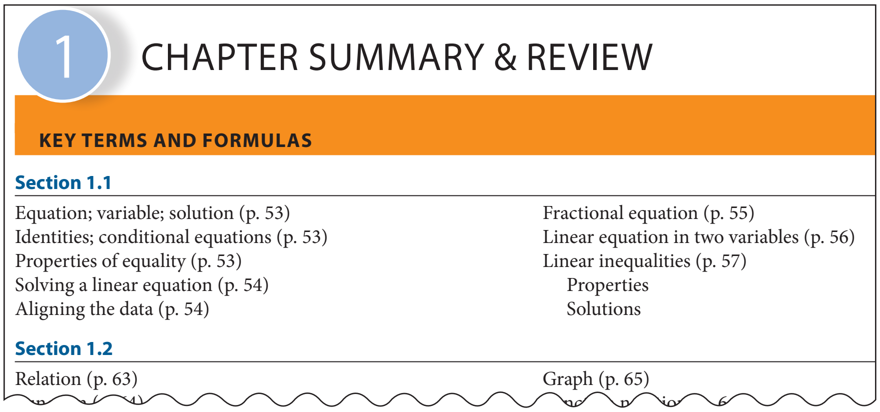
Procedure	Example
To solve a linear equation in one variable:	Solve $\frac{3x}{4} + 3 = \frac{x - 1}{3}$.
1. If the equation contains fractions, multiply both sides by the least common denominator (LCD) the fractions.	1. LCD is 12.
2. Remove any parentheses in the equation.	

PROPERTIES OF EQUALITY

Properties	Examples
Substitution Property	
The equation formed by substituting one expression for an equal expression is equivalent to the original equation.	$3(x - 3) - \frac{1}{2}(4x - 18) = 4$ is equivalen and to $x = 4$. We say that the solution se

Boxed Information. All important information is boxed for easy reference, and key terms are highlighted in boldface.

Key Terms and Formulas. At the end of each chapter, just before the Chapter Review Exercises, there is a section-by-section listing of that chapter's key terms and formulas, including their page references. This provides a well-organized core from which a student can build a review, both to consult while working the Review Exercises and to identify quickly any section needing additional study.



1 CHAPTER SUMMARY & REVIEW

KEY TERMS AND FORMULAS

Section 1.1

Equation; variable; solution (p. 53)	Fractional equation (p. 55)
Identities; conditional equations (p. 53)	Linear equation in two variables (p. 56)
Properties of equality (p. 53)	Linear inequalities (p. 57)
Solving a linear equation (p. 54)	Properties
Aligning the data (p. 54)	Solutions

Section 1.2

Relation (p. 63)	Graph (p. 65)

Review Exercises and Chapter Tests. At the end of each chapter, a set of Review Exercises offers students extra practice on topics in that chapter. These reviews cover each chapter's topics in their section order, with section references, so that students get a thorough, structured review and can readily find a section for further review if difficulties occur. A Chapter Test follows each set of Review Exercises. All Chapter Tests provide a mixture of problems that do not directly mirror the order of topics found within the chapter. This organization of the Chapter Test ensures that students have a firm grasp of material in the chapter. All answers to both the Review Exercises and Chapter Tests appear in the Answers section.

Changes in the Twelfth Edition

In the twelfth edition, the text continues to be characterized by complete and accurate pedagogy, mathematical precision, excellent exercise sets, numerous real and varied applications in the examples and exercises, and student-friendly exposition. The most significant changes to this edition follow.

1. More than 250 application examples and exercises that had become outdated have been updated or replaced with new applications.
2. A number of new Calculator Notes were added or expanded in the text.
3. Normal Distributions discussions were expanded to include more calculator usage and reduce reliance on tables. The number of examples and problems using inverse normal operations was increased.
4. Discussion and instruction were streamlined and clarified where appropriate.
5. Some skill exercise sets were reorganized and rewritten to improve variety, coverage, and grading.
6. More problems were added to exercises asking students to explain their answers.
7. Numerous new Spreadsheet Notes were added throughout the text, including in the discussion of statistical measures and binomial and normal distributions.
8. Many Spreadsheet Notes have been expanded to include specific Excel steps, along with references to Appendix B and the Online Excel Guide.
9. Appendix B has been updated by eliminating Part 1, Excel 2003, and replacing Part 2 with a new Appendix B discussing Excel 2010 and Excel 2016.
10. New Excel topics have been added in Appendix B.

Chapter Changes

Chapter 1:

- Difference Quotient definition was added, and its discussion was expanded.
- Calculator Note and Spreadsheet Note discussions for graphing equations and for solving linear equations were expanded throughout the chapter.
- "Solving Linear Equations" was added as a topic in Appendix B and referenced in the Section 1.4 Spreadsheet Note.
- The discussion of domains of function that result from function operations was expanded.

Chapter 2:

- Detailed Excel instructions for modeling data with functions were added to the Spreadsheet Notes.
- Section 2.3 now defines a monopoly market in the discussion of break-even points.
- More variety was added to the skill problems in Section 2.1.

Chapter 3:

- Detailed Excel instructions involving performing matrix operations and solving systems of equations with matrices were added to Spreadsheet Notes throughout the chapter.
- The Calculator Note with instructions for finding the inverse of a matrix was rewritten for clarity. A new Spreadsheet Note with instructions for finding the inverse of a matrix was added.
- The general solution of the matrix equation $AX = B$ by using A^{-1} was shortened.

Chapter 4:

- The introduction of the simplex method for solving linear programming problems was streamlined for clarity and efficiency.

Chapter 5:

- Section 5.1 was reorganized to improve the flow from general exponential functions to exponential functions with base e.
- The discussion of graphing exponential functions with Excel in the Spreadsheet Notes was expanded.

Chapter 6:

- The definition and discussion of Return on Investment was added to Section 6.1, and the return on investment was compared to the simple interest rate.
- Updated interest rates to bring them in line with the current financial community.
- Updated payment amounts to be more realistic.

Chapter 7:

- The exposition about use of probability trees to solve problems was clarified.
- Additional application problems that use Bayes theorem were added.
- Both the drill and application exercise sets were revised to improve balance, variety, and quality.

Chapter 8:

- Section 8.4 was rewritten to include more calculator usage and reduce the reliance on tables. Discussion of using Inverse Normal commands to solve normal probability problems, and an example were added. Calculator discussion of InvNorm was added along with problems using this skill.

- New Calculator Notes were added with detailed instructions for finding binomial probabilities and cumulative binomial probabilities.
- A new Spreadsheet Note with instructions for constructing bar graphs with Excel was added. The Calculator Note with instructions for graphing histograms was revised to provide detailed steps.
- The introduction to variance was revised. A Spreadsheet Note using the Excel formulas for mean, standard deviation was added.
- Applications of expected value were expanded to include $E(f(x))$.
- New Calculator Notes and Spreadsheet Notes with instructions for finding normal probabilities and for solving normal probability problems were added to Section 8.4.
- In the introduction to normal distribution, the relationship between areas under curve and percent of population between 2 scores was clarified.
- The Powerball Project was updated.
- A number of skills exercises were revised to improve variety.

Chapter 9:

- A new Spreadsheet Note with instructions for finding limits with Excel was added.
- The Calculator Note exploring the relationship between secant lines and tangent lines was rewritten for clarity.

Chapter 10:

- Details were added to the solution steps in applied max-min examples.
- An example to illustrate all possible cases for horizontal asymptotes of rational functions was added.
- A new Spreadsheet Note discussing the use of Excel in finding relative maxima and relative minima was added.
- Quantities and values were updated in various applications.

Chapter 11:

- The drill exercises in the Chapter Review were rebalanced to improve grading and variety.
- A new Spreadsheet Note has been added with instructions for finding relative maxima and minima for functions involving logarithms.

Chapter 12:

- The drill exercises were improved and expanded.
- Objectives for Section 12.1 and 12.2 were revised and expanded.

Chapter 13:

- Skill exercises were added and reorganized.
- Calculator Notes and Spreadsheet Notes with instructions for finding areas between two curves and finding numerical integrals with the Trapezoidal Rule were added.
- The discussion of probability calculations for continuous distributions was expanded, and more conditional probability problems were added.

Chapter 14:

- The exposition in the Test for Maximum and Minimum box as well as in examples and exercises was improved.
- The notation in the development of linear regression formulas was clarified.

Resources for the Student

Student Solutions Manual (978-1-337-63046-7)

This manual provides complete worked-out solutions to all odd-numbered exercises in the text, giving you a chance to check your answers and ensure you took the correct steps to arrive at an answer.

WEBASSIGN
From Cengage

www.webassign.com

Printed Access Card: 978-1-337-63054-2
Instant Access Code: 978-1-337-63055-9

Prepare for class with confidence using WebAssign from Cengage Harshbarger/Reynolds' *Mathematical Applications for the Management, Life, and Social Sciences, 12th Edition*. This online learning platform, which includes an interactive eBook, fuels practice, so you truly absorb what you learn—and are better prepared come test time. Videos and tutorials walk you through concepts and deliver instant feedback and grading, so you always know where you stand in class. Focus your study time and get extra practice where you need it most. Study smarter with WebAssign!

Ask your instructor today how you can get access to WebAssign, or learn about self-study options at www.webassign.com

Online Excel and Graphing Calculator Guides

To access the Online Technology Guides referenced throughout your textbook, you must first create a Cengage account at cengage.com/login. Once you are logged in, enter the ISBN for this book in the search tab: 9781337625340. This will pull up the product page for the book. Scroll down to the tabs near the bottom of the page and click on the "Free Materials" tab. Then hit the "Save to MyHome" button. The link to the free resources associated with your textbook, including the Online Technology Guides, will now be available on your My Home page.

Resources for the Instructor

Instructor Companion Site

Everything you need for your course in one place! This collection of book-specific lecture and class tools is available online via www.cengage.com/login. Access and download PowerPoint presentations, the complete solutions manual, technology guides, and more.

Complete Solutions Manual

The Complete Solutions Manual provides worked-out solutions of all exercises in the text. In addition, it contains solutions for the special features in the text, such as Extended Applications and Group Projects. This manual can be found on the Instructor Companion Site.

Cengage Learning Testing Powered by Cognero®

This flexible, online system allows you to author, edit, and manage test bank content, create multiple test versions in an instant, and deliver tests from your LMS, your classroom, or wherever you want. This is available online via www.cengage.com/login.

WEBASSIGN
From Cengage

www.webassign.com/cengage

Printed Access Card: 978-1-337-63054-2
Instant Access Code: 978-1-337-63055-9

WebAssign from Cengage for Harshbarger/Reynolds' *Mathematical Applications for the Management, Life, and Social Sciences, 12th Edition* is a fully customizable solution, including an interactive eBook and video lecture series for STEM disciplines that empowers you to help your students learn, not just do homework. CoursePacks make assigning homework fast and easy, and the ready-made projects make the task of collecting and grading projects easier. Insightful tools save you time and highlight exactly where your students are struggling. Decide when and what type of help students can access while working on assignments—and incentivize independent work so that help features aren't abused. Meanwhile, your students get an engaging experience, instant feedback, and better outcomes. A total win-win!

To try a sample assignment, learn about LMS integration, or connect with our digital course support visit www.webassign.com/cengage

Acknowledgments

We would like to thank the many people who helped us at various stages of revising this text. The encouragement, criticism, contributions, and suggestions that were offered were invaluable to us. We are deeply grateful to Edwin Herman for his skillful and thorough accuracy checking of the entire text and answer section.

For their reviews of this textbook and the many helpful comments that were offered, we would like to thank

Annie M. Stith-Willis, PhD	*Virginia Commonwealth University*
Elsie M. Campbell	*Angelo State University*
Dr. Helen Burn	*Highline Community College*
Barbara B. Hunter	*Highline Community College*
Gina P. Dunn	*Lander University*
Dr. Joseph McCollum	*Siena College*
Micah Chrisman, PhD	*Monmouth University*
Bracha Silverstone	*Wharton County Junior College*
John C. Esenwa	*Howard Community College*
Dr. William J. Arbuckle	*University Of Central Oklahoma*

Ronald J. Harshbarger
James J. Reynolds

©Andrey Bayda/Shutterstock.com

0 Algebraic Concepts

This chapter provides a brief review of the algebraic concepts that will be used throughout the text. You may be familiar with these topics, but it may be helpful to spend some time reviewing them. In addition, beginning with the next chapter, each chapter opens with a warm-up page that identifies prerequisite skills needed for that chapter. If algebraic skills are required, the warm-up cites their coverage in this chapter. Thus you will find that this chapter is a useful reference as you study later chapters.

The topics and some representative applications studied in this chapter include the following.

SECTIONS	APPLICATIONS
0.1 Sets Set operations Venn diagrams	Dow Jones Industrial Average, jobs growth, stocks
0.2 The Real Numbers Inequalities and intervals Absolute value	Income taxes, health statistics, average annual wage
0.3 Integral Exponents	Personal income, endangered species
0.4 Radicals and Rational Exponents Roots and fractional exponents Operations with radicals	Richter scale, half-life
0.5 Operations with Algebraic Expressions	Revenue, profit
0.6 Factoring Common factors Factoring trinomials	Simple interest, revenue
0.7 Algebraic Fractions Operations Complex fractions	Average cost, advertising and sales

SECTION 0.1 Sets

A **set** is a well-defined collection of objects. We may talk about a set of books, a set of dishes, a set of students, or a set of individuals with a certain blood type. There are two ways to tell what a given set contains. One way is by listing the **elements** (or **members**) of the set, in any order and usually between braces. We may say that a set A contains 1, 2, 3, and 4 by writing $A = \{1, 2, 3, 4\}$. To say that 4 is an element of set A, we write $4 \in A$. Similarly, we write $5 \notin A$ to denote that 5 is not an element of set A.

If all the elements of the set can be listed, the set is said to be a **finite set.** $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$ are examples of finite sets. When we do not want to list all the elements of a finite set, we can use three dots to indicate the unlisted elements of the set. For example, the set of even integers from 8 to 8952, inclusive, can be written as

$$\{8, 10, 12, 14, \ldots, 8952\}$$

Since we cannot list all the elements of an **infinite set**, we use the three dots to indicate that the list continues. For example, $N = \{1, 2, 3, 4, \ldots\}$ is an infinite set. This set N is called the set of **natural numbers.**

Another way to specify the elements of a given set is by description. For example, we may write $D = \{x: x \text{ is a Ford automobile}\}$ to describe the set of all Ford automobiles. Furthermore, $F = \{y: y \text{ is an odd natural number}\}$ is read "F is the set of all y such that y is an odd natural number."

EXAMPLE 1 Describing Sets

Write the following sets in two ways.

(a) The set A of natural numbers less than 6

(b) The set B of natural numbers greater than 10

(c) The set C containing only 3

Solution

(a) $A = \{1, 2, 3, 4, 5\}$ or $A = \{x: x \text{ is a natural number less than } 6\}$

(b) $B = \{11, 12, 13, 14, \ldots\}$ or $B = \{x: x \text{ is a natural number greater than } 10\}$

(c) $C = \{3\}$ or $C = \{x: x = 3\}$ ■

A set that contains no elements is called the **empty set** or the **null set,** and it is denoted by $\varnothing$ or by $\{\ \}$. The set of living veterans of the War of 1812 is empty because there are no living veterans of that war. Thus

$$\{x: x \text{ is a living veteran of the War of 1812}\} = \varnothing$$

Special relations that may exist between two sets are defined as follows.

RELATIONS BETWEEN SETS

Definition	Example
1. Sets X and Y are **equal** if they contain the same elements.	1. If $X = \{1, 2, 3, 4\}$ and $Y = \{4, 3, 2, 1\}$, then $X = Y$.
2. A is called a **subset** of B, which is written $A \subseteq B$ if every element of A is an element of B. The empty set is a subset of every set. Each set A is a subset of itself.	2. If $A = \{1, 2, c, f\}$ and $B = \{1, 2, 3, a, b, c, f\}$, then $A \subseteq B$. Also, $\varnothing \subseteq A$, $\varnothing \subseteq B$, $A \subseteq A$, and $B \subseteq B$.
3. If C and D have no elements in common, they are called **disjoint.**	3. If $C = \{1, 2, a, b\}$ and $D = \{3, e, 5, c\}$, then C and D are disjoint.

In the discussion of particular sets, the assumption is always made that the sets under discussion are all subsets of some larger set, called the **universal set** U. The choice of the universal set depends on the problem under consideration. For example, in discussing the set of all students and the set of all female students, we may use the set of all humans as the universal set.

We may use **Venn diagrams** to illustrate the various relationships among sets. A rectangle represents the universal set, and closed figures inside the rectangle represent the sets under consideration. Figures 0.1–0.3 show such Venn diagrams.

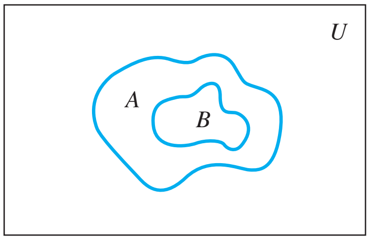


Figure 0.1
B is a subset of *A*; $B \subseteq A$.

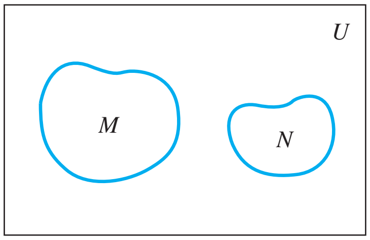


Figure 0.2
M and *N* are disjoint.

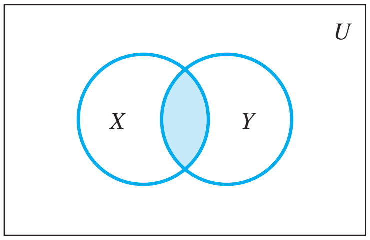


Figure 0.3
X and *Y* are not disjoint.

CHECKPOINT

Let $A = \{2, 3, 5, 7, 11\}$, $B = \{2, 4, 6, 8, 10\}$, and $C = \{6, 10, 14, 18, 22\}$. Use these sets to answer the following.

1. (a) Of which sets is 6 an element?
   (b) Of which sets is $\{2\}$ a subset?
2. Which of the following are true?
   (a) $2 \in A$
   (b) $\{2\} \in B$
   (c) $2 \in C$
   (d) $5 \notin A$
   (e) $5 \notin B$
3. Which pair of A, B, and C is disjoint?
4. Which of $\varnothing$, A, B, and C are subsets of
   (a) the set P of all prime numbers?
   (b) the set M of all multiples of 2?
5. Which of A, B, and C is equal to $D = \{x\colon x = 4n + 2 \text{ for natural numbers } 1 \le n \le 5\}$?

Set Operations

The set containing the members that are common to two sets is said to be the **intersection** of the two sets.

Set Intersection

The intersection of A and B, written $A \cap B$, is defined by

$$A \cap B = \{x\colon x \in A \text{ and } x \in \text{B}\}$$

EXAMPLE 2 Set Intersection

(a) If $A = \{2, 3, 4, 5\}$ and $B = \{3, 5, 7, 9, 11\}$, find $A \cap B$.
(b) Which of A, B, and $A \cap B$ is a subset of A?

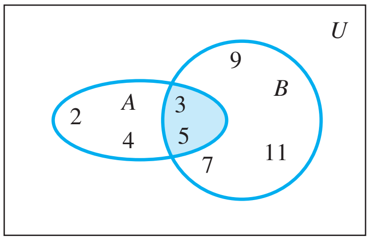


Figure 0.4

Solution

(a) $A \cap B = \{3, 5\}$ because 3 and 5 are the common elements of A and B. Figure 0.4 shows the sets and their intersection.
(b) $A \cap B$ and A are subsets of A.

The **union** of two sets is the set that contains all members of the two sets.

Set Union

The union of A and B, written $A \cup B$, is defined by

$$A \cup B = \{x: x \in A \text{ or } x \in B \text{ (or both)}\}^*$$

EXAMPLE 3 Set Union

If $X = \{a, b, c, f\}$ and $Y = \{e, f, a, b\}$, find $X \cup Y$.

Solution

$X \cup Y = \{a, b, c, e, f\}$

We can illustrate the intersection and union of two sets by the use of Venn diagrams. The shaded region in Figure 0.5 represents $A \cap B$, the intersection of A and B, and the shaded region in Figure 0.6—which consists of all parts of both circles—represents $A \cup B$.

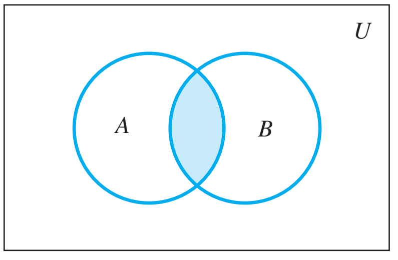


Figure 0.5
Intersection of *A* and *B*.

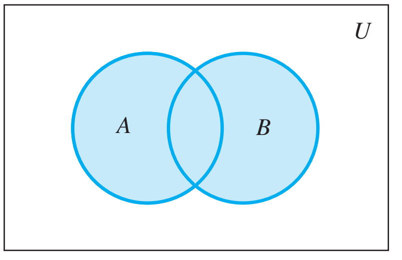


Figure 0.6
Union of *A* and *B*.

EXAMPLE 4 Set Intersection and Union

Let $A = \{x: x \text{ is a natural number less than } 6\}$ and $B = \{1, 3, 5, 7, 9, 11\}$.
(a) Find $A \cap B$.
(b) Find $A \cup B$.

Solution

Note that $A = \{1, 2, 3, 4, 5\}$.
(a) $A \cap B = \{1, 3, 5\}$
(b) $A \cup B = \{1, 2, 3, 4, 5, 7, 9, 11\}$

All elements of the universal set that are not contained in a set A form a set called the **complement** of A.

Set Complement

The complement of A, written A', is defined by

$$A' = \{x: x \in U \text{ and } x \notin A\}$$

We can use a Venn diagram to illustrate the complement of a set. The shaded region of Figure 0.7 represents A', and the *unshaded* region of Figure 0.5 represents $(A \cap B)'$.

* In mathematics, the word *or* means "one or the other or both."

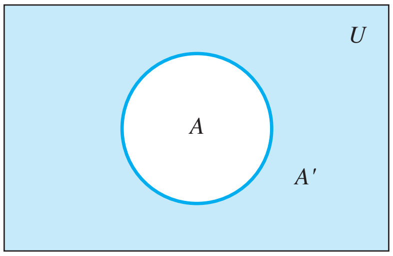


Figure 0.7

EXAMPLE 5 Operations with Sets

If U is the set of natural numbers less than 10, $A = \{1, 3, 6\}$, and $B = \{1, 6, 8, 9\}$, find the following.

(a) A'
(b) B'
(c) $(A \cap B)'$
(d) $A' \cup B'$

Solution

(a) $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, so $A' = \{2, 4, 5, 7, 8, 9\}$
(b) $B' = \{2, 3, 4, 5, 7\}$
(c) $A \cap B = \{1, 6\}$, so $(A \cap B)' = \{2, 3, 4, 5, 7, 8, 9\}$
(d) $A' \cup B' = \{2, 4, 5, 7, 8, 9\} \cup \{2, 3, 4, 5, 7\} = \{2, 3, 4, 5, 7, 8, 9\}$ ■

CHECKPOINT

Given the sets $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 3, 5, 7, 9\}$, $B = \{2, 3, 5, 7\}$, and $C = \{4, 5, 6, 7, 8, 9, 10\}$, find the following.

6. $A \cup B$
7. $B \cap C$
8. A'

EXAMPLE 6 Stocks

Suppose an investment adviser monitored several stocks for clients and on a certain day categorized 23 stocks according to whether

- their closing price on the previous day was less than \$50/share (set C)
- their price-to-earnings ratio was less than 20 (set P)
- their dividend per share was at least \$1.50 (set D).

Of these 23 stocks,

16 belonged to set P
12 belonged to set C
8 belonged to set D
3 belonged to both C and D
10 belonged to both C and P
7 belonged to both D and P
2 belonged to all three sets.

(a) How many stocks had closing prices of less than \$50 per share or price-to-earnings ratios of less than 20?
(b) How many stocks had none of the characteristics of set C, P, or D?
(c) How many stocks had only dividends per share of at least \$1.50?

Solution

We use a Venn diagram to organize the information. Note that the Venn diagram for three sets has eight separate regions [see Figure 0.8(a) on the next page]. To assign numbers from our data, we must begin with some information that refers to a single region—namely, that two stocks belonged to all three sets [see Figure 0.8(b)]. Because the region common to all three sets also is common to any pair, we can next use the information about stocks that belonged to two of the sets [see Figure 0.8(c)]. Finally, we can complete the Venn diagram [see Figure 0.8(d)].

(a) We need to add the numbers in the separate regions that lie within $C \cup P$. That is, 18 stocks closed under \$50 per share or had price-to-earnings ratios of less than 20.

(b) Five stocks are outside the three sets *C, D,* and *P.*

(c) Those stocks that had only dividends of at least \$1.50 per share are inside *D* but outside both *C* and *P.* There are no such stocks.

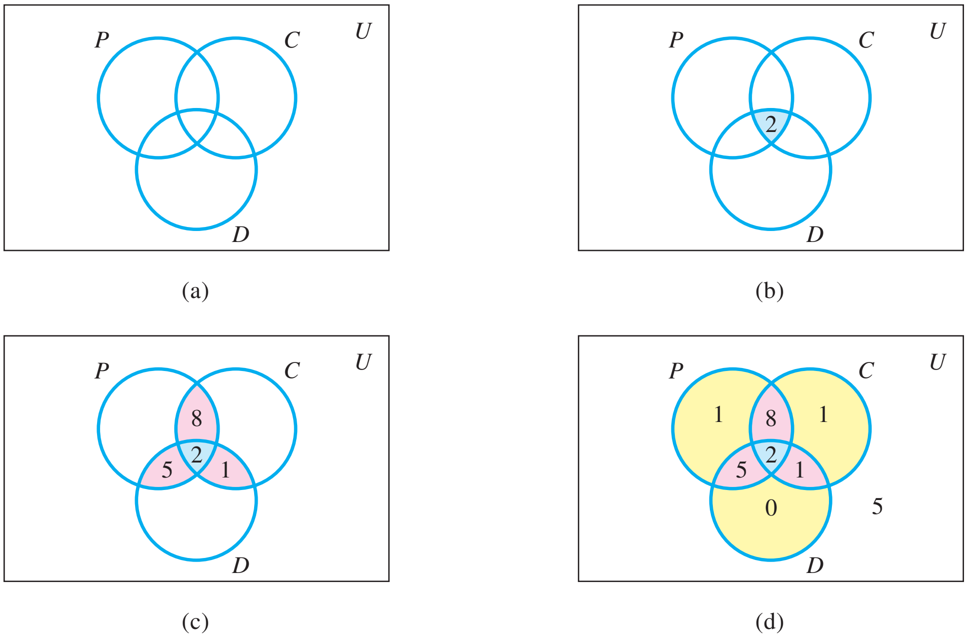


Figure 0.8

CHECKPOINT ANSWERS

1. (a) *B* and *C*
   (b) *A* and *B*
2. (a) True
   (b) False
   (c) False
   (d) False
   (e) True
3. *A* and *C*
4. (a) $\varnothing$ and *A*
   (b) $\varnothing$, *B*, and *C*
5. $C = D$
6. $\{1, 2, 3, 5, 7, 9\}$
7. $\{5, 7\}$
8. $\{2, 4, 6, 8, 10\}$

EXERCISES 0.1

In Problems 1–4, use $\in$ or $\notin$ to indicate whether the given object is an element of the given set.

1. 12 $\{1, 2, 3, 4, \ldots\}$
2. 5 $\{x: x$ is a natural number greater than $5\}$
3. 6 $\{x: x$ is a natural number less than $6\}$
4. 3 $\varnothing$

In Problems 5–8, write the following sets a second way.

5. $\{x: x$ is a natural number less than $8\}$
6. $\{x: x$ is a natural number greater than 6, less than $10\}$
7. $\{3, 4, 5, 6, 7\}$
8. $\{7, 8, 9, 10, \ldots\}$

In Problems 9 and 10, which of $\varnothing$, A, and B are subsets of B?

9. $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6\}$
10. $A = \{a, b, c, d\}$ and $B = \{c, d, a, b\}$?
11. Is $A \subseteq B$ if $A = \{a, b, c, d\}$ and $B = \{a, b, d\}$?
12. Is $A \subseteq B$ if $A = \{6, 8, 10, 12\}$ and $B = \{6, 8, 10, 14, 18\}$?

In Problems 13–16, use $\subseteq$ notation to indicate which set is a subset of the other.

13. $C = \{a, b, 1, 2, 3\}$, $D = \{a, b, 1\}$
14. $E = \{x, y, a, b\}$, $F = \{x, 1, a, y, b, 2\}$
15. $A = \{6, 8, 7, 4\}$, $B = \{8, 7, 6, 4\}$
16. $D = \{a, e, 1, 3, c\}$, $F = \{e, a, c, 1, 3\}$

In Problems 17–20, indicate whether the two sets are equal.

17. $A = \{a, b, \pi, \sqrt{3}\}$, $B = \{a, \pi, \sqrt{3}, b\}$
18. $A = \{x, g, a, b\}$, $D = \{x, a, b, y\}$
19. $D = \{x: x$ is a natural number less than $4\}$, $E = \{1, 2, 3, 4\}$
20. $F = \{x: x$ is a natural number greater than $6\}$, $G = \{7, 8, 9, \ldots\}$
21. From the following list of sets, indicate which pairs of sets are disjoint.

$$A = \{1, 2, 3, 4\}$$
$$B = \{x: x \text{ is a natural number greater than } 4\}$$
$$C = \{4, 5, 6, \ldots\}$$
$$D = \{1, 2, 3\}$$

22. If A and B are disjoint sets, what does $A \cap B$ equal?

In Problems 23–26, find $A \cap B$.

23. $A = \{2, 3, 4, 5, 6\}$ and $B = \{4, 6, 8, 10, 12\}$
24. $A = \{a, b, c, d, e\}$ and $B = \{a, d, e, f, g, h\}$
25. $A = \varnothing$ and $B = \{x, y, a, b\}$
26. $A = \{x: x$ is a natural number less than $4\}$ and $B = \{3, 4, 5, 6\}$

In Problems 27–30, find $A \cup B$.

27. $A = \{1, 2, 4, 5\}$ and $B = \{2, 3, 4, 5\}$
28. $A = \{a, e, i, o, u\}$ and $B = \{a, b, c, d\}$
29. $A = \varnothing$ and $B = \{1, 2, 3, 4\}$
30. $A = \{x: x$ is a natural number greater than $5\}$ and $B = \{x: x$ is a natural number less than $5\}$

In Problems 31–42, let

$$\mathbf{A = \{1, 3, 5, 8, 7, 2\}}$$
$$\mathbf{B = \{4, 3, 8, 10\}}$$
$$\mathbf{C = \{2, 4, 6, 8, 10\}}$$

and U be the universal set of natural numbers less than 11. Find the following.

31. A'
32. B'
33. $A \cap B'$
34. $A' \cap B'$
35. $(A \cup B)'$
36. $(A \cap B)'$
37. $A' \cup B'$
38. $(A' \cup B)'$
39. $(A \cap B') \cup C'$
40. $A \cap (B' \cup C')$
41. $(A \cap B')' \cap C$
42. $A \cap (B \cup C)$

The difference of two sets, $A - B$, is defined as the set containing all elements of A except those in B. That is, $A - B = A \cap B'$. Find $A - B$ for each pair of sets in Problems 43–46 if $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

43. $A = \{1, 3, 7, 9\}$ and $B = \{3, 5, 8, 9\}$
44. $A = \{1, 2, 3, 6, 9\}$ and $B = \{1, 4, 5, 6, 7\}$
45. $A = \{2, 1, 5\}$ and $B = \{1, 2, 3, 4, 5, 6\}$
46. $A = \{1, 2, 3, 4, 5\}$ and $B = \{7, 8, 9\}$

APPLICATIONS

47. ***Dow Jones Industrial Average*** The following table shows information about each year's Dow Jones Industrial average beginning value, ending value, and annual percent change for the years from 2005 to 2016. Let B be the set of years when the beginning value was less than 13,000. Let E be the set of years when the ending value was greater than 12,000. Let P be the set of years when the annual percent change was less than 7%.
    (a) List the elements of B, E, and P.
    (b) Is any of B, E, and P a subset of one of the others (besides itself)?
    (c) Write a verbal description of P'.
    (d) Find $B' \cap P'$ and describe it in words.
    (e) Find $E' \cap P$ and describe it in words.

Dow Jones Industrial Averages

Year	Beginning value	Ending value	Percent change
2005	10,783.01	10,717.50	−0.61%
2006	10,717.50	12,463.15	16.29%
2007	12,463.15	13,264.82	6.43%
2008	13,264.82	8776.39	−33.84%
2009	8776.39	10,428.05	18.82%
2010	10,428.05	11,577.51	11.02%
2011	11,577.51	12,217.56	5.53%
2012	12,217.56	13,104.14	7.26%
2013	13,104.14	16,576.66	26.50%
2014	16,576.66	17,823.07	7.52%
2015	17,823.07	17,425.03	−2.23%
2016	17,425.03	19,726.60	13.42%

Source: Dow Jones and Company

48. ***Job growth*** The number of jobs in 2000, the number projected in 2025, and the projected annual growth rate for jobs in some cities are shown in the following table. Consider the following sets.
   A = set of cities with at least 2,000,000 jobs in 2000 or projected in 2025
   B = set of cities with at least 1,500,000 jobs in 2000
   C = set of cities with projected annual growth rate of at least 2.5%
   (a) List A, B, and C (using the letters to represent the cities).
   (b) Is any of A, B, or C a subset of the other?
   (c) Find $A \cap C$ and describe the set in words.
   (d) Give a verbal description of B'.

Cities	Jobs in 2000 (thousands)	Projected Jobs in 2025 (thousands)	Annual Rates of Increase (%)
O (Orlando)	1098	2207	2.83
M (Myrtle Beach)	133	256	2.64
L (Atlanta)	2715	4893	2.38
P (Phoenix)	1953	3675	2.56
B (Boulder)	233	420	2.38

Source: Based on data from NPA Data Services, Inc.

Carbon emission controls **Suppose that the following table summarizes the opinions of various groups on the issue of carbon emission controls. Use this table for Problems 49 and 50.**

	Whites		Nonwhites		
Opinion	Rep.	Dem.	Rep.	Dem.	Total
Favor	100	250	30	200	580
Oppose	250	150	10	10	420
Total	350	400	40	210	1000

49. Identify the number of individuals in each of the following sets.
   (a) Republicans and those who favor carbon emission controls
   (b) Republicans or those who favor carbon emission controls
   (c) White Republicans or those who oppose carbon emission controls
50. Identify the number of individuals in each of the following sets.
   (a) Whites and those who oppose carbon emission controls
   (b) Whites or those who oppose carbon emission controls
   (c) Nonwhite Democrats and those who favor carbon emission controls
51. ***Languages*** A survey of 100 aides at the United Nations revealed that 65 could speak English, 60 could speak French, and 40 could speak both English and French.
   (a) Draw a Venn diagram representing the 100 aides. Use E to represent English-speaking aides and F to represent French-speaking aides.
   (b) How many aides are in $E \cap F$?
   (c) How many aides are in $E \cup F$?
   (d) How many aides are in $E \cap F'$?
52. ***Advertising*** Suppose that a survey of 100 advertisers in *U.S. News, These Times,* and *World* found the following.

   14 advertised in all three
   30 advertised in *These Times* and *U.S. News*
   26 advertised in *World* and *U.S. News*
   27 advertised in *World* and *These Times*
   60 advertised in *These Times*
   52 advertised in *U.S. News*
   50 advertised in *World*

   (a) Draw a Venn diagram representing the 100 advertisers.
   (b) How many advertised in none of these publications?
   (c) How many advertised only in *These Times*?
   (d) How many advertised in *U.S. News* or *These Times*?
53. ***College enrollments*** Records at a small college show the following about the enrollments of 100 first-year students in mathematics, fine arts, and economics.

   38 take math
   42 take fine arts
   20 take economics
   4 take economics and fine arts
   15 take math and economics
   9 take math and fine arts
   12 take math and economics but not fine arts

(a) How many take none of these three courses?
(b) How many take math or economics?
(c) How many take exactly one of these three courses?

54. ***Survey analysis*** In a survey of the dining preferences of 110 dormitory students at the end of the spring semester, the following facts were discovered about Adam's Lunch (AL), Pizza Tower (PT), and the Dining Hall (DH).

30 liked AL but not PT
21 liked AL only
63 liked AL
58 liked PT
27 liked DH
25 liked PT and AL but not DH
18 liked PT and DH

(a) How many liked PT or DH?
(b) How many liked all three?
(c) How many liked only DH?

55. ***Blood types*** Blood types are determined by the presence or absence of three antigens: A antigen, B antigen, and an antigen called the Rh factor. The resulting blood types are classified as follows:

type A if the A antigen is present
type B if the B antigen is present
type AB if both the A and B antigens are present
type O if neither the A nor the B antigen is present

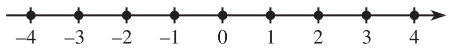
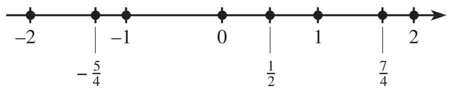
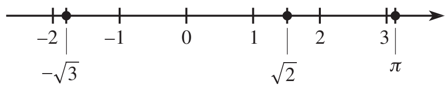
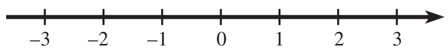
These types are further classified as *Rh-positive* if the Rh-factor antigen is present and *Rh-negative* otherwise.
(a) Draw a Venn diagram that illustrates this classification scheme.
(b) Identify the blood type determined by each region of the Venn diagram (such as A^+ to indicate type A, Rh-positive).
(c) Use Google or another source to find what percentage of the U.S. population has each blood type.

SECTION 0.2 The Real Numbers

In this text, we use the set of **real numbers** as the universal set. We can represent the real numbers along a line called the **real number line.** This number line is a picture, or graph, of the real numbers. Each point on the real number line corresponds to exactly one real number, and each real number can be located at exactly one point on the real number line. Thus, two real numbers are said to be equal whenever they are represented by the same point on the real number line. The equation $a = b$ (a equals b) means that the symbols a and b represent the same real number. Thus, $3 + 4 = 7$ means that $3 + 4$ and 7 represent the same number. Table 0.1 lists special subsets of the real numbers.

TABLE 0.1

SUBSETS OF THE SET OF REAL NUMBERS

	Description	Example (some elements shown)
Natural numbers	{1, 2, 3, . . .} The counting numbers.	
Integers	{. . ., −2, −1, 0, 1, 2, . . .} The natural numbers, 0, and the negatives of the natural numbers.	
Rational numbers	All numbers that can be written as the ratio of two integers, a/b, with $b \neq 0$. These numbers have decimal representations that either terminate or repeat.	
Irrational numbers	Those real numbers that *cannot* be written as the ratio of two integers. Irrational numbers have decimal representations that neither terminate nor repeat.	
Real numbers	The set containing all rational and irrational numbers (the entire number line).	

The following properties of the real numbers are fundamental to the study of algebra.

PROPERTIES OF THE REAL NUMBERS

Let a, b, and c denote real numbers.

1. The **Commutative Properties** for addition and multiplication.
$$a + b = b + a \qquad ab = ba$$
2. The **Associative Properties** for addition and multiplication.
$$(a + b) + c = a + (b + c) \qquad (ab)c = a(bc)$$
3. The **Additive Identity** is 0.
$$a + 0 = 0 + a = a$$
4. The **Multiplicative Identity** is 1.
$$a \cdot 1 = 1 \cdot a = a$$
5. Each real number a has an **Additive Inverse**, denoted by $-a$.
$$a + (-a) = -a + a = 0$$
6. The **Multiplicative Inverse** of the nonzero real number a is $1/a$, called the **reciprocal of *a*** and denoted by a^{-1}. (Note that $a^{-1} = 1/a$.)
$$a \cdot a^{-1} = a^{-1} \cdot a = 1$$
7. The **Distributive Law** for multiplication over addition.
$$a(b + c) = ab + ac \qquad (a + b)c = ac + bc$$

Note that there is a difference between a negative number and the negative of a number. For example, -5 is a negative number, but its negative is $-(-5) = 5$. Note also that Property 5 provides the means to subtract by defining $a - b = a + (-b)$ and Property 6 provides a means to divide by defining $a \div b = a \cdot (1/b)$. The number 0 has no multiplicative inverse, so division by 0 is undefined.

Inequalities and Intervals

We say that a is less than b (written $a < b$) if the point representing a is to the left of the point representing b on the real number line. For example, $4 < 7$ because 4 is to the left of 7 on the real number line. We also may say that 7 is greater than 4 (written $7 > 4$). We indicate that the number x is less than or equal to another number y by writing $x \le y$. We also indicate that p is greater than or equal to 4 by writing $p \ge 4$.

EXAMPLE 1 Inequalities

Use $<$ or $>$ notation to write the following.

(a) 6 is greater than 5. (b) 10 is less than 15.
(c) 3 is to the left of 8 on the real number line. (d) x is at most 12.

Solution

(a) $6 > 5$ (b) $10 < 15$ (c) $3 < 8$
(d) "x is at most 12" means that it must be less than or equal to 12. Thus, $x \le 12$. ■

The subset of the real numbers consisting of all real numbers x that lie between a and b, excluding a and b, can be denoted by the *double inequality* $a < x < b$ or by the **open interval** (a, b). It is called an open interval because neither of the endpoints is included in the interval. The **closed interval** $[a, b]$ represents the set of all real numbers x satisfying $a \le x \le b$. Intervals containing one endpoint, such as $(a, b]$ and $[a, b)$, are called **half-open intervals.**

We can use $[a, \infty)$ to represent the inequality $x \geq a$ and $(-\infty, a)$ to represent $x < a$. In each of these cases, the symbols ∞ and $-\infty$ are not real numbers but represent the fact that x increases without bound (∞) or decreases without bound $(-\infty)$. Table 0.2 summarizes the three types of intervals.

TABLE 0.2

INTERVALS

Type of Interval	Inequality Notation	Interval Notation	Graph
Open interval	$x > a$	(a, ∞)	
	$x < b$	$(-\infty, b)$	
	$a < x < b$	(a, b)	
Half-open interval	$x \geq a$	$[a, \infty)$	
	$x \leq b$	$(-\infty, b]$	
	$a \leq x < b$	$[a, b)$	
	$a < x \leq b$	$(a, b]$	
Closed interval	$a \leq x \leq b$	$[a, b]$	

Note that the interval $(-\infty, \infty)$, representing $-\infty < x < \infty$, is an open interval containing all real numbers (that is, the entire real number line).

CHECKPOINT

1. Decide which of the following are undefined.
   (a) $\frac{4}{0}$ (b) $\frac{0}{4}$
   (c) $\frac{4}{4}$ (d) $\frac{4-4}{4-4}$
2. For parts (a)–(d), write the inequality corresponding to the given interval and sketch its graph on a real number line.
   (a) $(1, 3)$ (b) $(0, 3]$
   (c) $[-1, \infty)$ (d) $(-\infty, 2)$
3. Express the following inequalities in interval notation and name the type of interval.
   (a) $3 \leq x \leq 6$ (b) $-6 \leq x < 4$

Absolute Value

Sometimes we are interested in the *distance* a number is from the origin (0) of the real number line, without regard to direction. The distance a number a is from 0 on the number line is the **absolute value** of a, denoted $|a|$. The absolute value of any nonzero number is positive, and the absolute value of 0 is 0.

EXAMPLE 2 Absolute Value

Evaluate the following.
(a) $|-4|$ (b) $|+2|$
(c) $|0|$ (d) $|-5-|-3||$

Solution

(a) $|-4| = +4 = 4$ (b) $|+2| = +2 = 2$
(c) $|0| = 0$ (d) $|-5-|-3|| = |-5-3| = |-8| = 8$

Note that if a is a nonnegative number, then $|a| = a$, *but if a is negative, then* $|a|$ *is the positive number* $(-a)$. Thus

Absolute Value

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

In performing computations with real numbers, it is important to remember the following rules for computations.

OPERATIONS WITH REAL (SIGNED) NUMBERS

Procedure	Example
1. (a) To add two real numbers with the same sign, add their absolute values and affix their common sign.	1. (a) $(5) + (6) = 11$ $\left(-\frac{1}{6}\right) + \left(-\frac{2}{6}\right) = -\frac{3}{6} = -\frac{1}{2}$
(b) To add two real numbers with unlike signs, find the difference of their absolute values and affix the sign of the number with the larger absolute value.	(b) $(5) + (-3) = 2$ $\left(-\frac{11}{7}\right) + (1) = -\frac{4}{7}$
2. To subtract one real number from another, change the sign of the number being subtracted and proceed as in addition.	2. $(-9) - (-8) = (-9) + (8) = -1$ $16 - (8) = 16 + (-8) = 8$
3. (a) The product of two real numbers with like signs is positive.	3. (a) $(-3)(-4) = 12$ $\left(\frac{3}{4}\right)(4) = 3$
(b) The product of two real numbers with unlike signs is negative.	(b) $5(-3) = -15$ $(-3)(4) = -12$
4. (a) The quotient of two real numbers with like signs is positive.	4. (a) $(-14) \div (-2) = 7$ $36/4 = 9$
(b) The quotient of two real numbers with unlike signs is negative.	(b) $(-28)/4 = -7$ $45 \div (-5) = -9$

When two or more operations with real numbers are indicated in an evaluation, it is important that everyone agree on the order in which the operations are performed so that a unique result is guaranteed. The following **order of operations** is universally accepted.

Order of Operations

First: Perform operations within parentheses or other symbols of grouping.
Second: Find indicated powers ($2^3 = 2 \cdot 2 \cdot 2 = 8$).
Third: Perform multiplications and divisions from left to right.
Fourth: Perform additions and subtractions from left to right.

EXAMPLE 3 Order of Operations

Evaluate the following.

(a) $-4 + 3$
(b) $4 + 3(-2)$
(c) $-4^2 + 3$
(d) $[2(3 - 4)]^2 + 3$
(e) $6 \div 2(2 + 1)$

Solution

(a) -1

(b) $4 + (-6) = -2$

(c) Note that with -4^2, the power 2 is applied only to 4, not to -4 (which would be written $(-4)^2$). Thus $-4^2 + 3 = -(4^2) + 3 = -16 + 3 = -13$.

(d) $[2(-1)]^2 + 3 = [-2]^2 + 3 = 4 + 3 = 7$

(e) $6 \div 2(3) = (6 \div 2)(3) = 3 \cdot 3 = 9$ ■

CHECKPOINT

True or false:

4. $-(-5)^2 = 25$
5. $|4 - 6| = |4| - |6|$
6. $9 - 2(2)(-10) = 7(2)(-10) = -140$

Often, we use letters to represent real numbers, such as in the formula for the area of a circle, $A = \pi r^2$. Formula evaluations also use the order of operations.

EXAMPLE 4 Average Annual Wage

Using Social Security Administration data from 2011 and projected to 2021, the U.S. average annual wage W (in dollars) can be approximated with the formula

$$W = 48t^2 + 1000(1.8t + 41)$$

where t is the number of years past 2010. Use the formula to approximate the U.S. average annual wage in 2020.

Solution

Note that 2020 is 10 years past 2010, so we use $t = 10$. Substituting $t = 10$ into the formula for W and using the order of operations give

$$\begin{aligned} W &= 48(10^2) + 1000(1.8(10) + 41) \\ &= 48(100) + 1000(18 + 41) = 4800 + 1000(59) \\ &= 4800 + 59{,}000 = \$63{,}800 \end{aligned}$$ ■

We will assume that you have a scientific or graphing calculator. Discussions of some of the capabilities of graphing calculators and Excel will be found throughout the text.

Most scientific and graphing calculators use standard algebraic order when evaluating arithmetic expressions. Working outward from inner parentheses, calculations are performed from left to right. Powers and roots are evaluated first, followed by multiplications and divisions, and then additions and subtractions.

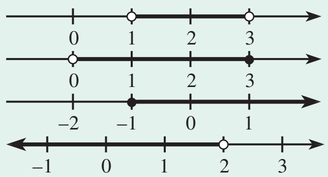
CHECKPOINT ANSWERS

1. Parts (a) and (d) are undefined because a denominator of zero means division by zero. Parts (b) and (c) are defined, and their values are 0 and 1, respectively.
2. (a) $1 < x < 3$

   (b) $0 < x \le 3$

   (c) $-1 \le x < \infty$ or $x \ge -1$

   (d) $-\infty < x < 2$ or $x < 2$



3. (a) [3, 6]; closed interval

   (b) [−6, 4); half-open interval
4. False; $-(-5)^2 = -25$.
5. False; $|4 - 6| = |-2| = 2$ and $|4| - |6| = 4 - 6 = -2$.
6. False; $9 - (2)(2)(-10) = 49$

EXERCISES 0.2

In Problems 1 and 2, indicate whether the given expression is one or more of the following types of numbers: rational, irrational, integer, natural. If the expression is meaningless, so state.

1. (a) $\frac{-\pi}{10}$
   (b) -9
   (c) $\frac{9}{3}$
   (d) $\frac{4}{0}$
2. (a) $\frac{0}{6}$
   (b) -1.2916
   (c) 1.414
   (d) $\frac{9}{6}$

Which property of real numbers is illustrated in each part of Problems 3–6?

3. (a) $8 + 6 = 6 + 8$
   (b) $5(3 + 7) = 5(3) + 5(7)$
4. (a) $-e \cdot 1 = -e$
   (b) $4 + (-4) = 0$
5. (a) $6(4 \cdot 5) = (6 \cdot 4)(5)$
   (b) $-15 + 0 = -15$
6. (a) $\left(\frac{3}{2}\right)\left(\frac{2}{3}\right) = 1$
   (b) $(12)\left(\frac{3}{4}\right) = \left(\frac{3}{4}\right)(12)$

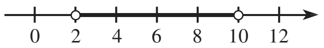
Insert the proper sign $<$, $=$, or $>$ to replace $\square$ in Problems 7–14.

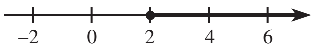
7. $-6 \square 0$
8. $2 \square -20$
9. $-14 \square -3$
10. $\pi \square 3.14$
11. $0.333 \square \frac{1}{3}$
12. $\frac{1}{3} + \frac{1}{2} \square \frac{5}{6}$
13. $|-3| + |5| \square |-3 + 5|$
14. $|-9 - 3| \square |-9| + |3|$

In Problems 15–26, evaluate each expression.

15. $-3^2 + 10 \cdot 2$
16. $(-3)^2 + 10 \cdot 2$
17. $\frac{4 + 2^2}{2}$
18. $\frac{(4 + 2)^2}{2}$
19. $\frac{16 - (-4)}{8 - (-2)}$
20. $\frac{(-5)(-3) - (-2)(3)}{-9 + 2}$
21. $\frac{|5 - 2| - |-7|}{|5 - 2|}$
22. $\frac{|3 - |4 - 11||}{-|5^2 - 3^2|}$
23. $\frac{(-3)^2 - 2 \cdot 3 + 6}{4 - 2^2 + 3}$
24. $\frac{6^2 - 4(-3)(-2)}{6 - 6^2 \div 4}$
25. $\frac{-4^2 + 5 - 2 \cdot 3}{5 - 4^2}$
26. $\frac{3 - 2(5 - 2)}{(-2)^2 - 2^2 + 3}$
27. What part of the real number line corresponds to the interval $(-\infty, \infty)$?
28. Write the interval corresponding to $x \geq 0$.

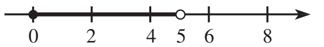
In Problems 29–32, express each inequality or graph using interval notation and name the type of interval.

29. $1 < x \leq 3$
30. $-4 \leq x \leq 3$
31. 

32. 

In Problems 33–36, write an inequality that describes each interval or graph.

33. $[-3, 5)$
34. $(-2, \infty)$
35. 

36. 

In Problems 37–44, graph the subset of the real numbers that is represented by each of the following and write your answer in interval notation.

37. $(-\infty, 4) \cap (-3, \infty)$
38. $[-4, 17) \cap [-20, 10]$
39. $x > 4$ and $x \geq 0$
40. $x < 10$ and $x < -1$
41. $[0, \infty) \cup [-1, 5]$
42. $(-\infty, 4) \cup (0, 2)$
43. $x > 7$ or $x < 0$
44. $x > 4$ and $x < 0$

In Problems 45–50, use your calculator to evaluate each of the following. List all the digits on your display in the answer.

45. $\dfrac{-1}{25{,}916.8}$
46. $\dfrac{51.412}{127.01}$
47. $(3.679)^7$
48. $(1.28)^{10}$
49. $\dfrac{2500}{(1.1)^6 - 1}$
50. $100\left[\dfrac{(1.05)^{12} - 1}{0.05}\right]$

APPLICATIONS

51. ***Take-home pay*** A salesclerk's take-home pay is found by subtracting all taxes and retirement contributions from gross pay (which consists of salary plus commission). Given the following information, complete parts (a)–(c).
    Salary = \$300.00 Commission = \$788.91
    Retirement = 5% of gross pay
    Taxes: State = 5% of gross pay
    Local = 1% of gross pay
    Federal withholding = 25% of (gross pay less retirement)
    Federal Social Security and Medicare = 7.65% of gross pay
    (a) Find the gross pay.
    (b) Find the amount of federal withholding.
    (c) Find the take-home pay.
52. ***Crude oil production*** Using data from 2010 and projected to 2030, the U.S. crude oil production, in billions of barrels, can be approximated by
    $$P = -0.00105t^2 + 0.0367t + 1.94$$
    where t is the number of years after 2010 (*Source:* U.S. Department of Energy).
    (a) What t-value represents 2020?
    (b) Actual production in 2014 was 2.10 billion barrels. What does the equation give as the 2014 approximation?
    (c) Estimate the production for 2020.
53. ***Worldwide Internet users*** Using data from 2014 and projected to 2019, the percent of people in the world who are Internet users can be approximated accurately by
    (1) $y = 2.14t + 32.2$ or by
    (2) $y = 0.00536t^2 + 2.07t + 32.4$
    with t equal to the number of years after 2010 (*Source:* statista.com).
    (a) Which equation is more accurate for 2016, when the percent was 45%?
    (b) Use both formulas to estimate the percent of users in 2025.
54. ***Health statistics*** Using data adapted from the National Center for Health Statistics, the height H in inches and age A in years for boys between 4 and 16 years of age are related according to
    $$H = 2.31A + 31.26$$
    To account for normal variability among boys, normal height for a given age is $\pm 5\%$ of the height obtained from the equation.
    (a) Find the normal height range for a boy who is 10.5 years old and write it as an inequality.
    (b) Find the normal height range for a boy who is 5.75 years old and write it as an inequality.
55. ***Income taxes*** Use the following federal tax table for a single person claiming one personal exemption.

Taxable Income *I*	Tax Due *T*
0–9275	10% *I*
9276–37,650	927.50 + 15%(*I* − 9275)
37,651–91,150	5183.75 + 25%(*I* − 37,650)
91,151–190,150	18,558.75 + 28%(*I* − 91,150)
190,151–413,350	46,278.75 + 33%(*I* − 190,150)
413,351–415,050	119,934.75 +35%(*I* − 413,350)
Over 415,050	120,529 + 39.6%(*I* − 415,050)

Source: Internal Revenue Service

(a) Write the last three taxable income ranges as inequalities.
(b) If an individual has a taxable income of \$37,650 calculate the tax due. Repeat this calculation for a taxable income of \$91,150.
(c) Write an interval that represents the amount of tax due for a taxable income between \$37,650 and \$91,150.

SECTION 0.3 Integral Exponents

If \$1000 is placed in a 5-year savings certificate that pays an interest rate of 10% per year, compounded annually, then the amount returned after 5 years is given by

$$1000(1.1)^5$$

The 5 in this expression is an *exponent.* Exponents provide an easier way to denote certain multiplications. For example,

$$(1.1)^5 = (1.1)(1.1)(1.1)(1.1)(1.1)$$

An understanding of the properties of exponents is fundamental to the algebra needed to study functions and solve equations. Furthermore, the definition of exponential and logarithmic functions and many of the techniques in calculus also require an understanding of the properties of exponents.

For any real number a,

$$a^2 = a \cdot a, \quad a^3 = a \cdot a \cdot a, \quad \text{and} \quad a^n = a \cdot a \cdot a \cdot \ldots \cdot a \quad (n \text{ factors})$$

for any positive integer n. The positive integer n is called the **exponent,** the number a is called the **base,** and a^n is read "a to the nth power."

Note that $4a^n$ means $4(a^n)$, which is different from $(4a)^n$. The 4 is the coefficient of a^n in $4a^n$. Note also that $-a^n$ is not equivalent to $(-a)^n$ when n is even. For example, $-3^4 = -81$, but $(-3)^4 = 81$.

Some of the rules of exponents follow.

Positive Integer Exponents

For any real numbers a and b and positive integers m and n,

1. $a^m \cdot a^n = a^{m+n}$
2. For $a \neq 0$, $\dfrac{a^m}{a^n} = \begin{cases} a^{m-n} & \text{if } m > n \\ 1 & \text{if } m = n \\ 1/a^{n-m} & \text{if } m < n \end{cases}$
3. $(ab)^m = a^m b^m$
4. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m} \quad (b \neq 0)$
5. $(a^m)^n = a^{mn}$

EXAMPLE 1 Positive Integer Exponents

Use rules of positive integer exponents to rewrite the following. Assume that all denominators are nonzero.

(a) $\dfrac{5^6}{5^4}$

(b) $\dfrac{x^2}{x^5}$

(c) $\left(\dfrac{x}{y}\right)^4$

(d) $(3x^2y^3)^4$

(e) $3^3 \cdot 3^2$

Solution

(a) $\dfrac{5^6}{5^4} = 5^{6-4} = 5^2$

(b) $\dfrac{x^2}{x^5} = \dfrac{1}{x^{5-2}} = \dfrac{1}{x^3}$

(c) $\left(\dfrac{x}{y}\right)^4 = \dfrac{x^4}{y^4}$

(d) $(3x^2y^3)^4 = 3^4(x^2)^4(y^3)^4 = 81x^8y^{12}$

(e) $3^3 \cdot 3^2 = 3^{3+2} = 3^5$ ■

For certain calculus operations, use of negative exponents is necessary to write problems in the proper form. We can extend the rules for positive integer exponents to all integers by defining a^0 and a^{-n}. Clearly $a^m \cdot a^0$ should equal $a^{m+0} = a^m$, and it will if $a^0 = 1$.

Zero Exponent

For any nonzero real number a, we define $a^0 = 1$. We leave 0^0 undefined.

In Section 0.2, we defined a^{-1} as $1/a$ for $a \neq 0$, so we define a^{-n} as $(a^{-1})^n$.

Negative Exponents

$$a^{-n} = (a^{-1})^n = \left(\frac{1}{a}\right)^n = \frac{1}{a^n} \qquad (a \neq 0)$$

$$\left(\frac{a}{b}\right)^{-n} = \left[\left(\frac{a}{b}\right)^{-1}\right]^n = \left(\frac{b}{a}\right)^n \qquad (a \neq 0,\quad b \neq 0)$$

EXAMPLE 2 Negative and Zero Exponents

Write the following without exponents.

(a) $6 \cdot 3^0$

(b) 6^{-2}

(c) $\left(\frac{1}{3}\right)^{-1}$

(d) $-\left(\frac{2}{3}\right)^{-4}$

(e) $(-4)^{-2}$

Solution

(a) $6 \cdot 3^0 = 6 \cdot 1 = 6$

(b) $6^{-2} = \frac{1}{6^2} = \frac{1}{36}$

(c) $\left(\frac{1}{3}\right)^{-1} = \frac{3}{1} = 3$

(d) $-\left(\frac{2}{3}\right)^{-4} = -\left(\frac{3}{2}\right)^4 = \frac{-81}{16}$

(e) $(-4)^{-2} = \frac{1}{(-4)^2} = \frac{1}{16}$

As we'll see in the chapter on the mathematics of finance (Chapter 6), negative exponents arise in financial calculations when we have a future goal for an investment and want to know how much to invest now. For example, if money can be invested at 9%, compounded annually, then the amount we must invest now (which is called the present value) to have \$10,000 in the account after 7 years is given by $\$10{,}000(1.09)^{-7}$. Calculations such as this are often done directly with a calculator.

Using the definitions of zero and negative exponents enables us to extend the rules of exponents to all integers and to express them more simply.

Rules of Exponents

For real numbers a and b and *integers* m and n,

1. $a^m \cdot a^n = a^{m+n}$
2. $a^m/a^n = a^{m-n} \quad (a \neq 0)$
3. $(ab)^m = a^m b^m$
4. $(a^m)^n = a^{mn}$
5. $(a/b)^m = a^m/b^m \quad (a, b \neq 0)$
6. $a^0 = 1 \quad (a \neq 0)$
7. $a^{-n} = 1/a^n \quad (a \neq 0)$
8. $(a/b)^{-n} = (b/a)^n \quad (a, b \neq 0)$

Throughout the remainder of the text, we will assume that all variable expressions are defined.

EXAMPLE 3 Using Rules of Exponents

Use the rules of exponents and the definitions of a^0 and a^{-n} to simplify the following with positive exponents.

(a) $2(x^2)^{-2}$ (b) $x^{-2} \cdot x^{-5}$

(c) $\dfrac{x^{-8}}{x^{-4}}$ (d) $\left(\dfrac{2x^3}{3x^{-5}}\right)^{-2}$

Solution

(a) $2(x^2)^{-2} = 2x^{-4} = 2\left(\dfrac{1}{x^4}\right) = \dfrac{2}{x^4}$

(b) $x^{-2} \cdot x^{-5} = x^{-2-5} = x^{-7} = \dfrac{1}{x^7}$

(c) $\dfrac{x^{-8}}{x^{-4}} = x^{-8-(-4)} = x^{-4} = \dfrac{1}{x^4}$

(d) $\left(\dfrac{2x^3}{3x^{-5}}\right)^{-2} = \left(\dfrac{2x^8}{3}\right)^{-2} = \left(\dfrac{3}{2x^8}\right)^2 = \dfrac{9}{4x^{16}}$ ■

CHECKPOINT

1. Complete the following.
   (a) $x^3 \cdot x^8 = x^?$ (b) $x \cdot x^4 \cdot x^{-3} = x^?$
   (c) $\dfrac{1}{x^4} = x^?$ (d) $x^{24} \div x^{-3} = x^?$
   (e) $(x^4)^2 = x^?$ (f) $(2x^4y)^3 = ?$
2. True or false:
   (a) $3x^{-2} = \dfrac{1}{9x^2}$ (b) $-x^{-4} = \dfrac{-1}{x^4}$ (c) $x^{-3} = -x^3$
3. Evaluate the following if possible. For any that are undefined, so state. Assume that $x > 0$.
   (a) 0^4 (b) 0^0 (c) x^0 (d) 0^x (e) 0^{-4} (f) -5^{-2}

EXAMPLE 4 Rewriting a Quotient

Write $(x^2y)/(9wz^3)$ with all factors in the numerator.

Solution

$$\frac{x^2y}{9wz^3} = x^2y\left(\frac{1}{9wz^3}\right) = x^2y\left(\frac{1}{9}\right)\left(\frac{1}{w}\right)\left(\frac{1}{z^3}\right) = x^2y \cdot 9^{-1}w^{-1}z^{-3}$$
$$= 9^{-1}x^2yw^{-1}z^{-3}$$ ■

EXAMPLE 5 Rewriting with Positive Exponents

Simplify the following so that all exponents are positive.

(a) $(2^3x^{-4}y^5)^{-2}$

(b) $\dfrac{2x^4(x^2y)^0}{(4x^{-2}y)^2}$

Solution

(a) $(2^3x^{-4}y^5)^{-2} = 2^{-6}x^8y^{-10} = \dfrac{1}{2^6} \cdot x^8 \cdot \dfrac{1}{y^{10}} = \dfrac{x^8}{64y^{10}}$

(b) $\dfrac{2x^4(x^2y)^0}{(4x^{-2}y)^2} = \dfrac{2x^4 \cdot 1}{4^2x^{-4}y^2} = \dfrac{2}{4^2} \cdot \dfrac{x^4}{x^{-4}} \cdot \dfrac{1}{y^2} = \dfrac{2}{16} \cdot \dfrac{x^8}{1} \cdot \dfrac{1}{y^2} = \dfrac{x^8}{8y^2}$ ■

CHECKPOINT ANSWERS

1. (a) x^{11} (b) x^2
(c) x^{-4} (d) x^{27}
(e) x^8 (f) $8x^{12}y^3$
2. (a) False; $3x^{-2} = \frac{3}{x^2}$
(b) True.
(c) False; $x^{-3} = \frac{1}{x^3}$
3. (a) $0^4 = 0$ (b) 0^0 is undefined.
(c) $x^0 = 1$ since $x \neq 0$ (d) $0^x = 0$ because $x > 0$
(e) 0^{-4} would be $\frac{1}{0^4}$, which is undefined.
(f) $\frac{-1}{25}$

EXERCISES 0.3

Evaluate in Problems 1–4. Write all answers without using exponents.

1. (a) $(-4)^4$ (b) -2^6
2. (a) -5^3 (b) $(-2)^5$
3. (a) 3^{-2} (b) $-\left(\frac{3}{2}\right)^2$
4. (a) 6^{-1} (b) $\left(\frac{2}{3}\right)^3$

In Problems 5–8, use a calculator to evaluate the indicated powers.

5. 1.2^4
6. $(-3.7)^3$
7. $(1.5)^{-5}$
8. $(-0.8)^{-9}$

In Problems 9–18, simplify the expressions with all exponents positive.

9. $6^5 \cdot 6^3$
10. $8^4 \cdot 8^2 \cdot 8$
11. $\frac{10^8}{10^9}$
12. $\frac{7^8}{7^3}$
13. $\frac{9^4 \cdot 9^{-7}}{9^{-3}}$
14. $\frac{5^4}{(5^{-2} \cdot 5^3)}$
15. $(3^3)^3$
16. $(2^{-3})^{-2}$
17. $\left(\frac{2}{3}\right)^{-2}$
18. $\left(\frac{-2}{5}\right)^{-4}$

In Problems 19–22, rewrite the expression with positive exponents ($x, y, z \neq 0$).

19. $-x^{-3}$
20. x^{-4}
21. $xy^{-2}z^0$
22. $4^{-1}x^0y^{-2}$

In Problems 23–36, use the rules of exponents to simplify so that only positive exponents remain.

23. $x^3 \cdot x^4$
24. $a^5 \cdot a$
25. $x^{-5} \cdot x^3$
26. $y^{-5} \cdot y^{-2}$
27. $\frac{x^8}{x^4}$
28. $\frac{a^5}{a^{-1}}$
29. $\frac{y^5}{y^{-7}}$
30. $\frac{y^{-3}}{y^{-4}}$
31. $(x^4)^3$
32. $(y^3)^{-2}$
33. $(xy)^2$
34. $(2m)^3$
35. $\left(\frac{2}{x^5}\right)^4$
36. $\left(\frac{8}{a^3}\right)^3$

In Problems 37–48, compute and simplify so that only positive exponents remain.

37. $(2x^{-2}y)^{-4}$
38. $(-32x^5)^{-3}$
39. $(-8a^{-3}b^2)(2a^5b^{-4})$
40. $(-3m^2y^{-1})(2m^{-3}y^{-1})$
41. $(2x^{-2}) \div (x^{-1}y^2)$
42. $(-8a^{-3}b^2c) \div (2a^5b^4)$
43. $\left(\frac{x^3}{y^{-2}}\right)^{-3}$
44. $\left(\frac{x^{-2}}{y}\right)^{-3}$
45. $\left(\frac{a^{-2}b^{-1}c^{-4}}{a^4b^{-3}c^0}\right)^{-3}$
46. $\left(\frac{4x^{-1}y^{-40}}{2^{-2}x^4y^{-10}}\right)^{-2}$
47. (a) $\frac{2x^{-2}}{(2x)^2}$ (b) $\frac{(2x)^{-2}}{(2x)^2}$
(c) $\frac{2x^{-2}}{2x^2}$ (d) $\frac{2x^{-2}}{(2x)^{-2}}$
48. (a) $\frac{2^{-1}x^{-2}}{(2x)^2}$ (b) $\frac{2^{-1}x^{-2}}{2x^2}$
(c) $\frac{(2x^{-2})^{-1}}{(2x)^{-2}}$ (d) $\frac{(2x^{-2})^{-1}}{2x^2}$

In many applications, it is necessary to write expressions in the form cx^n, where c is a constant and n is an integer. In Problems 49–56, write the expressions in this form.

49. $\frac{1}{x}$
50. $\frac{1}{x^2}$
51. $(2x)^3$
52. $(3x)^2$
53. $\frac{1}{4x^2}$
54. $\frac{3}{2x^4}$
55. $\left(\frac{-x}{2}\right)^3$
56. $\left(\frac{-x}{3}\right)^2$

APPLICATIONS

Compound interest **If \$$P$ is invested for n years at rate i (as a decimal), compounded annually, the future value that accrues is given by $S = P(1 + i)^n$ and the interest earned is $I = S - P$. In Problems 57–60, find S and I for the given P, n, and i.**

57. \$1200 for 5 years at 12%
58. \$1800 for 7 years at 10%
59. \$5000 for 6 years at 11.5%
60. \$800 for 20 years at 10.5%

Present value **If an investment has a goal (future value) of \$$S$ after n years, invested at interest rate i (as a decimal), compounded annually, then the present value P that must be invested is given by $P = S(1 + i)^{-n}$. In Problems 61and 62, find P for the given S, n, and i.**

61. \$15,000 after 6 years at 11.5%
62. \$80,000 after 20 years at 10.5%
63. ***Personal income*** The total U.S. personal income I (in billions of dollars) from 1960 and projected to 2024 can be approximated by

$$I = 533.6(1.065)^t$$

where t is the number of years after 1960 (*Source:* U.S. Bureau of Labor Statistics).
(a) What t-values correspond to the years 2000, 2014, and 2024?
(b) What does the formula estimate the total U.S. personal income to be in 2000, 2014, and 2024?
(c) The U.S. total personal income given by the U.S. Bureau of Labor Statistics for 2018 is \$19,129.6 billion. What does the formula estimate it to be in 2018?

64. ***China's shale gas*** For the years 2013 through 2020, the estimated annual production of shale-natural gas in China, in billions of cubic feet, can be approximated by the formula

$$y = 0.012(1.75)^t$$

where t is the number of years past 2010 (*Source:* Sanford C. Bernstein).
(a) What t-value corresponds to 2019?
(b) According to the formula, what is the production in 2019?
(c) What is the production in 2022 if this formula remains accurate?

65. ***Diabetes*** Using data from 2010 and projected to 2050, the number of millions of U.S. adults with diabetes, y, can be approximated by the formula

$$y = \frac{120}{1 + 5.25(1.066)^{-t}}$$

where t is the number of years after 2000 (*Source:* Centers for Disease Control and Prevention).
(a) The actual numbers of millions given by the CDC for selected years are as follows.

2010	2015	2020
32.3	37.3	50.0

For each of these years, find the number of U.S. adults with diabetes predicted by the formula. Round your answer to one decimal place.
(b) Using the number of U.S. adults with diabetes given above for 2015, how many more U.S. adults with diabetes does the formula predict will be added by 2025?
(c) Why is it reasonable that the formula approximating the number of U.S. adults with diabetes has an upper limit that cannot be exceeded? Use large t-values to discover this formula's upper limit.

66. ***U.S. population, ages 20–64*** Using Social Security Administration data for selected years from 1950 and projected to 2050, the U.S. population, ages 20–64, P (in millions) can be approximated by the equation

$$P = \frac{249.6}{1 + 1.915(1.028)^{-t}}$$

where t is the number of years past 1950.
(a) Some of the Social Security Administration data for this population (in millions) are as follows.

1980	2000	2020
134.0	169.8	198.2

For each of these years, use the equation to find the predicted U.S. population, ages 20–64.
(b) From 2000 to 2020, this population group is predicted to change by 28.4 million individuals. Find the population change predicted by the equation for 2025 to 2045. Is this greater or less than 2000–2020?
(c) Why is it reasonable for a formula such as this to have an upper limit that cannot be exceeded? Use large t-values to discover this formula's upper limit.

67. ***Health care expenditures*** The national health care expenditure H (in billions of dollars) can be modeled (that is, accurately approximated) by the formula

$$H = 738.1(1.065)^t$$

where t is the number of years past 1990 (*Source:* U.S. Department of Health and Human Services).
(a) What t-value corresponds to 2000?
(b) Approximate the national health care expenditure in 2000.
(c) Approximate the national health care expenditure in 2010.
(d) Estimate the national health care expenditure in 2018.

SECTION 0.4 Radicals and Rational Exponents

Roots

A process closely linked to raising numbers to powers is that of extracting roots. From geometry, we know that if an edge of a cube has a length of x units, its volume is x^3 cubic units. Reversing this process, we determine that if the volume of a cube is V cubic units, the length of an edge is the cube root of V, which is denoted

$$\sqrt[3]{V} \text{ units}$$

When we seek the **cube root** of a number such as 8 (written $\sqrt[3]{8}$), we are looking for a real number whose cube equals 8. Because $2^3 = 8$, we know that $\sqrt[3]{8} = 2$. Similarly, $\sqrt[3]{-27} = -3$ because $(-3)^3 = -27$. The expression $\sqrt[n]{a}$ is called a **radical,** where $\sqrt{}$ is the **radical sign,** n is the **index,** and a is the **radicand.** When no index is indicated, the index is assumed to be 2 and the expression is called a **square root;** thus $\sqrt{4}$ is the square root of 4 and represents the positive number whose square is 4.

Only one real number satisfies $\sqrt[n]{a}$ for a real number a and an odd number n; we call that number the **principal *n*th root** or, more simply, the ***n*th root.**

For an even index n, there are two possible cases:

1. If a is negative, there is no real number equal to $\sqrt[n]{a}$. For example, there are no real numbers that equal $\sqrt{-4}$ or $\sqrt[4]{-16}$ because there is no real number b such that $b^2 = -4$ or $b^4 = -16$. In this case, we say that $\sqrt[n]{a}$ is not a real number.
2. If a is positive, there are two real numbers whose nth power equals a. For example, $3^2 = 9$ and $(-3)^2 = 9$. To have a unique nth root, we define the (principal) nth root, $\sqrt[n]{a}$, as the *positive* number b that satisfies $b^n = a$.

We summarize this discussion as follows.

*n*th Root of *a*

The **(principal) *n*th root** of a real number is defined as

$$\sqrt[n]{a} = b \quad \text{only if} \quad a = b^n$$

subject to the following conditions:

	$a = 0$	$a > 0$	$a < 0$
n even	$\sqrt[n]{a} = 0$	$\sqrt[n]{a} > 0$	$\sqrt[n]{a}$ not real
n odd	$\sqrt[n]{a} = 0$	$\sqrt[n]{a} > 0$	$\sqrt[n]{a} < 0$

When we are asked for the root of a number, we give the principal root.

EXAMPLE 1 Roots

Find the roots if they are real numbers.

(a) $\sqrt[6]{64}$ (b) $-\sqrt{16}$ (c) $\sqrt[3]{-8}$ (d) $\sqrt{-16}$

Solution

(a) $\sqrt[6]{64} = 2$ because $2^6 = 64$ (b) $-\sqrt{16} = -\left(\sqrt{16}\right) = -4$ (c) $\sqrt[3]{-8} = -2$

(d) $\sqrt{-16}$ is not a real number because an even root of a negative number is not real. ■

Fractional Exponents

To perform evaluations on a calculator or to perform calculus operations, it is sometimes necessary to rewrite radicals in exponential form with fractional exponents.

We have stated that for $a \geq 0$ and $b \geq 0$,

$$\sqrt{a} = b \text{ only if } a = b^2$$

This means that $(\sqrt{a})^2 = b^2 = a$, or $(\sqrt{a})^2 = a$. To extend the properties of exponents to rational exponents, it is necessary to define

$$a^{1/2} = \sqrt{a} \text{ so that } (a^{1/2})^2 = a$$

Exponent 1/*n*

For a positive integer n, we define

$$a^{1/n} = \sqrt[n]{a} \quad \text{if} \quad \sqrt[n]{a} \text{ exists}$$

Thus $(a^{1/n})^n = a^{(1/n)\cdot n} = a$.

Because we want the properties established for integer exponents to extend to rational exponents, we make the following definitions.

Rational Exponents

For positive integer n and any integer m (with $a \neq 0$ when $m \leq 0$, with m/n in lowest terms, and with a nonnegative when n is even),

1. $a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m$
2. $a^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m}$

Throughout the remaining discussion, we assume that all expressions are real.

EXAMPLE 2 Radical Form

Write the following in radical form and simplify.

(a) $16^{3/4}$ (b) $y^{-3/2}$ (c) $(6m)^{2/3}$

Solution

(a) $16^{3/4} = \sqrt[4]{16^3} = (\sqrt[4]{16})^3 = (2)^3 = 8$

(b) $y^{-3/2} = \dfrac{1}{y^{3/2}} = \dfrac{1}{\sqrt{y^3}}$

(c) $(6m)^{2/3} = \sqrt[3]{(6m)^2} = \sqrt[3]{36m^2}$ ■

EXAMPLE 3 Fractional Exponents

Write the following without radical signs.

(a) $\sqrt{x^3}$ (b) $\dfrac{1}{\sqrt[3]{b^2}}$ (c) $\sqrt[3]{(ab)^3}$

Solution

(a) $\sqrt{x^3} = x^{3/2}$ (b) $\dfrac{1}{\sqrt[3]{b^2}} = \dfrac{1}{b^{2/3}} = b^{-2/3}$ (c) $\sqrt[3]{(ab)^3} = (ab)^{3/3} = ab$ ■

Our definition of $a^{m/n}$ guarantees that the rules for exponents will apply to fractional exponents. Thus we can perform operations with fractional exponents as we did with integer exponents.

EXAMPLE 4 Operations with Fractional Exponents

Simplify the following expressions.

(a) $a^{1/2} \cdot a^{1/6}$
(b) $a^{3/4}/a^{1/3}$
(c) $(a^3b)^{2/3}$
(d) $(a^{3/2})^{1/2}$
(e) $a^{-1/2} \cdot a^{-3/2}$

Solution

(a) $a^{1/2} \cdot a^{1/6} = a^{1/2+1/6} = a^{3/6+1/6} = a^{4/6} = a^{2/3}$

(b) $a^{3/4}/a^{1/3} = a^{3/4-1/3} = a^{9/12-4/12} = a^{5/12}$

(c) $(a^3b)^{2/3} = (a^3)^{2/3}b^{2/3} = a^2b^{2/3}$

(d) $(a^{3/2})^{1/2} = a^{(3/2)(1/2)} = a^{3/4}$

(e) $a^{-1/2} \cdot a^{-3/2} = a^{-1/2-3/2} = a^{-2} = 1/a^2$

CHECKPOINT

1. Which of the following are *not* real numbers?
   (a) $\sqrt[3]{-64}$ (b) $\sqrt{-64}$ (c) $\sqrt{0}$ (d) $\sqrt[4]{1}$ (e) $\sqrt[5]{-1}$ (f) $\sqrt[8]{-1}$
2. (a) Write as radicals: $x^{1/3}$, $x^{2/5}$, $x^{-3/2}$
   (b) Write with fractional exponents: $\sqrt[4]{x^3} = x^?$, $\dfrac{1}{\sqrt{x}} = \dfrac{1}{x^?} = x^?$
3. Evaluate the following.
   (a) $8^{2/3}$ (b) $(-8)^{2/3}$ (c) $8^{-2/3}$ (d) $-8^{-2/3}$ (e) $\sqrt[15]{71}$
4. Complete the following.
   (a) $x \cdot x^{1/3} \cdot x^3 = x^?$ (b) $x^2 \div x^{1/2} = x^?$ (c) $(x^{-2/3})^{-3} = x^?$
   (d) $x^{-3/2} \cdot x^{1/2} = x^?$ (e) $x^{-3/2} \cdot x = x^?$ (f) $\left(\dfrac{x^4}{y^2}\right)^{3/2} = ?$
5. True or false:
   (a) $\dfrac{8x^{2/3}}{x^{-1/3}} = 4x$ (b) $(16x^8y)^{3/4} = 12x^6y^{3/4}$
   (c) $\left(\dfrac{x^2}{y^3}\right)^{-1/3} = \left(\dfrac{y^3}{x^2}\right)^{1/3} = \dfrac{y}{x^{2/3}}$

Operations with Radicals

We can perform operations with radicals by first rewriting in exponential form, performing the operations with exponents, and then converting the answer back to radical form. Another option is to apply directly the following rules for operations with radicals.

Rules for Radicals	Example
Given that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real,*	
1. $\sqrt[n]{a^n} = (\sqrt[n]{a})^n = a$	1. $\sqrt[5]{6^5} = (\sqrt[5]{6})^5 = 6$
2. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$	2. $\sqrt[3]{2}\sqrt[3]{4} = \sqrt[3]{8} = \sqrt[3]{2^3} = 2$
3. $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$ $(b \neq 0)$	3. $\dfrac{\sqrt{18}}{\sqrt{2}} = \sqrt{\dfrac{18}{2}} = \sqrt{9} = 3$

*Note that this means that $a \geq 0$ and $b \geq 0$ if n is even.

Let us consider Rule 1 for radicals more carefully. Note that if n is even and $a < 0$, then $\sqrt[n]{a}$ is not real and Rule 1 does not apply. For example, $\sqrt{-2}$ is not a real number and

$$\sqrt{(-2)^2} \neq -2 \text{ because } \sqrt{(-2)^2} = \sqrt{4} = 2, \text{ which is not } -2$$

We can generalize this observation as follows: If $a < 0$, then $\sqrt{a^2} = -a > 0$, so

$$\sqrt{a^2} = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

This means that

$$\sqrt{a^2} = |a|$$

We can use the rules for radicals to simplify radical expressions. In general, a radical expression $\sqrt[n]{x}$ is considered simplified if x has no nth powers as factors.

EXAMPLE 5 Simplifying Radicals

Simplify the following radicals; assume that the expressions are real numbers.

(a) $\sqrt[3]{8^3}$ (b) $\left[\sqrt[7]{(3x^2+4)^3}\right]^7$ (c) $\sqrt{x^2}$

(d) $\sqrt{48x^5y^6}$ $(y \geq 0)$ (e) $\sqrt[3]{72a^3b^4}$

Solution

(a) $\sqrt[3]{8^3} = 8$ by Rule 1

(b) $\left[\sqrt[7]{(3x^2+4)^3}\right]^7 = (3x^2+4)^3$ by Rule 1

(c) $\sqrt{x^2} = |x|$ by the previous discussion

(d) To simplify $\sqrt{48x^5y^6}$, we first factor $48x^5y^6$ into perfect-square factors and other factors. Then we apply Rule 2.

$$\sqrt{48x^5y^6} = \sqrt{16 \cdot 3 \cdot x^4xy^6} = \sqrt{16}\sqrt{x^4}\sqrt{y^6}\sqrt{3x} = 4x^2y^3\sqrt{3x}$$

(e) We factor $72a^3b^4$ into factors that are perfect cubes and other factors. Then we apply Rule 2.

$$\sqrt[3]{72a^3b^4} = \sqrt[3]{8 \cdot 9a^3b^3b} = \sqrt[3]{8} \cdot \sqrt[3]{a^3} \cdot \sqrt[3]{b^3} \cdot \sqrt[3]{9b} = 2ab\sqrt[3]{9b}$$

Rule 2 for radicals also provides a procedure for multiplying two roots with the same index.

EXAMPLE 6 Multiplying Radicals

Multiply the following and simplify, assuming nonnegative variables.

(a) $\sqrt[3]{2xy} \cdot \sqrt[3]{4x^2y}$ (b) $\sqrt{8xy^3z} \cdot \sqrt{4x^2y^3z^2}$

Solution

(a) $\sqrt[3]{2xy} \cdot \sqrt[3]{4x^2y} = \sqrt[3]{2xy \cdot 4x^2y} = \sqrt[3]{8x^3y^2} = \sqrt[3]{8} \cdot \sqrt[3]{x^3} \cdot \sqrt[3]{y^2} = 2x\sqrt[3]{y^2}$

(b) $\sqrt{8xy^3z} \cdot \sqrt{4x^2y^3z^2} = \sqrt{32x^3y^6z^3} = \sqrt{16x^2y^6z^2} \cdot \sqrt{2xz} = 4xy^3z\sqrt{2xz}$

Rule 3 for radicals ($\sqrt[n]{a}/\sqrt[n]{b} = \sqrt[n]{a/b}$) indicates how to find the quotient of two roots with the same index.

EXAMPLE 7 Dividing Radicals

Find the quotients and simplify, assuming nonnegative variables.

(a) $\dfrac{\sqrt[3]{32}}{\sqrt[3]{4}}$ (b) $\dfrac{\sqrt{16a^3x}}{\sqrt{2ax}}$

Solution

(a) $\dfrac{\sqrt[3]{32}}{\sqrt[3]{4}} = \sqrt[3]{\dfrac{32}{4}} = \sqrt[3]{8} = 2$

(b) $\dfrac{\sqrt{16a^3x}}{\sqrt{2ax}} = \sqrt{\dfrac{16a^3x}{2ax}} = \sqrt{8a^2} = 2a\sqrt{2}$

Rationalizing

Occasionally, we want to express a fraction containing radicals in an equivalent form that contains no radicals in the denominator. This is accomplished by multiplying the numerator *and* the denominator by the expression that will remove the radical from the denominator. This process is called **rationalizing the denominator.**

EXAMPLE 8 Rationalizing Denominators

Express the following with no radicals in the denominator. (Rationalize each denominator.)

(a) $\dfrac{15}{\sqrt{x}}$ (b) $\dfrac{2x}{\sqrt{18xy}}$ $(x, y > 0)$ (c) $\dfrac{3x}{\sqrt[3]{2x^2}}$ $(x \neq 0)$

Solution

(a) We want to create a perfect square under the radical in the denominator.

$$\frac{15}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{15\sqrt{x}}{x}$$

(b) $\dfrac{2x}{\sqrt{18xy}} \cdot \dfrac{\sqrt{2xy}}{\sqrt{2xy}} = \dfrac{2x\sqrt{2xy}}{\sqrt{36x^2y^2}} = \dfrac{2x\sqrt{2xy}}{6xy} = \dfrac{\sqrt{2xy}}{3y}$

(c) We want to create a perfect cube under the radical in the denominator.

$$\frac{3x}{\sqrt[3]{2x^2}} \cdot \frac{\sqrt[3]{4x}}{\sqrt[3]{4x}} = \frac{3x\sqrt[3]{4x}}{\sqrt[3]{8x^3}} = \frac{3x\sqrt[3]{4x}}{2x} = \frac{3\sqrt[3]{4x}}{2}$$

CHECKPOINT

6. Simplify:
   (a) $\sqrt[3]{24x^3y}$
   (b) $\sqrt{12xy^2} \cdot \sqrt{3x^2y}$
   (c) $\dfrac{\sqrt{24x^4y^3}}{\sqrt{3y^2}}$

7. Rationalize the denominator of $\dfrac{x}{\sqrt{5x}}$ if $x \neq 0$.

It is sometimes useful, especially in calculus, to *rationalize the numerator* of a fraction. For example, in the following expression, we can rationalize the numerator by multiplying the numerator and denominator by $\sqrt[3]{2x}$, which creates a perfect cube under the radical:

$$\frac{\sqrt[3]{4x^2}}{3x} = \frac{\sqrt[3]{4x^2}}{3x} \cdot \frac{\sqrt[3]{2x}}{\sqrt[3]{2x}} = \frac{\sqrt[3]{8x^3}}{3x\sqrt[3]{2x}} = \frac{2x}{3x\sqrt[3]{2x}} = \frac{2}{3\sqrt[3]{2x}}$$

CHECKPOINT ANSWERS

1. Part (b) $\sqrt{-64}$ and part (f) $\sqrt[8]{-1}$ are not real numbers.
2. (a) $\sqrt[3]{x}$; $\sqrt[5]{x^2}$; $\dfrac{1}{\sqrt{x^3}}$ (b) $x^{3/4}$; $\dfrac{1}{x^{1/2}} = x^{-1/2}$
3. (a) 4 (b) 4 (c) $\dfrac{1}{4}$
   (d) $-\dfrac{1}{4}$ (e) ≈ 1.32867
4. (a) $x^{13/3}$ (b) $x^{3/2}$ (c) x^2
   (d) x^{-1} (e) $x^{-1/2}$ (f) $\dfrac{x^6}{y^3}$
5. (a) False; $8x$
   (b) False; $8x^6y^{3/4}$
   (c) True.
6. (a) $2x\sqrt[3]{3y}$
   (b) $6xy\sqrt{xy}$
   (c) $2x^2\sqrt{2y}$
7. $\dfrac{\sqrt{5x}}{5}$

EXERCISES 0.4

Unless stated otherwise, assume that all variables are nonnegative and that all denominators are nonzero.

In Problems 1–4, find the powers and roots, if they are real numbers.

1. (a) $\sqrt{256/9}$ (b) $\sqrt{1.44}$
   (c) $16^{3/4}$ (d) $(-16)^{-3/2}$
2. (a) $\sqrt[5]{-32^3}$ (b) $\sqrt[4]{-16^5}$
   (c) $-27^{-1/3}$ (d) $32^{3/5}$
3. (a) $\left(\frac{8}{27}\right)^{-2/3}$ (b) $64^{2/3}$ (c) $(-64)^{-2/3}$
4. (a) $\left(\frac{4}{9}\right)^{3/2}$ (b) $64^{-2/3}$ (c) $-64^{2/3}$

In Problems 5 and 6, rewrite each radical with a fractional exponent and then approximate the value with a calculator.

5. $\sqrt[9]{(6.12)^4}$ 6. $\sqrt[12]{4.96}$

In Problems 7–10, replace each radical with a fractional exponent. Do not simplify.

7. $\sqrt{m^3}$ 8. $\sqrt[3]{x^5}$ 9. $\sqrt[4]{m^2n^5}$ 10. $\sqrt[5]{x^3}$

In Problems 11–16, write in radical form. Do not simplify.

11. $2x^{1/2}$ 12. $12x^{1/4}$
13. $x^{7/6}$ 14. $y^{11/5}$
15. $-(1/4)x^{-5/4}$ 16. $-x^{-5/3}$

In Problems 17–30, simplify each expression so that only positive exponents remain.

17. $y^{1/4} \cdot y^{1/2}$ 18. $x^{2/3} \cdot x^{1/5}$ 19. $z^{3/4} \cdot z^4$
20. $x^{-2/3} \cdot x^2$ 21. $y^{-3/2} \cdot y^{-1}$ 22. $z^{-2} \cdot z^{5/3}$
23. $\dfrac{x^{1/3}}{x^{-2/3}}$ 24. $\dfrac{x^{-1/2}}{x^{-3/2}}$ 25. $\dfrac{y^{-5/2}}{y^{-2/5}}$
26. $\dfrac{x^{4/9}}{x^{1/12}}$ 27. $(x^{2/3})^{3/4}$ 28. $(x^{4/5})^3$
29. $(x^{-1/2})^2$ 30. $(x^{-2/3})^{-2/5}$

In Problems 31–36, simplify each expression.

31. $\sqrt{64x^4}$ 32. $\sqrt[3]{-64x^6y^3}$
33. $\sqrt{128x^4y^5}$ 34. $\sqrt[3]{54x^5x^8}$
35. $\sqrt[3]{40x^8y^5}$ 36. $\sqrt{32x^5y}$

In Problems 37–44, perform the indicated operations and simplify.

37. $\sqrt{12x^3y} \cdot \sqrt{3x^2y}$ 38. $\sqrt[3]{16x^2y} \cdot \sqrt[3]{3x^2y}$
39. $\sqrt{63x^5y^3} \cdot \sqrt{28x^2y}$ 40. $\sqrt{10xz^{10}} \cdot \sqrt{30x^{17}z}$
41. $\dfrac{\sqrt{12x^3y^{12}}}{\sqrt{27xy^2}}$ 42. $\dfrac{\sqrt{250xy^7z^4}}{\sqrt{18x^{17}y^2}}$
43. $\dfrac{\sqrt[4]{32a^9b^5}}{\sqrt[4]{162a^{17}}}$ 44. $\dfrac{\sqrt[3]{-16x^3y^4}}{\sqrt[3]{128y^2}}$

In Problems 45–48, determine a value for x that makes each statement true.

45. $(A^9)^x = A$ 46. $(B^{20})^x = B$
47. $(\sqrt[7]{R})^x = R$ 48. $(\sqrt{T^3})^x = T$

In Problems 49–54, rationalize each denominator and then simplify.

49. $\sqrt{2/3}$ 50. $\sqrt{5/8}$ 51. $\dfrac{\sqrt{m^2x}}{\sqrt{mx^2}}$
52. $\dfrac{5x^3w}{\sqrt{4xw^2}}$ 53. $\dfrac{\sqrt[3]{m^2x}}{\sqrt[3]{mx^5}}$ 54. $\dfrac{\sqrt[4]{mx^3}}{\sqrt[4]{y^2z^5}}$

In calculus, it is frequently important to write an expression in the form cx^n, where c is a constant and n is a rational number. In Problems 55–58, write each expression in this form.

55. $\dfrac{-2}{3\sqrt[3]{x^2}}$ 56. $\dfrac{-2}{3\sqrt[4]{x^3}}$ 57. $3x\sqrt{x}$ 58. $\sqrt{x} \cdot \sqrt[3]{x}$

In calculus problems, the answers are frequently expected to be in a form with a radical instead of a fractional exponent. In Problems 59–62, write each expression with radicals.

59. $\dfrac{3}{2}x^{1/2}$ 60. $\dfrac{4}{3}x^{1/3}$
61. $\dfrac{1}{2}x^{-1/2}$ 62. $\dfrac{-1}{2}x^{-3/2}$

APPLICATIONS

63. ***Richter scale*** The Richter scale reading for an earthquake measures its intensity (as a multiple of some minimum intensity used for comparison). The intensity I corresponding to a Richter scale reading R is given by

$$I = 10^R$$

   (a) A quake measuring 8.5 on the Richter scale would be severe. Express the intensity of such a quake in exponential form and in radical form.
   (b) Find the intensity of a quake measuring 9.0.
   (c) The San Francisco quake that occurred during the 1989 World Series measured 6.9, and the March 2011 quake that devastated Sendai, Japan, measured 9.0. Calculate the ratio of these intensities (larger to smaller).

64. ***Sound intensity*** The intensity of sound I (as a multiple of the average minimum threshold of hearing intensity) is related to the decibel level D (or loudness of sound) according to

$$I = 10^{D/10}$$

(a) Express $10^{D/10}$ using radical notation.
(b) The background noise level of a relatively quiet room has a decibel reading of 32. Find the intensity I_1 of this noise level.
(c) A decibel reading of 140 is at the threshold of pain. If I_2 is the intensity of this threshold and I_1 is the intensity found in part (b), express the ratio I_2/I_1 as a power of 10. Then approximate this ratio.

65. ***Investment*** If \$1000 is invested at r% compounded annually, the future value S of the account after two and a half years is given by

$$S = 1000\left(1 + \frac{r}{100}\right)^{5/2}$$

(a) Express this equation with radical notation.
(b) Find the value of this account if the interest rate is 6.6% compounded annually.

66. ***Life span*** Life expectancy in the United States can be approximated with the equation

$$L = 29x^{0.22}$$

where x is the number of years the birth year is past 1900 (*Source:* National Center for Health Statistics).
(a) Express this equation with radical notation.
(b) Use the equation to estimate the life expectancy for a person born in 2015.

67. ***Population*** The population P of India (in billions) for 2000–2050 can be approximated by the equation

$$P = 0.924t^{0.13}$$

where $t > 0$ is the number of years past 2000 (*Source:* United Nations).
(a) Express this equation with radical notation.
(b) Does this equation predict a greater increase from 2005 to 2010 or from 2045 to 2050? What might explain this difference?

68. ***Trust in government*** For the years after 1965, the percent of people who say they trust the government always or most of the time can be approximated by

$$y = \frac{154}{x^{0.5}}$$

where x is the number of years after 1960.
(a) Express this equation with radical notation.
(b) For the year 2020, what does this equation estimate as the percent of people who say they trust the government always or most of the time?
(c) What does this equation estimate as the change in percentage points during the decade from 1970 to 1980?

Half-life **In Problems 69 and 70, use the fact that the quantity of a radioactive substance after t years is given by $q = q_0(2^{-t/k})$, where q_0 is the original amount of radioactive material and k is its half-life (the number of years it takes for half the radioactive substance to decay).**

69. The half-life of strontium-90 is 25 years. Find the amount of strontium-90 remaining after 10 years if $q_0 = 98$ kg.

70. The half-life of carbon-14 is 5730 years. Find the amount of carbon-14 remaining after 10,000 years if $q_0 = 40.0$ g.

71. ***Population growth*** Suppose the formula for the growth of the population of a city for the next 10 years is given by

$$P = P_0(2.5)^{ht}$$

where P_0 is the population of the city at the present time and P is the population t years from now. If $h = 0.03$ and $P_0 = 30{,}000$, find P when $t = 10$.

72. ***Advertising and sales*** Suppose it has been determined that the sales at Ewing Gallery decline after the end of an advertising campaign, with daily sales given by

$$S = 2000(2^{-0.1x})$$

where S is in dollars and x is the number of days after the campaign ends. What are the daily sales 10 days after the end of the campaign?

73. ***Company growth*** The growth of a company can be described by the equation

$$N = 500(0.02)^{0.7^t}$$

where t is the number of years the company has been in existence and N is the number of employees.
(a) What is the number of employees when $t = 0$? (This is the number of employees the company has when it starts.)
(b) What is the number of employees when $t = 5$?

SECTION 0.5 Operations with Algebraic Expressions

In algebra, we are usually dealing with combinations of real numbers (such as 3, 6/7, and $-\sqrt{2}$) and letters (such as x, a, and m). Unless otherwise specified, the letters are symbols used to represent real numbers and are sometimes called **variables.** An expression obtained by performing additions, subtractions, multiplications, divisions, or extractions of roots with one or more real numbers or variables is called an **algebraic expression.**

Unless otherwise specified, the variables represent all real numbers for which the algebraic expression is a real number. Examples of algebraic expressions include

$$3x + 2y, \quad \frac{x^3y + y}{x-1}, \quad \text{and} \quad \sqrt{x-3}$$

Note that the variable x cannot be negative in $\sqrt{x-3}$ and that $(x^3y + y)/(x - 1)$ is not a real number when $x = 1$ because division by 0 is undefined.

Any product of a real number (called the **coefficient**) and one or more variables to powers is called a **term.** The sum of a finite number of terms with nonnegative integer powers on the variables is called a **polynomial.** If a polynomial contains only one variable x, then it is called a polynomial in x.

Polynomial in *x*

The general form of a **polynomial in *x*** is

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

where each coefficient a_i is a real number for $i = 0, 1, 2, \ldots, n$. If $a_n \neq 0$, the **degree** of the polynomial is n and a_n is called the **leading coefficient.** The term a_0 is called the **constant term.**

Thus $4x^3 - 2x - 3$ is a third-degree polynomial in x with leading coefficient 4 and constant term -3. If a term has two or more variables, the degree of the term is the sum of the exponents of the variables. The degree of a constant term is zero. Thus the degree of $4x^2y$ is $2 + 1 = 3$, the degree of $6xy$ is $1 + 1 = 2$, and the degree of 3 is 0. The **degree of a polynomial** containing one or more variables is the degree of the term in the polynomial having the highest degree. Therefore, $2xy - 4x + 6$ is a second-degree polynomial.

A polynomial containing two terms is called a **binomial,** and a polynomial containing three terms is called a **trinomial.** A single-term polynomial is a **monomial.**

Because polynomials represent real numbers, the properties of real numbers can be used to add, subtract, multiply, divide, and simplify polynomials. For example, we can use the Distributive Law to add $3x$ and $2x$.

$$3x + 2x = (3 + 2)x = 5x$$

Similarly, $9xy - 3xy = (9 - 3)xy = 6xy$.

Terms with exactly the same variable factors are called **like terms.** We can add or subtract like terms by adding or subtracting the coefficients of the variables. Subtraction of polynomials uses the Distributive Law to remove the parentheses.

EXAMPLE 1 Combining Polynomials

Compute (a) $(4xy + 3x) + (5xy - 2x)$ and (b) $(3x^2 + 4xy + 5y^2 + 1) - (6x^2 - 2xy + 4)$.

Solution

(a) $(4xy + 3x) + (5xy - 2x) = 4xy + 3x + 5xy - 2x = 9xy + x$

(b) Removing the parentheses yields

$$3x^2 + 4xy + 5y^2 + 1 - 6x^2 + 2xy - 4 = -3x^2 + 6xy + 5y^2 - 3$$

Using the rules of exponents and the Commutative and Associative Laws for multiplication, we can multiply and divide monomials, as the following example shows.

EXAMPLE 2 Products and Quotients

Perform the indicated operations.

(a) $(8xy^3)(2x^3y)(-3xy^2)$ (b) $-15x^2y^3 \div (3xy^5)$

Solution

(a) $8 \cdot 2 \cdot (-3) \cdot x \cdot x^3 \cdot x \cdot y^3 \cdot y \cdot y^2 = -48x^5y^6$

(b) $\dfrac{-15x^2y^3}{3xy^5} = -\dfrac{15}{3} \cdot \dfrac{x^2}{x} \cdot \dfrac{y^3}{y^5} = -5 \cdot x \cdot \dfrac{1}{y^2} = -\dfrac{5x}{y^2}$ ■

Symbols of grouping are used in algebra the same way they are used in the arithmetic of real numbers. Recall that when an expression has two or more symbols of grouping, we begin with the innermost terms and work outward.

EXAMPLE 3 Symbols of Grouping

Simplify $3x^2 - [2x - (3x^2 - 2x)]$.

Solution

$$\begin{aligned} 3x^2 - [2x - (3x^2 - 2x)] &= 3x^2 - [2x - 3x^2 + 2x] \\ &= 3x^2 - [4x - 3x^2] \\ &= 3x^2 - 4x + 3x^2 = 6x^2 - 4x \end{aligned}$$ ■

We can use the Distributive Law to multiply a polynomial by a monomial. For example,

$$x(2x + 3) = x \cdot 2x + x \cdot 3 = 2x^2 + 3x \qquad \text{and} \qquad 5(x + y + 2) = 5x + 5y + 10$$

EXAMPLE 4 Using the Distributive Law

Find the following products.

(a) $-4ab(3a^2b + 4ab^2 - 1)$ (b) $(4a + 5b + c)ac$

Solution

(a) $$\begin{aligned} -4ab(3a^2b + 4ab^2 - 1) &= -4ab(3a^2b) + (-4ab)(4ab^2) + (-4ab)(-1) \\ &= -12a^3b^2 - 16a^2b^3 + 4ab \end{aligned}$$

(b) $(4a + 5b + c)ac = 4a \cdot ac + 5b \cdot ac + c \cdot ac = 4a^2c + 5abc + ac^2$ ■

The Distributive Law can be used to multiply two polynomials. Consider the product of two binomials $(a + b)(c + d)$. If we begin by treating the sum $(a + b)$ as a single quantity, then two successive applications of the Distributive Law give

$$(a + b)(c + d) = (a + b) \cdot c + (a + b) \cdot d = ac + bc + ad + bd$$

Thus we see that the product can be found by multiplying $(a + b)$ by c, multiplying $(a + b)$ by d, and then adding the products.

EXAMPLE 5 The Product of Two Polynomials

Find the following products.

(a) $(x + 2)(x + 5)$ (b) $(4x^2 + 3xy + 4x)(2x - 3y)$

Solution

Notice that each part uses two successive applications of the Distributive Law.

(a) $(x + 2)(x + 5) = (x + 2)(x) + (x + 2)(5) = x^2 + 2x + 5x + 10 = x^2 + 7x + 10$

(b) $$\begin{aligned} (4x^2 + 3xy + 4x)(2x - 3y) &= (4x^2 + 3xy + 4x)(2x) + (4x^2 + 3xy + 4x)(-3y) \\ &= 8x^3 + 6x^2y + 8x^2 - 12x^2y - 9xy^2 - 12xy \\ &= 8x^3 - 6x^2y + 8x^2 - 9xy^2 - 12xy \end{aligned}$$ ■

Note in Example 5(a) that the product of two binomials has a special structure. We can obtain these products by finding the products of the First terms, Outside terms, Inside terms, and Last terms, and then adding the results. This is called the FOIL method of multiplying two binomials.

EXAMPLE 6 Products of Binomials

Multiply the following.

(a) $(x - 4)(x + 3)$
(b) $(3x + 2)(2x + 5)$

Solution

(a) $(x - 4)(x + 3) = \overset{\text{First}}{(x^2)} + \overset{\text{Outside}}{(3x)} + \overset{\text{Inside}}{(-4x)} + \overset{\text{Last}}{(-12)} = x^2 - x - 12$
(b) $(3x + 2)(2x + 5) = (6x^2) + (15x) + (4x) + (10) = 6x^2 + 19x + 10$

While all binomial products can be found with the Distributive Law (or FOIL), some products have special forms worth remembering.

Special Products

1. $(x + a)^2 = x^2 + 2ax + a^2$ binomial squared
2. $(x - a)^2 = x^2 - 2ax + a^2$ binomial squared
3. $(x + a)(x - a) = x^2 - a^2$ difference of two squares
4. $(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$ binomial cubed
5. $(x - a)^3 = x^3 - 3ax^2 + 3a^2x - a^3$ binomial cubed

EXAMPLE 7 Special Products

Multiply the following.

(a) $(x + 5)^2$ (b) $(3x - 4y)^2$
(c) $(x - 2)(x + 2)$ (d) $(x^2 - y^3)^2$
(e) $(x + 4)^3$

Solution

(a) $(x + 5)^2 = x^2 + 2(5)x + 25 = x^2 + 10x + 25$
(b) $(3x - 4y)^2 = (3x)^2 - 2(3x)(4y) + (4y)^2 = 9x^2 - 24xy + 16y^2$
(c) $(x - 2)(x + 2) = x^2 - 4$
(d) $(x^2 - y^3)^2 = (x^2)^2 - 2(x^2)(y^3) + (y^3)^2 = x^4 - 2x^2y^3 + y^6$
(e) $(x + 4)^3 = x^3 + 3(4)(x^2) + 3(4^2)(x) + 4^3 = x^3 + 12x^2 + 48x + 64$

CHECKPOINT

1. Remove parentheses and combine like terms: $9x - 5x(x + 2) + 4x^2$
2. Find the following products.
   (a) $(2x + 1)(4x^2 - 2x + 1)$ (b) $(x + 3)^2$
   (c) $(3x + 2)(x - 5)$ (d) $(1 - 4x)(1 + 4x)$

The techniques used to perform operations on polynomials and to simplify polynomials also apply to other algebraic expressions.

EXAMPLE 8 Operations with Algebraic Expressions

Perform the indicated operations.

(a) $3\sqrt{3} + 4x\sqrt{y} - 5\sqrt{3} - 11x\sqrt{y} - (\sqrt{3} - x\sqrt{y})$
(b) $x^{3/2}(x^{1/2} - x^{-1/2})$
(c) $(x^{1/2} - x^{1/3})^2$
(d) $(\sqrt{x} + 2)(\sqrt{x} - 2)$

Solution

(a) We remove parentheses and then combine the terms containing $\sqrt{3}$ and the terms containing $x\sqrt{y}$.

$$(3 - 5 - 1)\sqrt{3} + (4 - 11 + 1)x\sqrt{y} = -3\sqrt{3} - 6x\sqrt{y}$$

(b) $x^{3/2}(x^{1/2} - x^{-1/2}) = x^{3/2} \cdot x^{1/2} - x^{3/2} \cdot x^{-1/2} = x^2 - x$

(c) $(x^{1/2} - x^{1/3})^2 = (x^{1/2})^2 - 2x^{1/2}x^{1/3} + (x^{1/3})^2 = x - 2x^{5/6} + x^{2/3}$

(d) $(\sqrt{x} + 2)(\sqrt{x} - 2) = (\sqrt{x})^2 - (2)^2 = x - 4$ ■

In later chapters, we will need to write problems in a simplified form so that we can perform certain operations on them. We can often use division of one polynomial by another to obtain the simplification, as shown in the following procedure.

DIVISION OF POLYNOMIALS

Procedure	**Example**
To divide one polynomial by another:	Divide $4x^3 + 4x^2 + 5$ by $2x^2 + 1$.
1. Write both polynomials in descending powers of a variable. Include missing terms with coefficient 0 in the dividend.	1. $2x^2 + 1\overline{)4x^3 + 4x^2 + 0x + 5}$
2. (a) Divide the highest-power term of the divisor into the highest-power term of the dividend and write this partial quotient above the dividend. Multiply the partial quotient times the divisor, write the product under the dividend, and subtract, getting a new dividend.	2. (a) $\begin{array}{r} 2x \\ 2x^2+1\overline{)4x^3+4x^2+0x+5} \\ \underline{4x^3+2x} \\ 4x^2-2x+5 \end{array}$
(b) Repeat until the degree of the new dividend is less than the degree of the divisor. Any remainder is written over the divisor and added to the quotient.	(b) $\begin{array}{r} 2x+2 \\ 2x^2+1\overline{)4x^3+4x^2+0x+5} \\ \underline{4x^3+2x} \\ 4x^2-2x+5 \\ \underline{4x^2+2} \\ -2x+3 \end{array}$ Degree $(-2x + 3) <$ degree $(2x^2 + 1)$ Quotient: $2x + 2 + \dfrac{-2x + 3}{2x^2 + 1}$

EXAMPLE 9 Division of Polynomials

Divide $(4x^3 - 13x - 22)$ by $(x - 3)$, $x \neq 3$.

Solution

$$\begin{array}{r} 4x^2+12x+23 \\ x-3\overline{)4x^3+0x^2-13x-22} \\ \underline{4x^3-12x^2} \\ 12x^2-13x-22 \\ \underline{12x^2-36x} \\ 23x-22 \\ \underline{23x-69} \\ 47 \end{array}$$

Insert $0x^2$ so that each power of x is present.

The quotient is $4x^2 + 12x + 23$, with remainder 47, or

$$4x^2 + 12x + 23 + \frac{47}{x - 3}$$ ■

CHECKPOINT

3. Use long division to find $(x^3 + 2x + 7) \div (x - 4)$.

CHECKPOINT ANSWERS

1. $-x^2 - x$
2. (a) $8x^3 + 1$
   (b) $x^2 + 6x + 9$
   (c) $3x^2 - 13x - 10$
   (d) $1 - 16x^2$
3. $x^2 + 4x + 18 + \dfrac{79}{x - 4}$

EXERCISES 0.5

For each polynomial in Problems 1–4, (a) give the degree of the polynomial, (b) give the coefficient (numerical) of the highest-degree term, (c) give the constant term, and (d) decide whether it is a polynomial of one or several variables.

1. $10 - 3x - x^2$
2. $5x^4 - 2x^9 + 7$
3. $7x^2y - 14xy^3z$
4. $2x^5 + 7x^2y^3 - 5y^6$

The expressions in Problems 5 and 6 are polynomials with the form $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, where n is a positive integer. Complete the following.

5. For $2x^5 - 3x^2 - 5$,
   (a) $2 = a_?$
   (b) $a_3 = ?$
   (c) $-3 = a_?$
   (d) $a_0 = ?$
6. For $5x^3 - 4x - 17$,
   (a) $a_3 = ?$
   (b) $a_1 = ?$
   (c) $a_2 = ?$
   (d) $-17 = a_?$

In Problems 7–12, evaluate each algebraic expression at the indicated values of the variables.

7. $4x - x^2$ at $x = -2$
8. $10 - 6(4 - x)^2$ at $x = -1$
9. $10xy - 4(x - y)^2$ at $x = 5$ and $y = -2$
10. $3x^2 - 4y^2 - 2xy$ at $x = 3$ and $y = -4$
11. $\dfrac{2x - y}{x^2 - 2y}$ at $x = -5$ and $y = -3$
12. $\dfrac{16y}{1 - y}$ at $y = -3$
13. Evaluate $1.98T - 1.09(1 - H)(T - 58) - 56.8$ when $T = 74.7$ and $H = 0.80$.
14. Evaluate $R\left[\dfrac{0.083i}{1 - (1 + 0.083i)^{-n}}\right]$ when $R = 100{,}000$, $i = 0.07$, and $n = 360$.

In Problems 15–22, simplify by combining like terms.

15. $(16pq - 7p^2) + (5pq + 5p^2)$
16. $(3x^3 + 4x^2y^2) + (3x^2y^2 - 7x^3)$
17. $(4m^2 - 3n^2 + 5) - (3m^2 + 4n^2 + 8)$
18. $(4rs - 2r^2s - 11rs^2) - (11rs^2 - 2rs + 4r^2s)$
19. $-[8 - 4(q + 5) + q]$
20. $x^3 + [3x - (x^3 - 3x)]$
21. $x^2 - [x - (x^2 - 1) + 1 - (1 - x^2)] + x$
22. $y^3 - [y^2 - (y^3 + y^2)] - [y^3 + (1 - y^2)]$

In Problems 23–58, perform the indicated operations and simplify.

23. $(5x^3)(7x^2)$
24. $(-3x^2y)(2xy^3)(4x^2y^2)$
25. $(39r^3s^2) \div (13r^2s)$
26. $(-15m^3n) \div (5mn^4)$
27. $ax^2(2x^2 + ax + ab)$
28. $-3(3 - x^2)$
29. $(3y + 4)(2y - 3)$
30. $(4x - 1)(x - 3)$
31. $6(1 - 2x^2)(2 - x^2)$
32. $2(x^3 + 3)(2x^3 - 5)$
33. $(4x + 3)^2$
34. $(2y + 5)^2$
35. $(0.1 - 4x)(0.1 + 4x)$
36. $(x^3y^3 - 0.3)^2$
37. $9(2x + 1)(2x - 1)$
38. $3(5y + 2)(5y - 2)$
39. $\left(x^2 - \dfrac{1}{2}\right)^2$
40. $\left(\dfrac{2}{3} + x\right)\left(\dfrac{2}{3} - x\right)$
41. $(0.1x - 2)(x + 0.05)$
42. $(6.2x + 4.1)(6.2x - 4.1)$
43. $(x - 2)(x^2 + 2x + 4)$
44. $(a + b)(a^2 - ab + b^2)$
45. $(x^3 + 5x)(x^5 - 2x^3 + 5)$
46. $(x^3 - 1)(x^7 - 2x^4 - 5x^2 + 5)$
47. $(18m^2n + 6m^3n + 12m^4n^2) \div (6m^2n)$
48. $(16x^2 + 4xy^2 + 8x) \div (4xy)$
49. $(24x^8y^4 + 15x^5y - 6x^7y) \div (9x^5y^2)$
50. $(27x^2y^2 - 18xy + 9xy^2) \div (6xy)$
51. $(x + 1)^3$
52. $(x - 3)^3$
53. $(2x - 3)^3$
54. $(3x + 4)^3$
55. $(x^3 + x - 1) \div (x + 2)$
56. $(x^5 + 5x - 7) \div (x + 1)$
57. $(x^4 + 3x^3 - x + 1) \div (x^2 + 1)$
58. $(x^3 + 5x^2 - 6) \div (x^2 - 2)$

In Problems 59 and 60, simplify each expression.

59. (a) $(3x - 2)^2 - 3x - 2(3x - 2) + 5$
    (b) $(3x - 2)^2 - (3x - 2)(3x - 2) + 5$
60. (a) $(2x - 3)(3x + 2) - (5x - 2)(x - 3)$
    (b) $2x - 3(3x + 2) - 5x - 2(x - 3)$

In Problems 61–68, perform the indicated operations with expressions involving fractional exponents and radicals and then simplify.

61. $x^{1/2}(x^{1/2} + 2x^{3/2})$
62. $x^{-2/3}(x^{5/3} - x^{-1/3})$
63. $(x^{1/2} + 1)(x^{1/2} - 2)$
64. $(x^{1/3} - x^{1/2})(4x^{2/3} - 3x^{3/2})$
65. $(\sqrt{x} + 3)(\sqrt{x} - 3)$
66. $(x^{1/5} + x^{1/2})(x^{1/5} - x^{1/2})$
67. $(2x + 1)^{1/2}[(2x + 1)^{3/2} - (2x + 1)^{-1/2}]$
68. $(4x - 3)^{-5/3}[(4x - 3)^{8/3} + 3(4x - 3)^{5/3}]$

APPLICATIONS

69. ***Revenue*** A company sells its product for \$55 per unit. Write an expression for the amount of money received (revenue) from the sale of x units of the product.
70. ***Profit*** Suppose a company's revenue R (in dollars) from the sale of x units of its product is given by

$$R = 215x$$

Suppose further that the total costs C (in dollars) of producing those x units is given by

$$C = 65x + 15{,}000$$

(a) If profit is revenue minus cost, find an expression for the profit from the production and sale of x units.
(b) Find the profit received if 1000 units are sold.

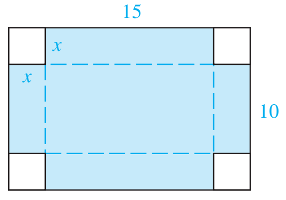
71. ***Rental*** A rental truck costs \$49.95 for a day plus 49¢ per mile.
(a) If x is the number of miles driven, write an expression for the total cost of renting the truck for a day.
(b) Find the total cost of the rental if it was driven 132 miles.

72. ***Cell phones*** Cell Pro makes cell phones and has weekly costs of \$1500 for rent, utilities, and equipment plus labor and material costs of \$18.50 for each phone it makes.
(a) If x represents the number of phones produced and sold, write an expression for Cell Pro's weekly total cost.
(b) If Cell Pro sells the phones to dealers for \$45.50 each, write an expression for the weekly total revenue for the phones.
(c) Cell Pro's weekly profit is the total revenue minus the total cost. Write an expression for Cell Pro's weekly profit.

73. ***Investments*** Suppose you have \$4000 to invest and you invest x dollars at 10% and the remainder at 8%. Write expressions in x that represent the
(a) amount invested at 8%,
(b) interest earned on the x dollars at 10%,
(c) interest earned on the money invested at 8%,
(d) total interest earned.

74. ***Medications*** Suppose a nurse needs 10 cc (cubic centimeters) of a 15.5% solution (that is, a solution that is 15.5% ingredient) of a certain medication, which must be obtained by mixing x cc of a 20% solution and y cc of a 5% solution. Write expressions involving x for
(a) y, the amount of 5% solution,
(b) the amount of ingredient in the x cc of 20% solution,
(c) the amount of ingredient in the 5% solution,
(d) the total amount of ingredient in the mixture.

75. ***Package design*** The volume of a rectangular box is given by $V =$ (length)(width)(height). If a rectangular piece of cardboard that is 10 in. by 15 in. has a square with sides of length x cut from each corner (see the figure) and if the sides are folded up along the dotted lines to form a box, what expression of x represents the volume?



SECTION 0.6 Factoring

Common Factors

We can factor monomial factors out of a polynomial by using the Distributive Law in reverse; $ab + ac = a(b + c)$ is an example showing that a is a monomial factor of the polynomial $ab + ac$. But it is also a statement of the Distributive Law (with the sides of the equation interchanged). To factor out a monomial factor, it must be a factor of each term of the polynomial, so it is frequently called a **common monomial factor.**

EXAMPLE 1 Monomial Factor

Factor $-3x^2t - 3x + 9xt^2$.

Solution

1. We can factor out $3x$ as follows.

$$-3x^2t - 3x + 9xt^2 = 3x\cdot(-xt) + 3x\cdot(-1) + 3x\cdot 3t^2 = 3x(-xt - 1 + 3t^2)$$

2. Or we can factor out $-3x$ (factoring out the negative will make the first term of the polynomial positive) and obtain

$$-3x^2t - 3x + 9xt^2 = -3x(xt + 1 - 3t^2)$$

We can use the Distributive Law to factor out common factors that are not monomials. For example, we can factor $(a + b)$ out of the polynomial $2x(a + b) - 3y(a + b)$ and get $(a + b)(2x - 3y)$. The following example demonstrates the **factoring by grouping** technique.

EXAMPLE 2 Factoring by Grouping

Factor $5x - 5y + bx - by$.

Solution

We can factor this polynomial using grouping. The grouping is done so that common factors (frequently binomial factors) can be removed. We see that we can factor 5 from the first two terms and b from the last two, which gives

$$5(x - y) + b(x - y)$$

This gives us two terms with the common factor $x - y$, so we get

$$(x - y)(5 + b)$$

Factoring Trinomials

We can use the formula for multiplying two binomials to factor certain trinomials. The formula

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

can be used to factor trinomials such as $x^2 - 7x + 6$.

EXAMPLE 3 Factoring a Trinomial

Factor $x^2 - 7x + 6$.

Solution

If this trinomial can be factored into an expression of the form

$$(x + a)(x + b)$$

then we need to find a and b such that

$$x^2 - 7x + 6 = x^2 + (a + b)x + ab$$

That is, we need to find a and b such that $a + b = -7$ and $ab = 6$. The two numbers whose sum is -7 and whose product is 6 are -1 and -6. Thus

$$x^2 - 7x + 6 = (x - 1)(x - 6)$$

A similar method can be used to factor trinomials such as $9x^2 - 31x + 12$. However, finding the proper factors for this type of trinomial may involve a fair amount of trial and error because we must find factors a, b, c, and d such that

$$(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd$$

When a second-degree trinomial can be factored but many possible factors need to be tested to find the correct factorization, an alternative method that uses factoring by grouping can be used. The procedure for this method of factoring second-degree trinomials such as $9x^2 - 31x + 12$ follows.

FACTORING A TRINOMIAL

Procedure	Example
To factor a trinomial into the product of its binomial factors:	Factor $9x^2 - 31x + 12$.
1. Form the product of the second-degree term and the constant term.	1. $9x^2 \cdot 12 = 108x^2$
2. Determine whether any two factors of the product of Step 1 will sum to the middle term of the trinomial. (If the answer is no, the trinomial will not factor into two binomials.)	2. The factors $-27x$ and $-4x$ give a sum of $-31x$.
3. Use the sum of these two factors to replace the middle term of the trinomial.	3. $9x^2 - 31x + 12 = 9x^2 - 27x - 4x + 12$
4. Factor this four-term expression by grouping.	4. $9x^2 - 31x + 12 = (9x^2 - 27x) + (-4x + 12)$ $= 9x(x - 3) - 4(x - 3)$ $= (x - 3)(9x - 4)$

In the example just completed, note that writing the middle term $(-31x)$ as $-4x - 27x$ rather than $-27x - 4x$ (as we did) will also result in the correct factorization. (Try it.)

EXAMPLE 4 Factoring a Trinomial

Factor $9x^2 - 9x - 10$.

Solution

The product of the second-degree term and the constant is $-90x^2$. Factors of $-90x^2$ that sum to $-9x$ are $-15x$ and $6x$. Thus

$$\begin{aligned} 9x^2 - 9x - 10 &= 9x^2 - 15x + 6x - 10 \\ &= (9x^2 - 15x) + (6x - 10) \\ &= 3x(3x - 5) + 2(3x - 5) = (3x - 5)(3x + 2) \end{aligned}$$

We can check this factorization by multiplying.

$$\begin{aligned} (3x - 5)(3x + 2) &= 9x^2 + 6x - 15x - 10 \\ &= 9x^2 - 9x - 10 \end{aligned}$$

Some special products that make factoring easier are as follows.

Special Factorizations

The perfect-square trinomials:

$$\begin{aligned} x^2 + 2ax + a^2 &= (x + a)^2 \\ x^2 - 2ax + a^2 &= (x - a)^2 \end{aligned}$$

The difference of two squares:

$$x^2 - a^2 = (x + a)(x - a)$$

EXAMPLE 5 Special Factorizations

(a) Factor $25x^2 - 36y^2$.
(b) Factor $4x^2 + 12x + 9$.

Solution

(a) The binomial $25x^2 - 36y^2$ is the difference of two squares, so we get

$$25x^2 - 36y^2 = (5x)^2 - (6y)^2 = (5x - 6y)(5x + 6y)$$

These two factors are called binomial **conjugates** because they differ in only one sign.

(b) Although we can use the technique we learned to factor trinomials, the factors come quickly if we recognize that this trinomial is a perfect square. It has two square terms, and the remaining term $(12x)$ is twice the product of the square roots of the squares $4x^2$ and 9 $(12x = 2 \cdot 2x \cdot 3)$. Thus

$$4x^2 + 12x + 9 = (2x + 3)^2$$

Most of the polynomials we factored were second-degree polynomials, or **quadratic polynomials.** Some polynomials that are not quadratic are in a form that can be factored in the same manner as quadratics. For example, the polynomial $x^4 + 4x^2 + 4$ can be written as $a^2 + 4a + 4$, where $a = x^2$.

EXAMPLE 6 Polynomials in Quadratic Form

Factor (a) $x^4 + 4x^2 + 4$ and (b) $x^4 - 16$.

Solution

(a) The trinomial is in the form of a perfect square, so letting $a = x^2$ will give us

$$x^4 + 4x^2 + 4 = a^2 + 4a + 4 = (a + 2)^2$$

Thus

$$x^4 + 4x^2 + 4 = (x^2 + 2)^2$$

(b) The binomial $x^4 - 16$ can be treated as the difference of two squares, $(x^2)^2 - 4^2$, so

$$x^4 - 16 = (x^2 - 4)(x^2 + 4)$$

But $x^2 - 4$ can be factored into $(x - 2)(x + 2)$, so

$$x^4 - 16 = (x - 2)(x + 2)(x^2 + 4)$$

CHECKPOINT

1. Factor the following.
   (a) $8x^3 - 12x$ (b) $3x(x^2 + 5) - 5(x^2 + 5)$ (c) $x^2 - 10x - 24$
   (d) $x^2 - 5x + 6$ (e) $4x^2 - 20x + 25$ (f) $100 - 49x^2$
2. Consider $10x^2 - 17x - 20$ and observe that $(10x^2)(-20) = -200x^2$.
   (a) Find two expressions whose product is $-200x^2$ and whose sum is $-17x$.
   (b) Replace $-17x$ in $10x^2 - 17x - 20$ with the two expressions in (a).
   (c) Factor (b) by grouping.
3. True or false:
   (a) $4x^2 + 9 = (2x + 3)^2$ (b) $x^2 + x - 12 = (x - 4)(x + 3)$
   (c) $5x^5 - 20x^3 = 5x^3(x^2 - 4) = 5x^3(x + 2)(x - 2)$

A polynomial is said to be factored completely if all possible factorizations have been completed. For example, $(2x - 4)(x + 3)$ is not factored completely because a 2 can still be factored out of $2x - 4$. The following guidelines are used to factor polynomials completely.

Guidelines for Factoring Completely

Look for: Common monomial factors first
Then for: Difference of two squares (if the expression is a binomial)
Then for: Trinomial squares
Then for: Other methods of factoring trinomials
Then for: Factoring by grouping (for 4-term polynomials)

EXAMPLE 7 Factoring Completely

Factor completely (a) $12x^2 - 36x + 27$ and (b) $16x^2 - 64y^2$.

Solution

(a) $12x^2 - 36x + 27 = 3(4x^2 - 12x + 9)$ Monomial
$= 3(2x - 3)^2$ Perfect Square

(b) $16x^2 - 64y^2 = 16(x^2 - 4y^2)$
$= 16(x + 2y)(x - 2y)$

Factoring the difference of two squares immediately would give $(4x + 8y)(4x - 8y)$, which is not factored completely (because we still could factor 4 from $4x + 8y$ and 4 from $4x - 8y$).

CHECKPOINT ANSWERS

1. (a) $4x(2x^2 - 3)$
   (b) $(x^2 + 5)(3x - 5)$
   (c) $(x - 12)(x + 2)$
   (d) $(x - 3)(x - 2)$
   (e) $(2x - 5)^2$
   (f) $(10 + 7x)(10 - 7x)$
2. (a) $(-25x)(+8x) = -200x^2$ and $-25x + 8x = -17x$
   (b) $10x^2 - 25x + 8x - 20$
   (c) $(2x - 5)(5x + 4)$
3. (a) False; $4x^2 + 9$ cannot be factored. In fact, sums of squares cannot be factored.
   (b) False; $(x + 4)(x - 3)$
   (c) True

EXERCISES 0.6

In Problems 1 and 2, factor out the common monomial factor.

1. (a) $9ab - 12a^2b + 18b^2$ (b) $4x^2 + 8xy^2 + 2xy^3$
2. (a) $8a^2b - 160x + 4bx^2$ (b) $12y^3z + 4yz^2 - 8y^2z^3$

In Problems 3–6, factor by grouping.

3. $7x^3 - 14x^2 + 2x - 4$
4. $5y - 20 - x^2y + 4x^2$
5. $6x - 6m + xy - my$
6. $x^3 - x^2 - 5x + 5$

Factor each expression in Problems 7–18 as a product of binomials.

7. $x^2 + 8x + 12$
8. $x^2 - 2x - 8$
9. $x^2 - 15x - 16$
10. $x^2 - 21x + 20$
11. $7x^2 - 10x - 8$
12. $12x^2 + 11x + 2$
13. $x^2 - 10x + 25$
14. $4y^2 + 12y + 9$
15. $49a^2 - 144b^2$
16. $16x^2 - 25y^2$
17. (a) $9x^2 + 21x - 8$ (b) $9x^2 + 22x + 8$
18. (a) $10x^2 - 99x - 63$ (b) $10x^2 - 27x - 63$
    (c) $10x^2 + 61x - 63$ (d) $10x^2 + 9x - 63$

In Problems 19–44, factor completely.

19. $4x^2 - x$
20. $2x^5 + 18x^3$
21. $x^3 + 4x^2 - 5x - 20$
22. $x^3 - 2x^2 - 3x + 6$
23. $x^2 - x - 6$
24. $x^2 + 6x + 8$
25. $2x^2 - 8x - 42$
26. $3x^2 - 21x + 36$
27. $2x^3 - 8x^2 + 8x$
28. $x^3 + 16x^2 + 64x$
29. $2x^2 + x - 6$
30. $2x^2 + 13x + 6$
31. $3x^2 + 3x - 36$
32. $4x^2 - 8x - 60$
33. $2x^3 - 8x$
34. $16z^2 - 81w^2$
35. $10x^2 + 19x + 6$
36. $6x^2 + 67x - 35$